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Incompleteness in Intuitionistic Metamathematics

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Abstract There are three main results; all are contributions to the intuitionistic metatheory of intuitionistic systems. First, pure intuitionistic predicate logic is provably incomplete with respect to ordinary model-theoretic semantics, provided that the metatheory is suitably intuitionistic. With the same proviso, intuitionistic propositional logic is also incomplete; in fact, the concept of validity for formulas in one propositional variable is not arithmetically definable. Also, one cannot prove—in standard intuitionistic metatheories—an existence theorem for countable models, even when the relevant theory is that of subfinite sets in the language of pure identity.

1 Introduction The three main results are all contributions to the intuitionistic metatheory of intuitionistic systems. In more detail, the first theorem shows the intuitionistic predicate logic to be, in the presence of a weak form of Church's Thesis (WCT), neither complete nor almost complete. The proof itself is fully constructive: there is a single formula ϕ such that Con(ϕ) is provable in a fragment of Heyting arithmetic while WCT implies that ϕ has no model. Although the conclusion of the proof, that intuitionistic predicate logic is incomplete, is not especially new, it seems that the proof given here is simpler and more straightforward than existing published proofs of similar results. Our proof joins the *incompleteness of intuitionistic logic* directly to standard classical proofs of the *incompleteness of first-order arithmetic*. Basically, WCT blocks the construction of classically possible structures $\mathfrak{A}$ because, in those structures, truth in $\mathfrak{A}$ would be definable—in contradiction to the Gödel-Tarski fixed-point theorem, which is correct both classically and constructively.

The second result, on intuitionistic propositional logic, shows that completeness theorems for propositional logic are almost as elusive as those for predicate logic. In the proof, the axioms of intuitionistic set theory together with Church's

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Thesis, Dependent Choice, and Markov's Principle (MP) are seen to be jointly inconsistent with the statement that propositional logic is complete with respect to Tarskian semantics. In fact, there is a relatively short *quantifier-free* formula Ψ which is, in the presence of the Extended Church's Thesis (ECT) and Markov's Principle, valid. Ψ, however, also has the following properties:

- It is not provable in intuitionistic propositional logic.
- Ψ has an arithmetic instance $\Psi_{\sigma,\tau}$ which is also valid but independent of any classically correct r.e. Harrop extension of Heyting arithmetic.
- Moreover, there is an instance Ψ_γ of Ψ in the language of set theory such that, if classical set theory is consistent, then Ψ_γ is unprovable in intuitionistic set theory.

Ψ is a formula in two propositional variables. As coda to the investigation of the metamathematical properties of Ψ, we observe that there may be no appreciable improvement in the "completeness picture" should attention be restricted to propositional formulas in a single variable only. By examining a topological model for intuitionistic set theory constructed over Sierpinski space, we find that intuitionistic set theory cannot prove that validity for single-variable formulas is arithmetically definable.

Given the classical associations between completeness and countable models theorems, we thought it reasonable to close with a brief remark on the intuitionistic status of the claim that every satisfiable theory in a countable first-order language has a countable model. It is shown that, even if restricted to theories of subfinite sets in the language of pure identity, the countable models claim implies Kripke's Scheme (KS), a principle which features large in discussions of Brouwer's *theory of the creative subject*. An obvious consequence is that Henkin-style maximal set constructions are unavailable in all the more familiar extensions of intuitionistic set theory.

Proofs of these results and the requisite formal details have been postponed until Section 4. In the meantime, there are matters of philosophical and historical orientation toward completeness issues in intuitionism. These take up the intervening sections.

2 Philosophical orientation In many respects, philosophy of mathematics is a study in the prejudices of an age. One might even say that it is a specially valuable avenue to that study. Rarely do the inadequacies of conventional philosophy come out in so plain a light as when they appear centerstage with the clarities of mathematics in the background. As far as our age is concerned, a principal prejudice is that mathematics is anagrammatic, that it is a clever or even demonic rearrangement or obscuring of the intelligible. Philosophers tell us that mathematics stands in dire need of a certain style of philosophical rewriting or reinterpretation, a new rendering of mathematics into an idiom that we do understand.

Behind or next to this prejudice lies another: that, paradoxical as it may sound, mathematics is of apiece with a family of languages and symbolic techniques and, yet, these languages and techniques do not speak to us directly. To put it another way, it is made to seem as if, even though we understand mathematical techniques, we do not understand them "really". According to the philosophers, if mathematics speaks at all, its ordinary organs of speech are defective. It appears virtually mute, since the voice of mathematics to which philosophers can harken seems so very subtle. It does not seem to be the voice to which we are accustomed, the voice, for instance, of the mathematics classroom.

Philosophers feel that they need to *explicate* mathematics, that it needs to be made plain to us. This feeling—and its wide acceptance—is a fair measure of the conceptual ground we have lost since the days in which Galileo could hold that the book of nature is written in the language of mathematics. In many ways, this book now seems closed. Many would have us believe that we can only guess at this book's contents by extrapolation from the contents of another, a book of philosophical elucidations, a book such as Frege's *Grundlagen*.

This situation is all the more strange when we reflect on the volume and clarity with which secondary-school mathematics speaks. What could be more direct and persuasive than the Euclidean proof that the sum of the angles of a triangle is 180 degrees? What could be more eloquent than Descartes' analysis of the conic sections? Such mathematical vocality is hardly limited to the more elementary branches of the subject. Galois theory is a fine example of mathematics which is of great beauty and import, yet (nowadays) hardly calls for a philosopher to tell us what it "really" means.

In a Heideggerian or Deweyan (re)history of mathematical thought might appear the statement that this pretended muteness, this "philosophical status" for mathematics, derives ultimately from a real social prejudice, the Platonistic prejudice against the artisan, the engineer, or the practitioner. Shoemaking and even bridge-building require a knowledge of special ways of speaking, special languages. They both call up specific techniques—formal and informal. Yet neither of these has a "philosophical status"; they do not impress us as mysteries. The relevant techniques and languages have been allowed to speak plainly, in their own voice. These are not the object of any "philosophy of".

But mathematics has been placed at a distance from these. Plato may have been the first to try to forge an alliance between (Greek) mathematics and other voices, the ones which philosophers pretend to speak for mathematics. These are generally voices concerned with the unvoiced grammatical structure of a sentence, hidden voices, such as the voice of the *daimon* which so obsessed Socrates. In Plato, the foremost mysterious voice is that of grammar or sentential structure, a voice in which Plato thought to hear the doctrine of the forms. It is obvious that there can be no word in a sentence which captures the sentential form of the sentence, one which guarantees that a group of words will be viewed as a sentence rather than as a mere list. With Plato, the work of the mathematician (and not of the shoemaker) is placed on the side of grammar and is assimilated with the unvoiced in the sentence. Perhaps it was originally a political or social need—to assimilate the mathematician and the grammarian. At any rate, mathematics came to be seen as needing a rewording, as fundamentally structural and, so, as calling for a new word to record those structural features. (It ought to be remarked that Wittgenstein was the first, in the present century, to realize that the only appropriate rendering of the silent voice of grammar is by silence, by making a gap in the course of standard philosophizing.)

As is characteristic of conventional philosophy, the outcome of the need to replace plain and indicative silence with metaphysics is paradox. This time, the paradox lies in the contrast between the end result and the starting point. The philosopher began the reinterpretation of mathematics by shunning the domain of the artisan but finishes by describing "the nature" of mathematics in terms which would only be suitable to the description of tedious domestic arrangements, trite scenarios such as the placement of pots on a countertop or the sorting of cards into a deck. In other words, the presumptive foundations of mathematics turn out, at the hands of the philosophers, to issue in terms which would be utterly insufficient to explain in detail the manufacture of a paper clip.

Wittgenstein, repeating an expression of Littlewood, referred to the pseudoexplanations of mathematical philosophy as "'gas". Our contention is not that all such gases are absolutely useless; a restatement of a mathematical result in intuitive terms may well be the key to a range of applications. It is the purely philosophical business of rereadings which interests us; it is the idea that what the philosopher has to say about mathematics is *deep*, that it exposes a form of reality which is hidden by the straightforward expression of a mathematical result. It is the presumptive *fundamentality* or *ultimacy* of these readings which give us most concern. The impression is created that the philosophical voicings and reexpressions of mathematics are *descriptions*, even accurate descriptions, of a realm which lies "below" mathematics and serves as a foundation for it. Rather than treating philosophy of mathematics as a collection of rhetorical flourishes and political slogans which belong more to the cultural neighborhood of mathematics than to the development of mathematics proper, the putative descriptions are compared with the fundamental principles of physics and accorded a correlative explanatory status. High on the list of principles which are often accorded this ultimacy or explanatory depth is the statement of completeness for a formal logic (cf. Dummett [7]). It is among the purposes of the present writing to bring this ultimacy into question, to bring the reader to wonder whether it might well be essential for the coherence of the intuitionistic *mathematical* enterprise that first-order predicate logic be *incomplete*.

We do not labor under the illusion that such a change in mind could be the result of new philosophical principles alone. A complete accounting for the state of contemporary philosophy is not to be found in "principles" or (what philosophers call) "positions." One must also look to deeply ingrained philosophical attitudes toward mathematical thinking. For example, it is very much a part of Frege's philosophy that he felt a prima facie need to coat the mathematical concept of a function by dipping it into the mysteries of *incomplete entities* and *unsaturatedness*. The recognition—in Frege and in the reader—of such a need is essential for understanding Frege's philosophy. But such a need is no part of Frege's "official" philosophy; it is not a consequence of any statement of the Fregean position. It is in the changing of such fundamental attitudes that intuitionistic mathematics has a special role to play.

2.1 Intuitionism and philosophy of mathematics The dangerous idea of a "natural reading" of a branch or result of mathematics is one kind of thinking which a familiarity with intuitionism might prevent. There is a bad tendency in philosophy to forgetfulness; philosophers of mathematics forget that they labor not in mathematics but in a land of readings or renditions of mathematics. Sometimes the readings are treated as if they were themselves the mathematical statements or even as if they were more important than the mathematics itself. (This is perhaps another symptom of the notion that philosophical readings of mathematics are fundamental.) How often have we worried over the "fact" that Gödel has demonstrated that there are ineffable truths of arithmetic, one of which may be the statement of the coherence of arithmetic itself? We rarely turn away from Gödel, even momentarily, to reflect that this is a reading of a mathematical technique and of the consequences of adopting that technique. We do not often question that reading by asking whether the relation between the truths of Hilbertian metamathematics and those of what we call 'ordinary arithmetic' is truly the identity or the inclusion the reading seems to presuppose. How often is it said that Löwenheim and Skolem proved that syntax does not determine semantics and, hence, that there must be a separate, nonformalistic study of interpretation? We rarely pause to ask whether what is here read as 'syntax', with its close attachment to the finite, bears so obvious a resemblance to words, the everyday expressions of meaning, with their close attachments to the unbounded.

