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Some Independence Results Related to the Kurepa Tree

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Abstract By an ω_1-tree we mean a tree of power ω_1 and height ω_1. Under the assumption of *CH* plus $2^{\omega_1} > \omega_2$ we call an ω_1-tree a Jech–Kunen tree if it has κ many branches for some κ strictly between ω_1 and 2^{ω_1}. We call an ω_1-tree being ω_1-anticomplete if it has more than ω_1 many branches and has no subtrees which are isomorphic to the standard ω_1-complete binary tree. In this paper we prove that: (1) It is consistent with *CH* plus $2^{\omega_1} > \omega_2$ that there exists an ω_1-anticomplete tree but no Jech–Kunen trees or Kurepa trees; (2) It is independent of *CH* plus $2^{\omega_1} > \omega_2$ that there exists a Jech–Kunen tree without Kurepa subtrees; (3) It is independent of *CH* plus $2^{\omega_1} >$ ω_2 that there exists a Kurepa tree without Jech–Kunen subtrees. We assume the existence of an inaccessible cardinal in some of our proofs.

Let T be a tree. For an ordinal α, T_α is the α-th level of T and $T|\alpha = \bigcup_{\beta<\alpha} T_\beta$. Let $ht(T)$, the height of T, be the smallest ordinal λ such that $T_\lambda = \emptyset$. By a branch of T we mean a linearly ordered subset of T which intersects every nonempty level of T. Let $\mathcal{B}(T) = \{B : B \text{ is a branch of } T\}$. For a $t \in T$ let $T(t) = \{s \in T : s \text{ and } t \text{ are comparable}\}$.

Let T be a tree. We recall that:

T is an ω_1-tree if $|T| = \omega_1$ and $ht(T) = \omega_1$. Without loss of generality we sometimes assume that $\langle T,\leq_T\rangle = \langle \omega_1,\leq_T\rangle$ with unique root 0 if T is an ω_1-tree.

An ω_1-tree T is called a *Kurepa tree* if $|T_\alpha| < \omega_1$ for any $\alpha < \omega_1$ and $|\mathcal{B}(T)| > \omega_1$.

An ω_1-tree T is called a *Jech–Kunen tree* if $\omega_1 < |\mathcal{B}(T)| < 2^{\omega_1}$.

T' is a *subtree of* T if $T' \subseteq T$ and $\leq_{T'} = \leq_T \cap\, T' \times T'$ (T' inherits the order of T). For an ordinal λ we call $\langle 2^{<\lambda},\subseteq\rangle$ a *standard* λ*-complete binary tree.* A tree is called a λ*-complete binary tree* if it is isomorphic to $\langle 2^{<\lambda},\subseteq\rangle$. A subtree T' of T is called *closed downward* if for any $t' \in T'$, $\{t \in T : t <_T t'\} \subseteq T'$.

An ω_1-tree T is called an ω_1*-anticomplete tree* if $|\mathcal{B}(T)| > \omega_1$ and T has no ω_1-complete binary subtrees.

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Facts

(1) Both Kurepa trees and Jech–Kunen trees are ω_1-anticomplete trees;
(2) Under CH and $2^{\omega_1} > \omega_2$, a Jech–Kunen tree is also a Kurepa tree if every level of it is countable;
(3) Under CH and $2^{\omega_1} > \omega_2$, a Kurepa tree is also a Jech–Kunen tree if it has fewer than 2^{ω_1} many branches.

The independence of the existence of Kurepa trees was proved by Silver (see [5]). In [2], Jech constructs a model of CH plus $2^{\omega_1} > \omega_2$, in which there is a Jech–Kunen tree. In fact, it is a Kurepa tree with fewer than 2^{ω_1} branches. The independence of the existence of Jech–Kunen trees under CH plus $2^{\omega_1} > \omega_2$ was given by Kunen in [4], in which he gave an equivalent form of Jech–Kunen trees in terms of compact Hausdorff spaces. The detailed proof can be found in [3], Theorem 4.8.

The technique used by Silver and Kunen to kill Kurepa trees and Jech–Kunen trees is to show that if an ω_1-tree T has a new branch in an ω_1-closed forcing extension, then T should have an ω_1-complete binary subtree. So in their models all ω_1-anticomplete trees are also killed.

In this paper we discuss two questions: (1) Assuming CH plus $2^{\omega_1} > \omega_2$, can we kill all Kurepa trees and Jech–Kunen trees without killing all ω_1-anticomplete trees? (2) How different are Kurepa trees and Jech–Kunen trees? For background in trees see Todorčević [6]; for background in forcing see Kunen [5]; and for Generalized Martin's Axiom see Weiss [7], §6. By an *inaccessible cardinal* we mean a strongly inaccessible cardinal.

Before proving theorems we need more notation of posets (partially ordered sets with largest elements). We always let $1_{\mathbb{P}}$ be the largest element of a poset $\mathbb{P}$.

Let I, J be two sets and λ be a cardinal.

$$Fn(I,J,\lambda) = \{f : f \text{ is a function}, f \subseteq I \times J \text{ and } |f| < \lambda\}$$

is a poset ordered by reverse inclusion. We omit λ if $\lambda = \omega$.

Let I be a subset of an ordinal κ and λ be a cardinal.

$$Lv(I,\lambda) = \{f : f \text{ is a function}, f \subseteq (I \times \lambda) \times \kappa,\ |f| < \lambda \text{ and } \forall\langle\alpha,\beta\rangle \in \mathrm{dom}(f)(f(\alpha,\beta) \in \alpha)\}$$

is a poset ordered by reverse inclusion.

In forcing arguments we let $\dot{a}$ be a name for a and $\ddot{a}$ be a name for $\dot{a}$. We always assume the consistency of ZFC and let M be a countable transitive model of ZFC.

