

The Axiom of Choice for Countable Collections of Countable Sets Does Not Imply the Countable Union Theorem

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Abstract A model for the theory ZFU (Zermelo–Fraenkel set theory weakened to permit the existence of atoms) is constructed in which the axiom of choice for countable collections of countable sets is true and the countable union theorem is false. By a transfer theorem of Pincus there is a model of Zermelo–Fraenkel set theory satisfying the same conditions.

Introduction and notation In what follows we will use CU to denote the countable union theorem: the union of a countable set of countable sets is countable. $C(\aleph_0, \aleph_0)$ will denote this statement: every countable set of nonempty countable sets has a choice function.

Sierpinski [7] noted in 1918 that CU implies $C(\aleph_0, \aleph_0)$. The argument given by Sierpinski is valid in ZFU, or Zermelo–Fraenkel set theory weakened to permit the existence of atoms (and without the axiom of choice). Sierpinski also showed that $C(\aleph_0, 2^{\aleph_0})$ (the axiom of choice for countable collections of sets each of cardinality 2^{\aleph_0}) implies CU, but he never solved the problem of whether or not $C(\aleph_0, \aleph_0)$ implies CU. This problem is mentioned again by Moore ([4], pp. 203, 324) and Shannon ([6], p. 570). In this paper we show that $C(\aleph_0, \aleph_0)$ does not imply CU in ZFU by constructing a Fraenkel–Mostowski model in which $C(\aleph_0, \aleph_0)$ is true and CU is false. Brunner and Howard [1] show that the implication $CU \rightarrow C(\aleph_0, 2^{\aleph_0})$ is not provable in ZF.

We now argue that $\neg CU$ and $C(\aleph_0, \aleph_0)$ are injectively boundable, and therefore by the results of Pincus ([5], Theorem 2A6) the existence of a Fraenkel–Mostowski model for $(\neg CU) \wedge C(\aleph_0, \aleph_0)$ implies the existence of a model of $ZF + (\neg CU) + C(\aleph_0, \aleph_0)$. (We refer the reader to [5] for definitions of the terms used in the remainder of this section, in particular “injectively boundable” and “the injective cardinality of a set x ” denoted $|x|_{\dots}$.) $\neg CU$ is clearly boundable

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and therefore injectively boundable. We will use the following lemma to show that $C(\aleph_0, \aleph_0)$ is injectively boundable.

Lemma 0 *If $|X|_- \geq \aleph_2$ then X is not the union of a countable set of countable sets.*

Lemma 0 follows from Theorem 1 of Jech [3]. It follows that the statement:

(*) $(\forall X)(|X|_- < \aleph_2 \rightarrow (\forall Y \in \mathcal{P}(\mathcal{P}(X))) [(Y \text{ is a countable set of countable sets} \wedge \cup Y = X) \rightarrow Y \text{ has a choice function}])$

is equivalent to $C(\aleph_0, \aleph_0)$, and (*) is injectively boundable.

The model Given a model M' of ZFU + AC with A as its set of atoms, a Fraenkel–Mostowski model M of ZFU is constructed by constructing a group G of permutations of A and a filter Γ of subgroups of G which satisfies

$$(\forall a \in A)(\exists H \in \Gamma)(\forall \phi \in H)(\phi(a) = a)$$

and

$$(\forall \phi \in G)(\forall H \in \Gamma)(\phi H \phi^{-1} \in \Gamma).$$

Then the model M is obtained as follows: each permutation of A extends uniquely to a permutation of M' by \in -induction, and for any $\phi \in G$ we identify ϕ with its extension. If H is a subgroup of G and $x \in M'$ and $(\forall \phi \in H)(\phi(x) = x)$ we say H fixes x . If it is also the case that $(\forall y \in x)(\forall \phi \in H)(\phi(y) = y)$ then we say H fixes x pointwise. The Fraenkel–Mostowski model M determined by G and Γ consists of all those $x \in M'$ such that for every y in the transitive closure of x , there is some $H \in \Gamma$ such that H fixes y . We refer the reader to Jech [2] for a proof that M is a model of ZFU.

