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On Power of Singular Cardinals

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Abstract Using elementary methods we find bounds for the function $2^{\aleph_\alpha}$ for $\aleph_\alpha = \alpha$. Using only ZFC without additional assumptions, when e.g., $\aleph_\alpha$ is strong limit of uncountable confinality:

(1) If there is no weakly inaccessible below $\aleph_\alpha$, then there is no such cardinal below $2^{\aleph_\alpha}$.
(2) If $\aleph_\alpha$ is the first cardinal such that $\lambda = \aleph_\lambda$ with $cf\lambda = \aleph_1$, then $2^{\aleph_\alpha} < \kappa$ when κ is the first cardinal such that $\kappa = \aleph_\kappa$ with cofinality $(2^{2^{\aleph_1}})^+$.

We shall also reprove some of Galvin and Hajnal's results. We do not require any knowledge of earlier results on the subject.

Introduction We shall deal with the following problem: Given a cardinal λ, what are the possible values of 2^λ? More exactly, given $\aleph_\alpha$, our task is to find an ordinal $\alpha(*)$ as small as possible which will satisfy $\aleph_{\alpha(*)} \geq 2^{\aleph_\alpha}$.

Let us write some basic facts concerning the power operation:

(0) $\alpha < \beta \Rightarrow 2^{\aleph_\alpha} \leq 2^{\aleph_\beta}$.
(1) For every α $2^{\aleph_\alpha} > \aleph_\alpha$ (Cantor's theorem).

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So only Sections 7, 8, and the last part of the Introduction were written by the author: the author gratefully thanks Avraham and Grossberg for their help.

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(2) For every α $cf(2^{\aleph_\alpha}) > \aleph_\alpha$ (follows from Zermelo Konig's).
(3) If $\aleph_\alpha$ singular, and there exist $\gamma < \alpha$ such that for every $\gamma \leq \beta < \alpha$ $2^{\aleph_\beta} = 2^{\aleph_\gamma}$ holds, then $2^{\aleph_\gamma} = 2^{\aleph_\alpha}$ (Bukovski-Hechler).

P. Cohen proved that on $2^{\aleph_0}$ there are no more restrictions.

Easton generalized Cohen's theorem and investigated the possible functions for $2^{\aleph_\alpha}$ when $\aleph_\alpha$ is regular. He proved that any function satisfying (0)–(2) is possible. Therefore the problem we are left with in this connection is what is the situation when $\aleph_\alpha$ singular? Are there any bounds to the size of $2^{\aleph_\alpha}$?

We shall deal here with the problem in ZFC without any additional assumptions and will find more bounds. We are not the first to face this problem—many people have worked on it including Scott, Solovay, Silver [11], Galvin and Hajnal [4], Magidor [8]–[10], Baumgartner and Prikry [1], Jech and Prikry [6], Devlin and Jensen [2], Dodd and Jensen [3], and Shelah [13].

We shall present our results independently of the above. We begin by presenting an elementary proof to our results.

As preliminary knowledge we assume less than the first 60 pages from Jech's book [5] or the basic part from Levi's basic book [7].

We shall reprove some of the results of Galvin and Hajnal [4]. In order to explain the motivation for our ideas we shall refer to their paper.

A disadvantage in the above-mentioned works is that they could not deal with cardinals of the form $\alpha = \aleph_\alpha$ in ZFC (without additional assumptions). From the results of Jensen, if $\aleph_{\omega_1}$ is a strong limit and $2^{\aleph_{\omega_1}} > \aleph_{\omega_1+1}$, then some strengthenings of ZFC are consistent. Moreover they have a natural model (class with the usual $\in$ relation). The results of [11] and [4] gave us this impression.

Smallness Thesis *When you have $\aleph_\alpha$ singular, $cf\alpha > \aleph_0$, and a strong limit is "small", then also $2^{\aleph_\alpha}$ is "small".*

We shall prove here some examples of this smallness thesis also for cardinals satisfying $\aleph_\alpha = \alpha$.

In Theorem 6.2 we prove that if below $\aleph_\alpha$ there is no weakly inaccessible cardinal, then such a cardinal also does not exist below $2^{\aleph_\alpha}$ (we assume $\aleph_\alpha$ singular, strong limit, $cf(\aleph_\alpha) > \aleph_0$, and for simplicity only assume that $cf\aleph_\alpha = \aleph_1$).

In Theorem 6.6 we prove: If $\lambda = \aleph_\lambda$ when λ is the first cardinal such that $cf\lambda = \aleph_1$ and $\aleph_\lambda = \lambda$, then $2^\lambda < \kappa$ when κ is the first cardinal such that $\kappa = \aleph_\kappa$ and $cf\kappa = (2^{2^{\aleph_1}})^+$.

In Section 1 we define some basic definitions and quote two theorems which we shall use in what follows.

In Section 2 we prove a theorem from [4]: For $\aleph_\alpha$ satisfying $cf\ \aleph_\alpha = \aleph_1$ and $\beta < \alpha \Rightarrow \aleph_\beta^{\aleph_1} < \aleph_\alpha$, the inequality $\aleph_\alpha^{\aleph_1} < \aleph_{(|\alpha|^{\aleph_1})^+}$ holds. The presentation largely follows [13].

Because our purpose is to prepare theorems for the rest by generalizing [4] almost disjoint transversals, our substitute is the cardinal $T_D(f)$.

We define a norm of ordinal functions into the ordinals with respect to a filter D and denote it by $\|f\|_D$. By $\|\alpha\|_D$ we denote the norm of the constant function whose value is α. $\|f\|_D$ is the least ordinal such that for all $g <_D f$, $\|g\|_D < \|f\|_D$ holds (the definition of $<_D$ is in Section 1.3. For the reason this definition is possible, see Lemma 2.9—we shall work with $\aleph_1$ complete filter D).

Galvin and Hajnal proved for the abovementioned function $\alpha(*)$ that $\alpha(*) \leq \|\alpha\|_D$ when D is the filter generated by the closed unbounded sets of $\aleph_1$ (we work here with cardinals of cofinality $\aleph_1$, therefore D will be over $\aleph_1$).

$\|f\|_D$ can be defined in an equivalent form by a game $G_m(D, f, \alpha)$ between two players I and II, which we define as follows: Player I begins and in his first step chooses a function $f_1 <_D f$. Player II in his first step chooses an ordinal $\alpha_1 < \alpha$. In general, player I in his n'th step chooses $f_n <_D f_{n-1}$ and player II replies by choosing $\alpha_n < \alpha_{n-1}$. Because $\alpha > \alpha_1 > \alpha_2 > \ldots$ is a decending sequence of ordinals it must be finite. So there is a stage when player II cannot continue. Player I cannot continue if the function from the previous step satisfies $\{i: f_{n-1}(i) = 0\}$ belongs to D or even is $\neq \emptyset$ *mod* D. Player I wins the game if player II cannot reply, and player II wins when player I cannot continue in his turn.

What is the connection between the above norm and the game $G_m(D, f, \alpha)$? It is easy to see that player I wins in the game $G_m(D, f, \alpha)$ if and only if $\|f\|_D \geq \alpha$.

Our method to improve Galvin-Hajnal's result is generalization of $\|f\|_D$; for this analysis we use games of types similar to the abovementioned game.

In Section 3 we define a game $G(D, f, \alpha)$ which is a little harder for player II than the previous game $G_m(D, f, \alpha)$, and later define the game $G(D, f)$ which is easier for player II than $G(D, f, \alpha)$.

We prefer player II because he provides us a bound on the norm. In Section 3 we shall see some basic properties of these games.

A variant of these games was studied (independently of our study) in an unpublished work of M. Magidor and R. Solovay. They used these games to prove the same bound Galvin and Hajnal found, before Galvin and Hajnal had the result. The disadvantage was that they used existence of Ramsey cardinals.

In Section 4 a filter D over $\aleph_1$ will be found such that player II has a winning strategy in every game $G(D, f)$ for every f: $\aleph_1 \rightarrow Ord$.

Our method is to use an additional assumption to ZFC (it is formulated at (4.1)) in order to: (i) prove the existence of a filter E over a set I; (ii) translate this filter to a filter D_E over $\aleph_1$; and (iii) show that D_E has the property we want from D. In the next stage we use Dodd and Jensen's result to show that if $2^{\aleph_\alpha}$ violates the continuum hypothesis, then Assumption 4.1 holds.

In Section 5 we study different notions of rank-functions and their interconnections, i.e., the function $\|f\|_D$ and the functions which we get by taking the first ordinal α such that player II wins the game $G(D, f, \alpha)$, and connect this to the cardinality of D—almost disjoint transversals from Section 2.

We also prove Theorem 5.5 which will help us to prove our theorems from Section 6. This theorem is similar in form and role to Theorem 2.10.

In Section 6 we prove the two theorems already reviewed and, to make our paper really independent from [4], we reprove one of their lemmas (6.5).

The results were announced in [16], where Theorem 6.6 was stated explicitly, and it was claimed that: (*) the method was strong enough to prove any instance of the thesis "let $\aleph_\alpha$ be strong limit of cofinality $> \aleph_0$, if $\aleph_\alpha$ is 'small' so is $2^{\aleph_\alpha}$". The author checked several cases: smaller than first inaccessible, first Mahlo; before $\aleph_\alpha$ there are few inaccessibles; $\aleph_\alpha$ is small in the class of fixed points of $\{\aleph_\alpha: \alpha\}$ (e.g., the first $\aleph_\alpha$, $\aleph_\alpha = \alpha$, $cf\alpha = \aleph_1$), similarly for fix points of fix points. However, the above claim (*) is wrong, as discovered by Hajnal.

For a class C let $C' = \{\alpha \in C$: the order type of $C \cap \alpha$ is $\alpha\}$. Let C_0 be the class of infinite cardinals, $C_{n+1} = C'_n$.

Now Hajnal, using partial information on the proof, reconstructed the proof for C_0, C_1, carried the induction for the C_n, and found out that it does not work for $C_\omega = \bigcap_{n<\omega} C_n$.

Hajnal's Question Suppose λ is the ω_1-th member of C_ω, is strong limit. Is 2^λ smaller than the $(2^{2^{\aleph_1}})^+$-th member of C_ω? Or can we have any bound better than the first inaccessible?[1]

We would be able to answer positively if we can prove that for every λ, not only for some D but for a majority of D (majority for an $\aleph_1$-complete filter on the family of suitable filters), Lemma 5.5 holds.

Other natural questions are: In Theorem 6.6 can we replace $(2^{2^{\aleph_1}})^+$ by $(2^{\aleph_1})^+$? Can we build models of ZFC in which the first $\lambda = \aleph_\lambda$ of cofinality $\aleph_1$ satisfies $2^{\aleph_1} < \lambda < 2^{2^{\aleph_1}}$, $(\forall \mu < \lambda)\ \mu^{\aleph_1} < \lambda$, and $\lambda^{\aleph_1}$ is bigger than the first inaccessible cardinal?

1 Preliminaries

1.1 Definitions and notations about filters Let D be a filter over I (see [5] or [7] for definition of filter), so $D \subseteq \mathcal{P}(I)$: D is a subset of the power set of I. For $B \subseteq I$ we say B is of *measure zero* and write $B = \emptyset$ *mod* D iff $I - B \in D$. B is of positive measure or $B \neq 0$ *mod* D iff $B \cap A \neq 0$ for all $A \in D$. And if $\{i \in I | Q(i)\} \in D$ where Q is some property and $Q(i)$ means Q holds for i, then we say: for almost all i, $Q(i)$ holds. A filter D is said to be *τ-complete* (where τ is an infinite cardinal) iff $\bigcap_{\alpha<\mu} A_\alpha \in D$ whenever $\mu < \tau$ and $A_\alpha \in D$ for all $\alpha < \mu$. In case $\tau = \aleph_1$ we say D is countably complete rather than $\aleph_1$-complete. For a set of ordinals B and a function f defined on B, f is *regressive* iff $f(\alpha) < \alpha$ for all nonzero $\alpha \in B$. In case $I = \lambda$, where λ is a cardinal, we say a filter D over λ is *normal* iff whenever $B \neq 0$ *mod* D and f is a regressive function on B, then for some $B' \subseteq B$, $B' \neq 0$ *mod* D and f is constant on B' (see 1.6).

Let $E \subseteq \mathcal{P}(I)$ and assume the intersection of any finite subset of E is nonvoid. Then *the filter generated by E* is the collection $\{B \subseteq I$ for some $X_1, \ldots, X_n$ in E $B \supseteq X_1 \cap \ldots \cap X_n\}$. (It is understood that $n < \omega$ and $\{X_1, \ldots, X_n\}$ is a finite subset of E.) Now for $B \subseteq I$, $B \neq 0$ *mod* D, $D + B$ denotes the filter generated by $\{B \cap X | X \in D\}$; i.e., $D + B$ is the filter generated by the set $D \cup \{B\}$.

We ask the reader to prove that the intersection of finitely many sets of the form $B \cap X$ where $X \in D$ is again of that form, hence that $D + B$ is a proper filter. The following lemma is also left to the reader.

1.2 Lemma
A. If D is a normal filter then $D + B$ is normal too.
B. If $A \neq \emptyset$ mod D and $B \neq \emptyset$ mod $D + A$ then $A \cap B \neq \emptyset$ mod D and $(D + A) + B = D + A \cap B$.
C. If $B - B' = \emptyset$ mod D (i.e., $B \subseteq B'$ "almost" holds) then $D + B' \subseteq D + B$.

1.3 Reduced products The Cartesian product $\Pi_{i\in I} A_i$ is the set of functions f on I such that $f(i) \in A_i$ for all $i \in I$. An equivalence relation is defined on the Cartesian product by (D is a filter over I) $f \equiv_D g$ iff $\{i \in I | f(i) = g(i)\} \in D$, $\Pi_{i\in I} A_i/D$ is the set of equivalence classes thus obtained.[2] If $\langle (A_i, \leq_i) | i \in I\rangle$ are partially ordered sets (posets for short) we can define a preorder (called also quasiorder) $\leq_D$ on the Cartesian product by $f \leq_D g$ iff $\{i \in I | f(i) \leq_i g(i)\} \in D$. $\leq_D$ is a preorder in the sense that $f \leq_D g$ and $g \leq_D f$ imply $f \equiv_D g$, but not necessarily $f = g$. Transitivity and reflexivity are easily checked. $\leq_D$ is a partial preorder; it might well be that none of $f \leq_D g$ or $g \leq_D f$ holds. In case $A_i = Ord$ (the class of ordinals) ${}^I Ord$ is the class of all functions $f: I \to Ord$. Taking the natural well order of the ordinals, we get a partial preorder $\leq_D$ defined above on ${}^I Ord$. An equivalence class now is not a set and this is why we prefer to deal with (partial) preorder rather than with a (partial) order defined on the equivalence classes. This means that we cannot speak about the least upper bound for a set (even if it exists, it is not necessarily unique). (Recall that $a \in P$ is a least upper bound of $A \subseteq P$ where $\leq$ is a partial preorder of P iff: (1) $e \leq a$ for all $e \in A$ and (2) if a' also satisfies $e \leq a'$ for all $e \in A$ then $a \leq a'$.)

We also define $f <_D g$ iff $\{i \in I | f(i) \leq_i g(i)$ and $f(i) \neq g(i)\} \in D$. Now $<_D$ is an irreflexive partial order. Remark that $f \leq_D g$ and $\neg(f \equiv_D g)$ do not imply $f <_D g$. Define $f \neq_D g$ iff $\{I \in I | f(i) \neq g(i)\} \in D$. Then $f \leq_D g$ and $f \neq_D g$ imply $f <_D g$. But if D will be an ultrafilter then $\neg(f \equiv_D g)$ and $f \neq_D g$ are equivalent.

1.4 We will use the Erdos-Rado Partition Theorem which says that if F is a function defined on $[(2^\lambda)^+]^2$ (the class of two elements subsets of the cardinal $(2^\lambda)^+$) and which takes values in λ, then there is $H \subseteq (2^\lambda)^+$, of cardinality λ^+ which is homogeneous for F, namely $F(\{\alpha, \beta\})$ is the same for any distinct α, $\beta \in H$. (See [5], [7], or any other textbook for the full Erdos-Rado theorem and its proofs.)

1.5 We shall also use (only in Section 6) Hajnal's theorem on free sets. We shall need it not only for regular cardinals.

Hajnal's theorem *If λ is regular cardinal and $\mu < \lambda$ a cardinal and f a function such that f's domain is λ and $(\forall\alpha < \lambda)$ $[|f(\alpha)| < \mu]$, then there exists $B \subseteq \lambda$, $|B| = \lambda$ and B is a free subset, i.e., $x \neq y \in B \Rightarrow x \notin f(y)$.*

A proof for this theorem is quoted in [15] in the combinatorial appendix as Theorem 2.8.

1.6 Recall that for an ordinal δ, $cf(\delta)$ is the minimal order type of an unbounded subset of δ. A cardinal λ is regular iff $cf\lambda = \lambda$; otherwise it is called singular. Also $cf\delta$ is always a regular cardinal, any successor cardinal is regular, and λ is an inaccessible cardinal if it is limit regular and $> \aleph_0$.

If $\lambda > \aleph_0$ is regular, by Fodor's Lemma, the family of closed unbounded subsets of λ generates a normal filter.

2 Generalized almost disjoint transversal In this section our purpose is to prepare the ground for the rest of our work. We first give a general description of this section. In Section 2.1 we begin the detailed exposition. In order to demonstrate these preparations, we shall prove the following theorem from Galvin-Hajnal [4]. *If* $\aleph_\alpha$ is singular, $\alpha < \aleph_\alpha$, $cf(\aleph_\alpha) > \aleph_0$ (for simplicity only we shall assume that $cf(\aleph_\alpha) = \aleph_1$, and is strong limit therefore satisfies $\beta < \alpha \Rightarrow \aleph_\beta^{\aleph_1} < \aleph_\alpha$; notice that from our assumption on $\aleph_\alpha$ follows that $2^{\aleph_\alpha} = \aleph_\alpha^{\aleph_1}$) *then* $2^{\aleph_\alpha} < \aleph_{(|\alpha|^{\aleph_1})^+}$.

Let us review the proof of this theorem (Theorem 2.11): Assume that $2^{\aleph_\alpha} \geq \aleph_{(|\alpha|^{\aleph_1})^+}$ and choose a regular cardinal λ such that $2^{\aleph_\alpha} \geq \lambda \geq \aleph_{(|\alpha|^{\aleph_1})^+}$ and $\lambda > 2^{\aleph_1}$. This can be done because if $2^{\aleph_\alpha} \leq 2^{\aleph_1}$ then the conclusion of our theorem is trivial and $2^{\aleph_\alpha} > \aleph_{(|\alpha|^{\aleph_1})^+}$ because $cf(2^{\aleph_\alpha}) > \aleph_\alpha > (|\alpha|^{\aleph_1})^+$, which holds as $\alpha < \aleph_\alpha$ and the hypothesis on $\aleph_\alpha$.

In Definition 2.7 for a filter D over $\aleph_1$, $f \in {}^{\aleph_1}Ord$ we define a cardinal $T_D(f)$—the power of a maximal set of functions less, by $<_D$, than f and distinct on a large set ($\in D$). This is not the definition in the section (Definition 2.7 gives a more general concept) but for our needs here this will be sufficient (by the second claim in the proof of Theorem 2.10).

In Theorem 2.3 we evaluate $T_{D_{ub}}(\aleph_\alpha)$. Here, instead of the function f, we take the constant function from $\aleph_1$ whose value is $\aleph_\alpha$. D_{ub} is the filter generated by the cobounded subsets of $\aleph_1$. We shall get $T_{D_{ub}}(\aleph_\alpha) = \aleph_\alpha^{\aleph_1}$ ($= 2^{\aleph_\alpha}$ when $\aleph_\alpha$ is strong limit).

Now we can apply Theorem 2.10 which for a given $\aleph_1$-complete filter D over $\aleph_1$ and a function $f \in {}^{\aleph_1}Ord$ such that $T_D(f) \geq \lambda$, gives us an $\aleph_1$-complete filter $D' \supseteq D$ (over $\aleph_1$) and an increasing sequence of length λ in ${}^{\aleph_1}Ord/D'$ below f. If we substitute $f = \aleph_\alpha$ (i.e., $f(i) = \aleph_\alpha$ for $i < \omega_1$), $D = D_{ub}$ we shall get an increasing sequence of length λ in the reduced power ${}^{\aleph_1}\aleph_\alpha/D'$; this will be a contradiction to Theorem 2.6 where it will be proved that in the reduced power ${}^{\aleph_1}\aleph_\alpha/D'$ there cannot be increasing sequence of length λ (remember $\lambda \geq \aleph_{(|\alpha|^{\aleph_1})^+}$).

The other theorems are preparations for Theorems 2.6, 2.8, 2.10. Most of the theorems appear in a more general form than we quoted above; instead of a filter over $\aleph_1$ we work with filter over a set I, and in part of the theorems we deal with reduced products of partial preorders instead of reduced power of ordinals.

Now we shall review Theorems 2.6, 2.8, and 2.10. In Theorem 2.6 we prove more than mentioned; we prove existence of an ordinal $\gamma < (|\alpha|^{|I|})^+$ such that in the reduced power ${}^I\aleph_\alpha/D$ (D a filter over I) there is no increasing chain of length $\aleph_\gamma$.

We prove this by contradiction. Assume that for every $\gamma < (|\alpha|^{|I|})^+$ there exists an increasing sequence of length $\aleph_{\gamma+1}$. Define a mapping $\gamma \to f_+^\gamma$ from a set of cardinality $(|\alpha|^{|I|})^+$ to a set of cardinality $|\alpha|^{|I|}$ as follows: For each γ choose an increasing sequence of length $\aleph_{\gamma+1}$ (exist by the assumption) take a supremum f^γ with the following additional property: If $g <_D f^\gamma$ then there exist f_ξ in the sequence such that $g <_D f_\xi$; the existence of such a supremum is promised by Theorem 2.3. Now define $f_+^\gamma(i) = cf(f(i))$ and our aim is to get a contradiction to the assumption by proving that $\gamma \mapsto f_+^\gamma$ is one to one.

