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The Cardinality of Powersets in Finite Models of the Powerset Axiom

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Abstract It is shown that in a finite model of the set-theoretical Powerset axiom a set s and its powerset $\mathcal{P}(s)$ have the same number of elements. Additional results are also derived.

Let (F,ϵ) be a finite model of the set-theoretical Powerset axiom, i.e., in (F,ϵ) every set has a powerset.

For instance, let us consider the finite model (M,ϵ) whose domain consists of the four sets a,b,c,d and where the ϵ-relation is defined by:

(1) $a = \{b\}, \qquad b = \{a\}, \qquad c = \{a,b,c\}, \qquad d = \{a,b,c,d\}.$

It can be readily verified that (M,ϵ) is a model of the Powerset axiom. To this end, we have only to verify that every one of the sets a,b,c,d of the model (M,ϵ) has a powerset in (M,ϵ). For instance, to show that the powerset $\mathcal{P}(c)$ of c exists in (M,ϵ), we must show that all the subsets of c which exist in (M,ϵ) are collected by a set of (M,ϵ). As (1) shows, $c = \{a,b,c\}$ and therefore, from the point of view of the standard ZF set theory, c has $2^3 = 8$ subsets given by: $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\}$. On the other hand, as (1) shows, of these 8 subsets of c only 3, namely $\{a\}$, $\{b\}$, $\{a,b,c\}$ are present in the model (M,ϵ). Again, as (1) shows, these 3 sets are respectively b,a,c and are collected in the model (M,ϵ) by the set c. Thus, we conclude that c is the powerset of c in the model (M,ϵ).

We observe that in the standard ZF set theory if a set has n elements then it has 2^n subsets. This is due to the fact that besides the Powerset axiom, ZF has other axioms which imply the existence of 2^n subsets for a set with n elements. By contrast, here we are considering finite set theoretical models and only the Powerset axiom, and we prove that in such models a set with n elements has n subsets.

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Indeed, as (1) shows, in the finite model (M, ϵ) we have:

(2) $\mathcal{P}(a) = b, \quad \mathcal{P}(b) = a, \quad \mathcal{P}(c) = c, \quad \mathcal{P}(d) = d$

where, as expected, $\mathcal{P}(x)$ stands for the powerset of x, i.e., the set of all subsets (of course, which exist in (M, ϵ)) of x.

In Abian and LaMacchia [2] it is shown that in a finite model of the Powerset axiom the set-theoretical Extensionality axiom also holds. Thus, the notion of "equality" and the notations introduced in (1) and (2) are justified. Moreover, it is shown in [2] that in a finite model of the Powerset axiom, besides the axiom of Extensionality, the axioms of Union and Choice also hold.

Furthermore, in [2] it is shown that in a finite model (F, ϵ) of the Powerset axiom, for every set x and y,

(3) $x \subseteq y$ iff $\mathcal{P}(x) \subseteq \mathcal{P}(y)$

and thus

(4) $x = y$ iff $\mathcal{P}(x) = \mathcal{P}(y)$

and

(5) every set of (F, ϵ) is a powerset of a unique set of (F, ϵ) and hence there is no empty set in (F, ϵ).

In what follows, for every set x and every positive integer n, we define the *n-th powerset* $\mathcal{P}^n(x)$ of x, recursively, as follows:

(6) $\mathcal{P}^1(x) = \mathcal{P}(x)$ and $\mathcal{P}^{n+1}(x) = \mathcal{P}(\mathcal{P}^n(x))$ for $n \geq 1$.

Lemma 1 *Let (F, ϵ) be a finite model of the Powerset axiom. For every set x in (F, ϵ), there exists a smallest positive integer m such that*

(7) $x = \mathcal{P}^m(x)$.

Proof: Since (F, ϵ) is a finite model, clearly for some positive integers n and k we have $\mathcal{P}^n(x) = \mathcal{P}^{n+k}(x)$. But then, in view of (4), we have $x = \mathcal{P}^k(x)$. Denoting by m the smallest such k, we establish (7).

Lemma 2 *Let (F, ϵ) be a finite model of the Powerset axiom. If s is a set in (F, ϵ) then there exists $s_1 \in s$ such that*

(8) $s = \mathcal{P}(s_1)$ *and such that* $x \subseteq s_1$ *for every* $x \in s$.

Proof: By (5) we have $s = \mathcal{P}(s_1)$, for some unique set s_1. Moreover, since $s_1 \in \mathcal{P}(s_1)$, we have $s_1 \in s$. Furthermore, for every $i = 1, \ldots, n$ if $s_i \in s$ then clearly $s_i \in \mathcal{P}(s_1)$ and therefore $s_i \subseteq s_1$, as required.

Next, we prove a key lemma.

Lemma 3 *Let (F, ϵ) be a finite model of the Powerset axiom. Then for every set t and s of (F, ϵ) it is the case that*

(9) $t \in s$ *iff* $\mathcal{P}(t) \in \mathcal{P}(s)$.