For our construction, we begin with a model M' of ZFU + AC with a countable set A of atoms. We first write A as a disjoint union $A = \cup_{i \in \omega} A_i$ where each A_i is countably infinite. For each $i \in \omega$ let F_i be a nonprincipal ultrafilter in $\mathcal{P}(A_i)$, and let I_i be the corresponding nonprincipal, maximal ideal $(= \{A_i - x \mid x \in F_i\})$. The group G of permutations of A is defined as follows:

$$G = \{ \phi \mid \phi \text{ is a permutation of } A \text{ and } (\forall i \in \omega) [\phi(A_i) = A_i \text{ and } (\exists u \in F_i) (\phi \text{ fixes } u \text{ pointwise})] \}$$

Γ is the filter of subgroups of G generated by the groups $G(g, e)$, where e is a finite subset of ω and g is a sequence of subsets of A such that $(\forall i \in \omega)(g(i) \in I_i)$ and $G(g, e)$ is defined by:

$$G(g, e) = \{ \phi \in G \mid (\forall i \in e) (\phi \text{ fixes } A_i \text{ pointwise}) \text{ and } (\forall i \in \omega) (\phi \text{ fixes } g(i) \text{ pointwise}) \}.$$

M is the permutation model determined by G and Γ .

It is easy to verify that $\{A_i \mid i \in \omega\}$ is a countable collection of countable sets (in M) whose union is not countable in M ($G(g, e)$ fixes a bijection of A_i and ω whenever $i \in e$ but no group in Γ fixes a well ordering of $\cup \{A_i \mid i \in \omega\} = A$). Hence the countable union theorem fails in M .

We now proceed to the proof that $C(\aleph_0, \aleph_0)$ holds in M .

Theorem *Suppose $Y \in m$ and $M \vdash$ “ Y is a countable collection of nonempty countable sets.” Then Y has a choice function in M .*

Proof: Assume the hypotheses. It follows that for some sequence $g : \omega \rightarrow \mathcal{P}(A)$ where $(\forall i \in \omega)(g(i) \in I_i)$ and some finite $e \subseteq \omega$ $G(g, e)$ fixes Y pointwise.

For each $i \in \omega$, let $g'(i)$ be an element of I_i such that $g(i) \subseteq g'(i)$ and $g'(i) - g(i)$ is infinite. We claim that $G(g', e)$ fixes a choice function for Y . Since $G(g', e)$ fixes Y pointwise, it suffices to show that for every $y \in Y$ there is a $z \in y$ such that $G(g', e)$ fixes z . Choose $y \in Y$.

Lemma 1 *If there is a $z \in y$ and a sequence $g'' : \omega \rightarrow \mathcal{P}(A)$ such that $(\forall i \in \omega)(g''(i) \in I_i$ and $g(i) \subseteq g''(i))$ and $G(g'', e)$ fixes z , then there is a $z' \in y$ such that $G(g', e)$ fixes z' .*

Proof: Assume the hypothesis and choose such a z and g'' . Then we can find a $\phi \in G(g, e)$ such that $(\forall i \in \omega - e)(\phi(g''(i)) \subseteq g'(i))$. (This depends on the facts that (1) $g'(i) - g(i)$ is infinite and (2) there is an element h of I_i containing $g''(i)$ such that $h - g''(i)$ is infinite.) Since y is fixed by $G(g, e)$ and $z \in y$ we have $\phi(z) \in y$. Further, $G(g', e)$ fixes $\phi(z)$, for if $\psi \in G(g', e)$ then $\phi^{-1}\psi\phi \in G(g'', e)$, and so $\phi^{-1}\psi\phi(z) = z$, that is $\psi\phi(z) = \phi(z)$. Therefore $z' = \phi(z)$ satisfies the conclusion of Lemma 1.

In this proof and in what follows, if η and σ are two permutations of A , we use $\eta\sigma$ to denote the composition defined by $(\eta\sigma)(a) = \eta(\sigma(a))$ and $\eta(\sigma)$ to denote the extension of η to M' applied to σ , $\eta(\sigma) = \eta\sigma\eta^{-1}$.