For this purpose in Definition 2.4 we introduce cofinality of a function relative to a filter (this is a direct generalization of cofinality of an ordinal), evaluate $cf(f_+^\gamma/D)$ and get $\aleph_{\gamma+1}$; this together with the technical Lemma 2.5 proves that $\gamma \mapsto f_+^\gamma$ one to one.

In the proof of Theorem 2.8 there are no problems; it is a direct computation.

Given an $\aleph_1$-complete filter D over $\aleph_1$, a regular cardinal $\lambda > 2^{\aleph_1}$, and a function $f \in {}^{\aleph_1}Ord$ such that $T_D(f) \geq \lambda$, Theorem 2.10 gives us an $\aleph_1$-complete filter $D' \supseteq D$ over $\aleph_1$ and an increasing sequence $\langle g_i : i < \lambda \rangle$ in ${}^{\aleph_1}Ord/D'$. The proof is divided into three steps:

In step zero choose $f^* \in {}^{\aleph_1}Ord$ such that $f^* \leq_D f$, $T_D(f^*) \geq \lambda$, and is $<_D$-minimal with these properties (possible as $<_D$ well founded by Lemma 9).

In the first step define the filter $D' \supseteq D$ and prove its $\aleph_1$ completeness.

In the second step find a set $H \subseteq {}^{\aleph_1}Ord$ such that $h \in H \Rightarrow h <_{D'} f^*$, $h_1 \neq h_2 \in H \Rightarrow h_1 \neq_{D'} h_2$, and $|H| \geq \lambda$.

In the third step define by induction an increasing (in $<_{D'}$) sequence of length λ of functions which are smaller (in $<_D$) than f^*; for this it suffices by regularity of λ to prove the following:

$$\text{If } g <_D f^* \text{ then } |\{h \in H : \neg(g <_{D'} h)\}| < \lambda.$$

2.2 Theorem *Let D be a filter over λ. Assume that for every $\xi < \lambda$, $(Q_\xi, \leq_\xi)$ is λ^+-well ordered then $(\Pi_{\xi\in\lambda} Q_\xi, \leq_D)$ is $(2^\lambda)^+$-well ordered.*[3]

Remark: A general treatment of τ-well ordered preorders (= quasiorder) can be found in [14].

In what follows λ is any cardinal.

2.2 Theorem *Let D be a filter over λ. Assume that for every $\xi < \lambda$, $(Q_\xi, \leq_\xi)$ is λ^+-well ordered then $(\Pi_{\xi\in\lambda} Q_\xi, \leq_D)$ is $(2^\lambda)^+$-well ordered.*[3]

Proof: Let $\langle f_i | i < (2^\lambda)^+ \rangle$ be a sequence of length $(2^\lambda)^+$ in $\Pi_{\xi\in\lambda} Q_\xi$. We want to find $i < j$ such that $f_i \leq_D f_j$. Actually we will get $i < j$ with even $\{\xi \in \lambda | f_i(\xi) \leq_\xi f_j(\xi)\} = \lambda$. Assume that no such $i < j$ exists. Then for any $i < j < (2^\lambda)^+$ there is $\xi \in \lambda$ such that $\neg(f_i(\xi) \leq_\xi f_j(\xi))$. Define $F(\{i, j\}) = \xi$ to be the minimal such $\xi < \lambda$. By the Erdos-Rado theorem we get $H \subseteq (2^\lambda)^+$ of cardinality λ^+ such that for some ξ_0 whenever $i, j \in H$, $i \neq j \Rightarrow F(\{i, j\}) = \xi_0$, hence $i \in H \wedge j \in H \wedge i < j \Rightarrow \neg(f_i(\xi_0) \leq_{\xi_0} f_j(\xi_0))$. In other words $\langle f_i(\xi_0) | i \in H \rangle$ contradict the assumption that $(Q_{\xi_0}, \leq_{\xi_0})$ is λ^+-well ordered.

The above theorem will be applied in the case where $(Q_\xi, \leq_\xi)$ is an ordinal with its well ordering $\in$. Surely no infinite decreasing sequence of ordinals can be found so that the theorem assumption holds in this case.

2.3 Theorem *Let D be a filter over λ, μ a cardinal whose cofinality at least $(2^\lambda)^+$, $\langle f_i \in {}^\lambda Ord | i < \mu \rangle$ an increasing sequence in $\leq_D$, i.e., $[i < j < \mu \Rightarrow f_i \leq_D f_j]$. Then $\{f_i | i < \mu\}$ has a least upper bound f^+ in $({}^\lambda Ord, \leq_D)$ such that for any $f \in {}^\lambda Ord$ $[f <_D f^+ \Rightarrow f <_D f_i$ for some $i < \mu]$.*

Proof: First, define by induction on α (as long as you can) functions $g_\alpha \in {}^\lambda Ord$ such that:

1. $f_i \leq_D g_\alpha$ for all $i < \mu$.
2. For $\beta < \alpha$ $\neg(g_\beta \leq_D g_\alpha)$.

g_0 can easily be defined by $g_0(\xi) = \bigcup\{f_i(\xi) | i < \mu\}$. By Theorem 2.2 there is no sequence $\langle g_i | i < (2^\lambda)^+ \rangle$ such that $\neg(g_\alpha \leq_D g_\beta)$ holds for $\alpha < \beta < (2^\lambda)^+$, so at some ordinal $\alpha_0 < (2^\lambda)^+$ there is no g_{α_0} satisfying conditions 1 and 2. The construction of the g_α is stopped at α_0, say. Let $A = \{g_\alpha(\xi) | \alpha < \alpha_0, \xi < \lambda\}$, then A is a set of no more than 2^λ ordinals. Because $Range(g_0) \subseteq A$, for any $i < \mu$ and $\xi < \lambda$ $f_i(\xi)$ is bounded by an ordinal in A, so that the following definition makes sense: $f_i^*(\xi) = min\{\beta \in A | f_i(\xi) \leq \beta\}$. So $f_i^* \in {}^\lambda A$ and $f_i \leq_D f_i^*$ of course.

Now $|{}^\lambda A| = |A|^\lambda \leq (2^\lambda)^\lambda = 2^\lambda$, remember $cf\mu \geq (2^\lambda)^+$, so as $\mu = \bigcup_{f^* \in {}^\lambda A} \{i < \mu | f_i^* = f^*\}$ for some $f^* \in A$ the set $\{i < \mu | f_i^* = f^*\}$ is unbounded in μ. We will now prove that this f^* is one that satisfies the requirements of the theorem.

f^* is an upper bound of $\{f_j | j < \mu\}$: For any $j < \mu$ there is $i > j$ such that $f_i^* = f^*$, as $f_j \leq_D f_i$ and $f_i \leq_D f_i^*$ we get $f_j \leq_D f^*$.

f^* is a least upper bound of $\{f_j | j < \mu\}$: If g is also an upper bound of $\{f_j | j < \mu\}$, then at stage α_0, g was also considered and failed. But it can fail only because of requirement 2. So there exists $\beta < \alpha_0$ such that $g_\beta \leq_D g$. Choose $i < \mu$ such that $f^* = f_i^*$, and let $C^a = \{\xi < \lambda : f_i(\xi) \leq g_\beta(\xi)\}$, $C^b = \{\xi < \lambda : g_\beta(\xi) \leq g(\xi)\}$. As we know that $f_i \leq_D g_\beta$ (by g_β's choice) clearly $C^a \in D$. As $g_\beta \leq_D g$, clearly $C^b \in D$. Now by the choice of A, and f_i^*, $[\xi < \lambda, f_i(\xi) \leq g_\beta(\xi) \Rightarrow f_i^*(\xi) \leq g_\beta(\xi)]$, hence for $\xi \in C^a \cap C^b$ $f^*(\xi) = f_i^*(\xi) \leq g_\beta(\xi) \leq g(\xi)$. As $C^a \cap C^b \in D$, clearly $f^* \leq_D g$ as required.

If $f \in {}^\lambda Ord$, $f <_D f^*$ then $(\exists i < \mu)$ $f <_D f^*$: For $i < \mu$ let $C_i = \{\xi < \lambda | f(\xi) < f_i(\xi)\}$. If $C_i \in D$ for some $i < \mu$ then $f <_D f_i$ and the proof is finished. So we assume $C_i \notin D$ (hence $\lambda - C_i \neq 0$ *mod* D) for all $i < \mu$ and seek a contradiction. Now $C_i \subseteq \lambda$, there are 2^λ subsets to λ and $2^\lambda < cf\mu$ hence for some $C^* \subseteq \lambda$ and an unbounded $U \subseteq \mu$, $C_i = C^*$ for all $i \in U$. And $C^* \notin D$ as we said before. Now define $f' \in {}^\lambda Ord$ by

$$f'(\xi) = \begin{cases} f(\xi) & \xi \in \lambda - C^* \\ f^*(\xi) & \xi \in C^* \end{cases}.$$

The contradiction will follow once we show that f' is an upper bound to $\{f_i | i < \mu\}$ and that $f^* \leq_D f'$ does not hold (because then f^* is not a least upper bound). For any $i \in U$ $f_i(\xi) \leq f^*(\xi)$ for almost all $\xi < \lambda$ and $f_i(\xi) \leq f(\xi)$ for all $\xi \in \lambda - C^*$ (by definition of $C_i = C^*$). This shows that $f_i \leq_D f'$ for *any* $i < \mu$ because U is unbounded in μ. But $f^* \leq_D f'$ does not hold, because if it did, then as $f <_D f^*$ we would get $f <_D f'$ — that is, $f(\xi) < f'(\xi)$ for almost all $\xi < \lambda$ — but $f(\xi) = f'(\xi)$ for $\xi \in \lambda - C^*$ and $\lambda - C^*$ is not a measure zero set. Contradiction.

2.3A Remark: Clearly in 2.3 f^+ is unique, i.e., if f' satisfies the conclusion then $f^+ =_D f$. Even a least upper bound, if it exists, is unique.

2.4 Definition For $f \in {}^I Ord$ and a filter D over I, $cf(f/D)$ is the least cardinality κ of a collection $\{f_\xi | \xi < \kappa\}$ with the following properties:

1. $f_\xi <_D f$ for all $\xi < \kappa$
2. If $g <_D f$ then $g \le_D f_\xi$ for some $\xi < \kappa$.

(Such a collection always exists [e.g., $\{g: g <_D f$ and $(\forall \xi \in I) g(\xi) \le f(\xi)\}$], hence one with a minimal cardinality can be found.)

2.4A Remark: (1) If we can find such $\{f_\xi: \xi < cf(f/D)\}$ with $[\zeta < \xi \Rightarrow f_\xi \le_D f_\zeta]$ then the cofinality is a regular cardinal. (See the proof of Theorem 2.6.)

2.5 Lemma *If $f, f' \in {}^I Ord$ and $\{i \in I | cf(f(i)) = cf(f'(i))\} \in D$ then $cf(f/D) = cf(f'/D)$.*

2.5A Remark: *If $\kappa = cf(f/D) > 2^{|I|}$ then κ is a regular cardinal.*

Proof: From the fact that $f_1 \equiv_D f_2 \Rightarrow cf(f_1/D) = cf(f_2/D)$ we conclude that it is possible in the proof to replace the functions f, f' by functions f_1, f_2 which satisfies $f \equiv_D f_1$ and $f' \equiv_D f_2$. Therefore we can assume that $I = \{i: cf(f(i)) = cf(f'(i))\}$. W.l.o.g. $\{i: f(i) \neq 0\} \in D$ (otherwise $cf(f/D) = cf(f'/D) = 0$), hence w.l.o.g. for every $i \in I$, $f(i) > 0$ and $f'(i) > 0$. Recall also that the cofinality of a successor ordinal is 1 and the cofinality of 0 is 0. For each $i \in I$ pick S_i of order type $cf(f(i))$ cofinal in f(i) (i.e, for any $x < f(i)$ there is $y \in S_i$, $x \le y$) and similarly pick $S'_i \subseteq f'(i)$ cofinal and of order type $cf(f'(i))$, let t_i: $S_i \to S'_i$ be an order isomorphism. It suffices, by symmetry, to prove $cf(f/D) \le cf(f'/D)$, and for this it suffices to show that: *whenever* κ is a cardinal with a collection $\{f_\xi | \xi < \kappa\}$ satisfying 1 and 2 of Definition 2.4 for f, *there* is a collection of the same cardinality for f'. Well, given $\{f_\xi | \xi < \kappa\}$ we can assume $f_\xi(i) < f(i)$ for *all* $i \in I$ and $\xi < \kappa$. Now define $\{f'_\xi | \xi < \kappa\}$ as follows: $f'_\xi(i) = t_i(\gamma)$ where $\gamma \in S_i$ is the first ordinal in S_i which is $\ge f_\xi(i)$. Of course $f'_\xi <_D f'$. Now, if $g' <_D f'$ it means that for almost all $i \in I$ $g'(i) < f'(i)$ and then for some $s \in S'_i$ $g'(i) \le s < f'(i)$. It follows that g^* can be found, $g' \le_D g^* <_D f'$ such that $g^*(i) \in S'_i$ for (almost) all $i \in I$. Now define g by $g(i) = t_i^{-1}(f^*(i))$. *Clearly* $g <_D f$, so by property 2 (from Definition 2.4) there is f_ξ with $g \le_D f_\xi$. This implies that $g^* \le_D f'_\xi$, so $g' \le f'_\xi$, as required to show that $\{f'_\xi | \xi < \kappa\}$ has properties 1 and 2 for f'.

For D a filter on I, we have

2.6 Theorem *If α is a cardinal ≥ 2 $({}^I\aleph_\alpha, <_D)$ does not contain an increasing sequence of length $\aleph_{(|\alpha|^{|I|})^+}$. Moreover there is $\gamma < (|\alpha|^{|I|})^+$ such that there is no increasing sequence of cardinality $\aleph_\gamma$.*

Proof: Assume to the contrary that for every ordinal $\gamma < (|\alpha|^{|I|})^+$ an increasing sequence $\langle f_\xi^\gamma | \xi < \aleph_{\gamma+1}\rangle$ of length $\aleph_{\gamma+1}$ exists in ${}^I\aleph_\alpha$. Now $|\gamma| \le \aleph_\gamma$ is true for any ordinal γ, so for $2^{|I|} < \gamma < (|\alpha|^{|I|})^+$, clearly $2^{|I|} < \aleph_{\gamma+1}$ holds. We know $\aleph_{\gamma+1}$ is a regular cardinal, and Theorem 2.3 can be applied to obtain a least upper bound f^γ to $\langle f_\xi^\gamma | \xi < \aleph_{\gamma+1}\rangle$ with the additional property stated in that theorem.

2.6A Claim $cf(f^\gamma/D) = \aleph_{\gamma+1}$.

Proof: Now $cf(f^\gamma/D) \le \aleph_{\gamma+1}$ because the family $\{f^\gamma_\xi | \xi < \aleph_{\gamma+1}\}$ shows it. [$f^\gamma_\xi <_D f^\gamma$ and if $g <_D f^\gamma$ then $g \le_D f^\gamma_\xi$ for some $\xi < \aleph_{\gamma+1}$]. To see that $cf(f^\gamma/D) = \aleph_{\gamma+1}$ we argue as follows: If $\{h_\xi|\xi < \kappa\}$ $\kappa < \aleph_{\gamma+1}$ is a family of functions exemplifying $cf(f^\gamma/D) \le \kappa < \aleph_{\gamma+1}$, then for any $\xi < \aleph_{\gamma+1}$ (as $f^\gamma_\xi <_D f^\gamma$) there is $\xi^* < \kappa$ with $f^\gamma_\xi \le_D h_{\xi^*}$. By regularity of $\aleph_{\gamma+1}$ we find an unbounded $U \subseteq \aleph_{\gamma+1}$ and $\xi_0 < \kappa$ such that for $\xi \in U$ $\xi^* = \xi_0$, i.e., $f^\gamma_\xi \le_D h_{\xi_0}$ for all $\xi \in U$. But then as $\langle f^\gamma_\xi | \xi < \aleph_{\gamma+1}\rangle$ is an increasing sequence, $f^\gamma_\xi \le_D h_{\xi_0}$ for all $\xi < \aleph_{\gamma+1}$, yet $h_{\xi_0} <_D f^\gamma$, contradicting the choice of f^γ as a least upper bound of that sequence.

The claim is proved and we continue the proof of the theorem. The functions f^γ_ξ are in ${}^I\aleph_\alpha$ so that f^γ can certainly be assumed to be in ${}^I(\aleph_\alpha + 1)$. Hence $cf(f^\gamma(i)) \in \{0, 1, \aleph_\beta | \beta \le \alpha\}$. This is why if we define $f^\gamma_+(i) = cf(f^\gamma(i))$ then $f^\gamma_+ \in {}^I\{0, 1, \aleph_\beta|\beta < \alpha\}$. Hence $\{f^\gamma_+ | 2^{|I|} < \gamma < (|\alpha|^{|I|})^+\}$ has cardinality $\le |\alpha + 2|^{|I|} = |\alpha|^{|I|}$. On the other hand, $cf(f^\gamma_+(i)) = cf(f^\gamma(i))$, for $i \in I$ implies $cf(f^\gamma_+/D) = cf(f^\gamma/D) = \aleph_{\gamma+1}$ (by Lemma 2.5 and the claim above). So $\gamma \ne \gamma' \Rightarrow f^\gamma_+ \ne f^{\gamma'}_+$ (they have different cofinalities), i.e., $\gamma \mapsto f^\gamma_+$ is one to one from a set of cardinality $(|\alpha|^{|I|})^+$ to a set of cardinality $\le |\alpha|^{|I|}$. Contradiction.

Notice that if $\alpha = \aleph_\alpha$ this theorem does not give any valuable information because then $|{}^I\aleph_\alpha = |\alpha|^{|I|} \le \aleph_{(|\alpha|^{|I|+})}$.

2.7 Definition For a filter D over I and $f \in {}^IOrd$, define $T_D(f) = sup\{|H| \,|H \subseteq {}^IOrd, [h \in H \Rightarrow h <_D f]$ and $[h_1 \ne h_2 \in H \Rightarrow h_1 \ne_D h_2]\}$.

2.7A Remark: $D \subseteq D'$ and $f \le_D f'$ implies $T_D(f) \le T_{D'}(f')$. (See Lemma 1.2 (c).)

For simplicity, from now on we will concentrate on the case $I = \aleph_1$ and D denotes a filter over $\aleph_1$ that extends the filter of co-bounded subsets of $\aleph_1$ ($X \subseteq \aleph_1$ is co-bounded iff $\aleph_1 - X$ is bounded). Let $\aleph_\alpha$ be singular cardinal, say $cf(\aleph_\alpha) = \aleph_1$. Assume $\beta < \alpha \Rightarrow \aleph_\beta^{\aleph_1} < \aleph_\alpha$. Note that if $\aleph_\alpha$ is strong limit then $\aleph_\alpha^{\aleph_1} = 2^{\aleph_\alpha}$. (See [5], Lemma 6.5.) As α is a limit ordinal and $cf(\alpha) = \aleph_1$, pick an increasing sequence $\langle \alpha_i | i < \omega_1\rangle$ such that $\alpha = \bigcup_{i<\omega_1} \alpha_i$, then we have $\prod_{i<\aleph_1} \aleph_{\alpha_i} = \aleph_\alpha^{\aleph_1}$. Define $h(i) = \aleph_{\alpha_i}^{\aleph_0} < \aleph_\alpha$.

2.8 Theorem *Let $\aleph_\alpha$, D, h be as above, then $T_D(h) = \aleph_\alpha^{\aleph_1}$.*

Proof: Let $A_i = \Pi_{j<i}\aleph_{\alpha_j}$ for $i < \aleph_1$. Then $|A_i| \le |{}^i\aleph_{\alpha_i}| \le \aleph_{\alpha_i}^{\aleph_0} = h(i)$, take t_i: $A_i \to h(i)$ a one-to-one map. Now for any $f \in \Pi_{i<\aleph_1}\aleph_{\alpha_i}$, define $g_f \in {}^{\aleph_1}\aleph_\alpha$ by $g_f(i) = t_i(f \restriction i) < h(i)$. If $f \ne f'$ then for some $j_0 < \aleph_1$ $f(j_0) \ne f'(j_0)$. Hence for all $i > j_0$ $g_f(i) \ne g_{f'}(i)$ so $\{i | g_f(i) \ne g_{f'}(i)\}$ is a co-bounded subset of $\aleph_1$ hence $g_f \ne_D g_{f'}$. The family $\{g_f | f \in \Pi_{i<\aleph_1}\aleph_\alpha\}$ is of cardinality $\aleph_\alpha^{\aleph_1}$ and this shows $T_D(h) \ge \aleph_\alpha^{\aleph_1}$. Of course $T_D(h) \le \aleph_\alpha^{\aleph_1}$.

2.9 Lemma *If D is $\aleph_1$-complete filter over $\aleph_1$ then there is no infinite descending sequence in $({}^{\aleph_1}Ord, <_D)$. Hence any nonvoid $A \subseteq {}^{\aleph_1}Ord$ has a minimal element. ($f \in A$ is minimal iff for $f' \in A$, $\neg(f' <_D f)$.)*

Proof: If $f_n \in {}^{\aleph_1}Ord$ and $f_{n+1} <_D f_n$ for $n < \omega$, then $\{i \in \aleph_1 | f_{n+1}(i) < f_n(i)\} = B_n \in D$ and $B = \bigcap_{n<\omega} B_n \in D$. But then $B \ne \phi$ and any $\alpha \in B$ gives a decending *sequence* $f_1(\alpha) < f_2(\alpha) \ldots$ of ordinals. Contradiction.