This is one regard—the investigation of "natural" presumptions—in which intuitionistic mathematics is of tremendous help. It teaches us not to be so sanguine about the seemingly casual or natural relation between the metamathematics of classical logic and its philosophical reexpressions. In the intuitionistic context, naturality is not so readily come by. A reading or intuitive expression of a significant intuitionistic theorem is well worth having, but it comes at the price of a salutary rethinking of what is "normal" or "natural" in the classical case. Among other things, we are thereby reminded that a theorem of mathematics does not carry any particular rendering as a matter of course. Also, we are encouraged to address the mathematics with a bit more respect when we come to interpret it. Intuitionism encourages us to treat its mathematics more in the way we treat quantum mechanics, where it is nowise obvious how to reexpress our formal findings in terms suitable for describing domestic arrangements and where our understanding suffers if we choose a reexpression unwisely.

In the face of intuitionistic mathematics, much that we have found "natural" in philosophy of mathematics withers away; if we insist upon applying classical ideas of normalcy or naturality to intuitionism, we often land in confusion. Here is an example of how confusion can stem from the desire to carry over, into intuitionistic mathematics, the "natural" distinction between the logical and nonlogical, that is, between scheme and content. Certain expositors make much of the fact that, as they describe it, intuitionists may accept an "axiom" of countable choice, namely,

$$\forall n \in \mathcal{N}\, \exists y \in A.\, \phi(n,y) \rightarrow \exists f : (\mathcal{N} \rightarrow A) \forall n.\, \phi(n, fn),$$

more or less as a matter of course, rather in the way in which the classical mathematician treats a theorem of logic. Viewed "naturally" by the expositor, logic is supposed to be "pure" and divorced from mathematical content, hence the intuitionistic attitude toward the so-called mathematical axiom seems strange. The conventional idea which is the cause of the seeming strangeness is that mathematics is just one kind of content with which the neutral schema of logic can be filled and that mathematical results ought not to be consequences of a pure logic. The difficulty here is the importation of a wholly anti-intuitionistic conception of logic, one on which mathematics has a merely incidental relation to logic, one on which mathematics is just one kind of content, among countless others, over which the schemes of logic can be interpreted. In intuitionism, what is called 'logic' is only one of the crucial features of mathematical reasoning.

Another fine (and more important) example is the almost ubiquitous chatter about the "weakness" of intuitionistic logic. Within classical confines, it is natural to make reasonable comparisons among systems of logic by looking to the inscriptions which they produce. But to extend such elementary comparisons across the gap between classical and intuitionistic mathematics is to court disaster. Admittedly, there are certain inscriptions that formal systems which purport to represent intuitionistic reasoning cannot produce; one such is the inscription '$p \vee \neg p$'. But that is not to say that there are any *laws* of classical logic or theorems of classical mathematics that intuitionistic mathematics does not respect, where a law or a theorem is not an inscription but might well, in a given context, be captured by such an inscription. For one thing, the negative translation results which stem from the original ideas of Gödel, Glivenko, and Gentzen (cf. Leivant [32]) have been read as a general permission for the intuitionist to speak of classical work as an abbreviation or truncation of intuitionistic thought. This appears to open up the prospect that all laws and theorems of classical set theory, for example, can be captured by inscriptions derivable in intuitionistic systems. Should we want to speak of mathematics as offering us a form of evidence for philosophical views, here is evidence that intuitionistic mathematics does not conceal weakness but exhibits tremendous strength, strength sufficient to incorporate classical set theory.

Of course, we need not wander as far as intuitionism to begin to question the naturalness of our graduate school renderings of metamathematical theorems. It was commonplace in traditional logic (and is still commonplace in some treatments of philosophy of mind) to speak as if propositions were containers filled up with bits of information. In close company with this container idea came the unpacking account of correct inference, according to which a valid argument serves to unpack some part of the information contained in the premises and bundle it up again in the conclusion. It is then but a short step to the idea that there is an "inference engine" in the mind, a faculty for making and evaluating inferences. It is conceived as a kind of spiritual or computational mailroom in which all this packing and unpacking can occur. But, unless one has so deformed the concept of information that it becomes indistinguishable from the concept of truth-value, it will be difficult to square even the simplest observations about inference with the unpacking metaphor. For instance, consider the invalidity of the scheme

$$p \vee \neg p \Rightarrow q \wedge \neg q.$$

On the containment model, this would have to count as a valid inference, since claims of the form $p \vee \neg p$ convey no appreciable information. Hence, despite the seeming naturalness of the container metaphor, it must be rejected as a true account of classical inference. Moreover, there is nothing particularly classical about the shortcomings of the container idea; the same sort of observations could be made with respect to the intuitionistic invalidity of the scheme

$$(p \rightarrow p) \Rightarrow q \wedge \neg q.$$

In fact, the status conventionally conferred upon the completeness theorem for first-order logic is a fine example of the mismatch between mathematics and its metaphors. It is standard in many treatments of logic for the author to wax almost poetical on the benefits which follow from completeness, when, in truth, it is nowise clear that completeness is a font of unalloyed benefice. One need only recall that an immediate consequence of the strong completeness theorem is the compactness theorem. Also, a direct result of Henkin's proof of completeness is the countable models theorem—or a simple form of downward Löwenheim-Skolem. In contemporary philosophy of classical mathematics, these latter two results stand most plainly for a "dark side" in metamathematics, chapter headings in the book of logic's shortcomings. Thanks to compactness, first-order means are said to be incapable of the expression of as salient a mathematical notion as "finite group" or "well-ordered set." Thanks to the countable models theorem and the observations of Skolem, the same means—applied to the language of set theory—are thought incapable of capturing adequately the notion of uncountable set. These shortcomings on the part of classical first-order logic are taken to show that, if we would like to retain for symbolic logic some vestige of Leibniz's ideal of a *characteristica universalis*, the attainment of which Gödel [14] thought not wholly chimerical, then we must extend the realm of the logical well beyond that of elementary or first-order logic. Therefore, if we rely only upon conventional standards, we ought to adopt a more balanced view of completeness, one on which it seems to do as much to weaken the claims of first-order logic to being a *mathematical* logic as it does to strengthen them.

Incidentally, there is an historical connection between the flawed logical theory based upon the "container" and the completeness issue in intuitionistic logic. The connection arose from articles such as Kreisel [27] and especially Troelstra [52]. Their idea was that, rather than fearing the prospect of incompleteness, we *ought to expect* it of intuitionism. Incompleteness was there viewed as a certification or verification of our ingrained suspicions—suspicions organized into the intuitive picture of the container—that, as "containers", intuitionistic propositions are far more capacious than their classical cousins. Hence, or so goes the suggestion, an intuitionistic statement of the form "scheme Φ is valid" is not likely to be reducible to the Σ_1^0 number-theoretic information supposedly contained in the statement "Φ is derivable" and, so, we ought to expect incompleteness. In this way, it was thought that a lack of completeness confirmed an intuitively plausible logical theory for intuitionism, that of the information container.

We agree that the incompleteness phenomena are not to be construed as symptoms of a mathematical debility in intuitionism. However, we will not agree, at least on the basis of the idea of the "information container", that incompleteness is to be expected. We refuse to console our feelings with this line of thought. First, it is thoroughly wed to the disenfranchised container metaphor. Second, even if we ignore the failings of the container idea, this treatment of completeness does damage to the schoolboy readings of constructive mathematics which give to the container what (little) credit it deserves. On such a reading, "constructive" Σ_1^0 statements are supposed to "contain more information" than their classical analogues. It is said that the constructive truth of a statement of the form $\exists n\phi(n)$ requires the provision both of a specific m and of a constructive proof of $\phi(m)$ and, hence, that the statement in question contains more information.

This is, or so the reading goes, what explains the relative difficulty in proving statements constructively. But, according to the considerations that are supposed to make us expect intuitionistic incompleteness, we can only come to expect it if we, at the same time, *belittle* the amount of information that might be contained in Σ^0_1 statements. These considerations would have us think that, even though such statements contain a great amount of information, this is not enough to encompass the information contained in a claim of validity. If we are being consistent and apply the container metaphor to constructive mathematics, we cannot both expect incompleteness for the above reasons and yet insist that extra information is contained in constructive existential claims. After all, the bonus of information putatively contained in constructive Σ^0_1 might well be enough to regain completeness intuitionistically.

There are independent grounds for refusing to take the container idea, as applied to intuitionism, very seriously. The "container" picture and the idea that intuitionistic propositions are relatively more capacious as containers than correlative classical propositions serves the conventional derogatory treatments of intuitionistic thought. They work together to allow one to think that there is some objective "information theoretic" ground on which intuitionistic logic is truly weaker than classical logic. The "more information" idea encourages one to think that an instance of a formula which is classically valid may fail to be intuitionistically valid because, when understood in the latter way, the formula would require more information for its truth and, hence, require more of its proof. It is supposed to follow that not every classically valid scheme is intuitionistically valid and that intuitionism is, thereby, inherently weaker than classical mathematics. Such reasoning also requires that the classical notion of a logical scheme carry over relatively unaltered from classical to intuitionistic contexts. However, there seems to be no such ready "carrying over": the classical concept of a scheme relies upon a particular construal of the specifically logical constants. In intuitionistic mathematics, there is no construal that compares with the satisfying analysis of logical constants in terms of Boolean functions.

Needless to say, the theorems of this paper are not intended to contribute (at least in the sense of 'contribute' usually in question) to the imaginary debate between classical and intuitionistic mathematicians. As everyone knows, there is either no such debate or, at best, very little of it. If debates between intuitionistic and nonintuitionistic mathematicians do ever occur, they are staged as an amusement or *raree* at a logic conference: people come to see the actors; they certainly don't come for the plot. What is called 'the debate' between classical and intuitionistic mathematics seems to be a political confrontation between the slogans or gases which surround one side or the other of the "dispute". The belief that such a confrontation is of real foundational importance is symptomatic of the idea that banalities about "building up numbers constructively" or "theories of meaning" stand in an enlightening rather than obscuring relation to mathematics. To think in terms of an important debate, one which needs to be resolved, is to think, in an alternative form, of philosophy as uncovering the subtle and significant underpinning to mathematics, one revealing conceptual problems to be solved by philosophical means.

Instead, the results in this paper are conceived as an aid to letting intuitionism speak for itself, without philosophical accompaniment. This, in turn, will require a general clearing of the philosophical air around intuitionism.

2.2 On the importance of being incomplete For the reader to gauge the bearing of our philosophical dealings with the incompleteness theorems, it will be necessary to get slightly ahead of the story and offer a preliminary outline of the proofs of the theorems and the ideas which they involve. It should be emphasized that this outline serves a very limited dialectical end: we are not trying to locate a foundation beneath our mathematics but are trying to draw a frame around it so as to set it off.

