On Type Definable Subgroups of a Stable Group

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Abstract We investigate the way in which the minimal type-definable subgroup of a stable group $G$ containing a set $A$ originates. We give a series of applications on type-definable subgroups of a stable group $G$.

1 Introduction It is not known how to construct a stable group "ab ovo". The stability of a given group structure is deduced usually from some stronger properties, for example the group's being abelian-by-finite, or definable in some stable structure. So at least one could wonder what type-definable subgroups of a stable group $G$ are possible to obtain. We address this problem here. In a way, our results generalize Zilber's ideas (cf. Zilber [12]) on generating subgroups by indecomposable subsets of an $\omega$-stable group $G$.

Throughout, we work with a stable group $G = (G, \cdot, e)$, which is sufficiently saturated (i.e., $G$ is a monster model). $L$ is the language of $G$. Given a type-definable subset $A$ of $G$ we know that there is $\bar{A}$, the minimal type-definable subgroup of $G$ containing $A$ (cf. Poizat [9]). We investigate here the relationship between $A$ and $\bar{A}$. For simplicity, usually we consider $A$ which is type-definable almost over $\emptyset$. A finite set $\Delta$ of formulas of $L$ is invariant under translation if it consists of formulas of the form $\varphi(u \cdot x \cdot v; \bar{y})$ ($u, v, \bar{y}$ are parameter variables here). Except in Section 2, $\Delta$ with possible subscripts will denote a finite set of formulas invariant under translation. One of the basic concepts of stable group theory is that of generic type, due to Poizat ([9]; see also Hrushovski [4]). Recall that if $H$ is a type-definable subgroup of $G$ then a strong 1-type $r$ of elements of $H$ is generic (for $H$) iff for every $\Delta$, $R_\Delta(r) = R_\Delta(H)$, where $R_\Delta$ is the Morley $\Delta$-rank (see Wagon [11]). Notice that as $\Delta$ is invariant under translation, $R_\Delta$ also is invariant under translation, meaning that for each definable subset $X$ of $G$ and $a \in G$, $R_\Delta(X) = R_\Delta(a \cdot X) = R_\Delta(X \cdot a)$. (This is the idea of "stratified order" from [9]; cf. also [4].) Let $\text{Mlt}_\Delta$ denote the Morley $\Delta$-multiplicity. $R_\Delta(a/A)$ abbreviates $R_\Delta(\text{tp}(a/A))$. Let $\check{R}(p)$ denote $(R_\Delta(p) : \Delta \subseteq L$ is finite and invariant.

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under translation). \( R(p) \leq R(q) \) means that for every \( \Delta \), \( R_\Delta(p) \leq R_\Delta(q) \). Let \( \text{gen}(H) \) denote the set of generic types of \( H \). \( H^0 \) is the connected component of \( H \). We give a description of \( \text{gen}(A) \) in topological terms, and prove some corollaries. We formulate also some open problems. Recall the following remark from [4], which can be taken as a definition of generic type.

1.1 Remark Assume \( H \) is a type-definable subgroup of \( G \). Then \( r \), a strong 1-type of elements of \( H \), is generic for \( H \) iff for every \( b \in H \) and a satisfying \( r\upharpoonright b \), \( a \cdot b \downarrow b \).

In our notation we usually follow Baldwin [1] and Wagon [11]. For background on stable groups see [9], [4], and Hrushovski [5]. By [11] we have

1.2 gives a rank equivalent for the forking relation. However this equivalent has one drawback. Condition \( R_\Delta(a/X) = R_\Delta(a) \) may involve formulas not in \( \Delta \), as it may happen that \( R_\Delta(tp_a(a)) > R_\Delta(a) \). In 1.3 we give another characterization of forking. Let \( R'_\Delta(p) = R_\Delta(p|\Delta) \) and \( \hat{R}'(p) = \langle R'_\Delta(p) : \Delta \subseteq L \rangle \). \( r \) in \( R'_\Delta \) stands for “restricted”.

1.3 Lemma Assume \( A \subseteq B \). If \( \hat{R}'(a/B) = \hat{R}'(a/A) \) then \( a \downarrow B(A) \). Moreover, if for some model \( M \subseteq A \), \( a \downarrow A(M) \), then \( a \downarrow B(A) \) implies \( \hat{R}'(a/B) = \hat{R}'(a/A) \).

Proof: The first part follows by [11], Section III. By Lachlan [7], if \( p \in S(M) \) then \( \text{Mlt}_\Delta(p|\Delta) = 1 \). This implies the “moreover” part.

2 A theorem For simplicity we work here with sets type-definable almost over the empty set of parameters, however all the proofs generalize immediately to the case of arbitrary set of parameters. “Type-definable” will always mean in this section “type-definable almost over \( \emptyset \)”. Let \( S \) be the set of strong 1-types over \( \emptyset \), with the standard topology \( \tau \). Notice that there is an obvious correspondence between closed subsets of \( S \) and type-definable subsets of \( G \). By the open mapping theorem, the mapping \( p \rightarrow \hat{p} = p|G \) is a homeomorphic embedding of \( S \) into \( S(G) \). We equip \( S \) with the following strong topology \( \tau' \). Let \( (I, \leq) \) be a directed set (i.e., \( \leq \) is a partial order on \( I \) and for all \( a, b \in I \) there is \( c \in I \) with \( c \geq a, b \)) and \( \hat{p} = \langle p_i, i \in I \rangle \) be a net of types from \( S \). We say that \( \hat{p} \) is strongly convergent to \( q \in S \) (or: \( q \) is a strong limit of \( \hat{p} \)) if for every \( \Delta \) there is \( i \in I \) such that for every \( j \in I, j \geq i \) implies \( \hat{p}_j|\Delta = \hat{q}|\Delta \). In particular, a strong limit of \( \hat{p} \) is a limit of \( \hat{p} \) in the usual sense. To distinguish between \( \tau \) and \( \tau' \), all topological notions regarding \( \tau' \) will be called strong. Notice that if \( q \) is a strong limit of \( \hat{p} \) then \( \hat{R}'(\hat{q}) \) is a pointwise limit of \( \hat{R}'(\hat{p}_i), i \in I \). For \( p \in S \) let \( R'_\Delta(p) = R'_\Delta(\hat{p}) \) and let \( \hat{R}'(p) = \langle R'_\Delta(p) : \Delta \subseteq L \rangle \).

We define binary operation \( * \) and unary operation \( -1 \) on \( S \) as follows. For \( p, q \in S \), \( p * q = \text{stp}(x \cdot y) \) and \( p^{-1} = \text{stp}(x^{-1}) \), where \( x, y \) are independent realizations of \( p \) and \( q \), respectively. Clearly this definition does not depend on a particular choice of \( x \) and \( y \). Similarly we define \( * \) on \( S(G) \). Notice that \( q = p * r \) iff \( \hat{q} = \hat{p} * \hat{r} \). Differing somewhat from the common notation, we let \( p^n \) denote \( p * \ldots * p \) (\( n \) times), and \( p^{-n} = p^{-1} * \ldots * p^{-1} \) (\( n \) times). If \( P \) is a set
of types then let $P(A)$ denote the set of elements of $A$ realizing some type from $P$. For $P \subseteq S$ let $\langle P \rangle$ be the minimal type-definable subgroup of $G$ containing $P(G)$. Clearly $\langle P \rangle$ is type-definable almost over $\emptyset$ anyway. If $P = \{p_1, \ldots, p_n\}$, then we write $\langle p_1, \ldots, p_n \rangle$ instead of $\langle P \rangle$. Theorem 2.3 below explains how $\langle P \rangle$ is formed. Let $\text{cl}(P)$ denote the topological closure of $P$, and let $*P$ denote the closure of $P$ under $*$. Let gen$(P)$ be the set of $r \in \text{cl}(*P)$ such that there is no $q \in \text{cl}(*P)$ with $R_\Delta(r) \leq R_\Delta(q)$, with some of the inequalities strict. As in [4] we have

2.1 Fact If $P \subseteq S$ is nonempty then gen$(P)$ is nonempty, too. Moreover, gen$(P)$ is a closed subset of $S$.

Following [4], for $p \in S$ and $x \in G$ let $^xp = r * p$, where $r = \text{stp}(x)$. For $P \subseteq S$ let $^xP = \{^xp : p \in P\}$.

2.2 Lemma
(a) $*$ is associative and continuous coordinate-wise.
(b) If $P \subseteq S$ is closed, then for every $x \in G$, $^xP$ is closed, too.
(c) $R_\Delta(p * q) \geq R_\Delta(p), R_\Delta(q)$.
(d) $R_\Delta(p * q) \geq R_\Delta(p), R_\Delta(q)$.