2.10 Theorem (1) *Let D be an $\aleph_1$-complete filter over $\aleph_1$, $\lambda > 2^{\aleph_1}$ a regular cardinal and $f \in {}^{\aleph_1}Ord$ with $T_D(f) \geq \lambda$. Then there is $D' \supseteq D$ an $\aleph_1$-complete filter over $\aleph_1$ and there is a $<_{D'}$-increasing sequence $\langle g_i | i < \lambda \rangle$ such that $g_i <_D f$*, (2) *Instead of* λ regular, $cf(\lambda) >_2^{|I|}$ suffices.

Proof: First we need a simple observation.

2.10A Claim *If $B = \bigcup_{i<\omega} B_i$ is a disjoint union and $B_i \neq \emptyset$ $Mod D$ and $g_i <_{D+B_i} g'$ and g satisfies $g \restriction B_i = g_i \restriction B_i$, then $g <_{D+B} g'$.*

Proof: $g_i <_{D+B_i} g'$ means that $\{\alpha \in \aleph_1 | g_i(\alpha) < g'(\alpha)\} \supseteq B_i \cap E_i$ for some $E_i \in D$. Set $E = \bigcap_{i<\omega} E_i \in D$ then for $\alpha \in B \cap E$, $g(\alpha) < g'(\alpha)$ so $g <_{D+B} g'$.

Next, from Lemma 2.9 we obtain a minimal element f^* in the class $\{g \in {}^{\aleph_1}Ord | g \leq_D f$ and $T_D(g) \geq \lambda\}$. So $f^* \leq_D f$, $T_D(f^*) \geq \lambda$ and if $g <_D f^*$ then $T_D(g) < \lambda$. Define now: $A \subseteq \aleph_1$ is *small* iff $A = \emptyset$ $mod D$, or there is $g <_{D+A} f^*$ with $T_{D+A}(g) \geq \lambda$.

2.10B Claim *Let D' denote the set of all $A \subseteq \aleph_1$ such that $\aleph_1 - A$ is small, then $D' \supseteq D$ is an $\aleph_1$-complete filter over $\aleph_1$.*

Proof: We rather show that the small sets form an $\aleph_1$-complete ideal extending the ideal dual to D. As for [$A' \subseteq A$ and A small $\Rightarrow A'$ is small], this follows from 2.7A. That $\aleph_1$ is not small is due to the minimality of f^* and $D + \aleph_1 = D$. If $A = \emptyset$ $mod D$ then A is small by definition. The only substantial claim is to show that if A_i, $i < \omega$ are small then $\bigcup_{i<\omega} A_i$ is small. Well, given a family of small subsets A_i, $i < \omega$, we can assume that the A_i are disjoint (otherwise, $A_i' = A_i - \bigcup_{j<i} A_j$ are small and $\bigcup_{i<\omega} A_i = \bigcup_{i<\omega} A_i'$), we can also assume that $A_i \neq 0$ $mod D$ (first take the union of those $A_i = \emptyset$ $mod D$ and then add it to some $A_j \neq \emptyset$ $mod D$, trivially $A \cup B$ is small if A is small and $B = \emptyset$ $Mod D$, because $D + A \cup B = D + A$). Let g_i exemplify that A_i is small, i.e., $g_i <_{D+A_i} f^*$ and $T_{D+A_i}(g_i) \geq \lambda$. Say $A = \bigcup_{i<\omega} A_i$, define $g \in {}^{\aleph_1}Ord$ by $g \restriction A_i = g_i$ and $g \restriction (\aleph_1 - A)$ is the constant function zero. We prove that g shows A is small. Now $g <_{D+A} f^*$ is due to 2.10A. To prove $T_{D+A}(g) \geq \lambda$ take any cardinal $\tau < \lambda$, we seek for a family $|H| > \tau$ as in Definition 2.7. For any $i < \omega$, $T_{D+A_i}(g_i) > \tau$ so there is a family $H^i \subset {}^{\aleph_1}Ord$, $|H^i| = \tau^+$, $[h \in H^i \Rightarrow h <_{D+A_i} g_i]$ and $\{h_1 \neq h_2$ in $H^i \Rightarrow h_1 \neq_{D+A_i} h_2]$. Enumerate $H^i = \{h^i_\xi | \xi < \tau^+\}$, then define $h_\xi \in {}^{\aleph_1}Ord$ by requiring $h_\xi \restriction A_i = h^i_\xi \restriction A_i$ and $h_\xi \restriction (\aleph_1 - A)$ is constantly zero. Now $h_\xi <_{D+A}$ g and $h_\xi \neq_{D+A} h_\zeta$ when $\xi \neq \zeta$. This ends the proof of the claim. (It can be proved that D' is normal if D is normal.)

We need a second claim.

2.10C Claim *Let D be a filter over $\aleph_1$, and $f \in {}^{\aleph_1}Ord$.*
(1) *If $T_D(f) > 2^{\aleph_1}$, then the supremum in Definition 2.7 is obtained.*
(2) *Moreover, if H is a family of functions such that $\{h \in H \Rightarrow h <_D f\}$, for h_1, $h_2 \in H$ $[h_1 \neq h_2 \Rightarrow h_1 \neq_D h_2]$, and H is maximal (with respect to inclusion), then $T_D(f) = |H|$.*

Proof: (1) follows from (2).

(2) If not, there exists a family of functions G such that $[g \in G \Rightarrow g <_D f]$, for $g_1, g_2 \in G$, $[g_1 \neq g_2 \Rightarrow g_1 \neq_D g_2]$, and $|G| > 2^{\aleph_1} + |H|$. Because of the maximality of H for every $g \in G$ there exists $h_g \in H$ which satisfies $\neg(g \neq_D h_g)$. Now by $|G| > 2^{\aleph_1} + |H|$ there exists $h^* \in H$ such that $G' = \{g \in G: h_g = h^*\}$ has greater cardinality than $2^{\aleph_1} + |H|$. For every $g \in G'$ define $A_g = \{\xi < \omega_1: g(\xi) = h^*(\xi)\}$, note that $A_g \neq \emptyset \ mod D$. $A_g \subseteq \omega_1$ so $|\{A_g: g \in G'\}| \leq 2^{\aleph_1}$, therefore there exists $g_1, g_2 \in G'$ $g_1 \neq g_2$ such that $Ag_1 = Ag_2$ so $\{\xi < \omega_1: g_1(\xi) = g_2(\xi)\} \supseteq A_g$ for some $g \in G'$. Hence g_1, g_2 cannot be distinct on a large set contradiction to the definition of G. This ends the proof of the claim.

Remember that $T_D(f^*) \geq \lambda$, so by the last claim let H be as in 2.10C relative to f^*. We continue the proof of Theorem 2.10 with a fourth claim.

2.10D Claim *If $g <_D f^*$ then $|\{h \in H | \neg(g <_{D'} h)\}| < \lambda$.*

Proof: Assume on the contrary that $S = \{h \in H | \neg(g <_{D'} h)\}$ has cardinality λ. For any $h \in S$ as $\neg(g <_{D'} h)$ there are sets $A, B \subseteq \aleph_1$ such that $A \cup B \neq 0$ *mod* D' and

(*) $A = \{i \in \aleph_1 | h(i) < g(i)\}$
$B = \{i \in \aleph_1 | h(i) = g(i)\}$.

But the number of such pairs A, B is $2^{\aleph_1}$ and $\lambda > 2^{\aleph_1}$ is regular, so there is *one* pair $A_0, B_0 \subseteq \aleph_1$ and $S_0 \subseteq S$, $|S_0| = \lambda$ such that (*) holds for any $h \in S_0$. Easily $B_0 = 0 \ mod D$: just take $h \neq h'$ in S_0, then $h \neq_D h'$ and if $B_0 \neq 0 \ mod D$ we get $i \in B_0$ such that $h(i) \neq h'(i)$ yet both are equal to $g(i)$. But $g <_D f^*$ (which implies $g <_{D+A_0} f^*$) and $T_{D+A_0}(g) \geq \lambda$ (by $|S_0| = \lambda$) implies A_0 is small, contradicting $A_0 \cup B_0 \neq 0 \ mod D'$.

The proof of the theorem itself now follows. Construct inductively an increasing sequence g_i, $i < \lambda$ such that $g_i <_D f^*$ and $i < j \Rightarrow g_i <_{D'} g_j$.

Assume $\{g_i | i < j\}$, $j < \lambda$ have been found for any $i < j$; the previous claim tells us that appart of less than λ many functions in H, $g_i <_{D'} h$ for all the rest. Now $|H| = \lambda$ is regular and $j < \lambda$, so we get that except a subset of H of cardinality $<\lambda$, all the rest of the functions satisfy $g_i <_{D'} h$ for all $i < j$, pick any such h to be g_j.[4]

2.11 Theorem *If $cf(\aleph_\alpha) = \aleph_1$ and $\alpha < \aleph_\alpha$, and $\beta < \alpha \Rightarrow \aleph_\beta^{\aleph_0} < \aleph_\alpha$ then $\aleph_\alpha^{\aleph_1} < \aleph_{(|\alpha|^{\aleph_1})^+}$.*

Proof: Assume $\aleph_\alpha^{\aleph_1} \geq \aleph_{(|\alpha|^{\aleph_1})^+}$. Then we have $2^{\aleph_1} < \aleph_\alpha$ (if $2^{\aleph_1} \geq \aleph_\alpha$ then $\aleph_\alpha^{\aleph_1} = 2^{\aleph_1} \leq \aleph_{(2^{\aleph_1})} < \aleph_{(|\alpha|^{\aleph_1})^+}$). Take D to be an $\aleph_1$ complete filter over $\aleph_1$ (for example the filter of the co-bounded subset of $\aleph_1$).

Let $\alpha = \bigcup_{i<\omega_1} \alpha_i$ be as in 2.8, define $h(i) = \aleph_{\alpha_i}^{\aleph_0}$, $T_D(h) = \aleph_\alpha^{\aleph_1}$ by Theorem 2.8 (and the supremum in the definition of $T_D(h)$ is actually obtained, see 2.10C(1)). So Theorem 2.10(2) can be applied by choosing a cardinal λ satisfying $\lambda = \aleph_{(|\alpha|^{\aleph_1})^+}$, we get $D' \supseteq D$ an $\aleph_1$-complete filter over $\aleph_1$ and $\langle g_i <_{D'} h | i < \lambda\rangle$ such that $[i < j \Rightarrow g_i <_{D'} g_j]$. But this contradicts Theorem 2.6 which just says ${}^{\aleph_1}(\aleph_\alpha)$ does not contain a $<_{D'}$-increasing sequence of length $\aleph_{(|\alpha|^{\aleph_1})^+}$ ($\leq \lambda$ by assumption).

Remark:
(1) If $\aleph_\alpha$ is a strong limit $2^{\aleph_\alpha} \neq \aleph_{(|\alpha|^{\aleph_1})^+}$ (compute their cofinalities).
(2) Remark that in case $\aleph_\alpha = \alpha$ this theorem is not informative and it is toward this case that we direct now our attentions.
(3) Note that the assumption of 2.11 implies:
(a) if $\aleph_\alpha$ is strong limit $|\alpha|^{\aleph_1} < \aleph_\alpha$ then $cf(\aleph_\alpha^{\aleph_1}) > \aleph_\alpha > (|\alpha|^{\aleph_1})^+$, hence $\aleph_\alpha^{\aleph_1} \neq \aleph_{(|\alpha|^{\aleph_1})^+}$
(b) if $|\alpha|^{\aleph_1} \geq \aleph_\alpha$ then $\aleph_\alpha^{\aleph_1} \leq (|\alpha|^{\aleph_1})^+ \leq \aleph_{(|\alpha|^{\aleph_1})^+}$.

3 Games $G(D, G, \alpha)$ and $G(D, g)$ As we explained in the introduction, in this section we define the games $G(D, g, \alpha)$ and $G(D, g)$.

We shall prove some easy lemmas, and the important conclusions are:

Conclusion 3.7 For given D, g, player II wins in $G(D, g)$ if and only if II wins in $G(D, g, \beta)$ for every or some $\beta \geq (2^{\aleph_1} + |\Pi_{i<\aleph_1} g(i)/D|)^+$.

In Lemma 3.8 for given D, g, if player I wins in $G(D, g)$ then I wins in $G(D, g')$ for some $g'\colon \omega_1 \to (2^{2^{\aleph_1}})^+$.

Conclusion 3.9 proves that if II wins in $G(D, (2^{2^{\aleph_1}})^+)$ then II wins in $G(D, g)$ for every $g \in {}^{\aleph_1}Ord$.

3.1 Definitions
(1) D will be from now on a normal filter on $\aleph_1$. For $g \in {}^{\aleph_1}Ord$ and $\alpha \in Ord$, *we define a game* $G(D, g, \alpha)$ played by two players as follows: Denote $D = D_0$, $g = g_0$, $\alpha = \alpha_0$. Player I begins, and in the first move chooses $A_1 \subseteq \aleph_1$, $A_1 \neq \emptyset \ mod D_0$ and $g_1 \in {}^{\aleph_1}Ord$, $g_1 <_{D_0+A_1} g_0$. Then player II for his first move chooses $D_1 \supseteq D_0 + A_1$ (D_1 is a normal filter on $\aleph_1$ by our convention about the use of the letter D) and also an ordinal $\alpha_1 < \alpha_0$.

In general, in the nth move, player I chooses $A_n \neq \emptyset \ mod D_{n-1}$ and $g_n <_{D_{n-1}+A_n} g_{n-1}$, and player II answers in his nth move with a normal filter $D_n \supseteq D_{n-1} + A_n$ and $\alpha_n < \alpha_{n-1}$. Of course it might be that player I or II cannot make any move; player I cannot move if $g_{n-1}(i) = 0$ for D_{n-1}-almost all $i \in \aleph_1$ and player II cannot move if $\alpha_{n-1} = 0$. A *play* in the game $G(D_0, g_0, \alpha_0)$ is a sequence of moves beginning with D_0, g_0, α_0 and ending when one of the players cannot make a move in his turn. Any sequence of moves played in this game must be finite because $\alpha_0 > \alpha_1 > \dots$ form a descending sequence of ordinals, so a play is finite. That player who cannot make a move in his turn *loses the play* and the other *wins*. A strategy for player I (player II) is a rule (i.e., a function) which tells I (II) what should be his nth move depending on the sequence of moves previously done by the players and (D, g, α). A strategy is a winning strategy in $G(D, g, \alpha)$ for I (II) if whenever I (II) plays in accordance with this strategy he wins all plays in $G(D, g, \alpha)$. A game $G(D, g, \alpha)$ is *determined* if either there is a winning strategy for I or a winning strategy for II in $G(D, g, \alpha)$.
(2) Now we define the game $G(D, g)$ which is played by two players as follows: Denote $D_0 = D$, $g_0 = g$. Player I begins, and in the first move chooses $A_1 \subseteq \omega_1$, $A_1 \neq \emptyset \ mod D_0$ and $g_1 <_{D+A_1} g_0$. Then player II in his first move chooses $D_1 \supseteq D_0 + A_1$ a normal filter on $\aleph_1$. In general, in the n'th move player I chooses $A_n \subseteq \omega_1$ $A_n \neq \emptyset \ mod D_{n-1}$ and $g_n <_{D_{n-1}+A_n} g_{n-1}$. Player II in his n'th

move chooses $D_n \supseteq D_{n-1} + A_n$ a normal filter on $\aleph_1$. Player I loses if he cannot go on, i.e., he cannot choose g_n. If the game continues ω steps then player I wins the game.

3.2 Remark: It is easy to see that the games $G(D, g, \alpha)$ and $G(D, g)$ are determined.

Note also that the game $G(D, g)$ is almost as $G(D, g, \alpha)$, the only difference is that it is easier for player II because he does not have to choose an ordinal.

3.3 Remark: If player I (II) wins the game then we can assume that the winning strategy does depend only on the last move made by the opponent and not on the previous moves [as this is true for every open (or closed) game]. Also, the fate of the play does not depend on A_n but on $D_{n-1} + A_n$ and not on g_n but on g_n/D_{n-1} (i.e., taking $g'_n \equiv_{D_{n-1}} g_n$ does not change).

3.4 Lemma *If II wins in $G(D, g, \alpha)$, $A \neq 0$ mod D and $g' \leq_{D+A} g$ and $\alpha' \geq \alpha$ then II wins in $G(D + A, g', \alpha')$.*

Proof: In Lemma 1.2 we stated that if $A \neq \emptyset \ mod D$ and $A_1 \neq \emptyset \ mod D + A$ then $(D + A) + A_1 = D + (A \cap A_1)$ and $D + A \subseteq D + (A \cap A_1)$. So if player I chooses $A_1 \neq \emptyset \ mod\,(D + A)$ and $g_1 <_{(D+A)+A_1} g'$ then player II "pretends" that he plays $G(D, g, \alpha)$ and that player I chooses $A \cap A_1$ and $g_1 <_{D+A\cap A_1} g$. His winning strategy provides him with a suitable answer: $D_1 \supseteq D + A \cap A_1$ and $\alpha_1 < \alpha \leq \alpha'$. Now II is in a winning position.

3.5 Lemma *If II wins in $G(D, g, \alpha)$ then for some $\alpha' < (2^{\aleph_1} + |\Pi_{i\in\aleph_1}(g(i) + 1)/D|)^+$ player II wins in $G(D, g, \alpha')$.*

Proof: Say $\mu = |\Pi_{i\in\aleph_1}(g(i) + 1)/D|$ (see 1.3), this means that there is a family $\{h_\alpha | \alpha < \mu\} \subseteq {}^{\aleph_1}Ord$ such that $h_\alpha(i) \leq g(i)$ for $i < \aleph_1$ and $(\forall h)$ $(\{i < \aleph_1 | h(i) \leq g(i)\} \in D \Rightarrow (\exists \alpha < \mu)\ h \equiv_D h_\alpha)$.

Now player II has a winning strategy for $G(D, g, \alpha)$. Look at all possible plays in which player II played in accordance with his strategy and player I played anyway. So the moves of player II are determined by its strategy. What freedom remains for player I? Choosing $A_n \subseteq \aleph_1$ and $g_n \in {}^{\aleph_1}Ord$. As all g_n satisfies $g_n <_{D_{n-1}+A_n} g$ we can assume that $g_n(i) \leq g(i)$ holds for all $i < \aleph_1$ (by Remark 3.3) and then find $h_\alpha \equiv_D g_n$, so of course $h_\alpha \equiv_{D_{n-1}+A_n} g_n$ (because $D \subseteq D_{n-1} + A_n$) hence we can assume player I makes his choices from $\{h_\alpha | \alpha < \mu\}$. Concluding, we see that there are $2^{\aleph_1} \cdot \mu$ possible situations, hence $\sum_{n\in\omega} (2^{\aleph_1} \cdot \mu)^n = 2^{\aleph_1} \cdot \mu$ plays. From this we deduce that

$$A = \{\beta | \text{ There is a play in which player II used his winning strategy in the game } G(D, g, \alpha) \text{ and } \beta \text{ was chosen by player II in one of his moves in that play.}\}$$

has cardinality $2^{\aleph_1} \cdot \mu$, hence order type $\alpha' < (2^{\aleph_1} \cdot \mu)^+$. Why does player II win in $G(D, g, \alpha')$? Well, let σ: $A \cup \{\alpha\} \to \alpha' + 1$ be an order-preserving map of $A \cup \{\alpha\}$ onto $\alpha' + 1$. Then provide player II with the following strategy: whenever you are presented with $D_{n-1} + A_n$, g_n, α_{n-1}, look at $D_{n-1} + A_n$, g_n,

$\sigma^{-1}(\alpha_{n-1})$ and ask what your original strategy says, if it gives you D_n, β and $\beta \in A$ then answer with D_n, $\sigma(\beta)$. Now, (A_1, g_1), (D_1, α_1), (A_2, g_2), $(D_2, \alpha_2)\ldots$ is a sequence of moves in $G(D, g, \alpha)$ in which player II used his original strategy iff (A_1, g_1), $(D_1, \sigma(\alpha_1))$ (A_2, g_2) $(D_2, \sigma(\alpha_2))\ldots$ is a sequence of moves in which player II used the new strategy for $G(D, g, \alpha')$: Hence II wins in $G(D, g, \alpha')$.

3.6 Lemma *For any D, g, α, if player I wins in $G(D, g, \alpha)$ and $\alpha \geq (2^{\aleph_1}\cdot|_{i\in\aleph_1}(g(i) + 1)/D|)^+$ then player I wins in $G(D, g)$.*

Proof: Again, let $\mu = |\Pi_{i\in\aleph_i}(g(i) + 1)/D$ and $H = \{h_\alpha|\alpha < \mu\} \subseteq {}^{\aleph_1}Ord$ as in the previous theorem's proof. We assume, by the argument given there that player I is directed by his winning strategy to play in his moves only with members of H. We describe now a winning strategy for player I in $G(D, g)$. While playing $G(D, g)$, player I speculates on plays in $G(D, g, \alpha)$. For the first move, player I gives A_1, g_1 as directed by his winning strategy for $G(D, g, \alpha)$. Now, if player II replies with $D_1 \supseteq D + A_1$ (they play $G(D, g)$), player I asks for each $\beta < (2^{\aleph_1}\cdot\mu)^+ \leq \alpha$, "What would my winning strategy for $G(D, g, \alpha)$ say if player II had replied with D_1, β to my first move A_1, g_1?" The answer is a pair A_2^β, g_2^β which depends on β. But the number of such pairs is $2^{\aleph_1}\cdot\mu < (2^{\aleph_1}\cdot\mu)^+$, so there is a fixed pair A_2, g_2, such that $\{\beta < (2^{\aleph_1}\cdot\mu)^+|A_2 = A_2^\beta, g_2 = g_2^\beta\}$ is unbounded in $(2^{\aleph_1}\cdot\mu)^+$ (a successor hence regular cardinal). In his new strategy for $G(D, g)$, player I answers with these A_2, g_2.