Proof: Let $t \in s$. But then by (8) there exists s_1 such that $s = \mathcal{P}(s_1)$ and $x \subseteq s_1$ for every $x \in s$. Consequently, $t \subseteq s_1$ which in turn by (3) implies $\mathcal{P}(t) \subseteq \mathcal{P}(s_1)$. Thus, $\mathcal{P}(t) \subseteq s$ and therefore $\mathcal{P}(t) \in \mathcal{P}(s)$. Hence we have proved:

(10) $t \in s$ implies $\mathcal{P}(t) \in \mathcal{P}(s)$.

However, from (10) it also follows that $\mathcal{P}(t) \in \mathcal{P}(s)$ implies $\mathcal{P}^2(t) \in \mathcal{P}^2(s)$, and therefore, by induction, we have:

(11) $\mathcal{P}(t) \in \mathcal{P}(s)$ implies $\mathcal{P}^k(t) \in \mathcal{P}^k(s)$ for every $k > 0$.

Next, let us observe that from (7) it follows that there exist positive integers m and n such that $t = \mathcal{P}^m(t)$ and $s = \mathcal{P}^n(s)$. Let v be the least common multiple of m and n. But then, obviously, we have:

(12) $t = \mathcal{P}^v(t)$ and $s = \mathcal{P}^v(s)$.

To prove the converse of (10), let $\mathcal{P}(t) \in \mathcal{P}(s)$. But then by (11) we have $\mathcal{P}^v(t) \in \mathcal{P}^v(s)$ which by (12) implies $t \in s$. Thus, the converse of (10) is established and the lemma is proved.

Based on Lemma 3, we prove the following theorem where $|x|$ denotes the number of elements of a set x (of course, counting from outside, i.e., in a standard model of ZF in which (F, ϵ) resides). That is, if $x \in F$ then $|x| = n$ if and only if in ZF the set x is equipollent to the natural number n.

Theorem 1 *Let (F, ϵ) be a finite model of the Powerset axiom. Then in (F, ϵ) a set and its powerset have the same number of elements, i.e.,*

(13) $|s| = |\mathcal{P}(s)|$ *for every set s of (F, ϵ).*

Proof: Let s be a set with n elements, i.e., $s = \{s_1, \ldots, s_i, \ldots, s_n\}$. Then by Lemma 3 it must be the case that $\mathcal{P}(s) = \{\mathcal{P}(s_1), \ldots, \mathcal{P}(s_i), \ldots, \mathcal{P}(s_n)\}$. However, from (4) it follows that distinct sets have distinct powersets. Thus, $|s| = |\mathcal{P}(s)|$, as desired.

Based on the above, we derive some additional results.

Corollary 1 *In (F, ϵ) for every set t and s we have*

(14) $t \in s$ *implies* $|t| \leq |s|$.

Proof: From (9) it follows that $t \in s$ implies $\mathcal{P}(t) \subseteq s$ and therefore $|\mathcal{P}(t)| \leq |s|$, which in turn by (13) implies $|t| \leq |s|$, as desired.

Remark The following statements which are proved in Abian and Amin [1] can be also proved based on Theorem 1 and Corollary 1.

Let (F, ϵ) be a finite model of the Powerset axiom. Then:

(a) Every element of a singleton of (F, ϵ) is itself a singleton.
(b) In (F, ϵ) at least one element of every set is a singleton.
(c) In (F, ϵ) there exists always a singleton.

Lemma 4 *Let (F, ϵ) be a finite model of the Powerset axiom. For every set r and t of (F, ϵ) if r is a proper subset of $\mathcal{P}(t)$ then $t \notin r$.*

Proof: Assume on the contrary that $t \in r$. But then, from (9) it follows that $\mathcal{P}(t) \subseteq r$ contradicting the fact that r is a proper subset of $\mathcal{P}(t)$. Thus, indeed, $t \notin r$.

Theorem 2 *In (F, ϵ) let s be a set with n elements. Then s has at most one subset with $n - 1$ elements.*

Proof: Assume on the contrary that s has two distinct subsets r_1 and r_2 such that $|r_1| = |r_2| = n - 1$. By (5) we see that $s = \mathcal{P}(t)$ for some t and by (9) we see that $|s| = |\mathcal{P}(t)| = n$. On the other hand, by Lemma 4 we have $t \notin r_1$ and $t \notin r_2$ and since $|r_1| = |r_2| = n - 1$, we must have $r_1 = r_2$, contradicting the fact that r_1 and r_2 are distinct. Thus, Theorem 2 is proved.

Finally we have:

Theorem 3 *In (F, ϵ) for every set s*

(15) $s \in s$ *iff* $s = \mathcal{P}(s)$.

Proof: Let $s \in s$. But then by (9) we have $\mathcal{P}(s) \subseteq s$ and by (13) we have $|\mathcal{P}(s)| = |s|$. Thus, $s = \mathcal{P}(s)$. Conversely if $s = \mathcal{P}(s)$ then since $s \in \mathcal{P}(s)$ we see that $s \in s$. Hence, (15) is established.

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