Proof: (a) That $*$ is continuous coordinate-wise follows by the open mapping theorem from Lascar and Poizat [8]. (b) follows from (a) and the fact that $S$ is compact. (c) and (d) are easy.

2.3 Theorem Assume $P$ is a nonempty subset of $S$. Then $\langle P \rangle = \{x \in G : \text{gen}(P) = \text{gen}(P)\}$. Also, gen$(P)$ is the set of generic types of $\langle P \rangle$.

The rest of this section is devoted to the proof of this theorem. So we fix a $P \subseteq S$. If $p, q \in S$ satisfy $p(G), q(G) \subseteq \langle P \rangle$, then also $p * q(G) \subseteq \langle P \rangle$. Also, if $Q \subseteq S$ and $Q(G) \subseteq \langle P \rangle$ then $\text{cl}(Q(G)) \subseteq \langle P \rangle$. Hence the set $\text{cl}(*P)$ is our first approximation of $\langle P \rangle$: we know that $\text{cl}(*P)(G) \subseteq \langle P \rangle$. It is surprising to find out that this is quite a good approximation: by 2.3 all generics of $\langle P \rangle$ belong to $\text{cl}(*P)$, hence 2.3 implies in fact $\langle P \rangle = \text{cl}(*P)(G) \cdot \text{cl}(*P)(G)$ ($X, Y$ is the complex product of $X, Y \subseteq G$). First notice that iteration of cl and $*$ does not increase $\text{cl}(*P)$ anymore.

2.4 Fact $\text{cl}(*P) = \text{cl}(*P)$.

Proof: Let $p, q \in \text{cl}(*P)$. It suffices to prove that within any open $U$ containing $p * q$, there is $r$ from $*P$. By 2.2, if $q'$ is close enough to $q$ then $p * q'$ belongs to $U$, and for fixed $q'$, if $p'$ is close enough to $p$ then $p' * q'$ belongs to $U$. We can choose $p'$ and $q'$ from $*P$, so we are done.

Let $\mu = |L|$, and let $\Delta_\alpha, \alpha < \mu$, be an enumeration of finite sets of formulas in $L$ invariant under translation. We define by induction on $\alpha \leq \mu$ closed subsets $P_\alpha$ of $\text{cl}(*P)$ as follows. $P_0 = \text{cl}(*P)$, $P_\delta = \bigcap_{\alpha < \delta} P_\alpha$ for limit $\delta$. $P_{\alpha + 1}$ is the set of $p \in P_\alpha$ such that $R_\Delta(p) = R_\Delta(P_\alpha(G))$. Notice that if we start with $P = S$, then this procedure leads to $P_\mu = \text{gen}(G)$ (cf. the introduction to [4]), whence $P_\mu$ does not depend on the particular choice of $\Delta_\alpha$'s in this case. We will see that this is always true, i.e. that $P_\mu = \text{gen}(\langle P \rangle)$, and so does not depend on the choice of $\Delta_\alpha$'s.
Let $n_\alpha = R_{\Delta_\alpha}(P_\alpha(G))$ and $k_\alpha = \text{Mlt}_{\Delta_\alpha}(P_\mu(G))$. Let $\varphi_{\alpha,i}(x), i < k_\alpha$, be disjoint formulas almost over $\emptyset$ of $\Delta_\alpha$-rank $n_\alpha$ and $\Delta_\alpha$-multiplicity 1 with $P_\mu(G) \subseteq \bigcup_i \varphi_{\alpha,i}(G)$. Define $\varphi_{\alpha,i,a}(x)$ as $\varphi_{\alpha,i}(a \cdot x)$. Let $X = \{ a \in G : aP_\mu = P_\mu \}$.

2.5 Claim \( X = \bigcap_{\alpha < \mu} \{ a \in G : \text{for each } i < k_\alpha, R_{\Delta_\alpha}(\varphi_{\alpha,i,a}(G) \cap P_\mu(G)) = n_\alpha \} \). In particular, \( X = \{ a \in G : aP_\mu \subseteq P_\mu \}, \) i.e. \( aP_\mu \subseteq P_\mu \) implies \( aP_\mu = P_\mu \).

Proof: Notice that if \( aP_\mu \subseteq P_\mu \) then for each \( \alpha \) and \( i, R_{\Delta_\alpha}(\varphi_{\alpha,i,a}(G) \cap P_\mu(G)) = n_\alpha \), hence \( a \in X \), and we are done.

Notice that \( \text{"} R_{\Delta_\alpha}(\varphi_{\alpha,i,a}(G) \cap P_\mu(G)) = n_\alpha \text{"} \) is a definable almost over \( \emptyset \) property of \( a \). Indeed, \( R_{\Delta_\alpha}(\varphi_{\alpha,i,a}(G) \cap P_\mu(G)) = n_\alpha \) iff for some (unique) \( j \), \( R_{\Delta_\alpha}(\varphi_{\alpha,j,a}(G)) = n_\alpha \), the latter property of \( a \) being definable over the parameters of \( \varphi_{\alpha,j}, j < k_\alpha \). Also, \( X \) is closed under taking inverses. In particular we get that \( X \) is a type-definable almost over \( \emptyset \) subgroup of \( G \). The next lemma concludes the proof of 2.3.

2.6 Lemma \( P(G) \subseteq X \), also \( P_\mu \) is the set of generic types of \( X \). In particular, \( X = \langle P \rangle \), \( P_\mu \) does not depend on the choice of \( \Delta_\alpha \)'s, \( n_\alpha = R_{\Delta_\alpha}(\text{cl}(\star P))(G) \) and \( P_\mu = \text{gen}(P) \).

Proof: If \( p \in P \) and \( q \in P_\mu \) then we have \( p \cdot q \in \text{cl}(\star P) = P_\emptyset \). By induction on \( \alpha < \mu \), by 2.2(c) we see that \( R_{\Delta_\alpha}(p \cdot q) = n_\alpha \), i.e. \( p \cdot q \in P_\mu \). This shows that \( P(G) \subseteq X \). \( X \) is type-definable, hence also \( \langle P \rangle \subseteq X \), and in particular \( P_\mu(G) \subseteq X \). If \( r \) is a generic type of \( X \) then we have \( r \cdot P_\mu = P_\mu \), hence by 2.2(c) and our definition of generic type, \( n_\alpha = R_{\Delta_\alpha}(X) = R_{\Delta_\alpha}(r) \), and each type from \( P_\mu \) is generic for \( X \). We need to show yet that every generic of \( X \) belongs to \( P_\mu \) (this will imply \( X \subseteq \langle P \rangle \), and finish the proof). Let \( r \in \text{gen}(X) \) and \( p \in P_\mu \). Let \( q = r \cdot p \). So \( q \in P_\mu \). Let \( a, b \) be independent realizations of \( r, p \) respectively and \( c = a \cdot b \). By 1.2, looking at the \( \Delta_\alpha \)-ranks of \( \text{tp}(c/b) \), we get \( b \downarrow c \), hence \( a = c \cdot b^{-1} \) satisfies \( q \cdot p^{-1} \), i.e. \( r = q \cdot p^{-1} \). We have \( P_\mu \cdot P_\mu = P_\mu \), hence \( P_\mu \cdot p \subseteq P_\mu \). Similarly as in 2.5 we get \( P_\mu \cdot p = P_\mu \), i.e. there is \( r' \in P_\mu \) with \( r' \cdot p = q \). Again we get \( r' = q \cdot p^{-1} \), hence \( r = r' \) and \( r \in P_\mu \). This proves the lemma.

3 Applications and corollaries Let \( T \) be a stable theory. Hrushovski proved in [5] that if \( p \) is a strong type and \( \cdot \) is a definable partial binary operation with some natural properties, defined for independent pairs of elements realizing \( p \), then (in \( \mathbb{C}^{eq} \)) there is a type-definable connected group \( (G, \cdot) \) and a definable embedding \( f : p (\mathbb{C}) \rightarrow G \) preserving \( \cdot \), such that \( f(p) \) is the generic type of \( G \).

In other words: a definite place plus less definite binary operation on it yields a definable group. Here we prove an analogous result: a definite group operation on a less definite place also yields a definable group, namely,

3.2 Theorem Assume \( T \) is stable, \( A \subseteq \mathbb{C} \) and \( \cdot \) is a definable binary operation such that \( (\Lambda, \cdot) \) is a group. Then (in \( \mathbb{C}^{eq} \)) there is a definable group \( H = (H, \cdot) \) and a definable group monomorphism \( h : A \rightarrow H \).