Theorem 1 *Assume the existence of an inaccessible cardinal. Then it is consistent with CH plus $2^{\omega_1} > \omega_2$ that there exists an ω_1-anticomplete tree but there are neither Kurepa trees nor Jech–Kunen trees.*

We need a lemma from Delvin [2].

Lemma 1 *Let $\mathbb{P}$, $\mathbb{P}'$ be two posets in M such that $\mathbb{P}$ has c.c.c. and $\mathbb{P}'$ is ω_1-closed in M. Let $G_{\mathbb{P}}$ be a $\mathbb{P}$-generic filter over M and $G_{\mathbb{P}'}$ be a $\mathbb{P}'$-generic filter over $M[G_{\mathbb{P}}]$. Let T be an ω_1-tree in $M[G_{\mathbb{P}}]$. If T has a new branch B in $M[G_{\mathbb{P}}][G_{\mathbb{P}'}] - M[G_{\mathbb{P}}]$, then T has a subtree T' in $M[G_{\mathbb{P}}]$, which is isomorphic to the tree $\langle 2^{<\omega_1} \cap M, \subseteq\rangle$ (standard ω_1-complete binary tree in M).*

Proof: First we work within M. In the proof we always let $i = 0, 1$. Without loss of generality we can assume that

$$1_{\mathbb{P}} \Vdash_{\mathbb{P}} (1_{\mathbb{P}'} \Vdash_{\mathbb{P}'} (B \text{ is a branch of } \dot{T})).$$

Claim 1 *Let $\alpha < \omega_1$ and $q \in \mathbb{P}'$. Then there is a $q' \leq_{\mathbb{P}'} q$ such that $1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Phi(\alpha, q', \dot{T}, \ddot{B}))$, where*

$$\Phi(\alpha, q, \dot{T}, \ddot{B}) =_{df} (\exists y \in \dot{T}_\alpha)(q \Vdash_{\mathbb{P}'} (y \in \ddot{B})).$$

Proof of Claim 1: See [1], Lemma 3.6.

Claim 2 *Let $\alpha < \omega_1$, $q \in \mathbb{P}'$ and $1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Phi(\alpha, q, \dot{T}, \ddot{B}))$. Then there is a $\beta < \omega_1$, $\beta > \alpha$ and $q^i \leq_{\mathbb{P}'} q$ such that $1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Psi(\alpha, \beta, q, q^0, q^1, \dot{T}, \ddot{B}))$, where*

$$\Psi(\alpha, \beta, q, q^0, q^1, \dot{T}, \ddot{B}) =_{df} [\textit{if } x \in \dot{T}_\alpha \textit{ and } q \Vdash_{\mathbb{P}'} (x \in \ddot{B}), \textit{ then there are}$$
$$x^i \in \dot{T}_\beta,\ x^0 \neq x^1 \textit{ and } x <_T x^i \textit{ such that } q^i \Vdash_{\mathbb{P}'} (x^i \in \ddot{B})].$$

Proof of Claim 2: See [1], Lemma 3.6.

Claim 3 *Let δ be an ordinal below ω_1. Let $q_\gamma : \gamma < \delta\rangle$ be a decreasing sequence in $\mathbb{P}'$ and $\langle \alpha_\gamma : \gamma < \delta \rangle$ be an increasing sequence in ω_1 such that $1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Phi(\alpha_\gamma, q_\gamma, \dot{T}, \ddot{B}))$ for all $\gamma < \delta$. Let $\alpha_\delta = \sup\{\alpha_\gamma : \gamma < \delta\}$. Then there is a $q \leq_{\mathbb{P}'} q_\gamma$ for all $\gamma < \delta$ such that $1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Phi(\alpha_\delta, q, \dot{T}, \ddot{B}))$.*

Proof of Claim 3: Since $\mathbb{P}'$ is ω_1-closed in M, there is a $q' \in \mathbb{P}'$ such that $q' \leq_{\mathbb{P}'} q_\gamma$ for all $\gamma < \delta$. By Claim 1 there is a $q \leq_{\mathbb{P}'} q'$ such that $1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Phi(\alpha_\delta, q, \dot{T}, \ddot{B}))$. This ends the proof of Claim 3.

We now prove the lemma. We construct a subset $\bar{\mathbb{P}} = \{p_s : s \in 2^{<\omega_1}\}$ of $\mathbb{P}'$ and a subset $O = \{\alpha_s : s \in 2^{<\omega_1}\}$ of ω_1 in M such that

(1) the map $s \mapsto p_s$ is an isomorphic imbedding from the standard ω_1-complete binary tree to $\mathbb{P}'$.
(2) $\forall s, t \in 2^{<\omega_1}$ ($s \subseteq t$ and $s \neq t \to \alpha_s < \alpha_t$).
(3) $\alpha_{s\frown\langle 0\rangle} = \alpha_{s\frown\langle 1\rangle}$ for all $s \in 2^{<\omega_1}$.
(4) $1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Phi(\alpha_s, p_s, \dot{T}, \ddot{B}))$ for all $s \in 2^{<\omega_1}$.
(5) $1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Psi(\alpha_s, \alpha_{s\frown\langle 0\rangle}, p_s, p_{s\frown\langle 0\rangle}, p_{s\frown\langle 1\rangle}, \dot{T}, \ddot{B}))$ for all $s \in 2^{<\omega_1}$.

Let $\alpha_{\langle\rangle} = 0$ and $p_{\langle\rangle} = 1_{\mathbb{P}'}$. Assume that we have α_s and p_s for all $s \in 2^{<\omega_1}$.

Case 1: $\alpha = \gamma + 1$.