One cannot begin a consideration of intuitionistic reasoning outside of a consideration of the objects of intuitionistic mathematics. Intuitionistic propositions are themselves collections, or better, *species* of abstract entities. Species may, unlike sets or Frege's *Wertverläufe*, be individuated nonextensionally. The abstract entities in question are always one or another of a kind of *data objects*. These are not to be thought of as intuitionistic proofs themselves — of either the formal or the informal sort. For purposes of this discussion, we can think of a typical data object as an abstract machine — not a syntactic entity, a mere program-form — but as a fully interpreted device. By 'fully interpreted', we mean that the machine serves as a presentation or intention for the graph of a transformation. In keeping with this, our concept of machine is very generous; the machines might be Turing machines, but there is no need to insist, from the start, that they are. We would also allow that a neighborhood assignment determining a continuous map from Baire space into Baire space counts as a suitable "machine". Or machines might be generators for the r.e. subsets of the Scott-Plotkin graph model. Of course, there are some features which a suitable abstract machine must have — but not many. For instance, it may be essential that the class of machines be able to accept (possibly coded) versions of themselves or other machines as inputs.

It is in terms of the mathematics of a class of generalized machines and of the transformations computed by them that we define validity. A scheme

$$\Phi \models \Psi$$

is to be valid when there is a data object or machine which, given any interpretation * of the scheme, computes a function mapping the species determined by Φ^* into that determined by Ψ^*. It is important that there is no a priori limit set on the mathematical complexity of the transformation which underwrites the validity of an inference; it might well be given by (or have as its intension) a Turing machine the convergence of which can only be proved in a theory the strength of intuitionistic set theory.

To further illustrate and elucidate the incompleteness phenomenon, we need to consider intuitionistic truth and its role in counterexamples to TND, the *law of the excluded third*. (We want to lend no credence to the unfortunate impression, one often suggested by the expository habits of the early intuitionists and of their contemporary commentators, that the very idea of intuitionistic mathematics is tightly linked to objections to classical logical laws. That the formula scheme which represents TND classically fails of universal intuitionistic validity is a feature, but not a foundational one, of intuitionistic mathematics and rests upon several nontrivial mathematical assumptions. This feature, which is sometimes elevated to the status of a defining characteristic of intuitionism, is not required for the cogency of intuitionistic reasoning.) The condition that an intuitionistic proposition be true is that there is a data object which belongs to its species. To take a specific example, a data object M belongs to the species determined by a fully interpreted formula

$$\forall x(\Psi(x) \vee \neg\Psi(x))$$

only if M, as a machine, accepts each item a in the range of the variable x as an input and outputs two data objects, a natural number $M(a)_1$ and a machine $M(a)_2$ such that either

- $M(a)_1 = 0$ and $M(a)_2$ belongs to $\Psi(a)$'s species or
- $M(a)_1 \neq 0$ and $M(a)_2$ belongs to $\neg\Psi(a)$'s species.

To put it another way, when a universally quantified disjunction is intuitionistically true, then, for each relevant item a, one or the other of the disjuncts is intuitionistically true on a and, uniformly in a, one must be able to determine which disjunct is true.

To see that $\phi \vee \neg\phi$ is not universally valid, we need only assume that our mathematical language suffices for the expression of "data object unsolvable properties": that there are intuitionistic propositions $\phi(x)$ such that no data object serves as a machine M which determines uniformly on possible substituends a whether or not $M(a)_2$ belongs to $\phi(a)$. If we are serious about the idea that the data objects are computing machines of a sort, the assumption represents no real limitation. If we had imagined that out data objects are one and all Turing machines—in other words, if we had adopted a version of Church's Thesis (*v.i.*)—then a suitable "data object unsolvable property" would be an expression, in the language of elementary arithmetic, of the halting problem for Turing machines.

As Heyting often stressed (cf. [17]), if we care to speak of a court of last appeal for intuitionistic mathematics, then it is neither syntactical nor metaphysical but mathematical. If you care to say that there is a foundation "beneath" intuitionistic mathematics, then it is just more mathematics; in particular, it will be, in part, the mathematics of data objects. Moreover, this will be an articulate mathematics. Whatever we may choose to serve as members in the category of data objects, those objects will be the objects of a suitable mathematical theory $\mathcal{T}$. One way in which to think of data objects as "machines" which can perform "computations" is to insist that the input–output behavior of the machines is expressible in $\mathcal{T}$: if machine M accepts a and outputs b, then $\mathcal{T} \vdash M(a) = b$. For the sake of exposition, we can take it that $\mathcal{T}$ extends elementary intuitionistic or *Heyting* arithmetic and, hence, that it extends an intuitionistic version of Robinson's theory $\mathfrak{Q}$.

With the mathematics of data objects as a background, we can describe a simple instance of the incompleteness phenomenon. Let $\mathcal{T}$ be the theory just described and assume that $\mathcal{T}^{cl}$ is a "classical version" of it. You can think of the classical version as standing to $\mathcal{T}$ in the way that Peano arithmetic stands to Heyting arithmetic; in other words, we can produce $\mathcal{T}^{cl}$ from $\mathcal{T}$ by inverting the negative translation. It is also assumed that $\mathcal{T}^{cl}$ is provably consistent—the proof being carried out in a suitable intuitionistic metatheory. We show that, if we work intuitionistically, $\mathcal{T}^{cl}$ has no models, and, hence, that $\mathcal{T}^{cl} \models \perp$ while $\mathcal{T}^{cl} \nvdash \perp$. It follows immediately that intuitionistic predicate logic is not strongly complete.

Assume that $\mathfrak{A} \models \mathcal{T}^{cl}$, i.e., that $\mathfrak{A}$ is, intuitionistically, a model for $\mathcal{T}^{cl}$. Since $\mathcal{T}^{cl}$ is classical, it satisfies TND with respect to the propositions expressed in the language of $\mathcal{T}$. Therefore, for any such ϕ,

$$\text{either } \mathfrak{A} \models \phi \quad \text{or } \mathfrak{A} \not\models \phi$$

holds in our intuitionistic metatheory. Since intuitionistic truth commutes with disjunction and we are thinking of data objects as generalized machines, it follows that there is a machine M, which, if ϕ is interpreted over $\mathfrak{A}$, determines whether ϕ is true in $\mathfrak{A}$ or not. We might as well say there is a machine M such that

$$M(\phi) = 0 \text{ iff } \mathfrak{A} \models \phi.$$

(Were we to suppose that M is a Turing machine, then Church's Thesis would have been invoked.) Since we can, by assumption, express the behavior of data objects in the theory $\mathcal{T}^{cl}$, $M(\phi) = 0$ is (equivalent to) a predicate of $\mathcal{T}^{cl}$ such that

$$\mathfrak{A} \models [M(\phi) = 0] \text{ iff } \mathfrak{A} \models \phi.$$

Therefore, the predicate "$M(\phi) = 0$" serves as an internal truth predicate for $\mathfrak{A}$. $\mathcal{T}$ extends Robinson's $\mathfrak{Q}$, so this is a violation of Tarski's indefinability theorem.

The proof is almost done. We now know that $\mathcal{T}^{cl}$ has no models or $\mathcal{T}^{cl} \models \perp$. But, since we can, by assumption, prove in the metatheory that $\mathcal{T}^{cl}$ is consistent, we cannot derive $\perp$ from $\mathcal{T}^{cl}$ or $\mathcal{T}^{cl} \nvdash \perp$. Therefore, intuitionistic predicate logic in the language of arithmetic is not strongly complete.

In order for the incompleteness results, the proofs of which follow the outline we have just given, to achieve the desired end, the fair expression of the character of intuitionistic mathematics and the loosening of the bonds of "naturalness", it will be necessary to prevent pedestrian, kneejerk reactions. One of those reactions is this: "*These incompleteness theorems simply confirm the widely held view that intuitionistic logic is inadequate*". Another is expressed by saying "*The incompleteness phenomena for intuitionistic logic with respect to model-theoretic semantics only shows that ordinary or Tarskian or model-theoretic semantics is ill-suited for the metamathematical study of intuitionistic entailment*". And, since our theorems seem to rely upon forms of (intuitionistic) Church's Thesis, a third would be "*These incompleteness theorems simply show what weird results (ones contradicting what we already know by classical means) will ensue if we adopt classically repugnant forms of Church's Thesis*".

The first reaction is easily dealt with. Incompleteness for intuitionistic logic does not reveal defects in intuitionistic reasoning, at least if one has not prejudged a variety of issues and proclaimed the parochial properties of classical logic to be requirements for any logic whatsoever. Just as, in the classical case, it would be hasty and shortsighted to declare that completeness confirms the "rightness" of the logic, so also would it be foolish to claim that incompleteness demonstrates the "wrongness" of intuitionistic logic. To claim that incompleteness is wrong or unnatural is, at best, to adopt a classical version of a dubious "naturalness".

Also, one should be reminded that intuitionistic logic does not bear to intuitionistic mathematics the same sort of foundational role as classical logic is sometimes thought to bear to classical mathematics. According to some philosophers, classical mathematics ought to look to logic, the general study of inference, for a foundation. So, if mathematics is in need of a justification, it is natural to shift the justificatory burden onto logic. Hence, the justification of a logic becomes, on this classical picture at least, an issue of real foundational concern. In intuitionism, there is no general study of inference; there is nothing more fundamental than the mathematics—and that includes logic. Logic is not a collection of principles more general and "less contentful" than mathematics itself but a motley collection of the fruits of mathematical work. Despite the many and inspiring efforts of Professer Dummett and colleagues (as in [7]), the prospect of justifying the "rightness" of an intuitionistic logic will not supply us with an epistemic basis on which to support a further intuitionistic mathematics. We know of no suitable development of what is called 'intuitionistic logic' on which it is anything else but a *branch* of intuitionistic mathematics. Now, that a logic is viewed as "inadequate" by someone who looks to the presumed "adequacies" of classical mathematics for a standard is of reduced foundational concern for the intuitionist. A justification of intuitionistic mathematics cannot be thought to rest upon a separate justification of logic, one without which the mathematics is somehow insecure. The claim that our logic is, at its stands, incomplete may just turn out to be the claim that, in logic, we have more mathematical work to do.

2.3 Intuitionism, truth and model theory To deal with the second sort of reaction, that there is some kind of disaffection or infelicity between intuitionistic mathematics and the sorts of semantical investigations represented by the work of Tarski, we need to address the putative relations between intuitionism and concepts of truth. First, as Dummett has rightly emphasized in his "Preface" to *Truth and Other Enigmas* [7], there is no conceptual conflict between intuitionistic mathematics and what is called "truth theoretic semantics". At one time, it was common in philosophy of mathematics to pretend that the very notion of a "truth predicate" which commutes with the logical signs and, so, satisfies the "T-schemes" which Tarski set down in his *Wahrheitsbegriff* is intuitionistically inappropriate. The popular impression was that intuitionism has a "proof" or "evidential" semantics and that this kind of semantical explanation is at odds with a concept of "Tarski style" truth. In response, we point out that there seems to be no incoherence in the idea that a "proof semantics" has room for a recognizable and familiar conception of truth, one on which the truth predicate commutes with the logical signs. Even traditional intuitionists were willing to offer an explication or definition of intuitionistic truth: that a proposition is true when, as we would say, its species of data objects has a member. One can, given the ordinary (or Heyting) explanation of the expression "data object a is a member of the species associated with proposition ϕ", use this definition of truth and axioms governing data objects to verify Tarskian "T-schemes" such as

$$\mathrm{True}(\phi \vee \psi) \text{ iff } \mathrm{True}(\phi) \text{ or } \mathrm{True}(\psi).$$

To those who like to think in terms of a debate between classical and intuitionistic mathematicians, it should be pointed out that, if the traditional intuitionistic attitude toward classical inference is to be granted any credibility, then the intuitionist and the classical mathematician must share at least the outlines of a concept of valid argument, namely, one on which the *truth* of the premises guarantees the *truth* of the conclusion. Also, it should be remarked that, in his arguments against the law of excluded middle, the intuitionist is already assuming that truth commutes with connectives such as disjunction and conjunction. One of the surest ways, then, to give cogency to the arguments in what is called the 'intuitionistic critique' of classical inference would be to grant that the intuitionist has an appreciable notion of truth, one which has the feature of commutation with the logical signs and in terms of which validity is to be explained. This will provide some notion of a base on which classical and intuitionistic interlocutors can stand to begin a productive discussion of logical laws.