Proof: The proof is an adaptation of the proof of Hrushovski's result from [5], modulo Section 2. Hence we give a sketch only. Wlog \( A \) is contained in the set of constants of the language of \( T \). As in Section 2, \( S \) denotes the set of strong \( 1 \)-types over \( \emptyset \). For \( a \in \mathbb{C} \) let \( p_a = \text{stp}(a) \), and let \( P = \{ p_a : a \in A \} \). First
we proceed as if we were acting within a group structure in Section 2. So for \( p, q \in S \) we define \( p \ast q \) as \( \text{stp}(x \cdot y) \), where \( x, y \) are independent realizations of \( p, q \) respectively, provided \( x \cdot y \) is defined. Notice that \( p*_a \ast p*_b \) is always defined for \( a, b \in A \), and equals \( p_{a,b} \). It follows that \( *P = P \), hence we can skip one step from the construction in Section 2, and consider just \( \text{cl}(P) \) (which equals \( \text{cl}(*P) \) here). By the open mapping theorem, if \( a, b \in \text{cl}(P)(S) \) are independent, then \( a \cdot b \) is defined, and also belongs to \( \text{cl}(P)(S) \) (see the proof of 2.4). In particular, \( * \) is defined on \( \text{cl}(P) \) and \( \text{cl}(P) = \text{cl}(P) \). Within \( \text{cl}(P) \) we look for "generic types" of the group we are going to define. We proceed as in the proof of 2.3; however, as in [4], we have to modify the meaning of \( \Delta \) from Section 2.

Wlog \( e \cdot x \) and \( x \cdot e \) are defined for every \( x \in S \), where \( e \) is the identity element of \( A \). Now \( \Delta \) ranges over sets of the form \( \{ \phi(a \cdot x \cdot b; y) : \phi(u \cdot x \cdot v; y) \in \Delta' \text{ and } a, b \in \text{cl}(P)(S) \} \) for some finite set \( \Delta' \) of formulas of \( L = L(T) \). Most importantly, for this new meaning of \( \Delta \), 1.2 continues to hold and 2.2(c) remains true for \( p, q \in \text{cl}(P) \); hence we are able to carry on reasonings typical for generic types in a stable group. Let \( \mu = |T| \), and let \( \Delta_{\alpha}, \alpha < \mu \) be an enumeration of the finite subsets of \( L(T) \) invariant under \( \cdot \)-translation.

We define \( \text{P}_{\mu} \) as in the proof of 2.3, and similarly as in Section 2 we prove the following claim.

3.2 Claim

(a) If \( p \in \text{cl}(P) \), then \( p \ast \text{P}_{\mu} = \text{P}_{\mu} \ast p = \text{P}_{\mu} \).

(b) \( \text{P}_{\mu} \) does not depend on the choice of \( \Delta_{\alpha} \)'s, and \( R_{\Delta_{\alpha}}(\text{P}_{\mu}(S)) = R_{\Delta_{\alpha}}(\text{cl}(P)(S)) \).

Let \( P' = \text{P}_{\mu} \). Notice that \( P' \) is a closed subset of \( \text{cl}(P) \). If \( P' \) consisted of a single type, the further proof would be nearly the same as in [5]. However, even if \( P' \) may have more elements than one, notice that:

(1) for each \( \Delta \), \( P'|\Delta \) is finite.

On the set of functions \( f \) from \( S \) uniformly definable by instances of some fixed formula, with \( \{ y \in P'(S) : y \downarrow f \} \subseteq \text{Dom}(f) \), we define an equivalence relation \( \sim \) by: \( f \sim f' \) iff for \( y \in P'(S) \) with \( y \downarrow f, f', f(y) = f'(y) \).

By (1), \( \sim \) is a definable equivalence relation, hence \( f/\sim \) is an element of \( \text{G}^{eq} \).

If \( g = f/\sim \) and \( y \in P'(S) \) is independent from \( g \), then \( g(y) \) is defined in an obvious way. In particular, every \( a \in \text{cl}(P)(S) \) determines a \( P' \)-germ \( g_a \) defined for \( c \downarrow a \) by \( g_a(c) = a \cdot c \). Let \( F_0 = \{ g_a : a \in \text{cl}(P)(S) \} \) and let \( F \) be the set of \( P' \)-germs of all definable functions \( f \in \text{G}^{eq} \) with \( \{ y \in P'(S) : y \downarrow f \} \subseteq \text{Dom}(f) \) such that for \( y \in P'(S) \) with \( y \downarrow f, f(y) \downarrow f \). Hence for \( g \in F \) and \( y \in P'(S) \) with \( y \downarrow g \) we have \( g(y) \downarrow g \). Notice that \( F_0 \) is type-definable almost over \( \emptyset \). By the choice of \( P' \), 3.2 and 1.2, \( F_0 \) is contained in \( F \).

For \( g_1, g_2 \in F \) let \( g_1 \circ g_2 \) be the \( P' \)-germ of the composition of \( g_2 \) and \( g_1 \). By the choice of \( F \), \( g_1 \circ g_2 \) is properly defined and belongs to \( F \). Now we define \( h \).

For \( a \in \text{cl}(P)(S) \) let \( h(a) = g_a \in F_0 \). We check that \( h \upharpoonright A \) is an embedding and maps \( \cdot \) to \( \circ \).

Indeed, if \( a \neq a' \in A \) then for any \( b \in P'(S) \) with \( b \downarrow a, a', a \cdot b \neq a' \cdot b \) (this follows by the open mapping theorem and the fact that \( A \) is a group, i.e. satisfies the right cancellation law). Hence \( h \upharpoonright A \) is an embedding.

Now let \( a, b \in \text{cl}(P)(S) \). We have trivially

(2) if \( a \downarrow b \) and \( c = a \cdot b \) then \( g_a \circ g_b = g_c \).
Of course $c \in \text{cl}(P)(G)$. (2) amounts to saying that for $d \in P'(G)$ with $d \downarrow a,b,c$, $(a \cdot b) \cdot d = c \cdot d$, which is trivial.

We need yet to find the type-definable group $H$ containing $F_0$. Let $F_1$ be the closure of $F_0$ under $\circ$. As in [5] we see that $F_1$ satisfies the right cancellation law (in the proof we use the fact that for each $g \in F_1$ and $r \in P'$ there is $y \in P'(G)$ with $y \downarrow g$ such that $g(y)$ satisfies $r$, this follows as in 3.2). Let $F_2$ be the closure of \{$g_a \cdot a \in P'(G)$\} under $\circ$. $F_2$ is a subset of $F_1$. We will show that $F_2$ is type-definable. As in [5] it suffices to prove that if $a,b,c \in P'(G)$ then for some $u,v \in P'(G)$ and $x \in P'(G)$ with $x \downarrow u$ there is $y \in P'(G)$ with $y \downarrow x$ and $y \downarrow u$ such that $u \cdot y = x$. Applying this to $x = b$, we can choose $u,v \in P'(G)$ such that $u \cdot v = b$, $u$ and $v$ are independent from $b$ and $u,v \downarrow a,b,c(b)$. It follows that $u \downarrow a,b,c$ and $v \downarrow a,b,c$. By (2), $g_a \cdot g_b \circ g_c = g_u \circ g_v \circ g_c = g_{a \cdot u} \circ g_{v \cdot c}$. $F_2$ is a type-definable semigroup with the right cancellation law, hence by [5], $F_2$ is a group. If $a \in \text{cl}(P)(G)$ and $b \in P'(G)$ are independent, then $a \cdot b = c \in P'(G)$, and by (2), $g_a \circ g_b = g_c$. As $F_2$ is a group, for some $u,v \in P'(G)$, $(g_u \circ g_v) \circ g_b = g_c = g_u \circ g_v$. By the right cancellation law in $F_1$ we get $g_a = g_u \circ g_v$. This shows that $F_1 = F_2$, and $H = F_2$ satisfies our demands.

As in [5] we can prove that $h$ is 1-1 on $P'(G)$, and the proof above shows that $h$ maps $P'$ onto $\text{gen}(H)$.

Another application of 2.3 consists in showing that existence of a subgroup of $G$ with some properties yields existence of type-definable subgroup of $G$ with these properties. Suppose $W(x_1,\ldots,x_n)$ is a formula of $L$. We say that a subset $A$ of $G$ satisfies $W$ if all $a \in A$ satisfy $W$. If $H$ is a type-definable subgroup of $G$ then we say that $H$ satisfies $W$ generically iff all independent tuples $a \in H$ of elements realizing generic types of $H$ satisfy $W$.

3.3 Corollary If a subgroup $A$ of $G$ satisfies $W$ then the minimal type-definable subgroup of $G$ containing $A$ satisfies $W$ generically.

Proof: Wlog $A$ is a set of constants. Let $P = \{\text{stp}(a) : a \in A\}$. Then obviously each independent tuple $a \in \text{cl}(P)(G)$ of suitable length satisfies $W$. By 2.3, the generic types of the minimal type-definable subgroups of $G$ containing $A$ belong to $\text{cl}(P)$, hence we are done.