In general, player I plays his new strategy in such a way that at the nth move the following holds:

For any $\gamma < \mu^+$ there are $\alpha_n < \alpha_{n-1} < \ldots < \alpha_2 < \alpha_1 < (2^{\aleph_1}\cdot\mu)^+$, $\gamma < \alpha_n$ such that if the play (in $G(D, g)$) until now is supplemented by the ordinals $\alpha_1, \alpha_2, \ldots, \alpha_n$ as though given by player II in his successive turns, *then* the resulting game in $G(D, g, \alpha)$ is one that was played by player I in his winning strategy. The argument given for the second move shows that player I can stick to this strategy. A play that continue ω moves is a victory for player I, hence we get a winning strategy for player I in $G(D, g)$.[5]

3.7 Conclusion *The following are equivalent for given D, g:*

1. *II wins in $G(D, g)$*
2. *II wins in $G(D, g, (2^{\aleph_1} + |\Pi_{i<\aleph_1}(g(i) + 1)/D|)^+)$*
3. *For some α, II wins in $G(D, g, \alpha)$*
4. *II wins in $G(D, g, \beta)$ for all $\beta \geq (2^{\aleph_1} + |\Pi_{i<\aleph_1}(g(i) + 1)|)^+$.*

Proof: $1 \Rightarrow 2$ by Lemma 3.6 (as the game is determined Lemma 3.6 says: not 2) $\Rightarrow$ not 1),) $2 \Rightarrow 3$ trivial. For $3 \Rightarrow 4$: by Lemma 3.5 if player II wins in $G(D, g, \alpha)$ then player II wins in $G(D, g, \alpha')$ for some $\alpha' < (2^{\aleph_1}\cdot|\Pi_{i<\aleph_1}(g(i) + 1)/D)^+$, hence by Lemma 3.4 player II wins in $G(D, g, \beta)$ for all $\beta \geq \alpha'$.

$4 \Rightarrow 1$ because $G(D, g)$ is easier for player II than $G(D, g, \beta)$.

3.7A Remark: We can add

5. II wins in $G(D, g, \beta)$ for some $\beta < (2^{\aleph_1} + |\Pi_{i<\aleph_1}(g(i) + 1)|)^+$.

3.8 Lemma *For any D, g, if player I wins in $G(D, g)$ then I wins in $G(D, g')$ for some g' satisfying $(\forall i \in \aleph_1)g'(i) < (2^{2^{\aleph_1}})^+$.*

Proof: Look at all plays in $G(D, g)$ where player I used a fixed winning strategy. At each move, II has $\leq 2^{2^{\aleph_1}}$ answers (this is the number of filters on $\aleph_1$), so the number of plays is $\leq (2^{2^{\aleph_1}})^{\aleph_0} = 2^{2^{\aleph_1}}$. Let E be the set of all $g^* \in {}^{\aleph_1}Ord$ *which appeared in one of the* $2^{2^{\aleph_1}}$ plays. Set $A = \bigcup_{g^* \in E} \text{Range}(g^*)$.

A is a set of ordinals of cardinality not greater than $2^{2^{\aleph_1}}$. Let $\sigma: A \to A' < (2^{2^{\aleph_1}})^+$ be the collapse of A to an ordinal A' (so σ is one to one order preserving onto A'). For any $g^* \in E$ we define $\sigma(g^*)$ by $(\sigma(g^*))(i) = \sigma(g(i))$. For $g' = \sigma(g)$ we get that player I has a winning strategy for $G(D, g')$: using σ, this strategy is just the translate of the winning strategy for $G(D, g)$. Instead of answering with A, g^*, player I answers A, $\sigma(g^*)$.

In the following conclusion we write $G(D, (2^{2^{\aleph_1}})^+)$ instead of $G(D, f)$ where f is the fixed function on $\aleph_1$ having the single value $(2^{2^{\aleph_1}})^+$.

3.9 Conclusion *If II wins in $G(D, (2^{2^{\aleph_1}})^+)$ then II wins in $G(D, g)$ for any $g \in {}^{\aleph_1}Ord$.*

Proof: Otherwise if for some g, player I wins in $G(D, g)$ then by Lemma 3.8, player I wins in $G(D, g')$ for some g' with $g'(i) < (2^{2^{\aleph_1}})^+$. That player I wins in $G(D, (2^{2^{\aleph_1}})^+)$ follows now easily by Lemma 3.4.

4 There exists a filter D such that player I wins in every G(D, g) Here follows a general description of Section 4. The detailed exposition starts in 4.0. Our aim in this section is to prove the following Theorem 4.15: Let $\aleph_\alpha$ be a singular cardinal $> 2^{2^{\aleph_1}}$, $cf(\aleph_\alpha) = \aleph_1$, and assume that $[\beta < \alpha \Rightarrow \aleph_\beta^{\aleph_1} < \aleph_\alpha]$; if $\aleph_\alpha^{\aleph_1} > \aleph_{\alpha+1}$, then there exists an almost nice D; i.e., a normal filter D over $\aleph_1$ such that II wins in $G(D, g)$ for every $g \in {}^{\aleph_1}Ord$. How shall we prove it? Assume by negation that for every D as above there exist a $g \in {}^{\aleph_1}Ord$ such that player I wins in $G(D, g)$. Hence by Lemma 3.8 there exist $g': \aleph_1 \to (2^{2^{\aleph_1}})^+$ such that player I wins in $G(D, g')$; the winning strategies of I (for all such D, g) can be encoded by a set A such that $A \subseteq (2^{2^{\aleph_1}})$ (this is the number of filters and functions in the game). Now apply a theorem of Dodd and Jensen and get a transitive model for ZFC V^+ which contains the ordinals and $A \in V^+$, and there exists a cardinal λ in V^+ such that it satisfies Assumption 4.1.

In a model which satisfies Assumption 4.1 we shall construct a filter E over I specified there (this is not $\aleph_1$) and prove that E is $\aleph_1$-complete and normal (we shall quote the definition of normality of a filter over a general set, this notation is an extension of the usual notation of normality for filters over ordinals).

Later define a filter D_E over $\aleph_1$ which is determined by E and a measure-preserving mapping F (relative to these two filters), and show that D_E normal filter.

In Theorem 4.13 we prove for $D = D_E$ that II has a winning strategy in every game $G(D, g, (2^{2^{\aleph_1}})^+)$ for every $g: \aleph_1 \to (2^{2^{\aleph_1}})^+$; therefore by Conclusion 3.9 II wins in $G(D, g)$ for every g (in particular in the model V^+); therefore II wins also $G(D, g')$ in V^+ but $A \in V^+$ so in V^+ there is a winning strategy for player I in the game $G(D, g')$. Contradiction.

4.0 For a cardinal λ we look at algebras of the form $M = (|M|, f_n)_{n \in \omega}$ where $|M| \subseteq \lambda$ and f_n is a k_n-place function from $|M|$ to $|M|$, ($|M|$ denote the universe of the algebra M and $\|M\|$ its cardinality). Let $A \subseteq M$ mean that A is a subalgebra of M, i.e., $|A| \subseteq |M|$ and $|A|$ is closed under the functions f_n; $f_n \restriction |A|: |A| \to |A|$. We assume that λ is a Ramsey cardinal, actually we only use and assume that λ satisfies the following:

4.1 Assumption on λ *For any algebra $M = (\lambda, f_n)_{n\in\omega}$, there is a subalgebra $A \subseteq M$ such that $\|A\| = (2^{2^{\aleph_1}})^+$ and $|A| \cap \omega_1$ is countable.*

4.1A Remark: If λ is Ramsey then Assumption 4.1 holds; we shall show that it follows even from $\lambda \to ((2^{2^{\aleph_1}})^+)_2^{<\omega}$.

Proof: For any $n < \omega$ define $g_n(\alpha_1, \ldots, \alpha_{k_n}) = sup\ [\omega_1 \cap$ the subalgebra of M generated by $\{\alpha_1, \ldots, \alpha_{k_n}\}]$. Now define $g'_n(\alpha_0, \ldots, \alpha_{2k_n-1})$ as the truth value of $\{g_n(\alpha_0, \ldots, \alpha_{k_n-1}) = g_n(\alpha_{k_n}, \ldots, \alpha_{2k_n-1})\}$. Now apply the above partition relation to the functions $\{g'_n: n < \omega\}$ and the subalgebra A^* of M generated by the homogeneous set A for these functions has the property required in Assumption 4.1.

4.2 Definitions

a. $I = \{S \subseteq \lambda \mid |S| = (2^{2^{\aleph_1}})^+$ and $S \cap \omega_1$ is countable$\}$

b. For an algebra $M = (\lambda, f_n)_{n<\omega}$, let $J(M) = \{|A| \mid A$ is a subalgebra of M and $|A| \in I\}$

c. $E = \{X \subseteq I \mid$ for some algebra $M = (\lambda, f_n)_{n<\omega}$, $X \supseteq J(M)\}$.

Remark: Such E were first considered (and their basic properties proved) in [10].

4.3 Lemma *E is a filter over I, countably closed.*

Proof: By our Assumption 4.1 on λ, E is not trivial ($\emptyset \notin E$). Assume $X_n \in E$, $n < \omega$, we want to prove $\bigcap_{n<\omega} X_n \in E$. For all $k < \omega$ there is an algebra $M_k = (\lambda, f_n^k)_{n<\omega}$ such that $X_R \supseteq J(M_k)$. Let $M = (\lambda, f_n^k)_{n,k<\omega}$, then $\bigcap_{k<\omega} X_k \supseteq J(M)$, because if $S \subseteq \lambda$ is closed under all functions f_n^k, n, $k < \omega$ surely, for any k, it is closed under f_n^k, $n < \omega$, so $S \in X_k$.

4.4 Definition For $X \subseteq I$ we say *f is a choice function on X* iff $f: X \to \lambda$ is such that $f(S) \in S$ for all $S \in X$.

4.5 Lemma *Let $X \subseteq I$, $X \neq \emptyset$ mod E and f be a choice function on X then for some $\alpha < \lambda$, $\{S \in X \mid f(S) = \alpha\} \neq \emptyset$ mod E.* (We say that a filter over I is normal if it satisfies the lemma, so Lemma 4.5 says that E is normal.)

Proof: Assume to the contrary that for any $\alpha < \lambda$ there is $M_\alpha = (\lambda, f_n^\alpha)_{n<\omega}$ such that $f^{-1}(\alpha) \cap J(M_\alpha) = \emptyset$: If $S \in J(M_\alpha)$ then either $S \notin X$ or $f(S) \neq \alpha$. By rearranging the functions and permitting dummy variables we can assume that f_n^α is an n place function on λ. Define an $n+1$ place function f_n^α by $f_n(\alpha, \alpha_1, \ldots, \alpha_n) = f_n^\alpha(\alpha_1, \ldots, \alpha_n)$ then look at $M = (\lambda, f_n)_{n<\omega}$.

Because $X \neq \emptyset$ mod E, we get $X \cap J(M) \neq \emptyset$. Pick $S \in X \cap J(M)$ then $S \in X$ and $f(S) = \alpha$ for some $\alpha \in S$. So $S \notin J(M_\alpha)$. But on the other hand, $S \in J(M)$ so S is closed under the functions f_n. Hence, as $\alpha \in S$, S is closed under the functions f_n^α, so $S \in J(M_\alpha)$. Contradiction.

4.6 Assertion: One can easily see that if E^* is a normal ideal over I and $X \neq \emptyset$ mod E^* then $E^* + X$ is normal.

4.7 Lemma *For any $\gamma < \omega_1$, $X_\gamma = \{S \in I | S \cap \omega_1$ is a limit countable ordinal above $\gamma\} \in E$.*

Proof: We have to find an algebra M rich enough so that $X_\gamma \supseteq J(M)$. Well, for any $i < \gamma + \omega$ throw to M the function which has the constant value i. Also, for any ordinal $\alpha < \omega_1$ pick an enumeration $\{\alpha_n | b < \omega\} = \alpha$, then define $f(\alpha, n) = \alpha_n$ if $\alpha < \omega_1$ and $n < \omega$ and $f(\alpha, n) = 0$ otherwise; define also $f(\xi) = \xi + 1$. Now if M has all these countably many functions described above then for $S \in J(M)$, $S \cap \omega_1$ is an initial segment closed under the successor function, hence a limit ordinal above α.

4.8 Definition Define a function F: $I \rightarrow \omega_1$ by $F(S) = sup\ (S \cap \omega_1)$. By the preceding lemma, for almost al S, $F(S) = S \cap \omega_1$.

4.9 Lemma *Let $E^* \supseteq E$ be a filter over I. Set $D_{E^*} = \{F''\ X | X \in E^*\}$ where $F''\ X = \{F(S)\colon S \in X\}$, then:*
(a) *D_{E^*} generate a filter over ω_1 which we call again D_{E^*}*
(b) *For $Y \subseteq \omega_1$, $Y \neq \varnothing\ mod D_{E^*} \Rightarrow F^{-1}(Y) = \{S \in I | F(S) \in Y\} \neq 0\ mod E^*$. And for $W \neq \varnothing\ mod E^*$, $F''\ W \neq \varnothing\ mod D_{E^*}$*
(c) *D_{E^*} is normal if E^* is normal.*

Proof: (a) Follows from $F''\ X_1 \cap F''\ X_2 \supseteq F''\ (X_1 \cap X_2)$. To prove (b) let $Y \neq 0\ mod D_{E^*}$. We have to show $F^{-1}(Y) \neq \varnothing\ mod E^*$, and this is true because for $X \in E^*$, $F''\ X \in D_{E^*}$. So $F''\ X \cap Y \neq \varnothing$ and hence $X \cap F^{-1}(Y) \neq \varnothing$. On the other hand if $W \neq \varnothing\ mod E^*$ then for any $Z \in D_{E^*}$, $Z \supseteq F''\ X$ for some $X \in E^*$. So $W \cap X \neq \varnothing$ and $F''\ W \cap F''\ X \neq \varnothing$. Hence $F''\ W \cap Z \neq \varnothing$. This proves $F''\ W \neq \varnothing\ mod D_{E^*}$.

(c) For the normality, let $Y \neq \varnothing\ mod D_{E^*}$ and f: $Y \rightarrow \omega_1$ a regressive function. $F^{-1}(Y) \neq \varnothing\ mod E^*$ is concluded by (b) above. Now define f^*: $f^*(S) = f(F(S))$ for $s \in F^{-1}(Y)$. Let X_0 be the set defined in Lemma 4.7, then $X_0 \in E^*$. For $S \in X_0 \cap F^{-1}(Y)$, $f^*(S) \in S$. Hence assuming E^* is normal and because $X_0 \cap F^{-1}(Y) \neq \varnothing\ mod E^*$, we get $\xi \in \omega_1$ such that $X = \{S \in F^{-1}(Y) | f^*(S) = \xi\} \neq \varnothing\ mod E^*$. Using (b) again, we see that $F''\ X \neq \varnothing\ mod D_{E^*}$, $F''\ X \subseteq Y$ and f is constantly ξ on $F''X$.

4.10 Notation: For g: $\omega_1 \rightarrow (2^{2^{\aleph_1}})^+$ define $\hat{g}$: $I \rightarrow \lambda$, a choice function for I, by $\hat{g}(S) =$ the $g(F(S))$-th members of S (in the natural order of $S \subseteq \lambda$).

4.11 Lemma *For a filter $E^* \supseteq E$ over I and g_1, g_2: $\omega_1 \rightarrow (2^{2^{\aleph_1}})^+$, $g_1 <_{D_{E^*}} g_2 \Leftrightarrow \hat{g}_1 <_{E^*} \hat{g}_2$.*

Proof: Trivial because $g_1(F(S)) < g_2(F(S))$ iff $\hat{g}_1(S) < \hat{g}_2(S)$.

4.12 Lemma *Let $E^* \supseteq E$ be a filter over I, $Y \subseteq \aleph_1$, $Y \neq \varnothing\ mod D_{E^*}$, then $D_{E^* + F^{-1}(Y)} = D_{E^*} + Y$.*

Proof: If $B \in D_{E^* + F^{-1}(Y)}$ then $B \supseteq F''\ X$ for some $X \in E^* + F^{-1}(Y)$. But $X \in (E^* + F^{-1}(Y))$ implies $X \supseteq X_1 \cap F^{-1}(Y)$ *for some* $X_1 \in E^*$. So $F''X \supseteq F''X_1 \cap Y$ hence $F''X \in D_{E^*} + Y$ but $B \supseteq F''X$ hence $B \in D_{E^*} + Y$. For the other direction, let $B \in D_{E^*} + Y$. This means $B \supseteq F''X \cap Y$ for some $X \in E^*$, so $F^{-1}(B) \supseteq F^{-1}(F''\ X \cap Y) \supseteq X \cap F^{-1}\ Y$. Getting $F^{-1}(B) \in E^* + F^{-1}(Y)$ we conclude $B \in D_{E^* + F^{-1}(Y)}$.

4.13 Theorem *Let $D = D_E$. For any g: $\aleph_1 \to (2^{2^{\aleph_1}})^+$ player II has a winning strategy in $G(D, g, (2^{2^{\aleph_1}})^+)$.*

Proof: By Lemma 3.5 it suffices to show that player II wins in $G(D, g, \lambda)$. Recall that player I is choosing in the n move A_n, g_n and player II has to answer with $D_n \supseteq D_{n-1} + A_n$ and $\alpha_n < \alpha_{n-1}$. In the first move, player I gives $A \neq \emptyset \ mod D$ and $g_1 <_{D+A_1} g$. Player II makes first some side calculations: he knows that $F^{-1} A_1 \neq \emptyset \ mod E$ (Lemma 4.9(b)), then he gets from Lemma 4.12 that setting $E_1 = E + F^{-1} A_1$, $D_{E_1} = D_E + A_1$, and E_1 is normal (this is by Assertion 4.6). Now $\hat{g}_1$ is a choice function on $F^{-1} A_1 \in E_1$ so II finds $X_1 \subseteq F^{-1} A_1$, $X_1 \neq \emptyset \ mod E_1$ such that $\hat{g}_1$ is constant on X_1 and α_1 is the unique value of $\hat{g}_1$ on X_1. II responds now with $D_1 = D_{E+X_1}$ and α_1. $D_{E+X_1} \supseteq D_E + A_1 = D_{E_1}$ is due to $E + X_1 \supseteq E + F^{-1}(A_1) = E_1$.

Generally, II chooses for himself at the n stage $X_n \subseteq F^{-1} A_n$, $X_n \neq \emptyset$ $mod E$ such that $\hat{g}_n$ has the constant value α_n on X_n and responds with $D_n = D_{E+X_n}$ and α_n. Let's see that this is a feasible strategy. Assume inductively the n move is done and player I gives in his $n + 1$ move $A_{n+1} \neq \emptyset \ mod D_n$ and $g_{n+1} <_{D_n+A_{n+1}} g_n$. Then player II knows that $D_n = D_{E+X_n}$, he sets $E_{n+1} = (E + X_n) + F^{-1}(A_{n+1})$ and concludes with Lemma 4.12 (where E^* is $E + X_n$) that $D_{(E+X_n)+F^{-1}(A_{n+1})} = D_{E+X_n} + A_{n+1} = D_n + A_{n+1}$. And $\hat{g}_{n+1} <_{E+X_n+F^{-1}(A_{n+1})} \hat{g}_n$ by Lemma 4.11, II then finds $X_{n+1} \neq \emptyset \ mod E$ $X_{n+1} \subseteq X_n \cap F^{-1}(A_{n+1}) \cap \{S \in I | \hat{g}_{n+1}(S) < \hat{g}_n(S)\}$ such that $\hat{g}_{n+1}$ has the constant value α_{n+1} on X_{n+1}. Now II answers with $D_{n+1} = D_{E+X_{n+1}}$ and α_{n+1}. $\alpha_{n+1} < \alpha_n$ because $\hat{g}_{n+1}$ and $\hat{g}_n$ are constant on X_{n+1} and $\hat{g}_{n+1}$ is less than $\hat{g}_n$ there (and $X_{n+1} \neq \emptyset$). $D_{n+1} \supseteq D_n + A_{n+1}$ because $X_{n+1} \subseteq X_n \cap F^{-1}(A_{n+1})$ implies $E + X_{n+1} \supseteq E + X_n \cap F^{-1}(A_{n+1}) = (E + X_n) + F^{-1}A_{n+1}$ (see Lemma 1.2). Hence $D_{E+X_{n+1}} \supseteq D_{(E+X_n)+F^{-1}A_{n+1}} = D_n + A_{n+1}$. Finally, as II can always answer and $\alpha_{n+1} < \alpha_n$ a decending sequence of ordinals, the victory for II is not late to come.

Theorem 4.13 was proved under the Assumption 4.1. Now we drop this assumption and get a general conclusion.

4.14 Conclusion *Suppose for any $A \subseteq (2^{2^{\aleph_1}})^+$ there is a transitive class $V^* = V^*_A$ which is a model of ZFC containing the ordinals, $A \in V^*$ and $V^* \models$ "There is a cardinal λ satisfying 4.1". Then there is a normal filter D over ω_1 such that player II wins in $G(D, g)$ for any g: $\omega_1 \to (2^{2^{\aleph_1}})^+$.*

Proof: If the conclusion is false then for any filter D over ω_1 for every function g: $\omega_1 \to (2^{2^{\aleph_1}})^+$ player I has a winning strategy in $G(D, g)$. The number of all possible pairs D, g is $(2^{2^{\aleph_1}})^+$ and a winning strategy for I in $G(D, g)$ is a function defined on filters D_n over $\aleph_1$ taking values pairs A_{n+1}, g_{n+1}. So we can find $A \subseteq (2^{2^{\aleph_1}})^+$ which encodes in some convenient way all such D, g and strategies for player I in $G(D, g)$, so that if $A \in V^*$ is a model of ZFC containing the ordinals then V^* contains $\mathcal{P}(\mathcal{P}(\aleph_1))$ and for any D, g in V^* player I has a winning strategy in $G(D, g)$. Let $V^* = V^*_A$, so by the hypothesis of 4.14. But in V^* there is a cardinal λ satisfying 4.1 by our assumption so that Theorem 4.13 applies in V^* and for $D_E = D$, player II has a winning strategy in $G(D, g, (2^{2^{\aleph_1}})^+)$ and a fortiori in $G(D, g)$ for any g. Contradiction.