Let $s \in 2^\gamma$. Since $1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Phi(\alpha_s, p_s, \dot{T}, \ddot{B}))$, then there is a $\beta < \omega_1$, $\beta > \alpha_s$ and $q^i \leq_{\mathbb{P}'} p_s$ such that $1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Psi(\alpha_s, \beta, p_s, q^0, q^1, \dot{T}, \ddot{B}))$ by Claim 2. Let $\alpha_{s\frown\langle i\rangle} = \beta$ and $p_{s\frown\langle i\rangle} = q^i$. (Note that q^0, q^1 are incompatible by Claim 2.)

Let G be any $\mathbb{P}$-generic filter over M. Then $M[G] \models [\Phi(\alpha_s, p_s, T, \dot{B})]$. Hence in $M[G]$ there is an $x \in T_{\alpha_s}$ such that $p_s \Vdash_{\mathbb{P}'} (x \in \dot{B})$. Since $M[G] \models [\Psi(\alpha_s, \alpha_{s\frown\langle 0\rangle}, p_s, p_{s\frown\langle 0\rangle}, p_{s\frown\langle 1\rangle}, T, \dot{B})$ and $x \in T_{\alpha_s}]$, then there are $x^i \in T_{\alpha_{s\frown\langle i\rangle}}$ such that $p_{s\frown\langle i\rangle} \Vdash_{\mathbb{P}'} (x^i \in \dot{B})$ in $M[G]$. This implies that $1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Phi(\alpha_{s\frown\langle i\rangle}, p_{s\frown\langle i\rangle}, \dot{T}, \ddot{B}))$.

Case 2: α is a limit ordinal below ω_1.

Let $s \in 2^\alpha$. Since $\langle \alpha_{s|\beta} : \beta < \alpha \rangle$ is increasing in ω_1, $\langle p_{s|\beta} : \beta < \alpha \rangle$ is decreasing in $\mathbb{P}'$ and $1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Phi(\alpha_{s|\beta}, p_{s|\beta}, \dot{T}, \ddot{B}))$ for all $\beta < \alpha$, then there is an

$\alpha_s = \sup\{\alpha_{s|\beta} : \beta < \alpha\}$ and a $p_s \leq_{\mathbb{P}'} p_{s|\beta}$ for all $\beta < \alpha$ such that $1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Phi(\alpha_s, p_s, \dot{T}, \ddot{B}))$ by Claim 3.

We now work within $M[G_{\mathbb{P}}]$ to construct a subtree $T' = \{t_s : s \in 2^{<\omega_1} \cap M\}$ of T such that

(1) the map $s \mapsto t_s$ is an isomorphic imbedding from $\langle 2^{<\omega_1} \cap M, \subseteq \rangle$ to T.
(2) $t_s \in T_{\alpha_s}$ and $p_s \Vdash_{\mathbb{P}'} (t_s \in \dot{B})$ for all $s \in 2^{<\omega_1} \cap M$.

Let $t_{\langle\rangle} = 0$, the root of T. Assume that we have t_s for all $s \in 2^{<\alpha} \cap M$.

Case 1: $\alpha = \beta + 1$.

Let $s \in 2^{\beta} \cap M$. Since $p_s \Vdash_{\mathbb{P}'} (t_x \in \dot{B})$ and $\Psi(\alpha_s, \alpha_{s\frown\langle 0\rangle}, p_s, p_{s\frown\langle 0\rangle}, p_{s\frown\langle 1\rangle}, T, \dot{B})$ is true, there are $t^i \in T_{\alpha_{s\frown\langle 0\rangle}}$ such that $t <_T t^i$, $t^0 \neq t^1$ and $p_{s\frown\langle i\rangle} \Vdash_{\mathbb{P}'} (t^i \in \dot{B})$.
Let $t_{s\frown\langle i\rangle} = t^i$ for $i = 0,1$.

Case 2: α is a limit ordinal below ω_1.

Let $s \in 2^{\alpha} \cap M$. Since $\Phi(\alpha_s, p_s, T, \dot{B})$ is true, there is an $x \in T_{\alpha_s}$ such that $p_s \Vdash_{\mathbb{P}'} (x \in \dot{B})$. Since $\forall\beta < \alpha\ (p_s \leq p_{s|\beta})$, then $p_s \Vdash_{\mathbb{P}'} (t_{s|\beta} \in \dot{B})$. Now $t_{s|\beta} <_T x$ because $\alpha_s > \alpha_{s|\beta}$ for all $\beta < \alpha$.
Let $t_s = x$.
We have now finished construction and T' is just the required subtree of T.

Proof of Theorem 1: Let κ be an inaccessible cardinal, $\mathbb{P}_1 = Lv(\kappa, \omega_1)$, $\mathbb{P}_2 = Fn(\kappa^+, 2, \omega_1)$ and $\mathbb{P}_3 = Fn(\omega_1, 2)$ in M. Let G_1 be a $\mathbb{P}_1$-generic filter over M, $M' = M[G_1]$, G_2 be a $\mathbb{P}_2$-generic filter over M', $M'' = M'[G_2]$, G_3 be a $\mathbb{P}_3$-generic filter over M'' and $M''' = M''[G_3]$. We want to show that $M''' \models [CH, 2^{\omega_1} = \omega_3$ and there exists an ω_1-anticomplete tree but there are neither Kurepa trees nor Jech–Kunen trees$]$.

We list some facts first:

(1) $M' \models [CH, 2^{\omega_1} = \omega_2 = \kappa$ and there are no Kurepa trees$]$. The proof can be found in [5], p. 261.
(2) $M'' \models [CH, 2^{\omega_1} = \omega_3 = \kappa^+$ and there exist neither Kurepa trees nor Jech–Kunen trees$]$. See Juhász [3], Theorem 4.8, for the proof.
(3) $M''' \models [CH, 2^{\omega_1} = \omega_3]$.