Notice that if $H$ is generically abelian then $H$ is abelian. In particular, we get another proof of an old result (cf. Baldwin and Pillay [2]).

3.4 Corollary If $A$ is an abelian subgroup of $G$ then $\bar{A}$ is also abelian.

Another application concerns the existence of free subgroups of $G$. Even if it is not known if a free group with $\geq 2$ generators is stable, at least we will see that there are “generically free” stable groups. Let $\mathcal{T}(I)$ denote the free group generated by the set $I$. We say that a type-definable subgroup $H$ of $G$ is generically free if for every $n < \omega$, for each nontrivial word $v(x_1,\ldots,x_n)$ in $\mathcal{T}(x_1,\ldots,x_n)$, $H$ satisfies generically $v(x_1,\ldots,x_n) \neq e$.

3.5 Lemma If $A$ is a free subgroup of $G$ with $\geq 2$ generators then $\bar{A}$ is generically free.
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Proof: Suppose I is the set of free generators of A, and wlog I is a set of constants of L. We say that a word w in letters from I is positive if a⁻¹ does not occur in w for any a ∈ I. Choose a ≠ b ∈ I. Let v_n = a⁻ⁿb⁻ⁿ, n > 0. We say that a word w(x_1, ..., x_n) in letters x_1, ..., x_n is nontrivial if it is nonempty and no x_i⁻¹x_j⁻¹ or x_j⁻¹x_i⁻¹ occurs in w. The following claim can be proved by induction on the length of w.

3.6 Claim Assume w(x_1, ..., x_m) is a nontrivial word in letters x_1, ..., x_m, n, k_i, i ≤ m, are natural numbers. If n_1, k_1, n_2, k_2, ..., n_m, k_m grows fast enough then for any positive words w_i, i ≤ m, of length k_i, w(v_n, w_1, ..., v_n, w_m) ≠ e holds in A.

Let A_0 be the semi-group generated by I. If c ∈ A_0 then c = c_1 ... c_n for some c_1, ..., c_n ∈ I. We define ℓ(c) = n. Applying 2.3 in the language expanded by adding constants for elements of A_0 we see that each generic type r of A is in the closure of {stp(c) : c ∈ A_0}. Also, as in 2.5, for every v, w ∈ A_0, the mappings r → stp(v) * r and r → r * stp(w) are permutations of gen(A). In particular, by 2.2(a), for every v, w ∈ A_0, gen(A) ⊆ cl({stp(vw) : c ∈ A_0}). Hence for every n, k we have

(1) gen(A) ⊆ cl({stp(vn)c : c ∈ A_0 and ℓ(c) ≥ k}).

Now suppose the lemma is false. This means that for some nontrivial word w(x_1, ..., x_m), w(x_1, ..., x_m) = e belongs to r_1(x_1) ⊗ ... ⊗ r_m(x_m) for some r_1, ..., r_m ∈ gen(A). By the open mapping theorem this means that ∃U_1 ∀p_1 ∈ U_1 ∃U_2 ∀p_2 ∈ U_2 ... ∃U_m ∀p_m ∈ U_m, w(x_1, ..., x_m) = e ∈ p_1(x_1) ⊗ ... ⊗ p_m(x_m), where U_i ranges over open neighborhoods of r_i. By (1) and 3.6 we get an easy contradiction.

It is well-known (cf. Shelah [10]) that there are two rotations of R^3 which generate a free group. By 3.5 we see that there is a type-definable subgroup H of the group of linear automorphisms of C^3, which is generically free. But the field of complex numbers is ω-stable, hence H is definable, and stable in itself.

4 On connected type-definable subgroups of G

From now on, "a subgroup of G" will always mean "a type-definable almost over ∅ subgroup of G". So if H is a subgroup of G then gen(H) is a subset of S. Suppose H is a connected subgroup of G and r ∈ gen(H). Then r * r = r and ⟨r⟩ = H. In fact, by 2.3 we have

4.1 Proposition Let r ∈ S. Then the following are equivalent.
(a) r * r = r
(b) ⟨r⟩ is connected and r is the generic type of ⟨r⟩. In particular, r * r = r implies r = r⁻¹.

Proposition 4.1 suggests the following problem. Is it possible to characterize, using only * and topological notions, the class of r ∈ S such that ⟨r⟩ is connected?

We can think of * and topology as our syntactical means, while ⟨r⟩ being connected is a kind of semantical notion. Another way to state this problem is as follows: What are the possible syntactical reasons that make ⟨r⟩ connected?
In this section we find an ample subset $\text{Con}$ of $S$ such that $\langle r \rangle$ is connected for $r \in \text{Con}$.

4.2 Remark  Let $H$ be a subgroup of $G$ and $p \in S$. Then
(a) $p(G) \subseteq H$ iff for some (every) $r \in \text{gen}(H)$, $p \cdot r \in \text{gen}(H)$
(b) $p(G) \subseteq H^0$ iff for some (every) $r \in \text{gen}(H)$, $p \cdot r = r$.

Proof: (a) $\rightarrow$ is obvious by 1.2. $\leftarrow$. Let $a, b$ be independent realizations of $p, r$ respectively. Then $c = a \cdot b \in H$, hence $a = c \cdot b^{-1} \in H$.

(b) Let $r_0$ be the generic type of $H^0$. Then by (a), $p(G) \subseteq H^0$ iff $p \cdot r_0 = r_0$.

$\leftarrow$. Suppose $p \cdot r = r$ for some $r \in \text{gen}(H)$. Let $a, b$ be independent realizations of $p, r$ respectively. Then $a \cdot b$ realizes $r, b$ and $a \cdot b$ are in the same $H^0$-coset of $H$. It follows that $a = (a \cdot b) \cdot b^{-1} \in H^0$.

Notice that by 2.2(d) and 4.2(a), if $p(G) \subseteq H$ and $r \in \text{gen}(H)$ then $\hat{R}'(p) \subseteq \hat{R}'(r)$, and $p \in \text{gen}(H)$ iff $\hat{R}'(p) = \hat{R}'(r)$. This again shows that any reasonable rank of a generic type is maximal possible. The next fact will be often used.

4.3 Fact  Let $H$ be a subgroup of $G$ and $p \in S$. Assume that for some $r \in \text{gen}(H)$, $\hat{R}(r) = \hat{R}(p \cdot r)$. Then $p^{-1} \cdot p(G) \subseteq H^0$ and for every $r \in \text{gen}(H)$, $\hat{R}(r) = \hat{R}(p \cdot r)$.

Proof: Choose a realizing $p$ and $b$ realizing $r$ with $a \downarrow b$, where $r \in \text{gen}(H)$ and $\hat{R}(r) = \hat{R}(p \cdot r)$. By 1.2, $a \cdot b \downarrow a$, hence $a \cdot b \downarrow a^{-1}$, i.e. $a \cdot b$ and $a^{-1}$ are independent realizations of $p \cdot r$ and $p^{-1}$ respectively. It follows that $b = a^{-1} \cdot (a \cdot b)$ realizes $p^{-1} \cdot (p \cdot r) = (p^{-1} \cdot p) \cdot r$ (i.e. $(p^{-1} \cdot p) \cdot r = r$ (* is associative).

By 4.2(b), $p^{-1} \cdot p(G) \subseteq H^0$. Hence, by 4.2(b) and 2.2(c), for every $r' \in \text{gen}(H)$, $\hat{R}(r') \subseteq \hat{R}(p \cdot r) \subseteq \hat{R}(p^{-1} \cdot p \cdot r') \subseteq \hat{R}(r')$, which gives $\hat{R}(r') = \hat{R}(p \cdot r')$.

Notice that $\hat{R}(r) = \hat{R}(p \cdot r)$ is equivalent by 1.2 and 1.3 to $\hat{R}'(r) = \hat{R}'(p \cdot r)$.

4.4 Corollary  $p \in S$ and $p(G) \subseteq \langle P \rangle$ then $p \cdot p^{-1} \cdot (G) \subseteq \langle P \rangle^0$.