One who questions the use of Tarskian semantics in an intuitionistic context may not be questioning the introduction of a concept of truth but, rather, casting doubt on the suitability to constructive mathematics of the ordinary conception of model structure, a relational structure having an inhabited set as its domain. In reply, it is worth emphasizing that no very abstruse property of model structures is presupposed in what we do here. It seems that, in showing as we do that there are formulas which are syntactically consistent and yet have no model, we are not making any real restriction on the sort of object we have in mind as a model structure, other than the fact that we can define satisfaction with respect to it in the usual way. For instance, satisfaction is defined so that it commutes with the logical signs. If $\mathfrak{A}$ is a model structure, then $\mathfrak{A} \models \neg\phi$ if and only if $\mathfrak{A} \not\models \phi$ for sentences ϕ. Whether we choose the domain of a structure to be an extensional set, an intensionally individuated species, or a collection of natural numbers on which an equivalence relation is defined seems to be of little moment to the result. Similiar remarks apply to the extensional conceptions of relation and function which feature in our understanding of model structure.

Second, even if we grant that Tarski's concept of a model is not the last word in intuitionistic semantics, we must, it seems, allow that it may be the *first* word. Recall that there must be an appreciable intuitionistic concept of valid scheme and, as we have set it out above, that concept relies upon the notion of a class of interpretations $*$ of certain elements of the scheme. Whatever kind of mathematical device we discover to fulfill the need for interpretations, it seems that it cannot be, in effect, much different from a Tarskian conception of model structure. We will require a domain over which quantifiers are to be interpreted and we will need functions and relations defined over the domain to act as values for scheme elements of type relation and type function. Hence, the model structure notion acts as a "least common denominator" concept for use in accounts and studies of formal validity. After all, the model structure concept is so convenient, attractive and streamlined that, were we ever to agree upon an ultimate explication of intuitionistic validity, we would either employ the concept of model directly or be forced to explain the connection between the explication and that notion.

Doubtless, the ultimate intuitionistic notion of interpretation must be far more articulate than Tarski's; the ultimate notion should make clear the bearing upon intuitionistic logic of any number of items missing from the Tarski account: constructive operations, constructive functions, and, perhaps, choice sequences and species. Nonetheless, there remain reasons for thinking that any acceptable interpretation of intuitionistic formalism could not be fundamentally at odds with Tarski's idea of interpretation with respect to a structure. One of the reasons is that the concept of validity as truth with respect to all assignments over all structures is so simple and persuasive that one would find it difficult to countenance any validity concept which is not consonant with it.

Please note also that the present paper includes results on propositional logic, results for which the notion of model drops out entirely. There, we ask only that truth commute with the connectives and that a formula of propositional logic be valid when it is satisfied by assignments of individual propositions to its variables. As always, a proposition will be a subset of $\{0\}$. There seems little one could object to here but these propositional results are, in a way, the most surprising of all.

2.4 Varieties of Church's Thesis Lastly, we come to the individual who objects that the negative completeness results confirm the already widespread impression that forms of Church's Thesis, on which some of the results rely, are "bad" or "weird". In general, what is called Church's Thesis in intuitionistic contexts can be represented as the assertion that the analysis of computation offered originally by Turing applies as well to the computations indicated by data objects. Put in mathematical terms, Church's Thesis (CT) is the claim that, with 'n,' 'm', and 'e' ranging over the natural numbers,

$$\forall n \forall m \phi(n,m) \rightarrow \exists e \forall n [(\{e\}(n)\downarrow \wedge\, \phi(n,\{e\}(n)))].$$

Here, e is an index for a Turing machine and the notation '$\{e\}(n)\downarrow$' means that the Turing machine with index e eventually halts when started on input n. Hence, CT implies that every total natural number function is computed by a data object which is a Turing machine.

Of the forms of CT which feature in the article, the strongest (which enters into the proof of incompleteness for propositional logic using Rose's formula) is Extended Church's Thesis (ECT). ECT is a relativized form of Church's Thesis:

$$\forall x[\phi(x) \rightarrow \exists y \psi(x,y)] \rightarrow \exists e \forall x[\phi(x) \rightarrow (\{e\}(x)\downarrow \wedge\, \psi(x,\{e\}(x)))].$$

Individual variables x, y, and z range over the natural numbers and ϕ ranges over the "ω-stable" predicates. A predicate $\Theta(x)$ is ω-stable when, for all natural numbers n,

$$\neg\neg\Theta(n) \rightarrow \Theta(n)$$

is constructively true. (McCarty [34] and Hyland [20] contain much more information on ω-stable sets and predicates.) Shortly put, ECT is a principle of relativized "effective choice" or "effective uniformization" for properties of the natural numbers, provided that the relativization is ω-stable.

A weaker form of CT and the form which features most prominently in our incompleteness arguments is WCT, Weak Church's Thesis:

[WCT] $\forall n(P(n) \vee \neg P(n)) \rightarrow \neg\neg\exists e.\forall n[P(n) \leftrightarrow \exists m.T(e,n,m)].$

$T(e, n, m)$ is the unary Kleene "T predicate". Note that what we call WCT is a variation on the principle which appears in the literature under the title 'Weak Church's Thesis'; details on both versions are obtainable from Beeson [1].

As far as a good deal of ordinary intuitionistic mathematics is concerned, there is little scope for objection to ECT; it is known to be consistent with powerful extensions of the intuitionistic set theory IZF, extensions which include Brouwer's Theorem on continuous functions and variations on the axiom of choice. Hence, there is a limited mathematical resistance to the identification of data objects with Turing machines. Moreover, whatever reasons one has for adopting Church's Thesis in the context of recursion theory ought to apply *ceteris paribus* to computations in intuitionistic mathematics generally. (One should recall that recursion theory is not a province solely of classical mathematics but can be developed intuitionistically as well.) As we have argued in McCarty [38], there is little room for telling objections of a nonmathematical sort; those which have appeared in the literature (cf. [1]) have been based upon an evaluation of intuitionistic mathematics from standards which are decidedly classical. For instance, that ECT allows one to derive results in real analysis which offend one's classical sensibilities is no objection whatsoever. On the contrary, given that Brouwer's Theorem, a hallmark of traditional intuitionism, is flagrantly "anticlassical", that CT extends our ability to derive "anticlassical" theorems is something of a compliment. Also, it should be emphasized that attributions of "weirdness" to intuitionistic results are often based upon the prejudices allied with judgments of intuitive naturalness for classical theorems. These are the sorts of prejudices we especially want to call into question.

WCT is, constructively and intuitionistically, a more plausible principle than Church's Thesis itself, as the intuitive semantical picture suggests. Were CT to be intuitionistically true, there would need to be available a uniform computable transformation which converts arbitrary proofs of, say, arithmetic $\forall\exists$ statements, into indices for effective uniformizations of the matrices of those statements. WCT does not require nearly so much. According to Beeson ([1], p. 57):

> WCT is a principle which is widely regarded as very plausible, although we have no proof of it. Since it contradicts classical mathematics, we will not find a proof for it in Bishop-style mathematics. Various metamathematical results suggest that we shall never prove it from something simpler.

Highly germane to this discussion is the metamathematical fact that historically important principles of Brouwerian intuitionism, such as the Fan Theorem, are consistent not with CT but with WCT (cf. Moschavakis [40]).

A further principle, one which is not intrinsically connected with computability, is Markov's Principle (MP). It also features in one of our proofs of incompleteness for propositional logic. We can represent MP as the claim that

$$\exists y.\, T(e, x, y) \text{ is } \omega\text{-stable.}$$

MP is not derivable in most standard intuitionistic formal systems. Beyond that, there has been little inclination to look askance at the influence upon intuitionistic mathematics of MP since it has been taken (on the basis of the syntax of its expression) to be "classically correct". One should note (since it enters into the proofs to come) that, under the influence of MP, the predicate

$$\forall x[\phi(x) \to (\{e\}(x)\downarrow \wedge \psi(x,\{e\}(x)))]$$

from the right side of ECT can be written as an ω-stable predicate provided that $\psi(x,y)$ can be.

*3 **Historical preliminaries*** There are, of course, intuitionistic completeness theorems for validity with respect to various more-or-less contrived concepts of interpretation. First, intuitionistic propositional logic is complete and constructively so with respect to interpretations on the frames in the Jaskowski sequence (Dummett [6]). Evert Beth [3] offered a proof of completeness for constructive predicate logic with respect to what is now called "Beth semantics". As proved by Dyson and Kreisel [8], Beth's proof was only constructively correct modulo a covert appeal to a form of Markov's Principle. Much more recently, Veldman and De Swart (De Swart [48], Veldman [54], and Lopez-Escobar and Veldman [33]) have discovered intuitionistic completeness proofs for predicate logic with respect to interpretations over generalized Beth and Kripke trees.

In contrast, it is known that unaided intuitionistic set theory is wholly incapable of proving the completeness of predicate and propositional logic for the structural semantics of Tarski. First, it should be noted that "Henkin-style" completeness proofs for classical predicate logic are themselves essentially nonconstructive. For one thing, it is impossible to prove in constructive metamathematics that all "nontheorems" of intuitionistic propositional logic IPL have countermodels, where a countermodel for sentence ϕ is a structure $\mathfrak{A}$ such that $\mathfrak{A} \nvDash \phi$. Famously,

$$\nvdash (\phi \vee \neg\phi)$$

and this will not imply that

$$\exists\mathfrak{A}.\ \mathfrak{A} \vDash \neg(\phi \vee \neg\phi).$$

The reason is that $\neg\neg(\phi \vee \neg\phi)$ is a theorem of IPL and, by soundness,

$$\forall\mathfrak{A}.\ \mathfrak{A} \nvDash \neg(\phi \vee \neg\phi).$$

Second, the familiar "maximal set" construction that produces a countable model from a countable consistent theory cannot be carried out in any of the ordinary constructive formalisms, including set theory plus choice principles. The countable models theorem, even if restricted to theories of subfinite structures in the language of pure identity, implies Kripke's Scheme, which is independent of those formalisms. (As we said, the last section of the present paper contains a proof of this fact.)

