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Concerted Instant-Interval Temporal Semantics I: Temporal Ontologies

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Abstract The general problem of the relationship between instant-based and interval-based temporal semantics is studied. The paper is in two parts. In this first part we consider instant and interval temporal structures and specify conditions for their mutual definability.

1 Introduction This paper belongs to the "research program" associated with the so-called interval-based temporal semantics (see, e.g., van Benthem [2], Burgess [5]–[6], Kamp [10]–[11], and others). These semantics have arisen mainly from the need of describing temporal expressions of natural languages. Although in many respects they have obvious advantages over ordinary instant- or point-based semantics, interval-based semantics are far less developed both formally and conceptually. Moreover, there are many reasons to think that a proper understanding of natural language expressions, and the more general task of adequately describing change while resolving well-known puzzles concerning change (see, e.g., Hamblin [8], Kamp [11], Kretzmann [13], and Sorabji [16]), could not be achieved on the "monistic" basis of either pure instant or pure interval semantics (although there are some reasons to think that points have in some sense dependent being on intervals). In this paper we will adopt an even-handed approach to these semantics, and show that there is a large area of "peaceful coexistence" in which they are mutually definable. In this first part we will consider temporal structures, comprising both points and intervals, and show how they can be reduced to pure point structures as well as to pure interval structures.[1] In the second part of the paper we will explore the possibilities of extending this area of agreement on temporal semantics proper, while also considering some natural logics of change that arise on the basis of these semantics.

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2 Point-interval structure A point-interval structure *PI* is defined as a triple $\langle P, I, < \rangle$ where *P* is a set of *points* (which we will denote by small Greek letters), *I* is a set of *intervals* (small Roman letters), and $<$ is an *incidence relation*: $\alpha < t$ means that a point α belongs to an interval t. Below we will make use of the following definitions:

Definition 1
(i) An interval t is a *part* of an interval s (in notation, $t \leq s$) iff $(\alpha)(\alpha < t \rightarrow \alpha < s)$
(ii) Intervals t and s *overlap* (in notation, $t \circ s$) iff they have a common part.

Definition 2 An interval t is *bounded by points* α and β (in notation, $\mathrm{C}(\alpha t \beta)$ iff $(\alpha \neq \beta)$ & $(\alpha < t)$ & $(\beta < t)$ & $(s)(\alpha < s \;\&\; \beta < s \rightarrow t \leq s)$.

We will say that a point-interval structure is *linear* if it satisfies the following five axioms:

PI1 $(\alpha)(\alpha < t \leftrightarrow \alpha < s) \rightarrow t = s$
PI2 $\alpha \neq \beta \rightarrow (\exists t)(\mathrm{C}(\alpha t \beta))$
PI3 $(t)(\exists \alpha\beta)(\mathrm{C}(\alpha t \beta))$
PI4 $(\alpha < t) \;\&\; (\alpha < s) \rightarrow (\exists u)(\beta)(\beta < u \leftrightarrow (\beta < t) \vee (\beta < s))$
PI5 $(\alpha)(\beta)(\gamma)(\beta \neq \gamma \rightarrow (\exists t)(\alpha < t \;\&\; \sim(\beta < t \leftrightarrow \gamma < t))$.

According to these axioms intervals having the same points coincide (PI1), any two different points determine an interval they bound (PI2), any interval is bounded by points (PI3), for any two intervals having some point in common there exists an interval containing all and only points of these intervals (PI4), and for two different points there is always an interval containing a given third point and only one point from this pair (PI5). We will say that a linear PI-structure is *open* (or *unbounded*) if it satisfies:

PI6 $(\alpha)(\exists t)(\exists s)(\alpha < t \;\&\; \alpha < s \;\&\; \sim(t \circ s))$.

It must be noted here that the above structure doesn't determine the *direction* of time. There are many reasons for not including this controversial feature in the definition of temporal structure, and besides, it is worthwhile to investigate how much can be described without such an assumption (cf. Needham [14]).