4.5 Definition  We define an increasing sequence of sets $\text{Con}_0 \subseteq \text{Con}_1 \subseteq \text{Con}_2 \subseteq S$. The definitions of $\text{Con}_0, \text{Con}_1, \text{Con}_2$ reflect more and more sophisticated reasons for $\langle r \rangle$ to be connected. Let $*$ denote the group operation in $T = T(\{x_n: n < \omega\})$. The expression $w(x_1, \ldots, x_n)$ of the form $a_1 \cdot \ldots \cdot a_k$, where each $a_i$ is either $x_j$ or $x_j^{-1}$ for some $j \leq n$, is called a $*$-tuple. If $r_1, \ldots, r_n \in S$ and $w(x_1, \ldots, x_n)$ is a $*$-tuple, then $w(r_1, \ldots, r_n)$ is the type from $S$ obtained by substituting in $w(x_1, \ldots, x_n)$ $r_i$ for $x_i$. We call $w$ a $0\cdot*-tuple$ if $w(\bar{x}) = e$ holds in $T$. Let

\begin{align*}
\text{Con}_0 &= \{w(r_1, \ldots, r_n): w(x_1, \ldots, x_n) \text{ is a } 0\cdot*-tuple, n < \omega \text{ and } r_1, \ldots, r_n \in S\}, \\
\text{Con}_1 &= \{p \in S: p = \text{stp}(a_1) \cdot \ldots \cdot \text{stp}(a_n) \text{ for some } n, a_i \in G \text{ and } a_1 \cdot \ldots \cdot a_n = e\} \text{ and} \\
\text{Con}_2 &= \{p \in S: \text{there is an infinite indiscernible set } I = \{a^1, a^2, \ldots\} \text{ with} \\
&\quad \quad a^1 = \{a^1_1, a^1_2, \ldots\}, a^1_1 \cdot \ldots \cdot a^1_n = e \text{ and } p = \text{stp}(a^1_1 \cdot a^2_2 \cdot \ldots \cdot a^n_n)\}.
\end{align*}

Finally, let $\text{Con} = \text{cl}(\text{Con}_2)$. 

It is easy to see that indeed $\text{Con}_0 \subseteq \text{Con}_1 \subseteq \text{Con}_2$. Also, $\text{Con}_0, \text{Con}_1, \text{Con}_2$ are all closed under $\ast$, hence by 2.4 $\text{Con}$ is closed under $\ast$. If $\langle r \rangle$ is connected and $r$ is the generic of $\langle r \rangle$ then $r \in \text{Con}_0$, hence $r \in \text{Con}$. The following was the motivation to define $\text{Con}_2$. Suppose we define $\text{Con}_1(G)$ in $S(G)$ like $\text{Con}_1$ in $S$. Assume some possibly forking extension $r \in S(G)$ of $p \in S$ belongs to $\text{Con}_1(G)$. Then $\langle r \rangle$ is connected (to be shown below), hence also $\langle p \rangle$ is connected. The definition of $\text{Con}_2$ grasps the syntactical meaning of the fact that there exists an $r \in S(G)$ extending $p$, which belongs to $\text{Con}_1(G)$.

In the next lemma we use local forking. However due to the remark after 1.2 we have to use $R'_\Delta$ instead of $R_\Delta$. Recall that for $q \in S$, $q = q|G$.

4.6 Lemma If $r \in \text{Con}$ and $R'_\Delta(q \ast r) = R'_\Delta(q)$ then $(q \ast r)|\Delta = q|\Delta$.

Proof: First assume $r \in \text{Con}_1$. Let $r = \text{stp}(a_1) \ast \ldots \ast \text{stp}(a_k)$ with $a_1 \cdot \ldots \cdot a_k = e$. Let $p_i = \text{stp}(a_i)$. Choose $b$ realizing $q$, independent from $a_1, \ldots, a_k$. Wlog $a_1, \ldots, a_k, b \downarrow G$. By 2.2(d) we have

1) $R'_\Delta(q) = R'_\Delta(q \ast p_1) = \ldots = R'_\Delta(q \ast p_1 \ast \ldots \ast p_k) = R'_\Delta(q \ast r)$.

By induction on $i \leq k$ we show

2) $b \cdot a_1 \cdot \ldots \cdot a_i$ realizes $(q \ast p_1 \ast \ldots \ast p_i)|\Delta$ and $R'_\Delta(b \cdot a_1 \cdot \ldots \cdot a_i/G \cup \{a_1, \ldots, a_k\}) = R'_\Delta(b \cdot a_1 \cdot \ldots \cdot a_i|G/G)$.

For $i = 0$, (2) holds vacuously. Suppose (2) holds for $i = t$, we will prove it for $i = t + 1$. We have $\text{Mlt}_\Delta((q \ast p_1 \ast \ldots \ast p_t)|\Delta) = 1$, hence if $c$ realizes $q \ast p_1 \ast \ldots \ast p_t$ and $c \downarrow a_1, \ldots, a_k(G)$, then $r = \text{tp}_\Delta(c/G \cup \{a_1, \ldots, a_k\}) = \text{tp}_\Delta(b \cdot a_1 \cdot \ldots \cdot a_t/G \cup \{a_1, \ldots, a_k\})$. We have $c \cdot a_{t+1}$ satisfies $q \ast p_1 \ast \ldots \ast p_t$. Clearly, $r$ determines $\text{tp}_\Delta(c \cdot a_{t+1}/G \cup \{a_1, \ldots, a_k\})$ (as $\Delta$ is invariant under translation).

Also, by (1) we have $R'_\Delta(c \cdot a_{t+1}/G \cup \{a_1, \ldots, a_k\}) = R'_\Delta(c \cdot a_{t+1}|G)$. Hence we get $\text{tp}_\Delta(c \cdot a_{t+1}/G \cup \{a_1, \ldots, a_k\}) = \text{tp}_\Delta(b \cdot a_1 \cdot \ldots \cdot a_t/a_{t+1}/G \cup \{a_1, \ldots, a_k\})$ and (2) holds for $i = t + 1$.

Applying (2) for $i = k$, using $a_1 \cdot \ldots \cdot a_k = e$, we get that $b$ realizes $(q \ast r)|\Delta$, i.e. $q|\Delta = (q \ast r)|\Delta$.

Now suppose $r \in \text{Con}_2$. Let $G'$ be a large saturated extension of $G$. Wlog we can choose $I = \{a^1, a^2, \ldots \}$, an indiscernible set witnessing $r \in \text{Con}_2$, such that $r = \text{stp}(a_1^1 \cdot \ldots \cdot a_n^2), I \downarrow G$ and $I$ is based on $G'$, so that $\{a^1, a^2, \ldots \}$ is independent over $G'$. Thus, $a_1^1 \cdot \ldots \cdot a_n^2$ realizes over $G$ the type $\hat{r}$. Choose $b$ realizing $q|G \cup I$. It suffices to prove that $\text{tp}_\Delta(b/G) = \text{tp}_\Delta(b \cdot a_1^1 \cdot \ldots \cdot a_n^2/G)$. We shall prove more, namely

3) $\text{tp}_\Delta(b/G') = \text{tp}_\Delta(b \cdot a_1^1 \cdot \ldots \cdot a_n^2/G')$.

Let $q' = \text{tp}(b/G')$, $r' = \text{tp}(a_1^1 \cdot \ldots \cdot a_n^2/G')$ and $p_i = \text{tp}(a_i^1/G')$. We see that $r' = p_1 \ast \ldots \ast p_n$ (in $S(G')$), and $a_1^1 \cdot \ldots \cdot a_n^2 = e$, hence $r' \in \text{Con}_1(G')$ defined in $S(G')$ like $\text{Con}_1$ in $S$. Also $\text{tp}(b \cdot a_1^1 \cdot \ldots \cdot a_n^2/G') = q' \ast r'$. But $b \cdot a_1^1 \cdot \ldots \cdot a_n^2$ realizes over $Gq \ast \hat{r}$, hence $q \ast r = q' \ast r'|G$. By the assumptions of Lemmas 2.2(d) and 1.3 we get

4) $R'_\Delta(q) = R'_\Delta(q') \preceq R'_\Delta(q' \ast r') \preceq R'_\Delta(q \ast r) = R'_\Delta(q)$. 

Thus \( \hat{R}_\Delta'(q') = \hat{R}_\Delta'(q' \ast * r') \). Now we can repeat the first part of the proof with \( r := r', q := q' \) and \( G := G' \) to get \( q' | \Delta = q' \ast * r'| \Delta \), i.e. (3).

Finally suppose that \( r \in \text{Con} \setminus \text{Con}_2 \) and \( R'_\Delta(q \ast * r) = R'_\Delta(q) \). For \( n < \omega \), the set of \( p \in S \) with \( R'_\Delta(p) \geq n \) is closed. By 2.2(a), for \( p \in \text{Con}_2 \) close enough to \( r \) we have \( R'_\Delta(q \ast p) = R'_\Delta(q) \), hence \( \hat{q} \ast \hat{p} | \Delta = \hat{q} | \Delta \). Again by 2.2(a), \( \hat{q} \ast \hat{r} | \Delta = \hat{q} | \Delta \).