We quote now a theorem of [3].

4.15 Theorem *If there is a cardinal $\aleph_\alpha > 2^{2^{\aleph_1}}$, $cf(\aleph_\alpha) = \aleph_1$ such that $\aleph_\alpha^{\aleph_1} > \aleph_\alpha^+$ then the assumption in Conclusion 4.14 holds (in fact in V^* there is a measurable cardinal).*

Remark: Dodd and Jenson proved the assumption of Theorem 4.15 implies existence of an inner model with a measurable cardinal λ. But since every measurable is also Ramsey cardinal (a well-known fact, proof of which can be found in Theorem 7.0 in [5]), so by the remark after Lemma 4.3 we have that λ exemplify requirement 4.1. What we are really doing here is repeating their proof of the covering lemma not for K and V but for $K(A)$ and V.

From now on let $\aleph_\alpha$ be a singular cardinal of cofinality $\aleph_1$, $\aleph_\alpha > 2^{2^{\aleph_1}}$, and $(\forall\beta < \alpha)\ \aleph_\beta^{\aleph_0} < \aleph_\alpha$. We seek some information on $\aleph_\alpha^{\aleph_1}$ even if $\alpha = \aleph_\alpha$. If $\aleph_\alpha^{\aleph_1} = \aleph_\alpha^+$ then we have an excellent estimate on $\aleph_\alpha^{\aleph_1}$, it has the lowest possible value. So we can assume

Hypothesis *For some α, $\aleph_\alpha \geq 2^{2^{\aleph_1}}$ and $\aleph_\alpha^{\aleph_1} > \aleph_\alpha^+$.*

and then the theorem quoted above yields by Conclusion 4.14 that:

4.16 Conclusion There is an almost nice D over ω_1 which shall mean: D a normal filter such that player II wins in $G(D, g)$ for any $g \in {}^{\omega_1}Ord$.

(Conclusion 4.14 speaks only about $g: \aleph_1 \to (2^{2^{\aleph_1}})^+$ but we know (Lemma 3.8) that if II wins in $G(D, g)$ for all $g: \aleph_1 \to (2^{2^{\aleph_1}})^+$ then II wins in $G(D, g)$ for any $g \in {}^{\aleph_1}Ord$.)

5 Rank of functions We give some general remarks and the detailed exposition begins in the paragraph before 5.1. The main Theorem 5.5 is in certain sense similar to Theorem 2.10: if τ a cardinal $cf(\tau) > (2^{2^{\aleph_1}})^+$, $g \in {}^{\aleph_1}Ord$, $T_D(g) \geq \tau$, and assume that Player II wins in $G(D, g)$, *then* there exists a normal filter $D' \supseteq D$ over $\aleph_1$ and an $<_{D'}$-increasing sequence $\langle g_\xi: \xi < \tau\rangle$ in ${}^{\aleph_1}Ord/D'$ such that $T_{D'}(g_\xi) < \tau$ and moreover $[A \neq \emptyset \ mod D' \Rightarrow T_{D'+A}(g_\xi) < \tau]$ for $\xi < \tau$.

For this proof we define a number of different notions of rank functions relatively to an $\aleph_1$-complete filter (this in addition to Galvin–Hajnal rank already introduced in the Introduction). We define $rk_D(g) = min\ \{\alpha$: II wins in $G(D, g, \alpha)\}$ $rk'_D(g) = min\ \{rk_{D^*}(g): D^* \supseteq D$ normal and II wins in $G(D^*, g)\}$. We shall show that rk' behaves in a similar form as Galvin–Hajnal's: $[f <_D g \Rightarrow rk'_D(f) < rk'_D(g)]$. This is Lemma 5.3(2).

We shall study also the relations between the distinct ranks: By Lemma 5.2(1) and Lemma 5.3(3) $\|f\|_D \leq rk'_D\ (f) \leq rk_D(f)$.

In Lemma 5.4 we connect these ranks with $T_D(f)$; and shall get $T_D(f) \leq \|f\|_D$.

If II wins in $G(D, g)$ then for all high enough β, II wins in $G(D, g, \beta)$ (Conclusion 3.7), so the following definitions make sense.

5.1 Definitions of rank Let D be a (normal) filter over $\aleph_1$, $g \in {}^{\aleph_1}Ord$ and assume II has a winning strategy in $G(D, g)$.

1. $rk_D(g) = min\ \{\alpha|$ II wins in $G(D, g, \alpha)\}$. So II wins in $G(D, g, rk_D(g))$ but not in $G(D, g, \alpha)$ for $\alpha < rk_D(g)$.
2. $rk'_D(g) = min\ \{rk_{D^*}(g)|\ D^* \supseteq D$ is a normal filter over ω_1 such that player II wins in $G(D^*, g)\}$.
So if $rk'_D(g) = \beta$ then II wins in $G(D^*, g, \beta)$ for some $D^* \supseteq D$ but for no $D' \supseteq D$ and $\alpha < \beta$ II wins in $G(D', g, \alpha)$.
3. We say that D *is g-good* if II wins in $G(D, g)$ and $rk_D(g) = rk'_D(g)$.
So D is g-good iff $rk_D(g) = \beta$ is the minimal ordinal such that II wins in $G(D^*, g, \beta)$ for some $D^* \supseteq D$.

5.2 Lemma *Let D and g be such that II wins in G(D, g) then*
1. $rk'_D(g) \leq rk_D(g)$.
2. *A winning strategy for II in* $G(D, g, rk_D(g))$ *is to choose at the n'th move* $D_n \supseteq D_{n-1} + A_n$ *such that* $rk_{D_n}(g_n)$ *is minimal and* $\alpha_n = rk_{D_n}(g_n)$. *(In this case* D_n *is* g_n*-good.)*
3. $rk_{D+A}(g) \leq rk_D(g)$ *for* $A \neq \emptyset$ *mod D, and equality holds in case D is g-good.*

Proof: 1. is immediate as the minimum in the definition of rk' is taken over a larger set of ordinals than that of rk.

2. II can play and win according to this strategy because if II wins in $G(D_{n-1}, g_{n-1}, \alpha_{n-1})$ then for any choice made by I of $A_n \neq 0$ $mod D_{n-1}$ and $g_n <_{D_{n-1}+A_n} g_{n-1}$, player II can use his winning strategy (the one we assume he got) to get $D^* \supseteq D_{n-1} + A_n$ and $\alpha < \alpha_{n-1}$ such that II wins in $G(D^*, g_n, \alpha)$. It follows that $rk_{D^*}(g_n) \leq \alpha < \alpha_{n-1}$, hence if II chooses $D_n \supseteq D_{n-1} + A_n$ with minimal $rk_{D_n}(g_n)$ then $\alpha_n = rk_{D_n}(g_n) < \alpha_{n-1}$ and (D_n, g_n, α_n) is a winning position for II.

3. Follows because if II wins in $G(D, g, \alpha)$ then II wins in $G(D + A, g, \alpha)$ (Lemma 3.4). If D is good for g then $rk_D(g) = rk'_D(g)$; by definition, this means $rk_D(g) \leq rk_{D'}(g)$ for any $D' \supseteq D$ (a normal filter with II winning in $G(D', g)$), in particular $rk_D(g) \leq rk_{D+A}(g)$.

5.3 Lemma *For a filter D and a function g such that II wins in G(D, g) the following holds*:
1. *If* $\leq_D g$ *then* $rk_D(f) \leq rk_D(g)$ *and* $rk'_D(f) \leq rk'_D(g)$.
2. *If* $f <_D g$ *then* $rk'_D(f) < rk'_D(g)$.
3. $rk'_D(g) \geq \|g\|_D$, *where* $\|g\|_D$ *is the Galvin-Hajnal degree* (see [4]) *defined by* $\|g\|_D = Sup\{\|f\|_D | f <_D g\}$. (In this definition we used the fact that $h <_D f$ is a well founded relation (Lemma 2.9). Note also that the use of the capital letter S in Sup means that if a set of ordinals A contains a maximal ordinal γ then $Sup A = \gamma + 1$.)

Proof: 1. II wins in $G(D, g, rk_D(g))$, so II also wins in $G(D, f, rk_D(g))$ (see Lemma 3.4); hence $rk_D(f) \leq rk_D(g)$. For rk'; $rk'_D(g) = rk_{D^*}(g)$ for some $D^* \supseteq D$, but we just saw that $rk_{D^*}(f) \leq rk_{D^*}(g)$, and $rk'_D(f) \leq rk_{D^*}(f)$ by definition, so $rk'_D(f) \leq rk'_D(g)$ follows.

2. For some D^* and α, $rk'_D(g) = rk_{D^*}(g) = \alpha$ where $D^* \supseteq D$ and for any $D' \supseteq D$, $\alpha \leq rk_{D'}(g)$. So player II has a winning strategy in $G(D^*, g, \alpha)$. Assume $f <_D g$ (and then $f <_{D^*} g$). Let player I play as first move in $G(D^*, g, \alpha)$, the set $A_1 = \aleph_1$ and the function $f_1 = f$. Player II uses his strategy and gives

back $D^{**} \supseteq D^*$ and ordinal $\alpha_1 < \alpha$. So II wins in $G(D^{**}, f, \alpha_1)$, from this $rk_{D^{**}}(f) \leq \alpha_1$ and $rk'_D(f) \leq \alpha_1$ follows. Hence $rk'_D(f) \leq \alpha_1 < \alpha = rk'_D(g)$.

3. Prove by induction on $\xi \in Ord$ that if $\|g\|_D = \xi$ then $rk'_D(g) \geq \xi$. This is immediate using part 2 above and the definition of $\|g\|_D$.

5.4 Lemma *If τ is a cardinal, $\tau > 2^{\aleph_1}$ and $g \in {}^{\aleph_1}Ord$ with $T_D(g) \geq \tau$ then $\geq \tau$.*

Proof: Assume $\|g\|_D = \xi < \tau$, we will find a contradiction. Set $\chi = |\xi| + 2^{\aleph_1} < \tau$. As $T_D(g) \geq \tau$ we can find, by the definition of $T_N(g)$ a set $H \subseteq {}^{\aleph_1}Ord$, $|H| = \chi^+$ such that $[h \in H \Rightarrow h <_D g]$ and for $h \neq h'$ in H, $h \neq_D h'$. But if $h <_D g$ then $\|h\|_D < \|g\|_D < \chi^+$, so we can find $H^* \subseteq H$, $|H^*| = \chi^+$ and an ordinal $\zeta < \|g\|_D$ such that $\|h\|_D = \zeta$ for $h \in H^*$. Set $H^* = \{h_i | i < \chi^+\}$. Now by Theorem 2.2 (that one which was an application of Erdos-Rado Theorem) as $\chi^+ \geq (2^{\aleph_1})^+$ we can find $i < j < \chi^+$ with $h_i \leq_D h_j$, but $h_i \neq_D h_j$, hence $h_i <_D h_j$. This clearly contradicts $\|h_i\|_D = \|h_j\|_D$.

The following theorem is central in our theory.

5.5 Theorem *Let τ be a cardinal $cf(\tau) > (2^{2^{\aleph_1}})^+$, $g \in {}^{\aleph_1}Ord$ and $T_D(g) \geq \tau$ or $\|g\|_D \geq T$* or *$\tau \leq rk_D(g)$. Assume that player II wins in $G(D, g)$. Then there are $g_\xi \in {}^{\aleph_1}Ord$* (for $\xi < \tau$) *and a normal $D' \supseteq D$ such that:*

1. *$g_\xi <_{D'} g$ for $\xi < \tau$*
2. *$\xi < \zeta < \tau \Rightarrow g_\xi <_{D'} g_\zeta$*
3. *$T_{D'}(g_\xi) < \tau$ for $\xi < \tau$*
4. *If $A \neq \emptyset$ modD' then $T_{D'+A}(g_\xi) < \tau$ for $\xi < \tau$*
5. *$\xi \leq rk_{D'}(g_\xi) = rk'_{D'}(g_\xi) < \tau$ for $\xi < \tau$.*

Proof: If $rk_D(g) \geq \tau$ let $D^* = D$. If $\|g\|_D \geq \tau$ by Lemma 5.4 and $rk'_D(g) \geq \|g\|_D$ by Lemma 5.3(3). Now $rk'_D(g) = rk_{D^*}(g)$ for some $D^* \supseteq D$ (by Definition 5.1). So always $rk_{D^*}(g) \geq \tau$. Look at the set K of all plays in $G(D^*, g, rk_{D^*}(g))$ in which II is conducted by the winning strategy described in 5.2, i.e., at the n move player II chooses the minimal ordinal α_n such that for some $D_n \supseteq D_{n-1} + A_n$ player II wins in $G(D_n, g_n, \alpha_n)$, in other words $\alpha_n = rk_{D_n}(g_n)$ is minimal. Set $\alpha_0 = rk_{D^*}(g)$.

5.5A Claim *Every ordinal $\gamma < rk_{D^*}(g) = \alpha_0$ is some α_n played by II in some play in K.*

Proof: Assume $\gamma < \alpha_0$ is not obtained in any play. If there is $\alpha > \gamma$ below α_0 *which is obtained in a move of II, let* α be the first such ordinal, otherwise set $\alpha = \alpha_0$. In either case, $\alpha = \alpha_n = rk_{D_n}(g_n)$ for some D_n, g_n which appear in a play in K. So we have a winning strategy for II in the game $G(D_n, g_n, \alpha)$ which furnishes only ordinals below γ (as no ordinal in the interval $[\gamma, \alpha)$ is obtained), so this is actually a winning strategy for $G(D_n, g_n, \gamma)$, contradicting $rk_{D_n}(g_n) = \alpha$.

Applying the claim we get for any $\alpha < \tau$ a filter $D_\alpha \supseteq D^*$ and a function $g_\alpha <_{D_\alpha} g$ such that $(D_\alpha, g_\alpha, \alpha)$ appears in a play in K (D_α as D_n, g_α as g_n, α as α_n in the n move in a play in K). By the strategy II is supposed to use we get that $rk'_{D_\alpha}(g_\alpha) = rk_{D_\alpha}(g_\alpha) = \alpha$. The number of filters over ω_1 is $\leq 2^{2^{\aleph_1}}$ and $cf(\tau) > 2^{2^{\aleph_1}}$; this implies that $S \subseteq \tau$ exists, $|S| = \tau$, and one filter D' can be found such that for $\alpha \in S$, $D_\alpha = D'$.

We now prove that the collection $\{g_\alpha | \alpha \in S\}$ satisfies the following 1-4.

1. $g_\alpha <_{D'} g$, because g_α is g_n and D' is D_n for $n > 0$ in some move in the game $G(D^*, g, rk_{D^*}(g))$.

2. Let $\beta < \alpha$ in S. *Claim:* $g_\beta <_{D'} g_\alpha$. *Proof:* Otherwise $A =^{def} \{i < \aleph_1 | g_\beta(i) \geq g_\alpha(i)\} \neq \emptyset \; mod D'$, $\alpha = rk_{D'}(g_\alpha)$ and D' is g_α-good, hence $\alpha = rk_{(D'+A)}(g_\alpha)$. Similarly $\beta = rk_{D'+A}(g_\beta)$. But $g_\alpha \leq_{D'+A} g_\beta$ hence $\alpha = rk_{D'+A}(g_\alpha) \leq rk_{D'+A}(g_\beta) = \beta$ by Lemma 5.3(1), contradicting $\beta < \alpha$.

3. $T_{D'}(g_\alpha) < \tau$. Because $T_{D'}(g_\alpha) \geq \tau$ would imply $\|g_\alpha\|_{D'} \geq \tau$ (Lemma 5.4). But $\|g_\alpha\|_{D'} \leq rk'_{D'}(g_\alpha)$ by Lemma 5.3(3) and $rk'_{D'}(g_\alpha) = \alpha$ as we know, so $\|g_\alpha\|_{D'} \leq \alpha < \tau$. Contradiction.

4. If D'' is any filter containing D' such that II wins in $G(D'', g_\alpha)$, the arguments in 3 above work again to deduce that $\alpha = rk'_{D''}(g_\alpha) \geq \|g_\alpha\|_{D''}$ and hence $T_{D''}(g_\alpha) < \tau$. But as D' is g_α-good player II wins $G(D' + A, g_\alpha)$ whenever $A \subseteq \aleph_1$, $A \neq \emptyset \; mod D'$ hence for such $A \, T_{D'+A}(g_\alpha) < \tau$.

Finally, as $|S| = \tau$ we can reenumerate the functions g_α so that the index set is of τ.

6 On the power of singular cardinals Here is a general description, and the detailed exposition begins with 6.1.

Here we shall present our main results which have meaning also when $\aleph_\alpha = \alpha$. In the first Theorem 6.2 we prove for $\aleph_\alpha$ singular $> 2^{2^{\aleph_1}}$, $cf(\aleph_\alpha) = \aleph_1$, $[\beta < \alpha \Rightarrow \aleph_\beta^{\aleph_0} < \aleph_\alpha\}$ that if there is no weakly inaccessbile cardinal below $\aleph_\alpha$ *then* there is no such cardinal also below $\aleph_\alpha^{\aleph_1}$.

Assume by negation that there exist a weakly inaccessible τ below $\aleph_\alpha^{\aleph_1}$ $(= 2^{\aleph_\alpha}$ if $\aleph_\alpha$ is strong limit).

Our assumptions satisfy Theorem 5.5 (for the filter D from 4.15—the winning filter for II in all the games $G(D, g)$).

Take the sequence of functions of length τ from the theorem. By Theorem 2.3 we can define $g_\tau: \aleph_1 \to \aleph_\alpha$ which is their supremum. Now we assumed that below $\aleph_\alpha$ there is no weakly inaccessible, so for $i < \aleph_1$, $g_\tau(i)$ can be successor ordinal, zero, $\aleph_0$, singular ordinal, or a successor cardinal. Denote the sets of ordinals $i < \aleph_1$ such that $g(i)$ is one of the five types of ordinals as mentioned above by A_1, A_2, A_3, A_4, A_5, respectively; since $\aleph_1 = \bigcup_{n=1}^{5} A_n$ there must be $1 \leq k \leq 5$ such that $A_k \neq 0 \; mod D$. We check each of the five possibilities and shall get contradictions.

By $\aleph_\lambda(\aleph_0)$ denote the first cardinal $\aleph_\beta$ such that $\aleph_\beta = \beta$, and $cf(\aleph_\beta) = \lambda$. In order to be independent from other works we shall introduce a lemma of Galvin and Hajnal from [4]. We then prove Theorem 6.6 which for $\aleph_{\omega_1}(\aleph_0)$ strong limit will give us the inequality $2^{\aleph_{\omega_1}(\aleph_0)} < \aleph_{(2^{2^{\aleph_1}})^+}(\aleph_0)$. The proof will be simple by using again Theorem 5.5 and the mentioned lemma.

6.1 Definition A cardinal τ is called weakly inaccessible iff τ is a limit regular uncountable cardinal.

6.2 Theorem *Let $\aleph_\alpha > 2^{2^{\aleph_1}}$ be of cofinality $\aleph_1$, and $\aleph_\beta^{\aleph_0} < \aleph_\alpha$ for all $\beta < \alpha$. Then: There is no weakly-inaccessible $\tau < \aleph_\alpha \Rightarrow$ There is no weakly-inaccessible $\tau \leq \aleph_\alpha^{\aleph_1}$.*

Proof: $\aleph_\alpha$ is a singular cardinal hence α is a limit ordinal and $cf(\aleph_\alpha) = cf(\alpha) = \aleph_1$; let $\langle \alpha_i < \alpha | i < \omega_1 \rangle$ be an increasing sequence cofinal in α, i.e., $\alpha = \bigcup_{i<\omega_1} \alpha_i$. Let D be the filter we obtained in 4.15: II wins in $G(D, g)$ for any $g \in {}^{\aleph_1}Ord$.

Define g: $\aleph_1 \to \aleph_\alpha$ by $g(i) = \aleph_{\alpha_i}^{\aleph_0}$, then $T_D(g) = \aleph_\alpha^{\aleph_1}$ (Theorem 2.8).

Suppose by contradiction that although there is no weakly inaccessible cardinal below $\aleph_\alpha$, there is a weakly inaccessible $\tau \leq \aleph_\alpha^{\aleph_1}$. Let $\langle g_\xi | \xi < \tau \rangle$ and $D' \supseteq D$ be given by Theorem 5.5, i.e.,

1. $g_\xi <_{D'} g$ for $\xi < \tau$
2. $\xi < \zeta \Rightarrow g_\xi <_{D'} g_\zeta$
3. $T_{D'+A}(g_\xi) < \tau$ for any $A \neq \emptyset$ *mod* D' and $\xi < \tau$.

Let $g_\tau \in {}^{\aleph_1}Ord$ be the least upper bound of $\{g_\xi | \xi < \tau\}$ given by Theorem 2.3, so $g_\xi \leq_{D'} g_\tau$ (we can assume $g_\xi(i) < g_\tau(i)$ for all $i < \aleph_1$) and not only g_τ is a least upper bound of $\{g_\xi | \xi < \tau\}$, but also if $h <_{D'} g_\tau$ then $h <_{D'} g_\xi$ for some $\xi < \tau$. Because $g_\tau \leq_{D'} g$ we can assume $(\forall i \in \aleph_1)$ $(g_\tau(i) < \aleph_\alpha)$ so that $g_\tau(i)$ is either a successor ordinal, or zero or $\aleph_0$, or a singular ordinal, or a successor cardinal (one of the form μ^+). But $g_\tau(i)$ is never a weakly inaccessible cardinal as there are none in $\aleph_\alpha$. So there are just five possibilities which will be ruled out, bringing a contradiction.

Possibility I $A_1 = \{i < \aleph_1 | g_\tau(i)$ is a successor ordinal$\} \neq \emptyset$ mod D'.