Claim 1 *There exists an ω_1-anticomplete tree in M'''.*

Proof of Claim 1: Let T be an ω_1-complete binary tree in M''. We want to show that T is an ω_1-anticomplete tree in M'''. Since in M''', $|\mathcal{B}(T)| \geq |(\mathcal{B}(T))^{M''}| = \omega_3$, it suffices to show that T has no ω_1-complete binary subtrees in M'''.

Suppose that is not true. Then T has an ω_1-complete binary subtree $T' = \{t_s : s \in 2^{<\omega_1}\}$ in M'''. Since $T' \restriction \omega$ is countable and $T' \subseteq T = \omega_1$, then there is a $\delta < \omega_1$ such that $T' \restriction \omega \in M''[G_3 \cap Fn(\delta,2)]$. Let $f \in 2^{\omega}$ be a new function in $M''' - M''[G_3 \cap Fn(\delta,2)]$. Then $C_f = \{t_{f|n} : n \in \omega\}$ is not in $M''[G_3 \cap Fn(\delta,2)]$. But $C_f = \{t \in T' \restriction \omega : t <_T t_f\}$ which is in $M''[G_3 \cap Fn(\delta,2)]$. This contradiction ends the proof of Claim 1.

Claim 2 *There exist neither Kurepa trees nor Jech–Kunen trees in M'''.*

Proof of Claim 2: Let T be an ω_1-tree in M'''. Then there is a $\theta < \kappa$ and a subset $I \subseteq \kappa^+$ of power ω_1 such that

$$T \in M[G_1 \cap Lv(\theta, \omega_1)][G_2 \cap Fn(I,2,\omega_1)][G_3].$$

Let $\mathbb{P}_1' = Lv(\theta, \omega_1)$, $\mathbb{P}_1'' = Lv(\kappa - \theta, \omega_1)$, $\mathbb{P}_2' = Fn(I,2,\omega_1)$, $\mathbb{P}_2'' = Fn(\kappa^+ - I,2,\omega_1)$. Then $\mathbb{P}_1 = \mathbb{P}_1' \times \mathbb{P}_1''$, $\mathbb{P}_2 = \mathbb{P}_2' \times \mathbb{P}_2''$ and all of these posets mentioned here are ω_1-closed. Let $G_1' = G_1 \cap \mathbb{P}_1'$, $G_1'' = G_1 \cap \mathbb{P}_1''$, $G_2' = G_2 \cap \mathbb{P}_2'$ and $G_2'' = G_2 \cap \mathbb{P}_2''$. Then $G_1 = G_1' \times G_1''$, $G_2 = G_2' \times G_2''$ and

$$M''' = M[G_1'][G_1''][G_2'][G_2''][G_3] = M[G_1'][G_2'][G_3][G_1''][G_2''].$$

Since

$$M[G_1'][G_2'][G_3] \models [|\mathcal{B}(T)| < \kappa],$$

then there is a new branch of T in $M''' - M[G_1'][G_2'][G_3]$ if T has more than ω_1 many branches in M'''. Since $\mathbb{P}_3$ has *c.c.c.* and $\mathbb{P}_1'' \times \mathbb{P}_2''$ is ω_1-closed in $M[G_1'][G_2']$, then there is a subtree T' of T in $M[G_1'][G_2'][G_3]$, which is isomorphic to $\langle 2^{<\omega_1} \cap M[G_1'][G_2'],\subseteq\rangle$ by Lemma 1.

This is impossible if T is a Kurepa tree because $T' \restriction \omega + 1$ is uncountable. This is also impossible if T is a Jech–Kunen tree because $2^{<\omega_1} \cap M[G_1'][G_2'] = 2^{<\omega_1} \cap M[G_1][G_2]$ and $|\mathcal{B}(T)| \geq |\mathcal{B}(T')| \geq (2^{\omega_1})^{M[G_1][G_2]} = \kappa^+ = 2^{\omega_1}$ in M'''.

Theorem 2 *Assume the existence of an inaccessible cardinal. Then it is consistent with CH plus $2^{\omega_1} > \omega_2$ that there exists a Jech–Kunen tree which has no Kurepa subtrees.*

Proof: Assume that κ is an inaccessible cardinal, $\mathbb{P}_1 = Lv(\kappa, \omega_1)$, $\mathbb{P}_2 = Fn(\omega_1,2)$ in M. Let G_1 be a $\mathbb{P}_1$-generic filter over M, $M' = [G_1]$, G_2 be a $\mathbb{P}_2$-generic filter over M' and $M'' = M'[G_2]$. Let $\mathbb{P}_3 = Fn(\omega_3,2,\omega_1)$ in M'', G_3 be a $\mathbb{P}_3$-generic filter over M'' and $M''' = M''[G_3]$. We want to show that $M''' \models [CH,\ 2^{\omega_1} = \omega_3$ and there exists a Jech–Kunen tree which has no Kurepa subtrees$]$.

We list some facts first:

(1) $M' \models [CH,\ 2^{\omega_1} = \omega_2$ and there are no Kurepa trees$]$.
(2) $M'' \models [CH,\ 2^{\omega_1} = \omega_2$ and every ω_1-complete binary tree in M' is an ω_1-anticomplete tree$]$. This was proved in Theorem 1.
(3) $M''' \models [CH,\ 2^{\omega_1} = \omega_3$ and every ω_1-complete binary tree in M' is a Jech–Kunen tree$]$. This is because an ω_1-closed forcing extension does not add any new branches to an ω_1-anticomplete tree.

Let T be an ω_1-complete binary tree in M'. Then T is a Jech–Kunen tree in M''' by Fact (3). We now want to show that T has no Kurepa subtrees in M'''.