When \( G \) is categorical, Zilber proved in [12] that if \( \{ A_i : i < \omega \} \) is a family of indecomposable definable subsets of \( G \), then \( \bigcup \{ A_i : i < \omega \} \) generates a definable subgroup of \( G \). This result was generalized to the superstable context in Berline and Lascar [3]. Unfortunately, in the stable case we do not have such a measure of types as Morley rank in the \( \omega \)-stable case or \( U \)-rank in the superstable case. Here we consider the following problem. Suppose \( H_i, i \in I \), are connected subgroups of \( G \). We know that \( H \), the minimal type-definable subgroup containing all the \( H_i \)'s, is connected. How is \( H \) related to the \( H_i \)'s? As a surrogate for Zilber's result, given \( p_i \in \text{Con} \) such that \( H \vdash q \), we describe topologically how to find \( p \in \text{Con} \) with \( H \vdash q \).

4.7 Theorem
(a) If \( r \in \text{Con} \) then \( \langle r \rangle \) is connected, moreover \( \langle r^n, n < \omega \rangle \) strongly converges to the generic type of \( \langle r \rangle \). So if \( q \) is the generic type of \( \langle r \rangle \) then \( \hat{R}'(q) \) is the pointwise limit of \( \hat{R}'(r^n) \), \( n < \omega \).

(b) If \( P \subseteq S \) and \( r \in \text{Con} \) then \( \langle P \cup \{ r \} \rangle = \langle r \ast * P \rangle \). Also, \( \langle P \ast * r \rangle = \langle r \ast * P \rangle \).

(c) If \( p_1, \ldots, p_n \in \text{Con} \) then \( \langle p_1, \ldots, p_n \rangle = \langle q \rangle \), where \( q = p_1 \ast \ldots \ast p_n \in \text{Con} \).

Proof: (a) By 2.2(d), for each \( \Delta \), \( \langle R'_\Delta(r^n), n < \omega \rangle \) is nondecreasing, and bounded by \( R'_\Delta(x = x) \), which is finite. Hence there is \( n(\Delta) \) such that for \( n > n(\Delta) \), \( R'_\Delta(r^n) = R'_\Delta(r^{n(\Delta)}) \) and by 4.6, \( \hat{r}^{n(\Delta)} | \Delta = \hat{r}^{n(\Delta)} | \Delta \). Thus \( \langle r^n, n < \omega \rangle \) strongly converges to some \( q \in S \). Also, \( r \ast * q = q \). By Theorem 2.3, \( q \) is a generic of \( \langle r \rangle \).

By 4.2(b), \( r(G) \subseteq \langle r \rangle^0 \), hence \( \langle r \rangle = \langle r \rangle^0 \) is connected.

(b) Let \( p \in P \). It suffices to prove that \( r(G), p(G) \subseteq \langle r \ast * P \rangle \). Let \( q \) be a generic of \( \langle r \ast * P \rangle \). By 2.2 we have

\[
* \quad \hat{R}'(q) \leq \hat{R}'(q \ast * r) \leq \hat{R}'(q \ast * p)
\]

\( r \ast * p \in r \ast * P \), hence by 4.2(a), \( q \ast (r \ast * p) \in \text{gen}(\langle r \ast * P \rangle) \). It follows that \( \hat{R}'(q \ast * r \ast * p) = \hat{R}'(q) \), and in (*) equalities hold. By 4.6, \( q = q \ast * r \), hence by 4.2(a), \( r(G) \subseteq \langle r \ast * P \rangle \). Also, \( q \ast * p = q \ast (r \ast * p) \) is a generic of \( \langle r \ast * P \rangle \), hence by 4.2(a) again, \( p(G) \subseteq \langle r \ast * P \rangle \). Similarly, we show \( \langle P \cup \{ r \} \rangle = \langle P \ast * r \rangle \).

(c) follows from (b).

4.8 Corollary
Assume \( P = \{ p_i : i \in I \} \subseteq \text{Con} \). If \( j = \{ i_1, \ldots, i_n \} \subseteq I \) then we define \( q_j = p_{i_1} \ast \ldots \ast p_{i_n} \). Assume \( q \in R = \bigcap_{i \in I} cl(\{ q_j : j \in I \text{ and } j \text{ is finite} \}) \). Then \( q \in \text{Con} \) and \( \langle q \rangle = \langle P \rangle \).

Proof: Clearly, \( \langle q \rangle \subseteq \langle P \rangle \). Suppose \( H \) is an almost-\( \emptyset \)-definable subgroup of \( G \) containing \( \langle q \rangle \). By 4.7(c), for every \( i, \langle p_i \rangle \leq H \), hence \( \langle P \rangle \subseteq H \). It follows that \( \langle q \rangle = \langle P \rangle \).

Notice that if \( q_j \) in 4.8 were defined as generic of \( \langle p_{i_1} \ast \ldots \ast p_{i_n} \rangle \), then any \( q \in R \) would be the generic of \( \langle P \rangle \), hence in fact \( R \) would be a singleton in such
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a case. We can say more. By 4.7 and 4.6, if \( r \) is the generic of \( \langle P \rangle \) then \( R'(r) \) is the pointwise supremum of \( \{R'(p) : p \in \ast P\} \). Also, \( r \) is the strong limit of some net of types from \( \ast P \).

In case when the \( U \)-rank of \( G \) is finite, we get a more exact counterpart of Zilber's result.

4.9 Corollary Assume \( G \) is a superstable group with finite \( U \)-rank and \( p \in \text{Con} \). Then for some \( n \), \( p^n \) is the generic type of \( \langle p \rangle \). In particular, \( \langle p \rangle = p^n(G) \cdot p^n(G) \).

Proof: From 2.2(c) and 1.2 it follows that for \( q \in S \), \( U(q \ast r) \geq U(q), U(r) \). Hence we can choose \( n \) such that for \( m > n \), \( U(p^m) = U(p^n) \). It follows that also \( R'(p^n) = R'(p^m) \), and by 4.6, \( p^m = p^n \). By Theorem 2.3 we are done.

5 A special case In this section we focus our attention on the special case of \( \langle p \rangle \) for a single type \( p \in S \). For \( P \subseteq S \) in Theorem 2.3 we explain where the generic types of \( \langle P \rangle \) lie. However, in some respect, the results of Section 3 improved greatly Theorem 2.3: if \( p \in \text{Con} \) and \( q \) is the generic of \( \langle p \rangle \) then \( q = \liminf \{p^n : n < \omega \} = \{q \in S : \) every open \( U \) containing \( q \) contains \( p^n \) for cofinally many \( n < \omega \} \).

(C) For \( p \in S \), \( \text{gen}_{\langle p \rangle} = \mathcal{E}(p) \).

By Theorem 2.3 we have of course \( \text{gen}(\langle p \rangle) \subseteq \mathcal{E}(p) \). Unfortunately Hrushovski found an easy counterexample to (C). Namely, let \( G = (Q, +, 1, P) \), where \( P = \{2^n : n < \omega \} \subseteq Q \). \( \text{Th}(G) \) is \( \omega \)-stable with Morley rank \( \omega \), \( P(x) \) is strongly minimal, \( \langle \text{stp}(1) \rangle = \) all of \( G \), but the strongly minimal type in \( P \) is in \( \mathcal{E}(\text{stp}(1)) \) and is not a generic of \( G \).

We show however that (C) is true for several cases, for example for all stable groups of bounded exponent. In a way we shall answer positively question 5.1 in case when \( P \subseteq S \) is a singleton, in the double step Theorem 5.12 below. We start with comparing \( \langle p \rangle \) and \( \langle q \rangle \) for various \( p, q \in \text{Con}_0 \). We need some additional notation. Let \( w(x_1, \ldots, x_n) = a_1 \ast \cdots \ast a_k \) be a \( \ast \)-tuple. For \( i \leq k \) let \( w_i \) be the shortest \( \ast \)-tuple such that in \( \forall \{x_n : n < \omega \} \), \( a_1 \ast \cdots \ast a_i = w_i \) holds. Let \( \text{In}_0(w) = \{w_i : i \leq k\} \) and \( \text{In}(w) = \{v \in \text{In}_0(w) : v \) is not a proper initial segment of any \( v' \in \text{In}_0(w)\} \). As an example notice that if \( w = w(x_1) \), then \( \text{In}(w) \) has at most two elements which are of the form \( x_1 \ast \cdots \ast x_1 \) or \( x_1^{-1} \ast \cdots \ast x_1^{-1} \).

5.2 Theorem Assume \( w(x_1, \ldots, x_n), v(x_1, \ldots, x_n) \) are 0-\( \ast \)-tuples and \( r_1, \ldots, r_n \in S \).

(a) \( \langle w(r_1, \ldots, r_n) \rangle = \{w'(r_1, \ldots, r_n) \ast w'(r_1, \ldots, r_n)^{-1} : w' \in \text{In}(w)\} \).
(b) If every \( w' \in \text{In}(w) \) is an initial segment of some \( v' \in \text{In}(v) \), then \( \langle w(r_1, \ldots, r_n) \rangle \subseteq \langle v(r_1, \ldots, r_n) \rangle \).
Proof: (a) $\supseteq$. First we prove that for each $w' \in \text{In}_0(w)$, $\langle w'(r_1, \ldots, r_n) * w'(r_1, \ldots, r_n)^{-1} \rangle \subseteq \langle (w_1, \ldots, w_r) \rangle$.