Most telling of extant results is the fact originally discovered by Gödel and described by Kreisel in [26]: that general validity for formulas of intuitionistic predicate logic implies the truth of the primitive recursive form of Markov's Principle. The latter is independent of set theory (and was not in general use by traditional intuitionists), so the completeness theorem for the attendant validity concept is unprovable in set theory.

Kreisel, in [27], showed that, under the assumption of Church's Thesis, intuitionistic validity for predicate formulas is not recursively enumerable. It follows that there is no axiomatization for the internal logic of the Kleene realizability structure. Kreisel's theorem received improvement at the hands of van Dalen [5] and of Leivant [29], who showed that a weaker form of CT can be used to get Kreisel's result. In [37], there is a proof that, under Weak Church's Thesis and Markov's Principle, first-order arithmetic is categorical and, hence, that validity is nonarithmetic. Among other things, this means that, even if we were to add as axioms sets of formulas defined by arithmetic predicates of arbitrary complexity, we still could not capture the logic internal to Kleene's realizability.

Kreisel had earlier demonstrated, in [26], that, when interpretations are parameterized with choice sequences, there are formulas of predicate logic which are valid but are not intuitionistic theorems. In the note [11], Friedman claimed to have used a form of Church's Thesis for a similar constructive incompleteness theorem with respect to Tarski-style interpretations without choice parameters. So far as we know, a complete proof for Friedman's result has yet to appear (cf. [31] and [32]). The sketchy remarks of [11] suggest that the proof method intended there differs from that described in the present account. It remains to be seen whether these results require CT or merely WCT.

There have also been a number of limited intuitionistic completeness theorems for Tarski-like semantics. In Kreisel [25], there is a proof that, if one makes nonclassical assumptions concerning lawless sequences, various fragments of intuitionistic predicate logic are provably complete. In [11], again under axioms for lawless sequences, a proof of completeness for the negation-free fragment of intuitionistic logic is sketched. A similar result can be found in [6] as a corollary of the completeness theorems of [48] and [54].

4 Classical arithmetic is semantically inconsistent

4.1 Detailed prospectus of results What follows in this and the next main sections is a simple proof—along the outlines of the informal arguments recently sketched—that WCT, a weak form of Church's Thesis, entails that the standard intuitionistic (and classical) predicate logics are both incomplete with respect to Tarskian semantics. The classical reader who is nonplussed by such an outcome is reminded that WCT is a so-called "anticlassical" axiom of intuitionistic mathematics. When added to intuitionistic arithmetic or set theory, WCT will have consequences which, if interpreted naively, appear to contradict well-known theorems of classical mathematics. In the present section, we see that it has as a consequence a seeming contradiction to Gödel's completeness theorem. The mainstay of the proof is the construction, from the Gödel-Tarski fixed-point theorem, of a modest formula which is valid but not provable in either Heyting's predicate logic IPL or, for that matter, in the classical predicate logic CPL. Intuitively, the formula is a register of the recursion-theoretic reasons for the intuitionistic failure of the law of excluded middle. Section 6 of the present paper then further refines the basic theorem by showing that there is room for incompleteness even within the closer confines of pure predicate logic.

It is not our contention that results such as these are entirely new. Similar theorems have been proved by Kreisel, as reported in [26], and have been claimed by Friedman in [10] and [11]. Friedman's results are described, without proofs, in [32]. It is our contention, however, that there is value in the relative simplicity and directness of our proofs of these theorems.

The incompleteness theorem for propositional logic (appearing in Sections 7 and 8) is an application of a theorem of Rose [44] to the question of the completeness of propositional logic under Tarskian interpretations. Here, extended Church's Thesis (ECT) and Markov's Principle (MP) come into play. The consequence is this: there is a valid propositional formula Ψ in two variables such that, within second-order arithmetic, ECT and MP imply that Ψ is not provable in Heyting's propositional logic. It will follow that intuitionistic Zermelo-Fraenkel set theory, extended with realizably true principles such as ECT and MP, is inconsistent with the claim that the intuitionistic propositional calculus is complete in the conventional sense of the term. Also, we show that there is a single arithmetic instance of Ψ which is independent of all classical Harrop extensions of Heyting arithmetic and there is a set-theoretic instance of Ψ which is independent of intuitionistic set theory.

We have chosen to include material on the completeness of propositional logic because, insofar as incompleteness is concerned, predicate logic had received a surfeit of attention. It is certainly worth redressing the imbalance and emphasizing the fact that the intuitionistic failure of completeness has as much to do with propositions as with predicates or quantifiers. Rose's theorem, from which we extracted the independent propositional formula Ψ, is well-known in a certain classical context: it falsifies Kleene's conjecture that intuitionistic propositional logic is complete with respect to arithmetical substitutions under standard realizability. It has not, at least as far as we know, been put to work to give an independence result for Tarskian completeness within an intuitionistic metatheory.

Rose's formula contains two propositional variables. The natural question arises: *Is it possible to prove that intuitionistic logic is complete with respect to Tarskian semantics for propositional formulas containing at most* one *propositional variable?* The correct answer turns out to be negative: as we see in Section 9, the intuitionistic set theory IZF, together with Brouwer's Theorem, cannot prove that propositional logic in one variable is complete. In fact, it is consistent with IZF to assume that validity for propositional formulas in one variable is not arithmetically definable.

Our tour of incompleteness concludes with a remark on a surprising connection between a countable models theorem for theories of pure identity and Kripke's Scheme, a principle proposed for the axiomatization of Brouwer's ideas on the creative subject.

4.2 Strong incompleteness Classical first-order Peano arithmetic (PA) is a finite extension neither of Heyting arithmetic (HA)—PA's constructive fragment—nor a fortiori of intuitionistic logic. Consequently, an intuitionistic proof that Peano arithmetic has no models directly entails only an infinitary or *strong* incompleteness theorem for constructive logic. Such a strong incompleteness result is the subject of the present section. It stands as a useful prolegomenon to the next section, where we use the same proof idea to give a proof of single-formula or weak incompleteness, which is, of course, a stronger result.

For the moment, IPL will stand for constructive first-order predicate logic with identity; each of [6], [22], and [50] offers a menu of alternative formalizations of it. Throughout the sequel, the derivability relation, $\vdash$, will be that of IPL. CPL will refer to the classical system obtained by adjoining to IPL the general principle of the excluded third. The object language for the first incompleteness theorems will be that of Peano arithmetic, which is assumed to contain a sign $\perp$ for distinguished logical falsehood.

The metatheory, which we call T, need not be precisely given. It should contain arithmetic and sufficient resources to manipulate arbitrary sets as the universes of structures and to define satisfaction for an arbitrary structure along Tarskian lines. The arithmetic should be sufficient to prove something like the Gödel–Glivenko translation of PA into HA so that $T \vdash (\mathrm{Con(PA)} \leftrightarrow \mathrm{Con(HA)})$. We also take it as given that T proves Con(HA). IZF (intuitionistic Zermelo Fraenkel) would do as T, as would HAS (intuitionistic second-order arithmetic) or any number of alternative constructive systems. Letters such as $\mathfrak{A}$ range over structures of appropriate signature. $\mathcal{N}$ is the standard model of arithmetic. The function $\lambda x. \lceil x \rceil$ is a primitive recursive coding of formulas as integers. Hence, we say that $\lceil \psi \rceil$ is the Gödel number of formula ψ. The sole anticlassical assumption of the incompleteness proof is Weak Church's Thesis (WCT), as described above.

The results of this section give incompleteness results with respect to some of the completeness concepts canvassed in the literature.

Definitions

1. IPL is *strongly complete* if and only if, for all sets Γ of formulas and individual formulas ϕ,

$$\Gamma \vdash \phi \text{ whenever } \Gamma \models \phi.$$

2. IPL is *strongly almost complete* if and only if, for all sets Γ of formulas and formulas ϕ,

$$\neg\neg(\Gamma \vdash \phi) \text{ whenever } \Gamma \models \phi.$$

These are some of the more familiar of the completeness notions described in sources such as [6] and [51], to which the reader is referred for further details. Terminology is nowise standard; the notions are named to suit ourselves. The relation between strong completeness and strong almost completeness is very simple, as the next proposition shows.

Proposition 4.1 *If* IPL *is strongly complete, then it is strongly almost complete. The converse fails in general.*

In the following theorem, the metatheory T plus WCT implies that PA has no models and, hence, is semantically inconsistent. By assumption, T also proves that PA is syntactically consistent. Therefore, in T plus WCT, IPL is sound but not strongly almost complete while CPL is both unsound and not strongly almost complete.

Theorem 4.1 (*In T plus* WCT) PA $\models \perp$.

Proof: Let $\mathfrak{A} \models \text{PA}$. The law of excluded third is a theorem of PA; so, for any sentence ϕ,

$$\text{either } \mathfrak{A} \models \phi \text{ or } \mathfrak{A} \not\models \phi.$$

As the intuitionists would say, this means that the predicate $\{\ulcorner \phi \urcorner : \mathfrak{A} \models \phi\}$ is decidable over those integers which are Gödel numbers of sentences. Hence, CT implies that the predicate is recursive. Therefore, there is a total recursive function with index e such that, for all Gödel number $\ulcorner \phi \urcorner$,

$$\{e\}(\ulcorner \phi \urcorner) = 0 \text{ iff } \mathfrak{A} \models \phi \text{ and}$$

$$\{e\}(\ulcorner \phi \urcorner = 1 \text{ iff } \mathfrak{A} \models \neg\phi.$$

By the property of numeralwise representability for PA (or its intuitionistic version HA), we know that there is a formula $\Sigma(x)$ in one free variable such that, for any arithmetic sentence ϕ,

$$\mathfrak{A} \models \Sigma(\ulcorner \phi \urcorner) \text{ just in case } \mathfrak{A} \models \phi.$$

However, the Gödel–Tarski fixed-point theorem for PA shows that there can be no such formula. Therefore, PA has no models or $\text{PA} \models \bot$.

This conclusion, that PA has no models, is stable; in other words, as a proposition, "PA has no models" is closed under double negation or "PA has no models" is equivalent to "$\neg\neg$PA has no models". Now, as the argument we have just completed shows, the latter statement will follow just as smoothly from the assumption that the predicate $\{\ulcorner \phi \urcorner : \mathfrak{A} \models \phi\}$ is not not ($\neg\neg$) recursive. The latter is just the consequent of WCT. Therefore, the claim that PA has no models follows as properly from WCT as it does from CT.

Corollary 4.1 (*In T plus* WCT) IPL *is not strongly almost complete and, therefore, not strongly complete.*

Proof: By [13], T proves that

$$\text{Con(PA)} \leftrightarrow \text{Con(HA)}.$$

Therefore, with WCT, T proves

$$\text{PA} \not\models \bot \text{ and } \text{PA} \models \bot.$$

We have the same result, in our intuitionistic metatheory, for classical predicate logic CPL:

Corollary 4.2 (*In T plus* WCT) CPL *is consistent, but neither sound nor strongly almost complete. Hence, it is not strongly complete.*

Proof: Immediate.