Suppose that there is a Kurepa subtree T' of T in M'''. Without loss of generality we can assume that T' is closed downward.

Since $\mathcal{B}(T) = (\mathcal{B}(T))^{M''}$, then $\mathcal{B}(T') \subseteq (\mathcal{B}(T))^{M''}$ in M'''. Since $T' \subseteq T$, there is a subset I of ω_3 in M'' such that $|I| = \omega_1$ and $T' \in M''[G_3 \cap Fn(I,2,\omega_1)]$. T' is still a Kurepa tree in $M''[G_3 \cap Fn(I,2,\omega_1)]$. Let $p_0 \in G_3 \cap Fn(I,2,\omega_1)$ such that

$$p_0 \Vdash (\dot{T}' \text{ is a Kurepa tree}).$$

For any $B \in \mathcal{B}(T')$ there is a $p_B \leq p_0$ such that $p_B \Vdash (B \in \mathcal{B}(\dot{T}'))$. Let

$$\mathcal{C} = \{B \in \mathcal{B}(T) : \exists\, p \leq p_0(p \Vdash (B \in \mathcal{B}(\dot{T}')))\}.$$

Since T' is a Kurepa tree in $M''[G_3 \cap Fn(I,2,\omega_1)]$, then $|\mathcal{C}| > \omega_1$ in M''. $|Fn(I,2,\omega_1)| = \omega_1$ because CH is true in M''. So there is a $p' \leq p_0$ in $Fn(I,2,\omega_1)$ such that

$$\mathcal{C}' = \{B \in \mathcal{C} : p' \Vdash (B \in \mathcal{B}(\dot{T}'))\}$$

has power $> \omega_1$.

Let $T'' = \bigcup \mathcal{C}'$ which is in M''. Then $p' \Vdash (T'' \subseteq \dot{T}')$ and that implies every level of T'' is at most countable. Since $\mathcal{C}' \subseteq \mathcal{B}(T'')$, then T'' is a Kurepa tree and this contradicts that there are no Kurepa trees in M''.

Theorem 3 *It is consistent with CH plus $2^{\omega_1} > \omega_2$ that there exists a Kurepa tree which has no Jech–Kunen subtrees.*

The following proof is due to K. Kunen.

Proof: Let M be a model of CH. In M, let κ be a regular cardinal such that $\omega_2 < \kappa$ and $2^{\omega_1} \leq \kappa$. Let $\mathbb{P} \in M$ be a partial order such that a condition $p \in \mathbb{P}$ is a pair $\langle T_p, l_p\rangle$, where T_p is a downward closed countable normal subtree of $\langle 2^{<\omega_1}, \subseteq\rangle$ of height $\alpha_p + 1$ for some countable ordinal α_p and l_p is a one-to-one function from some countable subset of κ into the top level of T_p. For two conditions $p, q \in \mathbb{P}$, $p \leq q$ iff $T_p \mid ht(T_q) = T_q$, $\mathrm{dom}(l_p) \supseteq \mathrm{dom}(l_q)$ and for all $\xi \in \mathrm{dom}(l_q)$, $l_q(\xi) \subseteq l_p(\xi)$.

$\mathbb{P}$ is the partial order used in Jech [2] and [6] to force a Kurepa tree, where $\mathbb{P}$ is shown to be ω_1-closed and have ω_2-c.c.

Let G be a $\mathbb{P}$-generic filter over M, $T_G = \bigcup\{T_p : p \in G\}$ and $B(\xi) = \{t \in T_G : \exists p \in G(t \subseteq l_p(\xi))\}$. In $M[G]$, CH holds, $2^{\omega_1} = \kappa > \omega_2$, T_G is a Kurepa tree with κ many branches and $\mathcal{B}(T_G) = \{B(\xi) : \xi < \kappa\}$ (see [2] or [6] for the detail).

Claim *There are no Jech–Kunen subtrees of T_G.*

Proof of Claim: Let $T \subseteq T_G$ and $\mathcal{B}(T) = \lambda < \kappa$ in $M[G]$. Without loss of generality we assume that T is closed downward. Let $\dot{T} = \bigcup\{\{s\} \times A_s : s \in 2^{<\omega_1}\} \in M^{\mathbb{P}}$ be a nice name for T (see [5], p. 208 for the definition of a nice name). Let $p_0 \in \mathbb{P}$ such that $p_0 \Vdash (\dot{T} \subseteq T_{\dot{G}}$ and $|\mathcal{B}(\dot{T})| = \lambda < \kappa)$. Since $\mathbb{P}$ has ω_2-*c.c.*, then the set

$$S = \{\xi < \kappa : \exists p \leq p_0\ (p \Vdash \dot{B}(\xi) \in \mathcal{B}(\dot{T}))\}$$

has the cardinality $\leq \omega_1\lambda < \kappa$. Defining

$$\mathrm{supt}(\dot{T}) = \{\xi < \kappa : \exists \langle s, p\rangle \in \dot{T}(\xi \in \mathrm{dom}(l_p))\}.$$

Since $|2^{<\omega_1}| = \omega_1$ in M and for every $s \in 2^{<\omega_1}$, $|A_s| \leq \omega_1$, then $|\mathrm{supt}(\dot{T})| \leq \omega_1$. Now pick a $\xi_0 \in \kappa$ such that $\xi_0 \notin S \cup \mathrm{supt}(\dot{T}) \cup \mathrm{dom}(l_{p_0})$. Since $\xi_0 \notin S$, we have $p_0 \Vdash \dot{B}(\xi_0) \notin \mathcal{B}(\dot{T})$.