Choose a $z \in y$. The remainder of the proof is carried out under the assumption:

(*) $(\forall g'' : \omega \rightarrow \mathcal{P}(A))$ [If $(\forall i \in \omega - e)(g''(i) \in I_i$ and $g''(i) \supseteq g(i))$ then $(\exists \phi \in G(g'', e))(\phi(z) \neq z)$].

If (*) is false then Lemma 1 applies.

We now prove Lemma 2, which says roughly that if (*) is true then there is some $j \in \omega - e$ such that (*) is still true when we restrict ourselves to ϕ that are the identity outside of A_j .

Lemma 2 *Assume (*); then there is some $j \in \omega - e$ such that for every $x \in I_j$ there is a $\phi \in G(g, e)$ such that $(\forall i \in \omega)(i \neq j$ implies ϕ fixes A_i pointwise) and ϕ fixes x pointwise and $\phi(z) \neq z$.*

Proof: Since $z \in M$, $\exists g'' : \omega \rightarrow \mathcal{P}(A)$ with $g''(i) \in I_i$ and a finite $e' \subseteq \omega$ such that $G(g'', e')$ fixes z . We may assume $(\forall i \in \omega)(g(i) \subseteq g''(i))$ and that $e \subseteq e'$. Further by (*) $e' - e \neq \emptyset$. Suppose $e' - e = \{i_1, i_2, \dots, i_n\}$ where $i_1 < i_2 < \dots < i_n$. Let j be the least i_k such that for some $h : \omega \rightarrow \mathcal{P}(A)$ with $h(i) \in I_i$ and $g(i) \subseteq h(i)$ for all $i \in \omega$, $G(h, e \cup \{i_1, i_2, \dots, i_k\})$ fixes z . Fix such an h . Then for any $x \in I_j$ if we define h' by

$$h'(i) = \begin{cases} h(i) & \text{if } i \neq j \\ h(j) \cup x & \text{if } i = j \end{cases}$$

then $G(h', e \cup \{i_1, i_2, \dots, i_{k-1}\})$ (where $j = i_k$) does not fix z by the minimality of $i_k = j$.

Therefore there is a $\phi' \in G(h', e \cup \{i_1, i_2, \dots, i_{k-1}\})$ such that $\phi'(z) \neq z$. Define $\phi^* \in G(h, e \cup \{i_1, i_2, \dots, i_k\})$ by

$$\phi^*(a) = \begin{cases} a & \text{if } a \in A_i \text{ for some } i \in e \cup \{i_1, \dots, i_k\} \\ \phi^{-1}(a) & \text{otherwise} \end{cases}$$

then $\phi^*(z) = z$. Let $\phi = \phi^* \phi'$, then (i) $\phi(z) \neq z$, (ii) $(\forall i \neq j = i_k)$ (ϕ fixes A_i pointwise), and (iii) ϕ fixes x pointwise (since $\phi|_{A_j} = \phi'|_{A_j}$ and ϕ' fixes x pointwise). This completes the proof of Lemma 2.

Fix a j that satisfies Lemma 2 and let

$$G_j = \{ \phi \in G(g, e) \mid (\forall i \in \omega)(i \neq j \text{ implies } \phi \text{ fixes } A_i \text{ pointwise}) \}.$$

We will identify G_j with $\{ \phi|_{A_j} \mid \phi \in G_j \}$. For each $x \subseteq A_j$ such that $x \in I_j$ and $g(j) \subseteq x$ define

$$G_{j,x} = \{ \phi \in G_j \mid \phi \text{ fixes } x \text{ pointwise} \}$$

and

$$P_x(z) = \{ \phi \in G_{j,x} \mid \phi(z) = z \}.$$

We note that using this notation, $G_j = G_{j,g(j)}$, $P_\emptyset(z) = P_{g(j)}(z)$ and in general $P_x(z) = P_{x \cup g(j)}(z)$. We also note that by Lemma 2, for every such x there is a ϕ in $G_{j,x} - P_x(z)$ and further that $P_x(z)$ is a subgroup of $G_{j,x}$.