By the definition of C and Axiom PI2 any pair of different points uniquely determines an interval they bound. Hence we may use the following description:

$$[\alpha\beta] =_{def} (\iota t)(\mathrm{C}(\alpha t \beta)).$$

We also define the *betweenness* relation among points:

$$\mathrm{b}(\alpha\beta\gamma) \equiv (\alpha \neq \beta \neq \gamma \neq \alpha) \;\&\; (\beta < [\alpha\gamma]).$$

And finally, the *fusion* of intervals t and s is defined as an interval containing exactly points of t and s:

$$t \nabla s =_{def} (\iota u)(\alpha)(\alpha < u \leftrightarrow \alpha < t \vee \alpha < s).$$

In view of PI1 this is a correct description. Note that by PI4 a fusion of two intervals exists if they have a common point.

Now we are in a position to prove our representation theorem. We first show that the betweenness relation among points satisfies all the axioms of linear order without a first or last element.

Theorem 1 *The betweenness relation on points for any open linear* PI-*structure satisfies the following properties:*
(P1) $b(\alpha\beta\gamma) \rightarrow b(\gamma\beta\alpha)$
(P2) $\alpha \neq \beta \neq \gamma \neq \alpha \rightarrow b(\alpha\beta\gamma) \vee b(\alpha\gamma\beta) \vee b(\beta\alpha\gamma)$
(P3) $b(\alpha\beta\gamma) \rightarrow \sim b(\alpha\gamma\beta)$
(P4) $b(\alpha\beta\gamma)\ \&\ (\delta \neq \alpha)\ \&\ (\delta \neq \beta)\ \&\ (\delta \neq \gamma) \rightarrow b(\alpha\beta\delta) \vee b(\delta\beta\gamma)$
(P5) $(\alpha)(\exists\beta\gamma)(b(\beta\alpha\gamma))$.

Proof: (P1) follows from the definition of the betweenness relation. For (P2) we build the folowing three intervals: $t = [\alpha\beta] \bigtriangledown [\beta\gamma]$, $s = [\alpha\gamma] \bigtriangledown [\gamma\beta]$, and $u = [\beta\alpha] \bigtriangledown [\alpha\gamma]$ (all exist by Axiom PI4). It is easy to show that these intervals contain the same points and hence coincide. Suppose now that $t = [\mu\eta]$. By the definition of fusion we have that both μ and η must belong to at least two intervals from the set $\{[\alpha\beta], [\alpha\gamma], [\beta\gamma]\}$ and hence they must belong to at least one of these intervals. Suppose that μ and η belong to $[\alpha\beta]$. Then $t \leq [\alpha\beta]$ and therefore $\gamma < [\alpha\beta]$.

For (P3), if $\gamma < [\alpha\beta]$ and $\beta < [\alpha\gamma]$, then $[\alpha\beta] = [\alpha\gamma]$ and therefore any interval containing α will contain β iff it contains γ, which contradicts Axiom PI5.

For (P4), since both α and γ belong to $t = [\alpha\delta] \bigtriangledown [\delta\gamma]$, we have that $\beta < t$ and therefore β belongs either to $[\alpha\delta]$ or to $[\delta\gamma]$.

Finally, for (P5), by Axiom PI6 there are t,s such that they are not overlapped and both contain α. Consider $t \bigtriangledown s$ and suppose that it has boundary points β and γ. If $\alpha = \beta$, then $[\beta\gamma] = [\alpha\gamma]$ and since $\gamma < t \bigtriangledown s$ we have that $\gamma < t$ or $\gamma < s$. But if γ belongs to t, then $[\alpha\delta] \leq t$ and therefore $s \leq t$. Similarly, it can be shown that γ cannot belong to s. Therefore $\alpha \neq \beta$, $\alpha \neq \gamma$, and $\alpha < [\beta\gamma]$. This completes the proof.