5 A consistent formula which implies $\bot$ To apply the reasoning of the preceding section to a single formula, one need only locate a finitely axiomatizable theory in which arbitrary recursive functions are represented and for which the internal fixed-point property holds. The candidate theory, which we call S, has equality as its only basic predicate. Its set of primitive function signs contains a nullary sign $\underset{\sim}{0}$ for zero and a unary sign for successor. Consequently, it includes the usual numerals. We take $\underline{n}$ to be the numeral denoting the number n. The language for S also includes a binary function sign *Sub* for the metamathematical substitution function, a 4-ary sign T for Kleene's computation, and result-extraction functions and signs for whatever primitive recursive functions would appear in typical Gödel–Herbrand derivations for *Sub* and T.

Sub, which represents the Gödel substitution function has the property that, for all n,

$$Sub(\ulcorner\phi\urcorner, n) = \ulcorner\phi[x/\underline{n}]\urcorner,$$

provided that ϕ has at most the variable x free. Our T predicate represents a primitive recursive computation-and-output function, in other words, a logical conjunction of Kleene's computation and result-extraction functions. We assume that, for all n, m, p, and q,

$$\text{either } T(n,m,p,q) = 0 \text{ or } T(n,m,p,q) = 1,$$

and $T(n,m,p,q) = 0$ precisely when p codes the complete computation of the Turing machine n on input m and the output of that computation is q.

Only finitely many equational axioms are needed to guarantee for S the obvious representation properties. Specifically, we insist that for all natural numbers n, m, p and q,

$$Sub(n,m) = p \text{ iff } S \vdash Sub(\underline{n},\underline{m}) = \underline{p} \text{ and}$$

$$T(n,m,p,q) = 0 \text{ iff } S \vdash T(\underline{n},\underline{m},\underline{p},\underline{q}) = \underline{0}.$$

(Here, the occurrences of *Sub* and T on the lefthand sides of each of the preceding biconditionals refer to the nonformal, intuitive versions of the substitution and computation functions, respectively.) The axioms required to afford such a guarantee should also include an axiom forcing T to be functional in its third and fourth arguments, i.e., that

$$T(x,y,z,w) \wedge T(x,y,a,b) \rightarrow (z = a \wedge w = b).$$

The last axiom of S expresses the *universal testability* (another instance of traditional intuitionistic terminology) of the halting problem:

$$\forall x \forall y[\neg\exists z T(x,y,z,\underline{0}) = \underline{0} \vee \neg\neg\exists z T(x,y,z,\underline{0}) = \underline{0}].$$

Note Historically, a statement ϕ of intuitionistic mathematics was considered *testable* whenever

$$\neg\phi \vee \neg\neg\phi$$

is true. Similarly, a predicate $P(n)$ was taken to be *universally testable* if

$$\forall n[\neg P(n) \vee \neg\neg P(n)].$$

Brouwer and Heyting found the notion of testability to arise naturally in the process of giving weak counterexamples to theorems of classical analysis (cf. [17]).

Definition Any sentence of the form

$$\neg\exists z T(\underline{n}, Sub(\underline{m},\underline{m}), z, \underline{0}) = \underline{0}.$$

is said to be an *instance* of T or, simply, an *instance*.

Because S contains the axiom of testability for the halting problem, any model for S will contain a recursive truth predicate for all instances. Therefore, it follows from the fixed-point theorem that S, like PA, is semantically inconsistent. This is the content of the next theorem.

Theorem 5.1 (*In T plus* WCT) $S \models \bot$.

Proof: We follow a proof-idea similar to that used in the case of PA. Let $\mathfrak{A} \models S$. From the testability of the halting problem, we know that, for any instance ϕ,

$$\text{either } \mathfrak{A} \models \phi \text{ or } \mathfrak{A} \not\models \phi.$$

CT implies that the predicate $\{ \ulcorner \phi \urcorner : \mathfrak{A} \models \phi \}$ is recursive on the Gödel numbers of instances. Because of representability and functionality for T, there is an n such that, for all instances ϕ,

$$\mathfrak{A} \models \exists z.\, T(\underline{n}, \ulcorner \phi \urcorner, z, \underline{0}) = \underline{0} \text{ iff } \mathfrak{A} \models \phi.$$

We now construct a fixed point for the formula

$$\neg \exists z T(\underline{n}, Sub(x, x), z, \underline{0}) = \underline{0}.$$

Let m be Gödel number of the above formula and let σ be the instance which results from "self-referential" substitution:

$$\neg \exists z T(\underline{n}, Sub(\underline{m}, \underline{m}), z, \underline{0}) = \underline{0}.$$

It should be clear that

$$S \vdash \sigma \leftrightarrow \neg \exists z T(\underline{n}, \ulcorner \sigma \urcorner, z, \underline{0}) = \underline{0}.$$

and, hence, that

$$\mathfrak{A} \models \sigma \text{ iff } \mathfrak{A} \not\models \sigma.$$

Therefore, S has no models.

This conclusion that S has no models is stable, so it follows not only from CT but from WCT as well.

To each of the infinitary completeness notions, there corresponds a notion for finite sets of formulas:

Definitions

1. IPL is *complete* if and only if, for all formulas ϕ,

$$\vdash \phi \text{ whenever } \models \phi$$

2. IPL is *almost complete* if and only if, for all formulas ϕ,

$$\neg\neg(\vdash \phi) \text{ whenever } \models \phi.$$

Corollary 5.1 (*In T plus* WCT) IPL *is neither complete nor almost complete for single sentences.*

Proof: In PA, there is a proof of Con(S) and Con(S) is a negative sentence. Therefore, by the Gödel-Gentzen translation theorem [13], there is a proof of $S \not\models \bot$ in T. Hence, there is a proof of $\not\models \neg S$ in T. But, by the theorem above,

$T +$ WCT proves that $\neg S$ is valid. Hence, T plus WCT proves that IPL is neither complete nor almost complete.

By the same token, there is a similar result for classical logic:

Corollary 5.2 *(In T plus* WCT*)* CPL *is consistent but neither sound nor complete for single sentences. It also fails to be almost complete.*

Just for the record, we list results which follow immediately from the last corollary but one. ECT is Extended Church's Thesis, MPS is Markov's Principle for sets, RDC is Relativized Dependent Choice, BP is Brouwer's Principle for Numbers, UP is Troelstra's Uniformity Principle, PAC is the Blass–Aczel Presentation Axiom of Choice. ECT has already been introduced; fairly authoritative descriptions of the others can be had from [1].

Corollary 5.3 *The almost completeness of* IPL *for single sentences is inconsistent with each the theories*

- IZF + ECT + MPS + RDC + PAC + UP *and*
- IZF + WCT + MPS + RDC + BP.

Proof: The first claim is proved by noting that the axioms there mentioned hold under the "1945" Kleene realizability interpretation for set theory and that ECT implies WCT. For the second, one checks that the axioms listed hold under the "realizability" interpretation which draws realizability witnesses from the r.e. substructure of the Plotkin–Scott graph model. Full details are available in [1], from which it will be clear that other principles, e.g., for local and uniform continuity, could well be added to these lists.

6 Incompleteness in pure predicate logic The incompleteness theorem just obtained for first-order logic with functions and identity can be transfered to pure constructive predicate logic. The proof of this fact, which proceeds by axiomatizing a theory of primitive recursive functions as predicates, is standard, so our exposition will be brief. A reader who needs more convincing is advised to consult either [26] or [6].

An appropriate language will contain as primitives a unary predicate Z and binary predicates Su and E, for zero, successor, and equality, respectively. Then, for each primitive recursive function $f(x_0, \dots, x_{n-1})$ other than successor which featured in the old system S, there will be an $n + 1$-ary predicate symbol P_f. The theory in this language which corresponds to S will be called S_P and is axiomatized (finitely) by the sentences of the following five groups. All open formulas are to be given a universal reading.

I. Axioms for equality
   1. Exx
   2. $(Exy \wedge Ezy) \rightarrow Ezx$.

II. Axioms for zero and successor
   1. $\exists x Zx$
   2. $\forall x \exists y Sxy$

3. $(Zx \wedge Zy) \rightarrow Exy$
4. $(Zx \wedge Syz) \rightarrow \neg Exz$
5. $(Sxy \wedge Sxz) \rightarrow Eyz$
6. $(Syx \wedge Szx) \rightarrow Eyz.$

III. Axioms for substitution

For each primitive n-ary predicate P and $i < n$,

$$(P(x_0, \ldots, x_i, \ldots, x_{n-1}) \wedge Ex_i y) \rightarrow P(x_0, \ldots, y, \ldots, x_{n-1}).$$

IV. Axioms for primitive recursive functions

For each primitive recursive function sign f occurring in the axioms for S, there is a set consisting of either two or three axioms governing the correlative predicate P_f. These ensure P_f's representability. For example, if f is a binary primitive recursive function, then the axioms will entail that, for all n, m, and p, $f(n, m) = p$ if and only if

$$[Zx_0 \wedge Sx_0x_1 \wedge \ldots \wedge Sx_{r-1}x_r] \rightarrow \forall z [P_f(x_n, x_m, z) \leftrightarrow Ezx_p]$$

is derivable. Here, $x_0, x_1, \ldots x_{r-1}, x_r$ is the first sequence of distinct variables which is longer than the largest integer value which features in a minimal Gödel–Herbrand derivation for $f(n, m) = p$. Recall that open formulas in this section are to be given a universal reading.

We will not set out all the requisite axioms; a single example should suffice to get the idea across. If f, g, and h are primitive recursive functions featuring in S and if, for all n,

$$f(n) = h(g(n)),$$

then the relevant axioms will be

1. $(P_f xy \wedge P_f xz) \rightarrow Eyz$
2. $(P_g xy \wedge P_h yz) \rightarrow P_f xz.$

Lastly, there are

V. Axioms for the functionality and testability of the computation predicate
1. $(P_T xyzw \wedge P_T xyuv) \rightarrow (Ezu \wedge Ewv)$
2. $Zu \rightarrow (\neg \exists z P_T xyzu \vee \neg\neg \exists z P_T xyzu).$

Let S_P be the conjunction of the axioms from groups I through V. Using representability for the substitution function, one can easily see that S_P gives definable fixed-points:

Definition For each r, $N(r)$ abbreviates the formula

$$Zx_0 \wedge Sx_0x_1 \wedge \ldots \wedge Sx_{r-1}x_r$$

where $x_0, x_1, \ldots x_{r-1}, x_r$ are the first $r + 1$ distinct variables.

Proposition 6.1 *For any formula $\Sigma(x)$ in the single free variable x, there is a sentence σ and a number m such that the Gödel number q of σ is $Sub(m, m)$ and S_P proves that*

$$N(r) \rightarrow [\sigma \leftrightarrow \Sigma(x_q)].$$

Here, r is the least integer greater than all those appearing in a minimal derivation of $Sub(m, m) = q$.

Proof: The usual proof of the fixed-point theorem is easily converted into a purely predicate form.

Once again, our principal interest will be the fixed-point of an instance of the computation predicate:

Definition A sentence of the form

$$N(r) \rightarrow \forall y(P_{Sub}x_m x_m y \rightarrow \neg\exists z P_T x_n yzx_0),$$

where r is chosen sufficiently large, is called an *instance* of T.