Our plan is now to show that under the assumption (*) the orbit of z under G_j is uncountable in M' and therefore that y is uncountable in M , contradicting our hypothesis. (Although it is not needed for the proof of the theorem, we note here that the orbit of z under G_j is well-orderable in M' . This fact, together with the observation made later that the orbit of z under G_j has cardinality $\geq 2^{\aleph_0}$ in M' , yields a slightly stronger theorem mentioned in the final section.) We break the argument into two cases handled by Lemmas 3 and 5 respectively.

Lemma 3 *Assume (*) and assume that for some $x \in I_j$, $P_x(z)$ is a normal subgroup of $G_{j,x}$, then the orbit of z under G_j is uncountable.*

Proof: The proof of Lemma 3 requires the following lemma, the proof of which we postpone until after we complete the proof of Lemma 3.

Lemma 4 *Suppose B is a countable set and F is a nonprincipal ultrafilter in $\mathcal{P}(B)$ and I is the corresponding ideal. Let $\mathcal{G} = \{ \phi \mid \phi \text{ is a permutation of } B \text{ and } (\exists u \in F) (\phi \text{ fixes } u \text{ pointwise}) \}$ and assume that \mathcal{K} is a proper normal subgroup of \mathcal{G} . Then every $\phi \in \mathcal{K}$ is the identity on a cofinite subset of B .*

Assume the hypotheses of Lemma 3 and choose an $x \in I_j$ such that $P_x(z)$ is a normal subgroup of $G_{j,x}$. Since $F = \{ u \cap (A_j - x) \mid u \in F_j \}$ is a nonprincipal ultrafilter in the power set of $B = A_j - x$ and $G_{j,x}$ is \mathcal{G} from Lemma 4, Lemma 4 applies with $\mathcal{K} = P_x(z)$ and we conclude that every $\phi \in P_x(z)$ is the identity on a cofinite subset of $(A_j - x)$.

Let C be an uncountable set of elements of $G_{j,x}$ such that

$$(\forall \psi_1, \psi_2 \in C)(\psi_1^{-1}\psi_2 \in P_x(z) \text{ implies } \psi_1 = \psi_2).$$

(For example C could be obtained as follows: Take an infinite $w \subseteq (A_j - g_j) - x$ such that $w \in I_j$. Write $w = \bigcup_{i \in \omega} w_i$ as the disjoint union of infinite sets. Let η_i be an infinite cycle moving exactly the elements of w_i . For each $s \subseteq \omega$, let ψ_s be $\bigcup_{i \in s} \eta_i$. Then $C = \{\eta_s \mid s \subseteq \omega\}$ satisfies the required conditions, and in fact $|C| = 2^{*\omega}$.)

The set $\{\psi(z) \mid \psi \in C\}$ is uncountable since $\psi_1(z) = \psi_2(z)$ implies $\psi_1^{-1}\psi_2(z) = z$ implies $\psi_1^{-1}\psi_2 \in P_x(z)$ implies $\psi_1 = \psi_2$. This completes the proof of Lemma 3.

Lemma 5 *Assume (*) and $\forall x \in I_j P_x(z)$ is not a normal subgroup of $G_{j,x}$, then the orbit of z under G_j is uncountable.*

Proof: Clearly, if $\phi \in G_j$ moves $P_{g_j}(z) = P_\emptyset(z)$ then $\phi(z) \neq z$. We note here that

$$\phi(P_\emptyset(z)) = \{\phi(\psi) \mid \psi \in P_\emptyset(z)\} = \{\phi\psi\phi^{-1} \mid \psi \in P_\emptyset(z)\} = \phi P_\emptyset(z) \phi^{-1}.$$

So it suffices to show that the orbit of $P_{g_j}(z) = P_\emptyset(z)$ under G_j is uncountable. In order to do this we define a sequence of triples $\langle \phi_i, \eta_i, x_i \rangle$, $i \in \omega$ by induction such that for every $i \in \omega$:

- (1) $x_i \in I_j$ and $x_i \cap g_j = \emptyset$
- (2) $x_i \cap (\bigcup_{k < i} x_k) = \emptyset$
- (3) $\phi_i \in G_{j, \bigcup_{k < i} x_k}$, $\psi_i \in P_{\bigcup_{k < i} x_k}$
- (4) $(\forall a \in A)((\phi_i(a) \neq a \text{ or } \psi_i(a) \neq a) \text{ implies } a \in x_i)$
- (5) $\phi_i(\psi_i) \notin P_\emptyset(z)$.