Subclaim *For any $\xi \in \kappa - (\mathrm{supt}(\dot{T}) \cup \mathrm{dom}(l_{p_0}))$, $p_0 \Vdash \dot{B}(\xi) \notin \mathcal{B}(\dot{T})$.*

The claim follows from the subclaim because

$$p_0 \Vdash \mathcal{B}(\dot{T}) \subseteq \{\dot{B}(\xi) : \xi \in \operatorname{supt}(\dot{T}) \cup \operatorname{dom}(l_{p_0})\}$$

implies

$$p_0 \Vdash |\mathcal{B}(\dot{T})| = \lambda \leq \omega_1.$$

Proof of Subclaim: We define an isomorphism i from $\mathbb{P}$ to itself induced by π, a permutation of κ such that $\pi(\xi) = \xi_0$, $\pi(\xi_0) = \xi$ and $\pi(\alpha) = \alpha$ if $\alpha \in \kappa - \{\xi, \xi_0\}$. For any $p \in \mathbb{P}$, let $i(p) = \langle T_p, i(l_p)\rangle$, where

$$i(l_p) = \begin{cases} l_p & \text{if } \xi, \xi_0 \notin \operatorname{dom}(l_p) \\ (l_p - \{\langle \xi, l_p(\xi)\rangle\}) \cup \{\langle \xi_0, l_p(\xi)\rangle\} & \text{if } \xi \in \operatorname{dom}(l_p) \text{ and } \xi_0 \notin \operatorname{dom}(l_p) \\ (l_p - \{\langle \xi_0, l_p(\xi_0)\rangle\}) \cup \{\langle \xi, l_p(\xi_0)\rangle\} & \text{if } \xi_0 \in \operatorname{dom}(l_p) \text{ and } \xi \notin \operatorname{dom}(l_p) \\ (l_p - \{\langle \xi_0, l_p(\xi_0)\rangle, \langle \xi, l_p(\xi)\rangle\}) \cup \{\langle \xi_0, l_p(\xi)\rangle, \langle \xi, l_p(\xi_0)\rangle\} & \\ & \text{if } \xi, \xi_0 \in \operatorname{dom}(l_p) \end{cases}$$

let i_* be a map from $M^{\mathbb{P}}$ to $M^{\mathbb{P}}$ induced by i (see [5], p. 222 for the definition of i_*). Then $i(p_0) \Vdash i_*(\dot{B}(\xi_0)) \notin \mathcal{B}(i_*(\dot{T}))$. Since ξ and ξ_0 are not in $\operatorname{supt}(\dot{T}) \cup \operatorname{dom}(l_{p_0})$, then $i(p_0) = p_0$, $i_*(\dot{T}) = \dot{T}$ and $i_*(\dot{B}(\xi_0)) = \dot{B}(\xi)$, hence $p_0 \Vdash \dot{B}(\xi) \notin \mathcal{B}(\dot{T})$.

Remark The author's original proof of Theorem 3 involves the existence of two inaccessible cardinals.

In next two theorems we show the negative sides of Theorem 2 and Theorem 3. Before that we should introduce some properties of poset and Generalized Martin's Axiom. We take the form of Generalized Martin's Axiom from [7] in which they call it GMA($\aleph_1$-centered).

Let $\mathbb{P}$ be a poset. A subset Q of $\mathbb{P}$ is called centered if every finite subset of Q has a lower bound in $\mathbb{P}$. A poset is called ω_1-centered if it is the union of ω_1 many centered subsets. A poset is called countably compact if every countable centered subset of it has a lower bound.

GMA (Generalized Martin's Axiom) is the statement: Suppose $\mathbb{P}$ is an ω_1-centered and countably compact poset. Suppose $\kappa < 2^{\omega_1}$. If D_α is a dense subset of $\mathbb{P}$ for each $\alpha < \kappa$, then there exists a filter G of $\mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$.

We now define a poset in terms of a tree and its branches. Let T be a tree and $\mathcal{B}$ be a subset of $\mathcal{B}(T)$. We let

$\mathbb{P}(T,\mathcal{B}) = \{\langle A,\mathcal{C}\rangle : A$ is a countable subtree of T which is closed downward, $\mathcal{C}$ is a nonempty countable subset of $\mathcal{B}$ such that for every C in $\mathcal{C}$, $ht(C \cap A) = ht(A)\}$

be a poset ordered by:

$$\langle A_1,\mathcal{C}_1\rangle \leq \langle A_2,\mathcal{C}_2\rangle \text{ iff } \mathcal{C}_2 \subseteq \mathcal{C}_1 \text{ and } A_1 \mid ht(A_2) = A_2$$

for any $\langle A_1,\mathcal{C}_1\rangle,\langle A_2,\mathcal{C}_2\rangle \in \mathbb{P}(T,\mathcal{B})$.

Lemma 2 *Let T be an ω_1-tree and $\mathcal{B} \subseteq \mathcal{B}(T)$. Then*
(a) *for any $\langle A_1,\mathcal{C}_1\rangle$ and $\langle A_2,\mathcal{C}_2\rangle \in \mathbb{P}(T,\mathcal{B})$, $\langle A_1,\mathcal{C}_1\rangle$ and $\langle A_2,\mathcal{C}_2\rangle$ are compatible if and only if either $A_1 | ht(A_2) = A_2$ and for each $C \in \mathcal{C}_2$, $ht(C \cap A_1) = ht(A_1)$ or $A_2 | ht(A_1) = A_1$ and for each $C \in \mathcal{C}_1$, $ht(C \cap A_2) = ht(A_2)$;*
(b) *$\mathbb{P}(T,\mathcal{B})$ is ω_1-centered and countably compact if assuming CH.*

Proof: (1): "$\Leftarrow$": Easy.