Now we can simply follow—using predicates—the argument of the last section to prove

Theorem 6.1 (*In T plus* WCT) $S_P \models \perp$.

The incompleteness results recorded in the corollaries of Section 4 also apply to pure IPL and pure CPL.

Finally, we note that a minor change to the axioms of S_P suffices to put the incompleteness theorem into a very sharp form.

Theorem 6.2 (T *plus* WCT) *There is a sentence ϕ of the language of pure predicate logic which is negated prenex and contains only one appearance of $\vee$ such that ϕ is valid but unprovable in both* IPL *and* CPL.

Proof: To the axioms for S_P one adds the decidability of zero,

$$Zx \vee \neg Zx$$

and replaces the testability of the halting problem with its decidability,

$$Zu \rightarrow (\exists z P_T xyzu \vee \neg\exists z P_T xyzu).$$

Let S_{P1} stand for the conjunction of the resulting axioms. Since S_{P1} implies S_P and the latter entails $\perp$, so does the former. But, neither $\neg S_{P1}$ nor $\neg S_P$ is classically or intuitionistically provable. As is familiar, the $\vee$-subformula of the displayed decidability condition on P_T can be replaced by an equivalent formula written solely in terms of $\exists$, $\rightarrow$, and $\wedge$. After such replacements are made, S_{P1} is readily put into prenex form.

*7 **Incompleteness for propositional logic*** From Markov's Principle (MP) and ECT, T will prove that there is a simple propositional formula Ψ which is valid but derivable neither in constructive propositional logic nor in any of its classically correct Harrop extensions. Ψ is the following formula in two variables from the $\{\rightarrow, \neg, \vee\}$ fragment:

$$[(\neg\neg\phi \rightarrow \phi) \rightarrow (\neg\neg\phi \vee \neg\phi)] \rightarrow (\neg\neg\phi \vee \neg\phi)$$

where ϕ is

$$\neg p \vee \neg q$$

for atoms p and q. Ψ was first employed by G. F. Rose in [44] as a counterexample to Kleene's conjecture that a propositional form is a theorem of constructive logic if all its arithmetic substitutions are realizable. Rose's original argument was classical; the plan of our proof is to constructivize Rose's argument by applying ECT and MP (restricted to primitive recursive predicates). For the purpose at hand, we need to assume that the metatheory T extends HA and includes quantification over arbitrary subsets of $\mathcal{N}$.

As we said earlier, ECT is a relativized form of Church's Thesis:

$$\forall x[\phi(x) \to \exists y \psi(x,y)] \to \exists e \forall x[\phi(x) \to (\{e\}(x){\downarrow} \wedge \psi(x,\{e\}(x)))].$$

Individual variables x, y and z range over the natural numbers and ϕ ranges over the "ω-stable" predicates. The notation '$\{e\}(x){\downarrow}$' means that the Turing machine with index e eventually halts when started on input x. Again, a predicate $\Theta(x)$ is ω-stable when, for all natural numbers n,

$$\neg\neg\Theta(n) \to \Theta(n)$$

is constructively true.

The reader will also recall that MP is Markov's Principle restricted to primitive recursive predicates and is expressible as the single statement that $\exists y. T(e,x,y)$ is ω-stable.

IPropL is the propositional fragment of Heyting's predicate logic. Under standard Tarskian semantics, a propositional formula

$$\Theta(p,q,r)$$

in the atoms p, q, and r will be valid provided that, for every assignment v of subsets of $\{0\}$ to p, q, and r, the usual extension v^* of v assigns $\{0\}$ to Θ.

Theorem 7.1 $(T + \text{ECT} + \text{MP})$ Ψ *is valid but provable neither in* IPropL *nor in* IPL.

Proof: In proving that Ψ is valid, we drop explicit reference to the assignment function v.

Under interpretation, the antecedent of Ψ expresses the claim that

$$0 \in [(\neg\neg\phi \to \phi) \to (\neg\neg\phi \vee \neg\phi)].$$

Obviously, $\neg\neg\phi$ (under any specific interpretation) defines an ω-stable property of natural numbers. Therefore, ECT (in T) proves the embedded antecedent $(\neg\neg\phi \to \phi)$ of the above-displayed statement equivalent to

$$\begin{aligned}&\exists e \forall n[n \in \neg\neg\phi \to \{e\}(n){\downarrow}\\ &\quad \wedge (\{e\}(n) < 2) \wedge (\{e\}(n) = 0 \to 0 \in \neg p) \wedge (\{e\}(n) = 1 \to 0 \in \neg q)].\end{aligned}$$

This formula, minus its leading existential quantifier, we call $\chi(e)$ and remark that, by our earlier note on the influence of MP on ECT, it is ω-stable. Consequently, ECT is again applicable and, together with MP, it proves the original antecedent of Ψ equivalent to the formula

$$\begin{aligned}\exists g \forall e[\chi(e) \to (\{g\}(e){\downarrow} \wedge (\{g\}(e) < 2) \wedge (\{g\}(e) = 0 \to 0 \in \neg\neg\phi)\\ \wedge (\{g\}(e) = 1 \to 0 \in \neg\phi))].\end{aligned}$$

The result of deleting $\exists g$ from this formula will be called $\Delta(g)$. With MP added to T, it is also ω-stable.

Now, we assume that the antecedent of Ψ is true under v and argue by cases. First, if ϕ is false under v, then

$$v(\phi) = \varnothing = v(\neg\neg\phi)$$

and both $\Lambda n.0$ and $\Lambda n.1$ satisfy $\chi(e)$.

Note $\Lambda n.\rho(n)$ is Kleene's notation for an index of a machine computing the recursive function $\lambda n.\rho(n)$. $\Lambda n.0$ and $\Lambda n.1$ are computable in a standard fashion from 0 and 1, respectively.

Therefore, with g as given in the last display but one, both

$$g(\Lambda n.0)\downarrow \wedge\, g(\Lambda n.0) = 1$$

and

$$g(\Lambda n.1)\downarrow \wedge\, g(\Lambda n.1) = 1.$$

On the other hand, when ϕ is true with respect to v, then either $\neg p$ or $\neg q$ is true. If the former is true, then

$$g(\Lambda n.0)\downarrow \wedge\, g(\Lambda n.0) = 0.$$

If the latter is true, then

$$g(\Lambda n.1)\downarrow \wedge\, g(\Lambda n.1) = 0.$$

Independently of the argument by cases, one can define a recursive search procedure $\sigma(g)$ on g such that σ dovetails through the computation sequences for g on $\Lambda n.0$ and $\Lambda n.1$, determines which of the three possible cases obtains and computes according to the following receipe:

$$\sigma(g) = \begin{cases} 1 \text{ if } g(\Lambda n.0) = 1 = g(\Lambda n.1) \\ 0 \text{ o.w.} \end{cases}$$

The classical argument by cases sketched above gives assurance that $\sigma(g)$ terminates; a parallel constructive argument shows that it is not absurd that $\sigma(g)$ terminate. One application of MP then proves termination outright.

Another application of ECT shows that the original formula Ψ, under assignment v, is equivalent to

$$\begin{aligned}\exists h \forall g[\Delta(g) \to \{h\}(g)\downarrow \wedge (\{h\}(g) < 2) \wedge (\{h\}(g) = 0 \to 0 \in \neg\neg\phi) \\ \wedge (\{h\}(g) = 1 \to 0 \in \neg\phi)].\end{aligned}$$

If we set

$$\{h\}(g) = \langle\sigma(g), 0\rangle,$$

then h will satisfy the matrix of the preceding display and witness the fact that Ψ is true under v.

Therefore, since the assignment v was chosen arbitrarily, Ψ is valid. But any of the standard decision procedures for IPropL is constructively available and these show that Ψ is not derivable.

Note IPL is provably conservative over IPropL, so there are predicate instances of Ψ which are independent of IPL.

The following corollaries to the theorem require the notion of a *Harrop formula* of propositional or of predicate logic.

Definition A formula of a propositional language is *Harrop* whenever it has no strictly positive part with $\vee$ as its principal operator. A formula of a first-order language is *Harrop* whenever it has no strictly positive part with either $\exists$ or $\vee$ as principal operator.

An alternative definition and examples appear in Section 1.10.5 of [50].

Corollary 7.1 (T + ECT + MP) *If E is any classically correct extension of* IPropL *by Harrop formulas or any classically correct extension of* IPL *by Harrop formulas, then there is an appropriate instance of Ψ which is valid but independent of E.*

Proof: It suffices to observe that the obvious five-stage Kripke countermodel for Ψ can be constructed from models of classical logic by two applications of Smorynski gluing. (Cf. [46].)

8 Incompleteness and independence in arithmetic and set theory A proof of De Jongh's theorem (as in [46]) entails that there is an instance of Ψ which is independent of any extension of HA by Harrop formulas. There are also instances of Ψ which are unprovable when the schemes of transfinite induction, uniform reflection, and (arithmetic) Markov's Principle are added to HA. Moreover, Ψ has a set-theoretic substitution instance Ψ_γ which is independent of extensions of standard constructive set theory.

Theorem 8.1 (T + ECT + MP) *There is an arithmetic Σ_1 substitution instance of Ψ which is valid but which is not provable in any extension of* HA *by an r.e. set of Harrop formulas consistent with classical Peano arithmetic.*

Proof: By results of Kreisel and Friedman (cf. [50]), we may assume that there are theories U and U^{cl} extending HA and such that

1. U is a subtheory of our metatheory T,
2. U^{cl} proves Henkin's strong completeness theorem and is sufficient for the treatment of Kripke models for HA as in [46] and
3. U^{cl} is conservative over U with respect to Π^0_2 sentences.

Working in U^{cl}, we assume that E is r.e., consistent with PA and axiomatized by Harrop formulas. Then, by the incompleteness results of [28], [41], and [42], we know that there are Σ^0_1 sentences σ and τ which are independent of PA and such that each of

- $E_0 = E + \sigma + \tau$
- $E_1 = E + \neg\sigma + \tau$ and
- $E_2 = E + \sigma + \neg\tau$

is consistent with PA. By the Henkin completeness theorem, there is, for each E_i, a nonstandard model $\mathfrak{A}_i$ such that $\mathfrak{A}_i \models E_i$. The $\mathfrak{A}_i$ can be glued to form a five-stage Kripke model $\mathcal{K}$ which is demonstrably a model for HA + E. First, glue $\mathfrak{A}_1$ and $\mathfrak{A}_2$ and then glue the result to $\mathfrak{A}_0$.

Now, let $\Psi_{\sigma\tau}$ be the sought Σ_1 instance of Ψ formed by replacing p by σ and q by τ. It is a simple observation that

$$\mathcal{K} \nVdash \Psi_{\sigma\tau}.$$

As a consequence, we have HA + $E \nvdash \Psi_{\sigma\tau}$.