Suppose that for all $n < i$, $n \in \omega$, the triples $\langle \phi_n, \psi_n, x_n \rangle$ have been chosen and satisfy (1) through (5) with i replaced by n . By the hypotheses of Lemma 5 with $x = \bigcup_{k < i} x_k$ there is a $\phi'_i \in G_{j,x}$ such that $\phi'_i(P_x(z)) \neq P_x(z)$ and therefore there is a $\phi_i \in G_{j,x}$ and a $\psi_i \in P_x(z)$ such that $\phi_i(\psi_i) \notin P_x(z)$. (If $\phi'_i(P_x(z)) \not\subseteq P_x(z)$ let $\phi_i = \phi'_i^{-1}$.) Let $x_i = \{a \in A \mid \phi_i(a) \neq a \text{ or } \psi_i(a) \neq a\}$. (1) through (4) are clear, as is $\phi_i(\psi_i) \notin P_x(z)$. To argue for (5) assume that $\phi_i(\psi_i) \in P_\emptyset(z)$, then by (3) $\phi_i(\psi_i)$ fixes $\bigcup_{k < i} x_k = x$ pointwise; and so $\phi_i(\psi_i) \in P_x(z)$, a contradiction. Now we make two claims about the sequence $\langle \phi_i, \psi_i, x_i \rangle$, $i \in \omega$:

Claim 1 *If $\eta \in G_j$ agrees with ϕ_i on x_i then $\eta(\psi_i) = \phi_i(\psi_i)$.*

Proof: Assume $\eta \in G_j$ agrees with ϕ_i on x_i . Then we can show $\eta(\psi_i) = \eta\psi_i\eta^{-1}$ agrees with $\phi_i(\psi_i) = \phi_i\psi_i\phi_i^{-1}$ by considering two cases: If $a \in A - x_i$ then $\eta^{-1}(a) \notin x_i$ ($\eta^{-1}(a) \in x_i$ implies ϕ_i and η_i agree on $\eta^{-1}(a)$, hence $\phi_i(\eta^{-1}(a)) = a$; but x_i is closed under ϕ_i , a contradiction). So $\eta\psi_i(\eta^{-1}(a)) = \eta(\eta^{-1}(a)) = a = \phi_i\psi_i\phi_i^{-1}(a)$. On the other hand if $a \in x_i$ then say $a = \phi_i(b)$ where $b \in x_i$. This means $\eta(b) = a$ so

$$\eta\psi_i\eta^{-1}(a) = \eta\psi_i(b) = \eta\psi_i\phi_i^{-1}(a) = \phi_i\psi_i\phi_i^{-1}(a)$$

since η and ϕ_i agree on x_i .

Claim 2 *If $\eta \in G_j$ is the identity on x_i then $\phi_i(\psi_i) \notin \eta(P_\emptyset(z))$.*

Proof: Assume $\eta \in G_j$ is the identity on x_i and that $\phi_i(\psi_i) \in \eta(P_\emptyset(z))$. Then for some $\sigma \in P_\emptyset(z)$, $\phi_i(\psi_i) = \eta(\sigma)$ so $\eta^{-1}\phi_i(\psi_i) = \sigma$. Since η is the identity on x_i so is η^{-1} , hence $\eta^{-1}\phi_i$ agrees with ϕ_i on x_i . It follows by Claim 1 that $\eta^{-1}\phi_i(\psi_i) = \phi_i(\psi_i)$, which is not in $P_\emptyset(z)$ by (5), a contradiction.