Define, $h(i) = \begin{cases} g_\tau(i) - 1 & i \in A_1. \\ g_0(i) & i \notin A_1. \end{cases}$

As $g_0 <_{D'} g_\tau$, $h <_{D'} g_\tau$, so for some $\xi < \tau$, $h <_{D'} g_\xi <_{D'} g_\tau$, but this means that for D'-almost all $i < \aleph_1$, $h(i) < g_\xi(i) < g_\tau(i)$. Yet for $i \in A_1$ $h(i) = g_\tau(i) - 1$, contradiction. We can assume that $g_\tau(i)$ is never a successor ordinal.

Possibility II Not I but $A_2 = \{i < \aleph_1 | g_\tau(i) = 0\} \neq \emptyset$ *mod* D'. This case is impossible because $[g_0 <_{D'} g_\tau \Rightarrow g_\tau(i) > 0$ for almost all $i]$. We can hence assume that $g_\tau(i)$ is always a limit.

Possibility III $A_3 = \{i < \aleph_1 | g_\tau(i) = \aleph_0\} \neq \emptyset$ *mod* D'. Now $g_\xi <_{D'} g_\zeta$ for all $\xi < \tau$ and there are only $\leq 2^{\aleph_1}$ functions in ${}^{A_3}\aleph_0$. So for some $\xi < \zeta < \tau$: for all $i \in A_3$ $g_\xi(i) < \aleph_0$ iff $g_\zeta(i) < \aleph_0$ and then $g_\xi(i) = g_\zeta(i)$. Hence $g_\xi <_{D'} g_\zeta <_{D'} g_\tau$ rule out possibility III.

Possibility IV $A_4 = \{i < \aleph_1 | g_\tau(i)$ is a singular limit ordinal$\} \neq$ *mod* D'.

Define $e(i) = \begin{cases} cf(g_\tau(i)) & i \in A_4. \\ 0 & i \notin A_4. \end{cases}$

Then $e <_{D'} g_\tau$, therefore $(\exists \xi < \tau)$ $(e <_{D'} g_\xi)$ and as $T_{D'+A_4}(g_\xi) < \tau$, $T_{D'+A_4}(e) < \tau$. Say $\chi = T_{D'+A_4}(e)$. First pick c_i $cf(g_\tau(i)) \to g_\tau(i)$, for $i \in A_4$, each c_i increasing and continuous such that the range of c_i is cofinal in $g_\tau(i)$ and $c_i(0) = 0$. For each g_ξ define $g_\xi^* <_{D'} e$ as follows.

We define $g_\xi^*(i) = \beta$ iff: $i \in A_4$ and $\beta < e(i)$ is the unique ordinal such that $g_\xi(i)$ is in the interval $[c_i(\beta), c_i(\beta+1))$, or $i \notin A_4$, $i < \omega_1$, $\beta = 0$. Obviously $g_\xi^* <_{D'+A_4} e$: Let $H \subseteq \{g_\xi^* | \xi < \tau\}$ be maximal with respect to the property that $[h_1 \neq h_2 \Rightarrow h_1 \neq_{D'+A_4} h_2]$. Then $|H| \leq \chi = T_{D'+A_4}(e)$. For any $\xi < \tau$ there is $h \in H$ such that $\{i < \aleph_1 | g_\xi^*(i) = h(i)\} \neq \emptyset \ mod D + A_4$ (by maximality of H). Now $\tau > 2^{\aleph_1} \cdot \chi$ is regular and that implies the existence of $S \subseteq \tau$, $|S| = \tau$ and $A \neq \emptyset \ mod D' + A_4$ and $h \in H$ such that $g_\xi^* \restriction A = h \restriction A$ for $\xi \in S$. If now we define $\bar{h}(i) = c_i(h(i)+1)$ for $i \in A_4$ and $\bar{h}(i) = 0$ for $i \notin A_4$, then $g_\xi <_{D'+A_4+A} \bar{h}$ follows for all $\xi \in S$. But this implies $T_{D'+A_4+A}(\bar{h}) = |S| = \tau$ (because $[\xi \neq \xi' \Rightarrow g_\xi \neq_{D'} g_{\xi'}]$ and $D' \subseteq D' + A_4 + A$). Yet $\bar{h} <_{D'} g$; hence $\bar{h} <_{D'} g_\zeta$ for some $\zeta < \tau$, so that $T_{D'+A_4+A}(g_\zeta) \geq \tau$. Contradiction to 3 above.

Possibility V $\quad A_5 = \{i < \aleph_1 | g_\tau(i)$ is a successor infinite cardinal$\} \neq \emptyset \ mod D'$.

$$\text{Define } f(i) = \begin{cases} \mu \text{ where } g_\tau(i) = \mu^+ \text{ if } i \in A_5. \\ 0 \text{ if } i \notin A_5. \end{cases}$$

Then $f <_{D'} g_\tau$, and so $f <_{D'} g_{\xi'}$ for some $\xi < \tau$, so that $f <_{D'} g_{\xi'}$ for all $\xi' \geq \xi$ (as the sequence of g_ξ is increasing). For each $\xi' \geq \xi$ for $(D' + A_5)$-almost all $i < \aleph_1$ $f(i) < g_{\xi'}(i)$, $g_\tau(i) = f(i)^+$. Hence $|g_{\xi'}(i)| = |f(i)|$ for $(D' + A_5)$-almost all $i < \aleph_1$. This easily shows that $T_{D'+A_5}(g_{\xi'}) = T_{D'+A_5}(f)$. But the g_ζ for $\zeta < \tau$ are increasing, thus $T_{D'+A_5}(g_{\xi'}) \geq |\xi'|$. Yet $f <_{D'} g_\xi$ implies $\mu' \overset{def}{=} T_{D'+A_5}(f) \leq T_{D'+A_5}(g_\xi) < \tau$, and when $\xi \leq \xi' < \tau$, $|\xi'| \leq T_{D'+A_5}(g_{\xi'}) = T_{D'+A_5}(f) = \mu$, hence $\tau \leq (\mu')^+$, but τ is a limit cardinal contradiction.

6.3 Notation Let us define $\aleph_\beta(\aleph_\alpha)$ by induction on β: $\aleph_0(\aleph_\alpha) = \aleph_\alpha$, $\aleph_1(\aleph_\alpha) = \aleph_{\alpha+\aleph_\alpha}$ [so when $\alpha < \aleph_\alpha$, $\aleph_1(\aleph_\alpha) = \aleph_{\aleph_\alpha}$], $\aleph_{\beta+1}(\aleph_\alpha) = \aleph_1(\aleph_\beta(\aleph_\alpha))$, and for limit $\beta = \delta$, $\aleph_\delta(\aleph_\alpha) = \bigcup_{\gamma<\delta} \aleph_\gamma(\aleph_\alpha)$.

Note that $cf(\aleph_{\beta+1}(\aleph_\alpha)) = cf(\aleph_\beta(\aleph_\alpha))$, and for limit ordinal δ $cf(\aleph_\delta(\aleph_\alpha)) = cf\delta$; and also $\lambda = \aleph_\delta(\aleph_\alpha) \Rightarrow \lambda = \aleph_\lambda$.

6.4 Notation If f: $\omega_1 \to Ord$, define a function $\aleph_f$ by $\aleph_f(i) = \aleph_{f(i)}$, and let us define $\hat{f}$: $\hat{f}(i) = \aleph_{f(i)}(\aleph_0)$. For such f and ordinal α, $f' = \alpha + f$ if $Dom f = \omega_1$ and $f'(i) = \alpha + f(i)$.

6.5 Lemma (Galvin–Hajnal) *If $\mu = T_D(\aleph_\beta) \geq 2^{\aleph_1}$ then $T_D(\aleph_{\beta+f}) \leq \mu^{+\|f\|_D}$.*

Proof: By induction on $\|f\|_D = \alpha$. When $\alpha = 0$; it is easy because the choice of μ. For α positive denote $\lambda = \mu^{+\|f\|_D}$ and assume that $\{h_i: i < \lambda^+\}$ exemplify that $T_D(\aleph_f) \geq \lambda^+$. For every i let $A_i = \{j: h_j <_D h_i\}$.

First possibility There exists an i such that $|A_i| \geq \lambda > 2^{\aleph_1}$; therefore

(1) $T_D(h_i) \geq \lambda$

As $h_i < \aleph_{\beta+f}$, $\{i : f(i) = 0\} = \emptyset$. There is a function $g_i <_D fi$ such that $h_i \leq \aleph_{\beta+g_i}$. So

(2) $T_D(h_i) = T_D(\aleph_{\beta+g_i})$

from $\|g_i\|_D < \|f_i\|_D$ and the induction assumption it follows that:

(3) $T_D(\aleph_{\beta+h_i}) \leq \mu^{+\|h_i\|_D}$;

As $g_i <_D f_i$, $\|g_i\|_D < \|f_i\|_D$. Hence, $\mu^{+\|g_i\|_D} < \mu^{+\|f_i\|_D} = \lambda$

but (1) + (2) + (3) implies $\lambda < \mu^{+\|f\|_D}$, contradicting the choice of λ.

Second possibility For every i, $|A_i| < \lambda$, by Theorem 1.5 there exists $B \subseteq \lambda^+$, $|B| = \lambda^+$ such that $[i \neq j \in B \Rightarrow i \notin A_j]$ but $2^{\aleph_1} < \lambda^+$ and this is impossible by Theorem 2.2.

6.6 Theorem *Assume that* $\aleph_{\omega_1}(\aleph_0)$ *is a strong limit, then* $2^{\aleph_{\omega_1}(\aleph_0)} < \aleph_{(2^{2^{\aleph_1}})^+}(\aleph_0)$.

Proof: By Theorem 2.8 (take D as the filter generated by the closed unbounded subsets) there exists a function $f_0\colon \omega_1 \to \aleph_{\omega_1}(\aleph_0)$, $T_D(f_0) = \aleph_{\omega_1}(\aleph_0)^{\aleph_1}$. Without loss of generality there is $f\colon \omega_1 \to \omega_1$ such that $f_0 = \hat{f}$.

Assume that $T_D(\hat{f}) \geq \chi \stackrel{def}{=} \aleph_{(2^{2^{\aleph_1}})^+}(\aleph_0)$ and we shall get a contradiction; *by Lemmas* 5.3(3) *and* 5.4 $T_D(\hat{f}) \leq \|\hat{f}\|_D \leq rk'_D(\hat{f})$.

Now apply Theorem 5.5 for regular cardinals μ, $(2^{2^{\aleph_1}}) < \mu < \chi$, so we get $\{f^i_\mu\colon i < \mu\}$ and $D_\mu \supseteq D$ such that the following holds:

1. $|i| \leq T_{D_\mu}(f^i_\mu) < \mu$, D_μ is a normal filter on $\aleph_1$.
2. $i \leq rk'_D(f^i_\mu)$ and for $i < j < \mu$, $f^i_\mu <_{D_\mu} f^j_\mu$.
3. Replacing D_μ by $D_\mu + A$ for $A \neq 0$ $Mod\, D_\mu$ does not matter.
4. f^μ_μ is the supremum of $\{f^i_\mu\colon i < \mu\}$, and $(\forall f <_{D_\mu} f^\mu_\mu)$ $(\exists i < \mu)$ $f <_{D_\mu} f^i_\mu)$ (by Theorem 2.3).

For each μ let $g_\mu\colon \omega_1 \to \omega_1$ such that $\hat{g}_\mu \leq_{D_\mu} f^\mu_\mu \leq_{D_\mu} \widehat{(g_\mu + 1)}$.

For each μ we found a pair (D_μ, g_μ), the number of such pairs is $\leq 2^{2^{\aleph_1}}$. Denote by $S_0 = \{\mu < \chi\colon \mu$ a regular cardinal greater than $2^{2^{\aleph_1}}\}$, $|S_0| = \chi$, $cf\chi = (2^{2^{\aleph_1}})^+$, therefore there are $S_1 \subseteq S_0$, $|S_1| = \chi$, and g_*, D_* such that for all $\mu \in S_1$ $D_\mu = D_*$ and $g_\mu = g_*$.

There exists a first μ_0 in S_1 such that $\hat{g}_* \leq_{D_*} f^{\mu_0}_{\mu_0}$. Therefore $\|\hat{g}_*\|_{D_*} \leq \|f^{\mu_0}_{\mu_0}\|_{D_*} \leq \sup_{i<\mu_0} \|f^i_{\mu_0}\|_{D_*} \leq \sup_{i<\mu_0} rk'_{D_*}(f^i_{\mu_0}) \leq \sup_{i<\mu_0} T_{D_*}(f^i_{\mu_0}) = \mu_0$. (By the choice of g_*, μ_0; by (4); by Lemma 5.3(3) by Lemma 5.4; by (1) above respectively.) So there exists a bound on $\|g_*\|_{D_*}$.

Fact A: $\widehat{(g_* + 1)} = \aleph_{g_* + g_*}$ (by definition of $\widehat{(g_* + 1)}$)
Fact B: $T_{D_*}(\widehat{(g_* + 1)}) \geq \chi$.

For each $\mu \in S_1$ $f^\mu_\mu \leq_{D_*} \widehat{(g_* + 1)}$; if $\mu \neq \kappa \in S_1$ and $\neg(f^\mu_\mu \neq_{D^*} f^\kappa_\kappa)$ *then* there exists an A such that $f^\mu_\mu =_{D_*+A} f^\kappa_\kappa$ (remember that by 3, replacing D_* by $D_* + A$ does not matter) but

$$\mu = T_{D_*}(f^\mu_\mu) = T_{D_*+A}(f^\mu_\mu) = T_{D_*+A}(f^\kappa_\kappa) = T_{D_*}(f^\kappa_\kappa) = \kappa.$$

A contradiction arises because we have chosen $\mu \neq \kappa$. So $\{\mu < S_1\colon \neg(f^\mu_\mu \neq_{D^*} \widehat{(g_* + 1)})\}$ has cardinality $\leq 2^{\aleph_1}$ and hence $\{f^\mu_\mu\colon f^\mu_\mu <_{D_*} \widehat{(g_* + 1)}, \mu \in S_1\}$ has cardinality $|S_1| = \chi$ and exemplifies Fact B.

Fact C: $\|g_*\|_{D_*} \leq \aleph_{\mu_0} < \chi$ and $rk'(g_*) \leq \aleph_{\mu_0}$.

[The first inequality by the paragraph before Fact A, the second inequality as $\mu_0 \in S$ hence $\mu_0 \in \chi$ and $(\forall \kappa < \chi)$ $(\aleph_\kappa < \chi)$ and the third inequality like the first.]

Now we summarize our facts. By Fact A:

$$T_{D_*}(\widehat{(g_* + 1)}) \leq T_{D_*}(\aleph_{g_*+g_*}),$$

by Lemma 6.5. $T_{D_*}(\aleph_{g_*+g_*}) \leq \aleph_{\|g_*+g_*\|_{D_*}}$, easily $\|g_* + g_*\|_{D_*} \leq rk'_{D_*}(g_* + g_*) \leq rk'_{D_*}(g_*) + rk'_{D_*}(g_*)$ hence by Fact C $rk'_{D_*}(g_* + g_*) < \chi$, hence (as $\chi = \aleph_\chi$) $\aleph_{\|g_*+g_*\|_{D_*}} < \chi$. Together $T_{D_*}((\widetilde{g_* + 1})) < \chi$ which is a contradiction to fact B.

Open Problem Can the bound from the last theorem be improved? I.e., is it true that $2^{\aleph_{\omega_1}(\aleph_0)} < \aleph_{(2^{\aleph_1})^+}(\aleph_0)$; Jech and Prikry proved this inequality using an additional hypothesis to ZFC in [6].

*7 **The use of forcing*** Silver's proof [*S*] used forcing, and only after it, "elementary" proofs (giving stronger results) were found. In fact, our original observation was that we can eliminate the "elementarity" from the results of [4]. So we shall see here how things are done with forcing.

7.1 Lemma *Suppose P is a forcing notion, $\underset{\sim}{D}$ a P-name of an ultrafilter of the Boolean algebra $\mathcal{P}(\omega_1)^V = \{A \subseteq \omega_1 : A \in V\}$, such that:*
(1) for any $p \in P$, $D_p = \{A \subseteq \omega_1 : p \Vdash_P \text{“}A \in \underset{\sim}{D}\text{”}\}$ is a normal filter on ω_1
(2) for $A \subseteq \omega_1$, $A \neq \emptyset \bmod D_p$ there is $q \geq p$, such that $D_q \supseteq D_p + A$.

Let $G \subseteq P$ be the generic set, so $\underset{\sim}{D}[G] = \bigcup_{p \in G} D_p$

Suppose further that $\lambda > |P|$ is a regular cardinal (in V) then:
(A) *In $V[G]$, there is no decreasing sequence $f_\alpha/\underset{\sim}{D}[G]$, $f_\alpha \in V$, $\alpha < |P|^+$, f_α: $\omega_1 \to Ord$.*
(B) *If for each $\alpha < \lambda$, $f_\alpha \in V$, and (in $V[G]$) $f_\alpha/\underset{\sim}{D}[G] < f_\beta/\underset{\sim}{D}[G]$ $\alpha < \beta$ (and $\langle f_\alpha : \alpha < \lambda\rangle \in V[G]$); then $\langle f_\alpha/\underset{\sim}{D}[G] : \alpha < \lambda\rangle$ has a least upper bound $f/\underset{\sim}{D}[G]$, $f \in V$.*
(C) *If $f \in V$, f: $\omega_1 \to Ord$ then the power (in $V[G]$) of $\{f/\underset{\sim}{D}_G : f/\underset{\sim}{D}_G < g/\underset{\sim}{D}_G (f \in V)\}$ is $\underset{p \in G}{Max}\ T_{D_p}(g)$, provided it is $> |P|$.*
(D) *$V^{\omega_1}/\underset{\sim}{D}_G = \{f/\underset{\sim}{D}_G : f \in V, f: \omega_1 \to V\}$ is an elementary extension of V, but is not necessarily well founded.*

Proof: (A) we can replace $\langle f_\alpha : \alpha < |P|^+\rangle$ by a cofinal subsequence which belongs to V, and then as it is decreasing in $V[G]$, there is $p \in P$ which forces this. By condition (2) of Lemma 7.1 $f_\alpha/D_p < f_\beta D_p$ for $\beta < \alpha$, but D_p is normal hence is $\aleph_1$-complete. Contradiction.

(B) Choose (working in $V[G]$) by induction $g_i \in V$, such that: $f_\alpha/D_G < g_i/D_G < g_j/D_G$ for $\alpha < \lambda$, $j < i$. If there is a last g_i we finish, otherwise let g_i be defined for $i < \delta$; hence by part A, $\delta < |P|^+$. There is $p_0 \in G$ which forces this situation. Let

$$F = \{g \in Ord^{\omega_1} : \text{for some } \alpha < \delta,\ p \in P,\ p \Vdash_P \text{“}g = g_\alpha\text{”}\}.$$

As in (A) w.l.o.g. $\langle f_\alpha : \alpha < \lambda\rangle \in V$, hence f_α/D_p is increasing.

For $g \in F$, $\alpha < \lambda$ let $A_{g,\alpha} = \{i : f_\alpha(i) < g(i)\}$, so $A_{g,\alpha}/D_{p_0}$ is (not strictly) decreasing. It is eventually constant, otherwise as λ is regular $> |P|$ we get a contradiction to condition (2) of Lemma 7.1, so there is $\alpha_g < \lambda$ such that for $\alpha_g \leq \alpha < \lambda$, $A_{g,\alpha}/D_{p_0} = A_{g,\alpha_g}/D_{p_0}$. So $\alpha(*) = \underset{g \in F}{Sup}\ \alpha_g < \lambda$. Define g^*: $\omega_1 \to Ord$ by:

$$g^*(i) \text{ is } min\{g(i)\colon g \in F,\ g(i) > f_{\alpha(*)}(i)\}$$
$$\text{if defined, zero otherwise.}$$

It is easy to check that g^* is a least upper bound.

(C) Clearly if $T_{D_p}(g) \geq \lambda$ and $p \in G$ then the set $F = \{f/D_G\colon f/D_G < g/D_G\ (f \in V)\}$ has power $\geq \lambda$ ($> |P|$). Suppose $\langle \underset{\sim}{f}_i\colon i < \lambda\rangle$ is a P-name of λ distinct elements of F, λ regular $> |P|$. We can find $p_i \in G$, $f_i \in \Pi(g(i) + 1)$, $p_i \Vdash_P$ "$\underset{\sim}{f}_i = f_i$." For some $S \subseteq \lambda$, $|S| = \lambda$ and $p = p_i$ for every $i \in S$. So $p \Vdash_P$ "$f_i/D_G \neq f_j/D_G$" for $i \neq j \in S$.

By condition (2) of Lemma 7.1, this implies $f_i \neq_D f_j$ (i.e., $\{\alpha\colon f_i(\alpha) \neq f_j(\alpha)\} \in D_p$), hence $T_{D_p}(g) \geq \lambda$, so we finish.

7.2 Definition For a filter D on ω_1, $P(D)$ is the following forcing notion: the conditions are $A \subseteq \omega_1$, $A \neq \emptyset\ mod D$, and the order is inverse inclusion. This forcing gives naturally a name $\underset{\sim}{D}$ of an ultrafilter on $\mathcal{P}(\omega_1)^V$.

7.3 Fact: For D a normal filter on ω_1, $P = P(D)$ satisfies the assumptions (1), (2) of Lemma 7.1 when we choose $D_A = D + A$ for $A \in P$ so D_G is just G.

7.4 Lemma *If λ is regular $> 2^{2^{\aleph_1}}$, then there is a normal filter D on ω_1 such that in $V^{\omega_1}/D_G = V^{\omega_1}/G(G \subseteq P(D)$ generic) there is an "ordinal" which defines a λ-like initial segment. We call V^{ω_1}/D_G V_D.*

7.4 Remark: This applies to other suitable forcing.

Proof: By Theorem 4.16 player II wins in the game $G(D, \lambda)$ for some normal filter D on ω_1. By Theorem 5.5 there are $g_\xi \in {}^{\aleph_1}Ord$ (for $\xi < \lambda$) and normal filter D_1 on ω_1 extending D satisfying (1), (2), (3), (4) from Theorem 5.5. By Theorem 2.3 $\langle g_\xi/D_1\colon \xi < \lambda\rangle$ has a strict least upper bound which we name g/D_1.