This result shows that U^{cl} proves the purely arithmetic statement

$$[\mathrm{Con}_{PA}(E) \wedge \mathrm{Harrop}(E)] \to \mathrm{HA} + E \nvdash \Psi_{\sigma\tau}.$$

The latter is, in PA, provably equivalent to a Π^0_2 statement. Therefore, under the assumptions listed at the start of the proof, T + ECT + MP proves the theorem.

If $\langle\langle$ is a primitive recursive well-ordering of the natural numbers, then the corresponding scheme of transfinite induction, TI($\langle\langle$), is the set of formulas

$$\forall x[\forall y \langle\langle x.\phi(y) \to \phi(x)] \to \forall x\phi(x).$$

For an r.e. set of formulas E, the uniform reflection principle, RFN(E), is the scheme

$$\forall y[\exists x Pr_E(x, \ulcorner\phi(\underline{y})\urcorner) \to \phi(y)].$$

Pr_E is a primitive recursive proof predicate for HA + E; $\underline{y}$ is the Gödel number of the formal numeral denoting y. Again, we are assuming that E is classically consistent and axiomatized by Harrop formulas.

Theorem 8.2 (T + ECT + MP) *If* Φ = TI($\langle\langle$), RFN(E) *or* MP, *there is an arithmetic instance of* Ψ_γ *of* Ψ *which is valid but such that*

$$\mathrm{HA} + \Phi \nvdash \Psi_\gamma.$$

Proof: In the cases of TI($\langle\langle$) and RFN(E), we proceed much as in the last theorem, using the result of [46] that TI($\langle\langle$) and RFN(E) are preserved under arithmetic gluing. In the case of MP, we can use a classical proof of de Jongh's maximality theorem for HA + MP (as on page 384 of [46]) and then transfer the Π^0_2 result from U^{cl} to T.

Lastly, we prove that there is a simple, set-theoretically definable instance Ψ_γ of Ψ which is, in T + ECT + MP, provably independent of constructive set theory provided that set theory is consistent. Thanks to [12], we can assume that the axioms of the constructive set theory in question (we call it E) are essentially those of standard Zermelo-Fraenkel including extensionality; the background logic is, of course, IPL. Furthermore, we can take E to be, in T, provably equiconsistent with ZF. For a precise listing of the axioms of a suitable E, the interested reader may consult [12] or [1]. T + ECT + MP will prove that Ψ_γ is intuitionistically valid but that

$$\mathrm{Con}(E) \text{ implies } E \nvdash \Psi_\gamma.$$

Our proof exploits techniques familiar from the study of Heyting-valued models $\mathcal{V}(\tau)$, a concise introduction to which is [15]. One can, without impropriety, think of $\mathcal{V}(\tau)$ as a constructive analogue to a Scott–Solovay Boolean-valued universe $\mathcal{V}(\mathcal{B})$ as described, inter alia, in [2]. If τ is a topology, $\mathcal{V}(\tau)$ will be the cumulative universe of τ-valued sets; $\mathcal{V}(\tau) \models$ represents truth over $\mathcal{V}(\tau)$. If ϕ is a sentence of the language of set theory, $[\![\phi]\!]$ will be its Heyting value, an open set of τ, when interpreted over $\mathcal{V}(\tau)$.

We begin with an independence result proved in classical ZF:

Theorem 8.3 (ZF) *There is a topology τ and a definable instance Ψ_γ of Ψ such that*

$$\mathcal{V}(\tau) \not\models \Psi_\gamma.$$

Proof: Let $\mathcal{J}$ be the partial order which is constructed by first gluing together a copy of the natural numbers $\mathcal{N}$ – under their usual order – with a copy of the set of $2^{<\omega}$ of all finite binary sequences under the prefix order. Then, we glue the resulting ordered set together with a degenerate order, containing one element which we call a. In the set $\mathcal{J}$, we let b name the least element of the copy of $\mathcal{N}$ and c the least member of the copy of $2^{<\omega}$. Then τ will be the usual forcing topology on $\mathcal{J}$: the collection of sets closed upward under the glue-induced ordering. For any point x of $\mathcal{J}$, let $x\uparrow$ be the open set determined by x. With s and t variables restricted to range over $\mathcal{P}(\{0\})$, we can take the full *law of the excluded third* or TND to be the set theoretic claim

$$\forall s(0 \in s \vee 0 \notin s).$$

In the same vein, we take LIN to be the assertion that the subset order on $\mathcal{P}(\{0\})$ is linear:

$$\forall s, t(s \subseteq t \vee t \subseteq s).$$

Now, let Ψ_γ be the instance of Ψ obtained by substituting LIN for p and ($\neg$LIN $\vee$ TND) for q in Ψ. Using classical logic, one sees easily that the value of LIN in $\mathcal{V}(\tau)$ is

$$[\![\mathrm{LIN}]\!] = a\uparrow \cup b\uparrow$$

and that

$$[\![q]\!] = a\uparrow \cup c\uparrow.$$

It follows immediately that

$$\mathcal{V}(\tau) \not\models \Psi_\gamma.$$

Note Clearly, this theorem is a special case of the obvious extension of de Jongh's theorem (cf. [46]) to set theory. The extended theorem is easily proved from Jaskowski's theorem.

As a reward for formalizing this lemma we get a proof, in T, that, if E is consistent, then E does not derive Ψ_γ:

Theorem 8.4 *There is a set-theoretic substitution instance Ψ_γ of Ψ such that our intuitionistic metatheory T proves*

$$\mathrm{Con}(E) \textit{ implies that } E \nvdash \Psi_\gamma,$$

while T + ECT + MP proves that Ψ_γ is valid.

Proof: The preceding result tells us that there is a formal classical proof from the ZF axioms of

$$\mathcal{V}(\tau) \nVdash \Psi_\gamma.$$

Since T extends arithmetic, T will certify the fact that

$$\mathrm{ZF} \vdash (\mathcal{V}(\tau) \nVdash \Psi_\gamma).$$

We have assumed that T is also capable of proving formalized soundness theorems for interpretations over topological models such as $\mathcal{V}(\tau)$. So, for any set theoretic sentence ϕ,

$$T \vdash (E \vdash \phi \rightarrow \mathrm{ZF} \vdash \mathcal{V}(\tau) \vDash \phi).$$

It follows immediately that

$$T \vdash [\mathrm{Con(ZF)} \rightarrow E \nVdash \Psi_\gamma].$$

ZF and E are provably equiconsistent in T, so the proof of the theorem is complete.

9 Formulas in a single variable We return here to the subject of constructive propositional logic for formulas in a single propositional variable. We observe that there is an elementary semantical argument for the conclusion that completeness (again with respect to Tarski semantics) is not provable in T, or, indeed, in an intuitionistic set theory such as IZF. In fact, we show more: that it is consistent with intuitionistic set theory to assume that validity for formulas in one propositional variable is not arithmetically definable.

Theorem 9.1 *It is consistent with T to assume that constructive propositional logic restricted to formulas in one propositional variable is incomplete. It is consistent to assume that validity for this logic is not definable in arithmetic.*

Proof: Again, we employ the methods of Heyting-valued models but, for the moment, our metametalogic is classical. This time, let τ be the usual order topology on the two-element Sierpinski space $\{a, b\}$, where $a \leq b$. Let $\mathrm{Val}(\phi)$ be the set-theoretic statement that formula ϕ is valid—that is, it gets value $\{0\}$ under all assignments of subsets of $\{0\}$ to its constituent variables. For each $x \in \{a, b\}$, $x\uparrow$ is the set of all y such that $x \leq y$.

A trivial calculation shows that

$$[\![\mathrm{Val}(p \vee \neg p)]\!] = b\uparrow,$$

while

$$[\![\neg\mathrm{Val}(p \vee \neg p)]\!] = \varnothing.$$

Therefore,

$$[\![\mathrm{Val}(p \vee \neg p) \vee \neg\mathrm{Val}(p \vee \neg p)]\!] = b\uparrow.$$

If Φ is arithmetically definable, then its external truth is absolute with respect to $\mathcal{V}(\tau)$. It follows that it is internally decidable:

$$\mathcal{V}(\tau) \vDash \Phi \vee \neg\Phi.$$

In this case, that means that

$$[\![\Phi \vee \neg\Phi]\!] = a\uparrow.$$

Since the latter does not hold for $\mathrm{Val}(p \vee \neg p)$, it is not arithmetically definable. A fortiori, $\mathrm{Val}(\phi)$ is not arithmetically definable either.

10 Countable models and Kripke's Scheme Discussions of soundness and completeness in classical metamathematics are associated, conventionally, with considerations of results on countable models or *downward Löwenheim-Skolem Theorems.* Since these issues may well be connected in the reader's mind, it may not be amiss to describe, if only briefly, the status of countable models theorems in intuitionistic metamathematics. Also, it points up in a relatively dramatic fashion both the rigors to be faced and the rewards one might gain when transferring elements of classical model theory into traditionally intuitionistic territory. Although there is no such thing as *the* countable models theorem—classically or intuitionistically—for purposes of this section, the countable models theorem will be understood as the claim that every satisfiable theory in a countable first-order language has a countable model. Stated in this fashion, the countable models theorem bears, intuitionistically, a surprising connection to one of the more prominent foundational issues in Brouwer's analysis, the theory of the creative subject. One can easily show that the countable models theorem—even restricted to theories in the language of pure identity—implies Kripke's Scheme [KS].

One can think of KS as a comprehension principle for constructive functions. It says that, for each well-formed formula ϕ in which the variable f is not free, we have

$$\exists f[(\exists n.fn = 0) \leftrightarrow \phi].$$

Here, f ranges over constructive number-theoretic functions with values in $\{0,1\}$. Obviously, KS is inconsistent with Church's Thesis, for under CT, KS would imply that every predicate is r.e. It also clashes with forms of the axiom of choice which are otherwise constructively plausible. The result of this section was stated in [39] without proof.

Theorem 10.1 *The countable models theorem implies KS.*

Proof: Let ϕ be a proposition and let X be the subfinite collection

$$\{0\} \cup \{1 : \phi\}.$$

Recall that, intuitionistically, the concepts of finite and infinite take on a much more delicate logical shading and should be handled accordingly. A set is *finite* when it is in bijective correspondence with some natural number and *subfinite* when it is a subset of a finite set. The claim that every subfinite set is finite has a weak counterexample and, so, is not provable in any constructively correct extension of intuitionistic set theory. Cf. [16] and [34].

Let the structure $\mathfrak{A}$ be a countable model for the pure identity theory of X. Then, there is a function g that maps the set of natural numbers onto the universe A of $\mathfrak{A}$. Since equality on X is discrete, we may assume that g maps $\mathcal{N}$ into $\mathcal{N}$ or that $A \subseteq \mathcal{N}$.

Then, ϕ holds just in case $\mathfrak{A}$ satisfies the sentence

$$\exists x \exists y. \neg x = y.$$

This, in turn, obtains whenever $\exists n, m. g(n) \neq g(m)$. The appropriate instance of KS,

$$\exists f(\exists n. fn = 0 \leftrightarrow \phi),$$

now follows immediately.

Note The proof of this theorem requires a metatheory with a reasonable *Aussonderung* axiom; the constructive second-order arithmetic HAS would surely do.

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