"$\Rightarrow$": Let $\langle A,\mathcal{C}\rangle \leq \langle A_1,\mathcal{C}_1\rangle$ and $\langle A_2,\mathcal{C}_2\rangle$. Assume $ht(A_1) \geq ht(A_2)$. Then $A_1 | ht(A_2) = (A | ht(A_1)) | ht(A_2) = A | ht(A_2) = A_2$ and for each $C \in \mathcal{C}_2$, $ht(C \cap A_1) = ht(A_1)$ because $ht(C \cap A) = ht(A)$ and $A | ht(A_1) = ht(A_1)$.

(2): For any $A \subseteq T$ such that A is countable and closed downward, let

$$\mathbb{P}_A = \{\langle A,\mathcal{C}\rangle : \langle A,\mathcal{C}\rangle \in \mathbb{P}(T,\mathcal{B})\}.$$

Then $\mathbb{P}_A$ is a centered subset of $\mathbb{P}(T,\mathcal{B})$. We have only ω_1 many such A's if assuming *CH*. So $\mathbb{P}(T,\mathcal{B})$ is ω_1-centered.

Suppose $\{\langle A_n,\mathcal{C}_n\rangle : n \in \omega\}$ is a centered subset of $\mathbb{P}(T,\mathcal{B})$. Let $A = \bigcup_{n\in\omega} A_n$ and $\mathcal{C} = \bigcup_{n\in\omega} \mathcal{C}_n$.

Claim 1: $\langle A,\mathcal{C}\rangle \in \mathbb{P}(T,\mathcal{B})$.

Proof of Claim 1: If there is a $C \in \mathcal{C}$ such that $ht(C \cap A) < ht(A)$, then there are $m, n \in \omega$ such that $C \in \mathcal{C}_m$ and $ht(C \cap A_n) < ht(A_n)$. Since $\langle A_m,\mathcal{C}_m\rangle$ and $\langle A_n,\mathcal{C}_n\rangle$ are compatible, if $ht(A_n) \leq ht(A_m)$, then $ht(C \cap A_n) = ht(A_n)$ because $ht(C \cap A_m) = ht(A_m)$, a contradiction; if $ht(A_n) > ht(A_m)$, then $A_m | ht(A_n) \neq A_n$, hence $ht(C \cap A_n) = ht(A_n)$ by (1), also a contradiction.

Claim 2: $\langle A,\mathcal{C}\rangle$ *is a lower bound of* $\{\langle A_n,\mathcal{C}_n\rangle : n \in \omega\}$.

Proof of Claim 2: If there is an $n \in \omega$ such that $A | ht(A_n) \neq A_n$, then there is a $t \in A | ht(A_n) - A_n$. Let $t \in A_m$ for some $m \in \omega$. Since $\langle A_n,\mathcal{C}_n\rangle$ and $\langle A_m,\mathcal{C}_m\rangle$ are compatible, if $A_n | ht(A_m) = A_m$, then $t \in A_n$, a contradiction; if $A_m | ht(A_n) = A_n$, then $t \in A_m | ht(A_n)$ implies $t \in A_n$, also a contradiction.

So $\langle A,\mathcal{C}\rangle \leq \langle A_n,\mathcal{C}_n\rangle$ for all $n \in \omega$.

By Claim 1 and Claim 2, $\mathbb{P}(T,\mathcal{B})$ is countably compact.

Theorem 4 *Assume* GMA *and CH plus $2^{\omega_1} = \omega_3$. Then every Jech–Kunen tree has a Kurepa subtree.*

Proof: Let T be a Jech–Kunen tree with ω_2 many branches. Without loss of generality we can assume that $\forall t \in T\ (|\mathcal{B}(T(t))| = \omega_2)$. (We can make this by throwing away all t's with $|\mathcal{B}(T(t))| \leq \omega_1$.)

Let $\mathcal{B} = \mathcal{B}(T) = \{B_\alpha : \alpha < \omega_2\}$. For every $\beta < \omega_2$ let

$$D_\beta = \{\langle A,\mathcal{C}\rangle \in \mathbb{P}(T,\mathcal{B}) : \mathcal{C} \cap \{B_\alpha : \beta < \alpha < \omega_2\} \neq \emptyset\}.$$

For every $\gamma < \omega_1$ let

$$E_\gamma = \{\langle A,\mathcal{C}\rangle \in \mathbb{P}(T,\mathcal{B}) : ht(A) > \gamma\}.$$

Then D_β and E_γ both are dense subsets of $\mathbb{P}(T,\mathcal{B})$ for all $\beta < \omega_2$ and $\gamma < \omega_1$. By GMA there is a filter G of $\mathbb{P}(T,\mathcal{B})$ such that $G \cap D_\beta \neq \emptyset$ and $G \cap E_\gamma \neq \emptyset$ for all β and γ. Let

$$T' = \bigcup \{A : \langle A,\mathcal{C}\rangle \in G\}.$$

Then $ht(T') = \omega_1$ because $G \cap E_\gamma \neq \varnothing$ for all $\gamma < \omega_1$.

Claim 1 $|\mathcal{B}(T')| = \omega_2$.

Proof of Claim 1: If $|\mathcal{B}(T')| < \omega_2$, then there is a $\beta < \omega_2$ such that $\mathcal{B}(T') \subseteq \{B_\alpha : \alpha \leq \beta\}$. But this contradicts that $G \cap D_\beta \neq \varnothing$.

Claim 2 $\forall \alpha < \omega_1 (|T'_\alpha| \leq \omega)$.

Proof of Claim 2: Assume this is not true. Then there is an $\alpha < \omega_1$ such that $|T'_\alpha| = \omega_1$.