As for $\xi < \zeta < \lambda$, $g_\xi <_{D_1} g_\zeta <_{D_1} g$ clearly $T_{D_1}(g) \geq \lambda$. Hence $\Vdash_{\overline{P(D_1)}}$ "in $V^{\omega_1}/\underset{\sim}{D}_G$, there are $\geq \lambda$ 'ordinals' smaller than $g/\underset{\sim}{D}_G$". To conclude the proof we should show that:

(*) If $A \in P(D')$, $f\colon \omega_1 \to Ord$, ($f \in V$), then $A \Vdash_{\overline{P(D')}}$ "if $f/\underset{\sim}{D}_G < g/\underset{\sim}{D}_G$ then before $f/\underset{\sim}{D}_G$ there are (in $V^{\omega_1}/\underset{\sim}{D}_G$) less than λ 'ordinals'".

Suppose A, f is a counterexample then $A \Vdash_{\overline{P(D_1)}} f/\underset{\sim}{D}_G < g/\underset{\sim}{D}_G$, hence by Lemma 7.1(2) $f <_{D_1+A} g$; as $g \neq_{D_1} 0$ (remember $f_0 <_{D_1} g$) w.l.o.g. $f <_{D_1} g$. Now by Theorem 5.5(4) for every $B \neq 0\ mod D_1$ $T_{D_1+B}(f) < \lambda$. So by Lemma 7.1(2) we get that (*) holds.

7.5 Definition

(1) Let $\bar{g}$ denote a sequence of the form $\langle g_\eta\colon \eta \in u\rangle$, $u \in fc({}^\omega\omega)$ = the family of nonempty subsets of ${}^{\omega>}\omega$ closed under initial segment, $g_\eta \in {}^{\omega_1}Ord$. Let *Dom* $\bar{g} = u$, *Range* $\bar{g} = \{g_\eta\colon \eta \in u\}$.

(2) Let $\bar{\alpha}$ denote a sequence $\langle \alpha_\eta\colon \eta \in u\rangle$, α_η an ordinal, $u \in fc({}^\omega\omega)$. If $\alpha_\eta < \alpha_\nu$ when $\nu < \eta$, $\bar{\alpha}$ is called decreasing.

(3) We say $\bar{g}$ (or $(\bar{g}, \bar{\alpha})$) is decreasing for D if $\eta < \nu \Rightarrow g_\eta <_D g_\nu$ (and $\bar{\alpha}$ is decreasing).

7.6 Definition

(1) For D a normal filter on $\aleph_1$, $\bar{\alpha}$ decreasing, $\bar{g}$ decreasing, we define a game $G^*(D, \bar{g}, \bar{\alpha})$. It is played by two players, I and II, as follows. Denote $D = D_0$, $\bar{g} = \bar{g}_0$, $\bar{\alpha} = \bar{\alpha}_0$, $u_0 = Dom\, \bar{g}_0$.

Player I begins and in the first move he chooses $A_1 \subseteq \aleph_1$, $A_1 \neq \emptyset \bmod D_0$ and u_1, $u_0 \subseteq u_1 \in fc({}^{\omega<}\omega)$ and functions $g_\eta (\eta \in u_1 - u_0)$ from $\aleph_1$ to the ordinals such that $\langle g_\eta : \eta \in u_1 \rangle$ is $(D + A_1)$-decreasing. Then player II for the first move chooses a normal filter D_1 on $\aleph_1$ extending $D_0 + A$ and ordinals α_η $(\eta \in u_1 - u_0)$ such that $=\bar{\alpha}_1$ $\langle \alpha_\eta : \eta \in u_1 \rangle$ is decreasing.

In general in the nth move player I chooses $A_n \neq \emptyset \bmod D_{n-1}$ u_n, $u_{n-1} \subseteq u_n \in fc({}^{\omega>}\omega)$ and functions $g_\eta (\eta \in u_n - u_{n-1})$ from $\aleph_1$ to the ordinals such that $\langle g_\eta : \eta \in u_n \rangle$ is $(D_{n-1} + D_n)$-decreasing. Then player II chooses a normal filter D_n on $\aleph_1$ extending $D_{n-1} + A_n$ and ordinals α_η $(\eta \in u_n - u_{n-1})$ such that $\langle \alpha_\eta : \eta \in u_n \rangle$ is decreasing. This play is finished when player II has no legal move or after ω moves. (Player I always has a legal move.) If player II has no legal move, he loses. If the play lasts ω moves, player II wins.

Convention: We write g instead of $\bar{g}$ for $\bar{g} = \langle g_\eta : \eta \in \{\langle\rangle\}\rangle$ $g_{\langle\rangle} = g$, and α instead of g when g is constantly α. (This applies to Definitions 7.5 and 7.6.)

7.7 Definition For D (a normal ultrafilter on $\aleph_1$) and D-decreasing $\bar{g}$ we define a game $G^*(D, \bar{g})$. Let $D_0 = D$, $\bar{g}_0 = g$. In the nth move, player I chooses $A_n \subseteq \omega_1$, $A_n \neq \emptyset \bmod D_{n-1}$ and $\bar{g}_n$, extending g_{n-1} (i.e., $\bar{g}_{n-1} = \bar{g}_n \restriction Dom\, \bar{g}_{n-1}$) such that $\bar{g}_n$ is $(D_{n-1} + A_n)$-decreasing and player II chooses D_n extending $D_{n-1} + A_n$.

In the end player II wins if $\bigcup_{n<\omega} Dom\, g_n$ has no infinite branch.

7.8 Claim

(1) *Every game $G^*(D, \bar{g}, \bar{\alpha})$ is determined. Moreover, the winner has a winning strategy whose decision depends on the present situation only (and not on the series of moves leading to it).*

(2) *For $G^*(D, \bar{g}, \bar{\alpha})$ we can make player I choose $D_{n-1} + A_n$ instead of A_n, and $g_\eta/(D_{n-1} + A_n)$ instead of g_η (see comparison with Remark* 3.3).

(3) *If player II wins in $G^*(D, \bar{g}, \bar{\alpha})$, $A \neq \emptyset \bmod D$, $u = Dom\, g = Dom\, \bar{\alpha}$, $(\forall \eta \in u)$ $(\alpha'_\eta \geq \alpha_\eta \wedge g'_\eta \leq_D g_\eta)$, $\bar{g}' = \langle g'_n : \eta \in u \rangle$ is $(D + A)$-decreasing and $\bar{\alpha}' = \langle \alpha_\eta : \eta \in u \rangle$ is D-decreasing then player II wins $G^*(D + A, \bar{g}', \bar{\alpha}')$ (compare with Lemma* 3.4).

(4) *If player II wins in $G^*(D, \bar{g}, \bar{\alpha})$ then we can find a decreasing $\bar{\alpha}' = \langle \alpha'_\eta : \eta \in u \rangle$ $(u = Dom\, \bar{g} = Dom\, \bar{\alpha})$ such that $(\forall \eta)$ $[\Pi_{i<\omega_1}(g_{\langle\rangle}(i) + 1) + 2^{\aleph_1})^+ > \alpha'_\eta]$ and player II wins $G^*(D, \bar{g}, \bar{\alpha}')$ (compare with Lemma* 3.5).

(5) *Suppose D and $\bar{g} = \langle g_\eta : \eta \in u \rangle$ are given, and for any decreasing $\bar{\alpha} = \langle \alpha_\eta : \eta \in u \rangle$, satisfying $\alpha_{\langle\rangle} < (2^{\aleph_1} + \Pi_i(g_{\langle\rangle}(i) + 1))^+$ player I wins $G^*(D, \bar{g}, \bar{\alpha})$.*

Then player I wins in $G^*(D, \bar{g})$ (like Lemma 3.6).

(6) *If player II wins $G^*(D, \bar{g}, \bar{\alpha})$ then he wins $G^*(D, \bar{g})$ (player II will play "in the side" a play of $G^*(D, \bar{g}, \bar{\alpha})$ in which he uses his strategy).*

(7) *The following are equivalent for a given D and D-decreasing $\bar{g} = \langle g_\eta : \eta \in u \rangle$.*

(A) *Player II wins in* $G^*(D, \bar{g})$.

(B) *Player II wins in* $G^*(D, \bar{g}, \bar{\alpha})$ *for some decreasing* $\bar{\alpha} = \langle \alpha_\eta : \eta \in u \rangle$, $\alpha_{\langle\rangle} < (2^{\aleph_1} + \Pi_i(g_{\langle\rangle}(i) + 1))^+$

(C) *For some* $\bar{\alpha}$ *player II wins in* $G^*(D, \bar{g}, \bar{\alpha})$.

(D) *Player II wins* $G^*(D, \bar{g}, \bar{\alpha})$ *whenever* $\bar{\alpha}$ *is decreasing,* $\alpha_\eta \geq (2^{\aleph_1} + \Pi(g_{\langle\rangle}(i) + 1))^+$ *for each* η.

[*Proof:* (C) ⇒ (B) by Claim 7.8(4)
(D) ⇒ (C) trivial
(B) ⇒ (D) by Claim 7.8(3)
¬(B) ⇒ ¬(A) by Claim 7.8(5) as the games $G^*(D, \bar{g}, \bar{\alpha})$ are determined
(C) ⇒ (A) by Claim 7.8(6).]

(8) *Each game* $G^*(D, \bar{g})$ *is determined* (*by Claim* 7.8(7) or use Borel determinancy).

(9) *For any* D, $\bar{g}$ *if player I wins in* $G^*(D, \bar{g})$ *then I wins in* $G^*(D, \bar{g}')$ *for some* $\bar{g}'$ *satisfying* $(\forall \eta \in Dom\, \bar{g}')$ $(\forall i < \aleph_1)$ $[g'(i) < (2^{2^{\aleph_1}})^+]$ (*Proof:* like Lemma 3.8.)

(10) *If player II wins in* $G^*(D, \bar{g})$, $\bar{g} = g_{\langle\rangle}$, $g_{\langle\rangle} = \alpha$ *whenever* $\alpha < (2^{2^{\aleph_1}})$ *then player II wins in every* $G^*(D, \bar{g})$. (*Proof:* like Conclusion 3.9.)

7.9 Claim

(1) *If* $(2^{2^{\aleph_1}})^+ \leq_D g_{\langle\rangle}$, *and player II wins the game* $G^*(D, \bar{g})$ *then D is nice where*

7.10 Definition We say that (a normal filter) D (on ω_1) is *nice* if player II wins in every game $G^*(D, \bar{g})$.

7.11 Lemma

(1) *If for some* λ, E (*from Definition* 4.2) *is nontrivial* (*i.e.,* $\varnothing \notin E$) $E^* \supseteq E$ *is normal then* D_{E^*} (*see Lemma* 4.9) *is nice.* (*Proof:* like Theorem 4.13.)

(2) *If for any* $A \subseteq (2^{2^{\aleph_1}})^+$ *there is a transitive class* V_A^* *which is a model of ZFC containing the ordinals,* $A \in V_A^*$ *and* $V^* \models$ *"there is* λ *satisfying Assumption 4.1"* (*e.g.,* $\lambda \to ((2^{2^{\aleph_1}})^+)_2^{<\omega}$) *then there is a nice D.* (*Proof:* like Conclusion 4.14.)

(3) *For a universe V of set theory, if* $(\exists \lambda > 2^{2^{\aleph_1}})$ $\lambda^{\aleph_1} > \lambda^+$ *then there is a nice D.*[6]

7.12 Lemma *If player II wins* $G^*(D, \alpha, \langle\gamma\rangle)$ *P collapse the* $|\alpha|^{\aleph_1}$ *to* $\aleph_0$ *then there is a P-name* $\underset{\sim}{D}$ *so that* Lemma 7.1(1) *and* (2) *hold and* $\Vdash_P$ *"*$\{g/\underset{\sim}{D}_G : g \in V, g \leq_{\underset{\sim}{D}G} \alpha\}$ *is well ordered of order type* $\leq \gamma$*".*

Proof: For notational simplicity let P be the levi collapse of $|\alpha|^{\aleph_1}$ to $\aleph_0$.

Let $\mu = |\alpha|^{\aleph_1}$ and w.l.o.g. $\alpha \geq 2$. Let $\{f_i : i < \mu\}$ be a list of the functions from ω_1 to α. Let $\{A_i : i < 2^{\aleph_1}\}$ list all subsets of ω_1.

Note that $p \in P$ *iff* p is a function from some $n = n(p) < \omega$ into μ. Also, $p \leq q$ iff $p \subseteq q$. So $Dom\, p = \{0, \ldots, n(p) - 1\}$.

We now *define* by induction on $n < \omega$, for every $h \in P$ with $n(h) = n$, the following $\bar{g}^h$, A^h, $\bar{\alpha}^h$, D^h such that

(*)

(1) $\bar{g}^{h\restriction 1}$, $A^{h\restriction 1}$, $\alpha^{h\restriction 1}$, $D^{h\restriction 1}$; $g^{h\restriction 2}$, $A^{h\restriction 2}$, $\bar{\alpha}^{h\restriction 2}$, $D^{h\restriction 2}, \ldots$, is a play of $G^*(D, \alpha, \langle\gamma\rangle)$ in which player II uses a winning strategy.

(2) $A^{h\restriction 1}$ is $A_{h(0)}$ if $h(0) < 2^{\aleph_1}$ and $A_{h(0)} \neq \emptyset \ mod D$, and ω_1 otherwise. For $m > 0$, $A^{h\restriction(m+1)}$ is $A_h(m)$ if $h(m) < 2^{\aleph_1}$ and $A_{h(i)} \neq \emptyset \ mod D^{h\restriction m}$ and $A^{h\restriction(m+1)} = \omega_1$ otherwise.
(3) $\bar{g}^{h\restriction m}$ is $D^{h\restriction m}$-decreasing, $Rang(\bar{g}^{h\restriction m}) = \{f_i\colon i \in Rang(h)\}$ and if $\eta \in Dom(\bar{g}^{h\restriction m})$, $i \in Rang\, h$ *then either* $g_\eta^{h\restriction m} \leq_{D^{h\restriction m}} f_i$ *or* $f_i \in \{g^{h\restriction m}_{\eta\frown\langle l\rangle}\colon \eta^\frown\langle l\rangle \in Dom(\bar{g}^{h\restriction m})\}$.

The generic $G \subseteq P$ provide us with the ultrafilter $D_G =^{def} \bigcup_{P\in G} D_P$ on the *Boolean Algebra* $\mathcal{P}(\omega_1)^V$ and with a witness for the well foundedness of $\{g/D_G\colon g \in {}^{(\omega_1)}\alpha\}$.

7.13 Theorem (Galvin–Hajnal) *If δ is a limit ordinal of cofinality $\aleph_1$, $2^{\aleph_1} < \aleph_\delta$, $(\forall\alpha < \delta)\ \aleph_\alpha^{\aleph_0} < \aleph_\delta$, $|\delta|^{\aleph_1} < \aleph_\delta$ then $\aleph_\delta^{\aleph_1} < \aleph_{(|\delta|^{\aleph_1})^+}$.*

Proof: By Definition 7.10(3) there is a nice D (or the conclusion holds trivially). Let γ be such that player II wins $G^*(D, \delta, \langle\gamma\rangle)$. By Lemma 7.12 letting P be the collapse of $|\delta|^{\aleph_1}$ to $\aleph_0$ there is $\underset{\sim}{D}_G$ (a P-name) satisfying its conclusion. Let $G \subseteq P$ be generic over V in $V[G]$ we can compute the power $V^* = V^{\omega_1}/\underset{\sim}{D}_G$, and j be the natural embedding of V into V^*. This is a model of ZFC, not necessarily well-founded, but it is well-founded below $j(\delta)$, moreover it has order type $\leq\gamma$. Now we can prove by induction on a, $V^* \models$ "a an ordinal $\leq j(\delta)$" that $\{f\colon V^* \models \text{“}f < \aleph_a\text{”}\}$ has power $\leq \aleph_{\gamma(a)}$ where $\gamma(a)$ is the order type of $\{a'\colon V^* \models a' < a\}$.

7.14 Theorem *Suppose $[\beta < \delta \Rightarrow \aleph_\beta^{\aleph_0} < \aleph_\delta]$, δ a limit ordinal of cofinality $\aleph_1$ and $2^{2^{\aleph_1}} < \aleph_\delta$.*
(1) If $\zeta < \omega_1$ and there is no weakly inaccessible ζ-Mahlo cardinal $\mu < \aleph_\delta$, then there is no such cardinal $\leq \aleph_\delta^{\aleph_1}$.
(2) If $\zeta < \omega_1$ there are $\leq \kappa$ weakly inaccessible ζ-Mahlo cardinals $\leq \aleph_\delta$ then there are $\leq \kappa^{2^{\aleph_1}}$ such cardinals $\leq \aleph_\delta^{\aleph_1}$.
(3) If there is no weakly inaccessible ω_1-Mahlo cardinal $< \aleph_\delta$ then there is no weakly inaccessible $(2^{\aleph_1})^+$-Mahlo cardinal $\leq \aleph_\delta^{\aleph_1}$ (really ζ-Mahlo for some $\zeta < (2^{\aleph_1})^+$).

Proof: (1) Let $\lambda \leq \aleph_0^{\aleph_1}$ be a counterexample.

By Lemma 7.4, for some D (a normal ultrafilter on ω_1), letting $P(D)$ be as in Definition 7.2, $G \subseteq P(D)$ generic over V, in V^{ω_1}/G there is an "ordinal" a which defines a λ-like initial segment of the "ordinals" of V^{ω_1}/G. Now for a closed unbounded set of cardinals $\mu < \lambda$, if $cf\mu > 2^{\aleph_1}$ there is a "cardinal" a_μ V^{ω_1}/G, which defines a μ-like initial segment of the "ordinals" of V^{ω_1}/G (see Definition 7.7(B)). Now we can prove by induction on $\xi \leq \zeta$ that:

(*) If a_μ is defined, μ a weakly inaccessible ξ-Maho cardinal then $V^{\omega_1}/G \models$ "a_μ is a weakly inaccessible (ξ/G)-Mahlo cardinal".

For $\xi = \zeta$ we get a contradiction (to the relevant variant of last theorem).

(2) Left to the reader.

(3) Combine the proofs of Theorem 7.14.

7.14 Definition Let:

$$\aleph_\alpha^0(\lambda) = \lambda^{+\alpha}$$

$\aleph^{i+1}_\alpha(\lambda)$ is defined by induction on α:

$$\aleph^{i+1}_0(\lambda) = \lambda$$
$$\aleph^{i+1}_{\alpha+1}(\lambda) = \aleph^i_{(\aleph^{i+1}_\alpha(\lambda))^+}(\aleph^{i+1}_\alpha(\lambda))$$
$$\aleph^{i+1}_\delta(\lambda) = \bigcup_{\alpha<\delta} \aleph^{i+1}_\alpha(\lambda)$$

$\aleph^\xi_\alpha(\lambda) = \bigcup_{\zeta<\xi} \aleph^\zeta_\alpha(\lambda)$ (for ξ a limit ordinal).

7.15 Fact:
(1) $\aleph^i_\alpha(\lambda)$ is a monotonically increasing function of i, α, λ (but not necessarily strictly).
(2) $\aleph^i_\alpha(\lambda) \geq \lambda$, α, and when $\alpha > 0$, also $\geq i$.
(3) $\aleph^i_\alpha(\lambda)$ is strictly increasing in α.
(4) $\{\aleph^{i+1}_\delta(\lambda)$: δ a limit ordinal$\}$ is equal to $\{\mu$: $\aleph^i_\mu(\lambda) = \mu\}$ (i.e., set of fixed points of $\aleph^i_x(\lambda)$ (as a function in x).
(5) For ξ limit

$$\{\aleph^\xi_\delta(\lambda)\text{: } \delta \text{ a limit ordinal}\} \text{ is equal to } \{\mu\text{: } \aleph^i_\mu(\lambda) = \mu \text{ for every } i < \xi\}.$$

(6) $\aleph^i_{\alpha+\beta}(\lambda) = \aleph^i_\beta(\aleph^i_\alpha(\lambda))$.

7.16 Theorem *Suppose* $\forall\mu < \aleph^2_{\omega_1}(\aleph_0)$ $[\mu^{\aleph_0} < \aleph^2_{\omega_1}(\aleph_0)]$, $2^{2^{\aleph_1}} < \aleph^2_{\omega_1}(\aleph_0)$. *Then* $(\aleph^2_{\omega_1}(\aleph_0))^{\aleph_1} < \aleph^2_\xi(\aleph_0)$ *for some* $\xi < (2^{2^{\aleph_1}})^+$.

Proof: As usual we can assume that there is a nice D (normal filter on ω_1). Let $\xi_0 < \omega_1$ be such that $2^{2^{\aleph_1}} < \aleph^2_{\xi_0}(\aleph_0)$. For $\xi < (2^{2^{\aleph_1}})^+$ let $\mu_\xi = \aleph^2_\xi(\aleph_0)$, and so μ_ξ is increasing continuous, $\mu_{\xi+1} = \aleph^1{}_{(\mu_\xi)^+}(\mu_\xi)$. Let for $\xi < (2^{2^{\aleph_1}})^+$, $\zeta < \mu^+_\xi$, $\mu_{\xi,\zeta}$ be $\aleph^1_\zeta(\mu_\xi)$ so $\mu_{\xi,\zeta+1} = \aleph^0_{(\mu^+_{\xi,\zeta})}(\mu_{\xi,\zeta})$.

Let for $\xi < (2^{2^{\aleph_1}})^+$, $\zeta < \mu^+_\xi$, and $\gamma < \mu^+_{\xi,\zeta}$, $\mu_{\xi,\tau,\zeta}$ be $\aleph^0_\gamma(\mu_\xi) = \mu^{+\gamma}_\xi$.