The proof is similar to that of 4.6 and 4.7(b). Let $q$ be the generic of $\langle w(r_1, \ldots, r_n) \rangle$, $r = w'(r_1, \ldots, r_n)$, and it suffices to prove that $q * (r * r^{-1}) = q$. As $w' \in \text{In}_0(w)$, there is a $p \in S$ such that $q * r * p$ is the generic of $\langle w(r_1, \ldots, r_n) \rangle$. Hence, $\hat{R}(q) = \hat{R}(q * r)$. By 4.3, $\langle r * r^{-1} \rangle \subseteq \langle (w(r_1, \ldots, r_n)) \rangle$.

(a) $\subseteq$. Let $H = \langle (w'(r_1, \ldots, r_n) * w'(r_1, \ldots, r_n)^{-1} : w' \in \text{In}(w)) \rangle$, and let $q$ be the generic of $H$. Choose $b_1, \ldots, b_n \in G$ realizing $r_1, \ldots, r_n$ respectively, and if $w(r_1, \ldots, r_n) = p_1 * \ldots * p_k$, where $p_i = r_f$, $\epsilon = \pm 1$, then put $a_i = b_f$. Thus, $a_1 \cdots a_k = e$. Choose $c$ realizing $q$, independent from $b_1, \ldots, b_n$. As in 4.6 (the case $r \in \text{Con}_1$) we prove that for every $i \leq k$, $c \cdot a_1 \cdots a_i$ realizes $q * p_1 \cdots * p_i$ (the proof relies on the definition of $H$). This implies $\langle w(r_1, \ldots, r_n) \rangle \subseteq H$.

(b) follows from (a).

By 5.2 and 4.4 we get the following corollary.

5.3 Corollary Let $p \in S$. Then $\langle p^n * p^{-n} \rangle \subseteq \langle p^{n+1} * p^{-(n+1)} \rangle \subseteq \langle p \rangle^0$.

One could wonder whether $\langle p^n * p^{-n} \rangle = \langle p^{-n} * p^n \rangle$. This seems unlikely, although by 5.2 and 4.8 it is not hard to prove that $\langle \{ p^n * p^{-n} : n < \omega \} \rangle = \langle \{ p^{-n} * p^n : n < \omega \} \rangle$. In the next lemma we shall see that the relationship between $\{ p^n * p^{-n} : n < \omega \}$ and $\{ p^{-n} * p^n : n < \omega \}$ is even closer.

5.4 Lemma Let $q$ be the generic type of $\langle \{ p^n * p^{-n} : n < \omega \} \rangle = \langle \{ p^{-n} * p^n : n < \omega \} \rangle$.

(a) $q = \text{lim}_n p^n * p^{-n} = \text{lim}_n p^{-n} * p^n$.
(b) $\hat{R}(q) = \text{lim}_n \hat{R}(p^n)$ (the limit is pointwise here).

Proof: First notice that $\hat{R}(q) \geq \text{lim}_n \hat{R}(p^n)$, as $\hat{R}(p^n * p^{-n}) \geq \hat{R}(p^n)$. On the other hand we know that $q \in \text{cl}(\{p\})$, where $P = \{ p^n * p^{-n} : n < \omega \}$. For a finite $\Delta$ choose $m$ such that for $n \geq m$, $R\Delta(p^n) = R\Delta(p^m)$. As in the proof of 4.6, for every $r \in P$, $\hat{p}^m * r | \Delta = \hat{p}^m | \Delta$. By 2.2(a), $\hat{p}^m * \hat{q} | \Delta = \hat{p}^m | \Delta$. By 2.2(d), $R\Delta(p^m) = R\Delta(p^n * q) \geq R\Delta(q)$. This shows (b).

Now let $r \in \text{cl}(\{ p^n * p^{-n} : n > m \})$. Then $\hat{R}(r) \geq \text{lim}_n \hat{R}\Delta(p^n * p^{-n}) \geq \text{lim}_n \hat{R}(p^n) = \hat{R}(q)$. So by 4.2(a), $\hat{R}(r) = \hat{R}(q)$, and $r$ is the generic of $\langle \{ p^n * p^{-n} : n < \omega \} \rangle$. It follows that $q = r$, i.e. $q = \text{lim}_n p^n * p^{-n}$. But $\hat{R}(q) = \text{lim}_n \hat{R}(p^n * p^{-n})$, hence we see that $q$ is the strong limit of $\langle p^n * p^{-n} : n < \omega \rangle$.

5.5 Corollary Let $p \in S$. There is a connected type-definable almost over $\emptyset$ subgroup $H$ of $\langle p \rangle^0$ such that $\hat{R}(H) = \text{lim}_n \hat{R}(p^n)$.

The $q$ from Lemma 5.4 might be called $p^n * p^{-n}$ or $p^{-n} * p^n$. It is not hard to prove that $p * q * p^{-1} = q$, hence such a notation would imply $p^{(1+\omega) * p^{-1}} = p^{(1+\omega) * p^{-2}}$, which agrees well with $\omega = 1 + \omega$.

Now let us see what the connection is between $\langle P \rangle$ and $\langle P \rangle^0$ for $P \subseteq S$. First we deal with $P = \{ p \}$.

5.6 Lemma Let $p \in S$. Then $\langle \{ p \} : \langle p^n \rangle \rangle$ is finite for each $n > 0$. Also, $\langle p \rangle^0 = \cap_n \langle p^n \rangle$. In particular, $\langle \{ p \} : \langle p \rangle^0 \rangle \leq 2^{\aleph_0}$.

Proof: By 5.3, for $i \leq n$, $p^i \in \text{in}_1(G) \subseteq \langle p^n \rangle$, hence $p^i(G)$ is contained in one left (and one right) $\langle p^n \rangle$-coset of $\langle p \rangle$. Thus also for every $j < \omega$, $p^j(G)$ is contained in one left $\langle p^n \rangle$-coset and it follows that there are only finitely many left
\( \langle p^n \rangle \)-cosets containing some \( p^i(G) \). In particular, for \( q_0 \), the generic type of \( \langle p \rangle^0 \), \( q_0(G) \) is contained in one \( \langle p^n \rangle \)-coset of \( \langle p \rangle \). As \( q_0 = q_0 \ast q_0^{-1} \), we have \( q_0(G) \subseteq \langle p^n \rangle \) and \( \langle p \rangle^0 \subseteq \langle p^n \rangle \).

Thus if \( q \) is a generic of \( \langle p \rangle \) then \( q(G) \) is contained in one left \( \langle p^n \rangle \)-coset of \( \langle p \rangle \). Also, \( q \in \mathcal{L}(p) \) and there are only finitely many \( \langle p^n \rangle \)-cosets containing some \( p^i(G) \). Thus there are only finitely many \( \langle p^n \rangle \)-cosets containing \( q(G) \) for some \( q \in \text{gen}(\langle p \rangle) \). This implies \( \{ \langle p \rangle : \langle p^n \rangle \} \) is finite.

Now suppose that \( H \) is a relatively definable almost over \( \emptyset \) subgroup of \( \langle p \rangle \) with finite index in \( \langle p \rangle \). Then \( q_0(G) \subseteq H \), hence by 2.3 for some \( n, p^n(G) \subseteq H \). It follows that \( \langle p^n \rangle \subseteq H \), i.e. \( \langle p \rangle^0 = \bigcap_{n<\omega} \langle p^n \rangle \).

Notice that if \( X \) is a free group with \( k \) generators then there are \( (k + \aleph_0) \)-many normal subgroups of \( X \) with finite index in \( X \). Hence by a similar proof we get

**5.7 Corollary** If \( P \subseteq S \) then \( \{ \langle P \rangle : \langle P \rangle^0 \} \) \( \leq 2^{\mid P \mid + \aleph_0} \).

Suppose for some \( k, p(x) \vdash x^k = e \); that is, \( p \) is a type of elements of finite order. Then we have \( p^k \in \text{Con}_1 \); hence by 5.6 we get the following corollary.

**5.8 Corollary** If \( p(x) \vdash x^k = e \) then \( \{ \langle p \rangle : \langle p \rangle^0 \} \leq k \) and \( \text{gen}(\langle p \rangle) = \mathcal{L}(p) \) is finite. Let \( q \) be the generic of \( \langle p \rangle^0 \). Then \( q = \lim_n p^{nk} \). Also, for \( i < k \lim_n p^{nk+i} \) exists and is a generic of \( \langle p \rangle \), and every generic of \( \langle p \rangle \) is obtained in this way.