Now we claim that the sequence $\langle \phi_i, \psi_i, x_i \rangle, i \in \omega$ can be chosen so that

$$(6) \quad \bigcup_{i \in \omega} x_i \in I_j.$$

(If the sequence $\langle \phi_i, \psi_i, x_i \rangle, i \in \omega$, does not satisfy (6) then one of the sequences $\langle \phi_{2i}, \psi_{2i}, x_{2i} \rangle, i \in \omega$ or $\langle \phi_{2i+1}, \psi_{2i+1}, x_{2i+1} \rangle, i \in \omega$ will.) Assuming (6), for each subset K of ω define an element Δ_K of G_j by

$$\Delta_K(a) = \left\{ \begin{array}{ll} a & a \notin \bigcup_{i \in K} x_i \\ \phi_i(a) & a \in x_i \text{ where } i \in K \end{array} \right\} = \bigcup_{i \in K} \phi_i.$$

We complete the proof of Lemma 5 by showing that for all K and $K' \subseteq \omega$, if $K \neq K'$ then $\Delta_K(P_\emptyset(z)) \neq \Delta_{K'}(P_\emptyset(z))$. Assume the hypothesis and without loss of generality assume that $i \in K - K'$. It suffices to show that $\phi_i(\psi_i) \in \Delta_K(P_\emptyset(z)) - \Delta_{K'}(P_\emptyset(z))$.

By Claim 1, since Δ_K agrees with ϕ_i on x_i

$$\phi_i(\psi_i) = \Delta_K(\psi_i) \in \Delta_K(P_\emptyset(z)).$$

By Claim 2, since Δ_K is the identity on x_i ,

$$\phi_i(\psi_i) \in \Delta_{K'}(P_\emptyset(z)).$$

This completes the proof of Lemma 5.

Now we finish the proof of the theorem by proving Lemma 4. Assume the hypotheses of Lemma 4. For $\phi \in \mathcal{G}$ let $\text{type}(\phi)$ be the sequence $\alpha : \omega \rightarrow \omega \cup \{\omega\}$ where, when ϕ is written as a product of disjoint cycles, $\alpha(0)$ is the number of infinite cycles, $\alpha(1)$ is the number of elements of B fixed by ϕ and for $n > 1$, $\alpha(n)$ is the number of n cycles. (For every $\phi \in \mathcal{G}$, $\text{type}(\phi)(1) = \omega$.)

Claim 1 *If $\phi \in \mathcal{K}$ and $\psi \in \mathcal{G}$ and $\text{type}(\phi) = \text{type}(\psi)$ then $\psi \in \mathcal{K}$.*

Proof: Assuming the hypotheses, let η^* be a one-to-one function with domain $\{a \in B \mid \phi(a) \neq a\}$ and image $\{a \in B \mid \psi(a) \neq a\}$ such that for every cycle $(\dots, a_0, a_1, a_2, \dots)$ of ϕ (finite or infinite) $(\dots, \eta^*(a_0), \eta^*(a_1), \eta^*(a_2), \dots)$ is a cycle of ψ . Extend η^* to η where $\eta \in \mathcal{G}$. (This may require using an infinite subset of B in I disjoint from $\text{dom}(\eta^*) \cup \text{image}(\eta^*)$.)

We complete the proof of Claim 1 by showing that $\eta\phi\eta^{-1} = \psi$. Choose $b \in B$.

Case 1. $\psi(b) = b$. In this case, $\phi(\eta^{-1}(b)) = \eta^{-1}(b)$ so $\eta\phi\eta^{-1}(b) = b = \psi(b)$.

Case 2. $\psi(b) \neq b$. Assume $b = \eta^*(a)$ and occurs in the cycle $(\dots, \eta^*(a), \eta^*(a'), \dots)$ of ψ . Then $\phi(a) = a'$ so $\eta\phi\eta^{-1}(b) = \eta\phi\eta^{-1}(\eta^*(a)) = \eta\phi(a) = \eta(a') = \eta^*(a') = \psi(\eta^*(a)) = \psi(b)$.