Clearly for $\xi \geq \xi_0$, $\mu_{\xi,\zeta,\gamma} > 2^{2^{\aleph_1}}$ (when defined) we assume that the conclusion fails.

By Theorem 2.8 there is a function $f \in {}^{\aleph_1}\aleph_1$ such that $(\aleph^2_{\omega_1}(\aleph_0))^{\aleph_1} = T_{D_0}(\aleph^2_f(\aleph_0))$ $D_0 = \{A \subseteq \omega_1\text{: } |\omega_1 - A| \leq \aleph_0\}$. By Theorem 5.5, Theorem 2.3 for each $\xi < (2^{2^{\aleph_1}})^+$, $\xi \geq \xi_0$, $\zeta < \mu^+_\xi$, $\gamma < \mu^+_{\xi,\zeta}$ there are $g_{\xi,\zeta,\gamma}$ and a normal filter $D_{\xi,\zeta,\gamma}$ on ω_1 such that:

$$g_{\xi,\zeta,\gamma} \leq_{D_{\xi,\zeta,\gamma}} f,$$

$$T_{D_{\xi,\zeta,\gamma}+A}(g_{\xi,\zeta,\gamma}) = \mu^+_{\xi,\zeta,\gamma} \text{ for every } A \neq \emptyset \mod D_{\xi,\zeta,\gamma}$$

$$T_{D_{\xi,\zeta,\gamma}+A}(g) \leq \mu_{\xi,\zeta,\gamma} \text{ for every } g <_{D_{\xi,\zeta,\gamma}+A} g_{\xi,\zeta,\gamma}, \quad A \neq \emptyset \mod D_{\xi,\zeta,\gamma}.$$

For $\xi_0 \leq \xi < (2^{2^{\aleph_1}})^+$, $\zeta < \mu^+_\xi$, as $\mu_{\xi,\zeta} > 2^{2^{\aleph_1}}$ there is $D_{\xi,\zeta}$ such that:

$$S_{\xi,\zeta} = \{\gamma < \mu^+_{\xi,\zeta}\text{: } D_{\xi,\zeta,\gamma} = D_{\xi,\zeta}\} \text{ has power } \mu^+_{\xi,\zeta}$$

and for $\xi_0 \leq \xi < (2^{2^{\aleph_1}})^+$ there is D_ξ such that

$$S_\xi = \{\zeta < \mu^+_\xi\text{: } D_{\xi,\zeta} = D_\xi\} \text{ has power } \mu^+_\xi$$

and there is D such that

$$S = \{\xi\text{: } \xi_0 < \xi < (2^{2^{\aleph_1}})^+, D_\xi = D\} \text{ has power } (2^{2^{\aleph_1}})^+.$$

Now we use $V^* = V^{\omega_1}/\underset{\sim}{D}[G]$ (as in Lemma 7.12's proof: $G \subseteq P$ generic over V, $\underset{\sim}{D}$ a P-name of an ultrafilter on $\mathcal{P}(\omega_1)^V$ satisfying Lemma 7.11(1), (2).)

In it for each $\gamma \in S_{\xi,\zeta}$, $\zeta \in S_\xi$, $\xi \in S$: $\{a$: $V_D \models$ "$a < g_{\xi,\zeta,\gamma}/G$ a an ordinal"$\}$ is $\mu^+_{\xi,\zeta,\gamma}$-like. Let $g_{\xi,\zeta,\gamma}/D$ be the least upper bound of $\{g_{\xi,\zeta,\gamma}/D$: $\gamma \in S_{\zeta,\gamma}\}$ (clearly easily).

Now suppose $\xi \in S$, ζ_1, $\zeta \in S_\xi$ then from the outside we know that the number of $\{a$: $V^* \models$ "$a < g_{\xi,\gamma}/\underset{\sim}{D}[G]$ a is a cardinal"$\}$ is larger than $\{a$: $V^* \models a < g_{\xi,\zeta_1}/\underset{\sim}{D}[\underset{\sim}{G}]$ a an ordinal$\}$.

Hence $V^* \models$ "$g_{\xi,\zeta}/\underset{\sim}{D}[G] > g_{\xi,\zeta_1}/D[\underset{\sim}{G}]$". Continuing, we get $V^* \models$ "$f/\underset{\sim}{D}[G] \geq \aleph^2_{\omega_1}(\aleph_0)$" and $\{a$: $V_D \models a < \omega_1\}$ has power $\leq 2^{\aleph_1}$, a contradiction.

8 *Framework for preservative pairs*

8.0 Context Let V be our universe, let P be the forcing of collapsing of $2^{\aleph_1}$ to $\aleph_0$. Let $G \subseteq P$ be generic. For every normal filter D over ω_1 in V, we can find in $V[G]$ a filter D^* on the Boolean algebra $\mathcal{P}(\omega_1)^V = \{A$: $A \in V$, $A \subseteq \omega_1$ (of V)$\}$ extending D, such that D^* is a P-name satisfying Lemma 7.1(1), (2) (work as in Lemma 7.11). We let V_D be V^{ω_1}/D^* (i.e., the set is $\{f/D^*$: f a function in V from ω_1 to $V\}$.

So V_D is an elementary extension of V, but it is not well-founded. Let the natural embedding be j_D. We denote cardinals of V_D by θ, σ. We still know, that

(a) Its set of "ordinals" have quite large well founded initial segment (more than ω_1, in fact at least ω_2; because the αth element is f_α/D, where for some g: $\omega_1 \overset{\text{onto}}{\rightarrow} \alpha$, $f_\alpha(i)$ = order type of $\{g(j)$: $j < i\}$), and we can choose D^* such that $\{a$: $a < \omega_1\}$ $\omega_1 \in V \subseteq V_D$ will be well ordered, if D is good enough, and we can restrict ourselves to such D's but this is immaterial here).

(b) If α is an ordinal of V, $\{a \in V_D$: $a < j_D(\alpha)$, a an ordinal in $V_D\}$ has the power $\leq |\alpha|^{\aleph_1}$ (computed in V or equivalently in $V[G]$).

(c) By Fact 7.3, for every regular $\lambda > 2^{2^{\aleph_1}}$, for *some* D, V_D has a cardinal $\theta = \theta(\lambda, D)$, such that $\{a \in V^D$: $a < \theta\}$ is λ-like (= has power λ, but every initial segment has power $< \lambda$). We write $\theta = \theta(\lambda, D)$ also for singular. Clearly for each λ and D there is at most one such θ (this justifies writing $\theta = \theta(\lambda, D)$.) Also, for given θ and D for at most one λ, $\theta = \theta(\lambda, D)$ and then we write $\lambda = \lambda(\theta, D)$.

8.0A Definition We let $TC(V_D) = \{\theta \in V_D$: $\{a$: $a < \theta\}$ is λ-like for some $\lambda\}$, $TC'(V_D) = \{\lambda(\theta, D)$: $\theta \in V^D$, $\lambda(\theta, D)$ defined$\}$. So by Fact 7.3 $\bigcup_D TC'(V_D)$ include all regular cardinals of $V[G]$ above $(2^{2^{\aleph_1}})^V$.

We also prove (Lemma 7.1B):

(d) For every regular cardinal $\lambda > 2^{\aleph_1}$ of V, and an increasing sequence of ordinals of V_D of length λ which belong to $V[G]$, the sequence has a least upper bound (among the ordinals of V_D).

We shall use those propositions only [(c) is the main point].

For $a \in V_D$ let $pow(a) = |\{b \in V_D$: $b \in a\}|$ (power taken in $V[G]$).

8.1 Definition A pair of functions (f, g) of V (i.e., view it as a class of V or a definition) from cardinals to cardinals, is called *preservative* provided that f, g are monotonic $(\lambda \leq \mu \Rightarrow f(\lambda) \leq f(\mu),\ g(\lambda) \leq g(\mu))$ and for any regular cardinal $\lambda > (2^{2^{\aleph_1}})$ in V for some D there is $\theta \in TC\,(V_D)$, $\lambda \leq \lambda(\theta, D)$ and $pow[f(\theta)^{V_D}] \leq g(\lambda)$.

8.2 Claim *The pair (g,g) is preservation for the following function g (and similar others):*

$g_0(\lambda) = \lambda^+$, $g_1(\lambda) = Min\{\mu\colon \mu > \lambda$ *is weakly inaccessible*$\}$

$g_2(\lambda) = Min\{\mu\colon \mu > \lambda$, μ *weakly inaccessible Mahlo*$\}$ *for* $\alpha < \aleph_1$

$g_3^{\alpha}(\lambda) = Min\{\mu\colon \mu > \lambda$ μ *a weakly inaccessible* α*-Mahlo*$\}$.

Proof: Trivial for a reader who arrives here.

8.3 Definition For a (monotonic) function (from cardinals to cardinals) f, we define $f^{\langle\alpha\rangle}$ by induction on the ordinal α:

$$f^{\langle 0\rangle}(\lambda) = \lambda,\ f^{\langle\alpha+1\rangle}(\lambda) = [f(f^{\langle\alpha\rangle}(\lambda))]^+$$

(the $()^+$ is a technical point only, usually absorbed)

$$\text{for } \delta \text{ limit } f^{\langle\delta\rangle}(\lambda) = \bigcup_{\alpha<\delta} f^{\langle\alpha\rangle}(\lambda).$$

8.4 Definition For f (as usual) f^* is defined by

$$f^*(\lambda) = f^{\langle\lambda\rangle}(\aleph_0).$$

8.5 Claim *If (f, g) is preservative then so is $(f^{\langle\mu\rangle}, g^{\langle\mu\rangle})$ where $\mu = (2^{2^{\aleph_1}})^+$ and $(2^{2^{\aleph_1}})^+$ is interpreted as a member of V_D by j_D.*

Proof: Let $\lambda > 2^{2^{\aleph_1}}$, $\lambda = pow\ (b)$ and let $\lambda_\alpha = g^{\langle\alpha\rangle}(\lambda)$, so for α successor (ordinal) λ_α is a successor (cardinal). For each $\alpha + 1$ (by Definition 8.1) there are $D_{\alpha+1}$ and $\theta_{\alpha+1} \in TC\,(V_{D_{\alpha+1}})$, such that $\lambda(\theta_{\alpha+1}, D_{\alpha+1}) \geq \lambda_{\alpha+1}$, and $pow[f(\theta_{\alpha+1})^{V_D}] \leq g(\lambda_{\alpha+1})$.

The number of possible $D_\alpha s$ is $2^{2^{\aleph_1}}$, so for some D, $C = \{\alpha + 1\colon D_{\alpha+1} = D\}$ has power $(2^{2^{\aleph_1}})^+$. Now for $\alpha < \beta$ in C, look in V_D:

$$pow[f(\theta_\alpha)]^{V_D} \leq g(\lambda_\alpha) \leq \lambda_\beta,$$

but $\lambda_\beta \leq \lambda(\theta_\beta, D)$, $\theta_\beta \in TC(V_D)$; hence, as (f, g) is preservative,

(*) $V_D \models$ "$f(\theta_\alpha) \leq \theta_\beta$".

Now working in V_D, for each $\alpha \in C$, for some $\gamma(\alpha) < (2^{2^{\aleph_1}})^+$ in V_D $\theta_\alpha \in [f^{\langle\gamma(\alpha)\rangle}(\lambda), f^{\langle\gamma(\alpha)+1\rangle}(\lambda))^{V_D}$ (a close-open interval), as otherwise our conclusion holds and we finish.

Moreover by fact (d), θ_α $(\alpha \in C)$ has a least upper bound θ^*. Clearly, $\theta^* \in TC'(V_D)$, $\lambda(\theta^*, D) = \lambda_\mu$ (remember $\mu = (2^{2^{\aleph_1}})^+$), so our aim is to show that $V_D \models$ "$f^{\langle\mu\rangle}\ (\theta_\alpha) \leq \theta^*$" for $\alpha = Min\,C$. If this fails, we can have: $V_D \models$ "$\gamma(\alpha) \leq \gamma^*$" for every $\alpha \in C$ holds for some $\gamma^* \in V_D$ such that $V_D \models$ "$\gamma^* < \mu$".

So for some $\gamma \in V_D$, $C' = \{\alpha \in C\colon \gamma(\alpha) = \gamma\}$ has power μ (the power $2^{2^{\aleph_1}}$ is computed in V, C' is defined in $V[G]$, but it has a subset in V of the same power; we usually do not bother with such things). But this gives easy contradiction to the statement (*) above.

8.6 Claim *If (f, g) is preservative then so is*

$$((f^*)^{\langle(2^{2^{\aleph_1}})^+\rangle}, (g^*)^{\langle(2^{2^{\aleph_1}})^+\rangle})$$

Proof: Let λ be a regular cardinal $> 2^{2^{\aleph_1}}$. Let $\lambda_\alpha = g^*(\alpha)$ for $\alpha < (2^{2^{\aleph_1}})$, and $\mu_{\alpha,\beta} = g^\beta(\lambda_\alpha)$ for $\beta < \lambda_\alpha$. For each successor α, $\beta < \lambda_\alpha$, there is $D_{\alpha,\beta}$ and $\sigma_{\alpha,\beta}$ such that $\mu_{\alpha,\beta} \le \chi(\sigma_{\alpha,\beta}, D)$ and $pow[f(\sigma_{\alpha,\beta})^{V_D}] \le g(\mu_{\alpha,\beta})$.

For fixed α, successor α ($> 2^{2^{\aleph_1}}$ if you like), λ_α is successor $> 2^{2^{\aleph_1}}$, hence for some D_α, $C_\alpha = \{\beta: \beta < \lambda_{\alpha+1}$ successor, $D_{\alpha,\beta} = D_\alpha\}$ has power λ_α, and for some $C = \{\alpha: \alpha < (2^{2^{\aleph_1}})^+$ successor, $D_\alpha = D\}$ has power $(2^{2^{\aleph_1}})^+$. Now for $\alpha \in C$, $\beta \in C_\alpha$, $\gamma \in C_\alpha$, $\beta < \gamma$ as $pow(f(\sigma_{\alpha,\beta})^{V_D}] \le g(\mu_{\alpha,\beta}) \le \mu_{\alpha,\gamma}$ but $\lambda(\sigma_{\alpha,\beta}, D) \ge \mu_{\alpha,\beta}$, $\sigma_{\alpha,\beta} \in TC\ (V_D)$ hence $V_D \models$ "$f(\sigma_{\alpha,\beta}) \le \sigma_{\alpha,\gamma}$.

Let $\beta_\alpha = min\, C_\alpha$, $\theta_\alpha = \sigma_{\alpha,\beta_\alpha}$ so $\lambda_\alpha \le \lambda(\theta_\alpha, D)$. Now we can similarly prove that if $\alpha_1 < \alpha_2 < \alpha_3$ are in C then

(*) $\quad pow[f^*(\theta_{\alpha_1})^{V_D}] \le \lambda_{\alpha_3} \le \lambda(\theta_{\alpha_3}, D)$

the second inequality holds by choice, for the first look at $\sigma_{\alpha_2,\beta}$ ($\beta \in C_{\alpha_2}$) their number is $> = \lambda_{\alpha_2} > \lambda(\sigma_{\alpha,\beta_{\alpha_1}}, D) =$ etc.

So now, letting $C^e = \{\alpha \in C$: $\alpha \cap C$ has even order type$\}$, we get $\alpha_1 < \alpha_2$ in C^e *implies* "$f^*(\theta_{\alpha_1}) \le \theta_{\alpha_2}$" hold in V_D. The rest is as in Claim 8.5.

8.7 Claim *If (f, g) is preservative, and $f(\omega_1)$ is a limit cardinal* ($\forall\lambda < f(\omega_1)$) $\lambda^{\aleph_1} < f(\omega_\alpha)$, *then* $f(\omega_1)^{\aleph_1} < g((2^{2^{\aleph_1}})^+)$.

Proof: Look at V_D as in Definition 8.7. This has an ordinal α whose order type is ω_1 of V (but is considered countable). It is easy to check that $f(\alpha)^V$ is in TC (V_D) and its cardinality is $f(\omega_1)$. But as

$$V \prec V_D,\ V_D \models f(\alpha)^{\aleph_1} < f(\omega_1{}^{V_D}) \le f(2^{2^{\aleph_1}})^+)^{V_D} < g\ ((2^{2^{\aleph_1}})^+) \text{ but}$$
$$pow[f(2^{2^{\aleph_1}})^+)] \le g(2^{2^{\aleph_1}})^+), \text{ and}$$
$$pow(f(\alpha)^{\aleph_1{}^{V_D}}) = f(\omega_1)^{\aleph_1}$$

combining we finish.

Unfortunately, the Milner Rado [12] Paradox generalizes

8.8 Definition For a class C of cardinals we define by induction on n, a function Suc^n_C from cardinals to cardinals

$$Suc^0_C(\lambda) = min\{\mu \in C: \mu > \lambda\}$$
$$Suc^{n+1}_C = (Suc^n_C)^*.$$

8.9 Claim *If μ is smaller than the first inaccessible cardinal, then* $\{\lambda$: $\lambda \le \mu^*$ *a cardinal$\}$ can be decomposed to $C_n (n < \omega)$ (i.e.,* $\bigcup_{n<\omega} C_n = \{\lambda: \lambda <$ $\mu\}$) *such that* $Suc^n_{C_n}(\aleph_0) > \lambda$.

Proof: By induction on λ; for first $\lambda(\aleph_0)$ and for successor; no problem (w.l.o.g. $(\forall\chi < \lambda)$ $(\forall n)$ $[Suc^n_{\{\mu:\mu\le\lambda\}}(\chi) < \lambda]$.

For λ singular let $\lambda = \bigcup_{i<\mu} \lambda_i$ $\mu = cf\lambda$, λ $(i < \mu)$ increasing continuous, $\lambda_0 = \aleph_0$, $\lambda_\omega > \mu$. Let $\{\mu$: $\mu < \lambda_i\} = \bigcup_{n<\omega} C'_n$ as above (by the induction hypothesis).

Let $C_{2n} = C'_n$, $C_{2n+1} = \bigcup_{\mu>i>1} C^i_n$. Checking is easy.

NOTES

1. On this see [17]. On the powers of singular cardinals of countable confinality see [18]. On theorems similar to the ones presented here for λ of cofinality $\aleph_1$, such that $\mu < \lambda < \mu^{\aleph_0}$ for some μ, see [19].

   Note that Galvin and Hajnal's bound is based on: if $f(i) = g(i)^+$ (both cardinals), then $T_{Dw_1}(f) \leq (T_{Vw_1}(g))^+$. We get that for each regular λ_1 for some normal filter D on w_1, and g, $\{f : f <_D g\}$ is λ-like, and on this base our bounds. To get such D we use as a hypothesis that $\lambda^{\aleph_1} > \lambda^+$ for some $\lambda > 2^{2^{\aleph_1}}$ (using the covering lemma), but if this hypothesis is missing then our conclusions are trivial.

2. We want that $\{i \in I : A_i = B_i\} \in D$ implies $\prod_{i\in I} A_i/D = \prod_{i\in I} B_i/D$. So we should change the definition for the case $\emptyset \neq \{i \in I : A_i = \emptyset\} \in \{I \setminus A : A \in D\}$. So let $\prod_{i\in I} A_i/D = \{f/D : f \in \prod_{i\in I} (A_i \cup \{\emptyset\})/D$, and $\{i \in I : f(i) \in A_i\} \in D\}$.

3. Of course, if D is a filter on λ, $(Q_\xi, \leq_\xi)$ is κ-well ordered for $\xi < \lambda$ and $\mu \to (\kappa)^2_\lambda$, then $\prod_{\xi<\lambda} (\varphi_\xi, \leq_\xi)$ is μ-well ordered.

4. Proof of 2.10(2): So w.l.o.g. λ is singular. Still 2.10B holds. Let $\lambda = \sum_{\alpha < cf\lambda} \lambda\alpha$, $cf\lambda < \lambda_\alpha < \lambda$, $[\alpha < \beta \Rightarrow \lambda_\alpha < \lambda_\beta]$. As $cf(\lambda) > 2^{|I|}$, for every g $<_D f$ there is $\lambda_g < \lambda$ such that for every nonsmall $A \subseteq I$, $T_{D+A}(g) \leq \lambda g$. So for some $\alpha(g) < cf\lambda$, $\lambda_g \leq \lambda^+_{\alpha(g)}$. Let $H_\alpha = \{g \in H : \alpha(g) \leq \alpha\}$;

   (*) $$|H_\alpha| < \lambda \text{ for } \alpha < cf\lambda.$$

   Clearly $H = \bigcup \{H_\alpha : \alpha < cf\lambda\}$ and H_α increases with α. By the Hajnal free subset theorem, we now define by induction on $\beta < cf\lambda$ and ordinal $\gamma(\beta)$ and functions $g^\beta_j(j < \lambda_\beta)$ such that:

   (1) $\gamma(\beta) < cf\lambda$, and $[\beta_1 < \beta \Rightarrow \gamma(\beta_1) < \gamma(\beta)]$
   (2) $g^\beta_j \in H_{\gamma(\beta)}$
   (3) if $\beta_1 < \beta$, $j(1) < \lambda_{\gamma(\beta_1)}$ then $g_{j(1)} <_{D'} g^\beta_j$
   (4) if $j(1) < j$, then $g^\beta_{j(1)} <_{D'} g^\beta_j$.

   Arriving at β, as $H = \bigcup \{H_j : \gamma < cf(\lambda)\}$, (by(*)) for some γ, $\bigcup_{\zeta<\beta} \gamma(\zeta) < \gamma < cf\lambda$ and $|H_\gamma| > \left(\sum_{\zeta<\beta} (\lambda_{\gamma(\zeta)} + \lambda)\right)^+$, we now let $\gamma(\beta) = \gamma$ and choose $g^\beta_j(j < \lambda_\gamma)$ from $H_{\gamma(\beta)}$ by induction on j (as in the proof of 2.10(1)).

5. This is the proof of the omitting-types theorem (and nondefinability of well ordering) in model theory.

6. Of course, we can restrict ourselves to using only nice filters on $\aleph_1$ (in the strategy, and so in the game).

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