**5.9 Corollary** If \( \text{Th}(G) \) is small and \( P \subseteq S \) is finite then \( \langle P \rangle \) is connected-by-finite.

**Proof:** By adding a finite set of constants to \( L \) we can assume that \( P \subseteq S(\emptyset) \). By Theorem 2.3, every generic of \( \langle P \rangle \) is in \( \text{cl}(P) \), hence \( S(\emptyset) \) being countable implies that \( \text{gen}(\langle P \rangle) \) is countable, too, and \( \{ \langle P \rangle : \langle P \rangle^0 \} < \omega \).

The next theorem shows that in many cases (C) is true. For the definition of weakly normal groups, see [6]. Notice that any pure group which is abelian-by-finite is weakly normal.

**5.10 Theorem** Assume \( p \in S \) and \( G \) has bounded exponent or is weakly normal. Then \( \text{gen}(\langle p \rangle) = \mathcal{L}(p) \).

**Proof:** In case when \( G \) has bounded exponent the conclusion follows by 5.8. So suppose \( G \) is weakly normal. Choose any \( q \in \mathcal{L}(p) \). We will prove that \( q \in \text{gen}(\langle p \rangle) \). Let \( r \) be the generic of \( \langle p \rangle \) such that \( q^{-1} * r(G) \subseteq \langle p \rangle^0 \), that is \( q(G) \) and \( r(G) \) are in the same \( \langle p \rangle^0 \)-coset of \( \langle p \rangle \). We will prove that \( q = r \). By Hrushovski and Pillay [6], every definable subset of \( G \) is a Boolean combination of cosets of almost over \( \emptyset \) definable subgroups of \( G \). Hence, fix an almost-\( \emptyset \)-definable \( H < G \). It suffices to prove that for any \( a \in G, r(G) \subseteq aH \) iff \( q(G) \subseteq aH \).

Suppose \( r(G) \subseteq aH \). Then \( r^{-1} * r(G) \subseteq H \), hence \( \langle p \rangle^0 \subseteq H \). As \( q(G) \) and \( r(G) \) are in the same \( \langle p \rangle^0 \)-cosets, we get \( q(G) \subseteq aH \).

Now suppose \( q(G) \subseteq aH \). Then \( q(x) \vdash x \in aH \), and \( q \in \mathcal{L}(p) \), so there are infinitely many \( n \) with \( p^n(G) \subseteq aH \). Choose \( n, k > 0 \) with \( p^n(G), p^{n+k}(G) \subseteq aH \). It follows that \( p^k(G) \subseteq H \), hence again by 5.6 \( \langle p \rangle^0 \subseteq H \). As above we get \( r(G) \subseteq aH \).
It is easy to see that $\ast$ restricted to $\mathrm{gen}(\langle p \rangle)$ is continuous (as a binary function). Unfortunately, $\ast$ is not always continuous on $\mathcal{L}(p)$, because this implies (C) for $p$. Define $f_p : S \to S$ by $f_p(q) = p \ast q$, and similarly define $f_p^{-1}$.

**5.11 Lemma** $f_p|\mathcal{L}(p)$ is a permutation of $\mathcal{L}(p)$. Also, $f_p^{-1} \circ f_p|\mathcal{L}(p) = \text{id}_{\mathcal{L}(p)}$.

**Proof:** Suppose $\langle p^n : i \in I \rangle$ is a net converging to $q \in \mathcal{L}(p)$, and wlog $\langle p_i^{n_i} : i \in I \rangle$ converges to $q'$ in $\mathcal{L}(p)$. We see that $f_p(q') = q$, hence $\text{Rng}(f_p|\mathcal{L}(p)) = \mathcal{L}(p)$. For a fixed $\Delta$, as in the proof of 4.6 and 5.4, we see that if $n$ is large enough then $\hat{p}^{-1} \ast \hat{p} \ast \hat{p}^n | \Delta = \hat{p}^n | \Delta$. It follows that $\langle p^{-1} \ast p \ast p^n : i \in I \rangle$ also converges to $q$. But this means that $f_p^{-1} \circ f_p|\mathcal{L}(p) = \text{id}_{\mathcal{L}(p)}$, and we are done.

Let $p \in S$. Suppose we are given a task of getting a generic type of $\langle p \rangle$; we know topology, independent multiplication $\ast$, but cannot measure any ranks. The first guess would be to choose a $q_0 \in \mathcal{L}(p)$. We know that possibly $\text{gen}(\langle p \rangle) \neq \mathcal{L}(p)$. So it may happen that $q_0 \not\in \text{gen}(\langle p \rangle)$. However $q_0$ in some respect is more similar to a generic of $\langle p \rangle$ than any $p^n$, for example any rank of $q$ is $\geq$ that rank of $p^n$. Also, $\langle p^0 \rangle \subseteq \langle q_0 \rangle \subseteq \langle p \rangle$, gen($\langle q_0 \rangle$) $\subseteq$ gen($\langle p \rangle$) and $\mathcal{L}(q_0) \subseteq \mathcal{L}(p)$ (this is proved below). So maybe if we try again and choose $q_1 \in \mathcal{L}(q_0)$, then we are more lucky in getting a generic of $\langle p \rangle$. The next theorem confirms this guess.

**5.12 Double step theorem** Assume $p \in S$, $q \in \mathcal{L}(p)$ and $r \in \mathcal{L}(q)$. Then $r$ is a generic type of $\langle p \rangle$.

**Proof:** First notice that

1. $\langle p \rangle^0 \subseteq \langle q \rangle \subseteq \langle p \rangle$.

Indeed, any almost-$\emptyset$-definable subgroup $H$ of $G$ containing $\langle q \rangle$ contains $p^n(G)$ for some $n$, hence also $\langle p^n \rangle$. By 5.6, $\langle p \rangle^0 \subseteq \langle p^n \rangle \subseteq H$. Looking at ranks, (1) implies $\text{gen}(\langle q \rangle) \subseteq \text{gen}(\langle p \rangle)$. Also, $\mathcal{L}(p)$ is closed and closed under $\ast$, hence $\mathcal{L}(q) \subseteq \mathcal{L}(p)$. Now let $q_0 = q^{-1} \ast q$. We show that

2. $q_0 \in \mathcal{L}(p)$.

Choose a net $\langle p_i^n : i \in I \rangle$ converging to $q$. Then $\langle p_i^{-n} : i \in I \rangle$ converges to $q^{-1}$. It suffices to find within an arbitrary open $U$ containing $q_0$ a type from $\mathcal{L}(p)$. By 2.2(a) we can find an $i \in I$ such that $p_i^{-n} \ast q \in U$. By 5.11, the mapping $s \to p_i^{-n} \ast s$ is a permutation of $\mathcal{L}(p)$, hence $p_i^{-n} \ast q \in \mathcal{L}(p)$.

By (1), $\langle p \rangle^0 \subseteq \langle q_0 \rangle \subseteq \langle q \rangle \subseteq \langle p \rangle$, hence $\langle p \rangle^0 = \langle q_0 \rangle^0 = \langle q \rangle^0$. But $q_0 \in \text{Con}$, hence by 4.7, $\langle q_0^n, n < \omega \rangle$ is strongly convergent to $q_1$, the generic type of $\langle q_0 \rangle = \langle q \rangle^0$. By 5.3, 5.4 and 2.2(d) it follows that $\hat{R}(q_1) = \lim_n \hat{R}^\prime(q^n_0) = \lim_n \hat{R}^\prime(q^n)$. We know that any $s \in \mathcal{L}(p)$ is a generic of $\langle p \rangle$ iff $\hat{R}^\prime(s) = \hat{R}^\prime(q_1)$, and for every $s \in \mathcal{L}(p)$, $\hat{R}^\prime(s) \leq \hat{R}^\prime(q_1)$. On the other hand, by 2.2(d), $\hat{R}^\prime(r) \geq \lim_n \hat{R}^\prime(q^n) = \hat{R}^\prime(q_1)$, as $r \in \mathcal{L}(q)$. This implies $\hat{R}^\prime(r) = \hat{R}^\prime(q_1)$, hence $r$ is a generic type of $\langle p \rangle$.

Take $r$ from 5.12. By 5.6 we can define the generic type $r'$ of $\langle p \rangle^0$ as $\lim_n r'^n$. A similar argument yields the following corollary.
5.13. Corollary Let $p \in S$. The following conditions are equivalent.

(a) $p$ is a generic type of $\langle p \rangle$.
(b) $p = \lim_n p^{n+1}$
(c) $p = \lim_n p^{n+1}$.

A challenging problem is to generalize 5.12 for arbitrary $P \subseteq S$. This would tell us more about restrictions on the structure of $G$ imposed by the stability assumption.

REFERENCES