Claim 2 *If for some $\phi \in \mathcal{K}$ $\{a \mid \phi(a) \neq a\}$ is infinite then there is a $\psi \in \mathcal{K}$ of type α where $\alpha(2) = \omega$ and $(\forall n \notin \{1, 2\})(\alpha(n) = 0)$. (That is, ψ is the product of infinitely many 2 cycles.)*

Proof: Assume that ϕ satisfies the hypotheses of the claim and for each infinite cycle $c = (\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots)$ of ϕ (when ϕ is written as a product of disjoint cycles) let $T(c) = (\dots, a_6, a_4, a_2, a_0, a_{-2}, a_{-4}, \dots)$ and note that

$T(c) \circ c = \dots (a_{-2}, a_{-1})(a_0, a_1)(a_2, a_3)(a_4, a_5) \dots$. For each finite cycle $c = (a_1, a_2, \dots, a_n)$ of ϕ where $n > 3$, let $T(c) = (a_n, a_{n-1}, \dots, a_4, a_1, a_2, a_3)$, and for each three cycle $c = (a_1, a_2, a_3)$ of ϕ let $T(c) = (a_1, b_c, a_3)$ where the b_c 's are chosen from a set $v \in I$ disjoint from $\{a \in B \mid \phi(a) \neq a\}$ and are chosen so that if c and c' are two different 3-cycles of c , then $b_c \neq b_{c'}$. In each case $T(c)$ is a cycle of the same length as c , and $T(c) \circ c$ is a product of two cycles.

If we let ψ be the permutation with cycles $\{T(c) \mid c \text{ is a cycle of } \phi\}$ then by Claim 1 $\psi \in \mathcal{K}$ and therefore $\psi\phi$ is an infinite product of disjoint two cycles in \mathcal{K} .

Claim 3 *Every $\phi \in \mathcal{G}$ is the product of two elements of type α where α is the type defined in Claim 2 (infinite product of disjoint two cycles).*

Proof: It suffices to show that every cycle is a product of two such permutations.

Let ϕ be the infinite cycle $(\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots)$ then $\phi = \psi\eta$ where $\psi = (a_{-1}, a_1)(a_{-2}, a_2)(a_{-3}, a_3) \dots$ and $\eta = (a_0, a_{-1})(a_1, a_{-2})(a_2, a_{-3}) \dots$

Suppose $\phi = (a_{-n}, a_{-n+1}, \dots, a_{-1}, a_0, a_1, \dots, a_n)$ is a cycle of odd length, then $\phi = \psi\eta$ where $\psi = (a_1, a_{-1})(a_2, a_{-2}) \dots (a_n, a_{-n})$ and $\eta = (a_0, a_{-1})(a_1, a_{-2}) \dots (a_{n-1}, a_{-n})$. (Add infinitely many two cycles to ψ and their inverses to η to get two permutations of type α .)

Finally, suppose that $\phi = (a_{-n}, a_{-n+1}, \dots, a_{-1}, a_1, a_2, \dots, a_n)$ is a cycle with even length. Then $\phi = \psi\eta$ where $\psi = (a_1, a_{-1})(a_2, a_{-2}) \dots (a_n, a_{-n})$ and $\eta = (a_1, a_{-2})(a_2, a_{-3}) \dots (a_{n-1}, a_{-n})$. This completes the proof of Claim 3. Lemma 4 follows easily from the three claims.

Conjecture An examination of the above argument will show that we have proved a slightly stronger theorem, namely that the axiom of choice for well-ordered collections of sets, each of cardinality not greater than or equal to 2^{\aleph_0} ("If Y is a well-ordered set of sets such that $(\forall y \in Y) [\neg(|y| \geq 2^{\aleph_0})]$ then Y has a choice function"), does not imply the countable union theorem.

We conjecture that for any ordinal α , the statement, "If Y is a well-ordered set of sets such that $(\forall y \in Y) (\neg(|y| \geq 2^{\aleph_\alpha}))$ then Y has a choice function", does not imply that a countable union of sets of cardinality \aleph_α can be well-ordered.

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