Let $\langle A,\mathcal{C}\rangle \in G$ such that $ht(A) > \alpha$. Since A is countable, there is a $t \in T'_\alpha - A$. Let $\langle A',\mathcal{C}'\rangle \in G$ such that $t \in A'$. Since $\langle A,\mathcal{C}\rangle$ and $\langle A',\mathcal{C}'\rangle$ are compatible, then either $A \,|\, ht(A') = A'$ or $A' \,|\, ht(A) = A$. $A \,|\, ht(A') = A'$ is impossible because $t \notin A$. $A' \,|\, ht(A) = A$ is also impossible because $t \in A' \cap T'_\alpha$ and $\alpha < ht(A)$.

By Claim 1 and Claim 2, T' is a Kurepa subtree of T.

Theorem 5 *It is consistent with* GMA *and* $2^{\omega_1} > \omega_2$ *that there exist Kurepa trees with* 2^{ω_1} *many branches and every Kurepa tree has Jech–Kunen subtrees.*

We need a lemma to prove Theorem 5.

Lemma 3 *Let M be a model of CH plus $2^{\omega_1} > \omega_2$. Let T be an ω_1-tree such that for every $t \in T$, $|\mathcal{B}(T(t))| \geq \omega_2$ and let $\mathcal{B} \subseteq \mathcal{B}(T)$ such that $|\mathcal{B}| = \omega_2$ and for every $t \in T$, $|\mathcal{B}(T(t)) \cap \mathcal{B}| = \omega_2$. If G is a $\mathbb{P}(T,\mathcal{B})$-generic filter over M and $T_G = \bigcup\{A : \langle A,\mathcal{C}\rangle \in G\}$, then T_G is a Jech–Kunen subtree of T in $M[G]$.*

Proof: Let $\mathcal{B} = \{B_\alpha : \alpha < \omega_2\}$. Since

$$D_\beta = \{\langle A,\mathcal{C}\rangle \in \mathbb{P}(T,\mathcal{B}) : \mathcal{C} \cap \{B_\alpha : \beta < \alpha < \omega_2\} \neq \varnothing\}$$

is dense in $\mathbb{P}(T,\mathcal{B})$, then $|\mathcal{B}(T_G)| \geq \omega_2$ by the proof of Claim 1 of Theorem 4. We now need to show that $|\mathcal{B}(T_G)| = \omega_2$.

Suppose that is not true. Then there is a $B \in (\mathcal{B}(T))^M - \mathcal{B}$ such that $B \in \mathcal{B}(T_G)$ in $M[G]$ since ω_1-closed forcing extension adds no new branches of T. Let $\langle A_0,\mathcal{C}_0\rangle \Vdash (B \in \mathcal{B}(T_{\dot{G}}))$. Since $B \notin \mathcal{C}_0$, there is an $\alpha < \omega_1$, $\alpha > ht(A_0)$ such that B is different from C at α-th level for all $C \in \mathcal{C}_0$. Let

$$A_1 = \left(\left(\bigcup \mathcal{C}_0\right) \cup A_0\right) \cap (T \,|\, \alpha + 1).$$

Then $\langle A_1,\mathcal{C}_0\rangle \leq \langle A_0,\mathcal{C}_0\rangle$. Hence $\langle A_1,\mathcal{C}_0\rangle \Vdash (B \in \mathcal{B}(T_{\dot{G}}))$. But if H is a $\mathbb{P}$-generic filter over M such that $\langle A_1,\mathcal{C}_0\rangle \in H$, then $B \notin \mathcal{B}(T_H)$ in $M[H]$ since $ht(B \cap A_1) < ht(A_1)$, a contradiction.

Proof of Theorem 5: Let M be a model of CH plus $2^{\omega_1} = 2^{\omega_2} = \omega_3$ and there are Kurepa trees with ω_3 many branches. (See [6], p. 282 for such a model.) Let $\mathbb{P}$ be the ω_3 steps countable support iterated forcing poset for GMA in M and G be a $\mathbb{P}$-generic filter over M. We want to show that $M[G] \vDash [CH,\ 2^{\omega_1} = \omega_3$, there are Kurepa trees with ω_3 many branches and every Kurepa tree has Jech–Kunen subtrees$]$.

Let T be a Kurepa tree in $M[G]$. Without loss of generality we can assume that for every $t \in T$, $|\mathcal{B}(T(t))| \geq \omega_2$. Let $\mathcal{B} \subseteq \mathcal{B}(T)$ such that for every $t \in T$, $|\mathcal{B} \cap \mathcal{B}(T(t))| = \omega_2$. Then $\mathbb{P}(T,\mathcal{B})$ is ω_1-centered and countably compact by

Lemma 2. Let $\alpha < \omega_3$ such that T, $\mathcal{B}$ and $\mathbb{P}(T,\mathcal{B})$ are in $M[G_\alpha]$, which is the initial α steps iterated forcing extension of M in $M[G]$ and $\mathbb{P}(T,\mathcal{B})$ is the poset used at α-th step forcing extension for GMA. Let H be the $\mathbb{P}(T,\mathcal{B})$-generic filter over $M[G_\alpha]$ such that $M[G_{\alpha+1}] = M[G_\alpha][H]$. Then

$$T_H = \bigcup \{A : \langle A,\mathcal{C}\rangle \in H\}$$

is a Jech–Kunen subtree of T in $M[G_{\alpha+1}]$. T_H is still a Jech-Kunen tree in $M[G]$ because the poset for the rest of the forcing extension is ω_1-closed in $M[G_{\alpha+1}]$.

Remark All the results in this paper about trees can be translated into the results about linear orders. Among them the one related Jech–Kunen tree is most interested.

Let L be called a Jech–Kunen continuum iff L is a Dedekind complete dense linear order with density ω_1 and power strictly between ω_1 and 2^{ω_1}. Assume CH plus $2^{\omega_1} > \omega_2$. Then there exists a Jech–Kunen tree iff there exists a Jech–Kunen continuum.

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