Consider the following mapping f. For any instant $\alpha : f(\alpha) = \alpha$, while for any interval $t : f(t) = \{\alpha : \alpha < t\}$. Then it is obvious that f is an isomorphism between an open linear PI-structure $\langle P, I, < \rangle$ and a structure $\langle P, I(P), <_P \rangle$, where $I(P)$ is a set of all finite point-intervals (that is, sets of points of the form $\{\alpha : b(\beta\alpha\gamma)$ or $\alpha = \beta$ or $\alpha = \gamma\}$ for any pair $\{\beta,\gamma\}$ of different points) and $\alpha <_P t \equiv \alpha \in t$ for $t \in I(P)$. Therefore we have:[2]

Theorem 2 *The set of axioms* PI1-PI6 *characterizes point-interval structures, generated by a linear betweenness order without a first or last element.*

Note Checking the proof of Theorem 1 it is easy to see that postulates PI1-PI5 characterize point-interval structures, corresponding to arbitrary linear point structures (which are determined by postulates P1-P4).

Below we will give an analogous internal characterization of pure interval structures, corresponding to linear PI-structures.

3 Interval structures An *interval structure* is defined as a pair $\langle I, \leq \rangle$, where I is a set of intervals and $\leq$ is a relation of being a part on it. Below we will use the folowing definitions:

Definition 1 (*overlapping*) $t \circ s \equiv (\exists u)(u \leq t \;\&\; u \leq s)$.

Definition 2 (*touching*) $t * s \equiv (\exists u)(v)(v \circ u \leftrightarrow v \circ t \vee v \circ s)$.

Definition 3 (*abutment*) $t + s \equiv t * s \;\&\; {\sim} t \circ s$.

Definition 4 (*fusion*) $t \nabla s =_{def} (\iota u)(v)(v \circ u \leftrightarrow v \circ t \vee v \circ s)$.

In view of Axiom I1 below the above description is correct. Now we will say that an interval structure is *linear* if it satisfies the following axioms:

I0 '$\leq$' is a partial order on intervals
I1 $(u)(t \circ u \rightarrow s \circ u) \rightarrow t \leq s$
I2 $t \circ s \rightarrow t * s$
I3 $t * s \vee (\exists u)(t + u \;\&\; u + s)$
I4 ${\sim}(t \circ s) \;\&\; t + u \;\&\; u + s \;\&\; t \leq w \;\&\; s \leq w \rightarrow u \leq w$.

We say that the structure is *open* if it also satisfies:

I5 $(t)(\exists u)(\exists s)(t + u \;\&\; t + s \;\&\; {\sim}(u \circ s))$.

Theorem 3 *The set of postulates* I1–I5 *completely characterizes interval structures, corresponding to open linear point-interval structures (and thereby also interval structures, generated by open linear point structures).*

Proof: It is obvious that I1–I5 hold for finite intervals of open linear point structures and hence in all open linear point-interval structures. Now for pairs of abutting intervals we define the following equivalence relation:

$$\{t,s\} \approx \{u,v\} \equiv (w)(w * t \;\&\; w * s \leftrightarrow w * u \;\&\; w * v).$$

Let $\langle t,s \rangle$ be an equivalence class, containing $\{t,s\}$. We will identify points with such equivalence classes and define the incidence relation in the following way:

$$\langle t,s \rangle < u \equiv t * u \;\&\; s * u.$$

By the definition of the equivalence relation this definition is correct. Consider now the following lemma:

Lemma 1 ${\sim}(u \circ v) \;\&\; t * u \;\&\; t * v \;\&\; s * u \;\&\; s * v \rightarrow t * s$.

Proof: If t does not touch s, then by I3 for some $w : t + w \;\&\; w + s$. Now, by the definition of touching and I2 there exists an $e = (t \nabla u) \nabla (u \nabla s)$. Since $t \leq e$ and $s \leq e$ (by I1) we have by I4 that $w \leq e$. But taking into account that w overlaps neither t nor s we obtain (again using I1) that $w \leq u$. In the same way it can be shown that $w \leq v$ and hence $u \circ v$ —a contradiction.

As a consequence of this lemma we have the following alternative description of our equivalence relation:

$$\{t,s\} \approx \{u,v\} \equiv t * u \;\&\; t * v \;\&\; s * u \;\&\; s * v.$$

To prove the theorem we now must check all the postulates PI1–PI6.

PI1. Suppose that t is not a part of s. By I5 there are u, v such that $t + u$, $t + v$, and ${\sim}(u \circ v)$. If both u and v touch s, then there must exist a $w = (u \nabla s) \nabla (v \nabla s)$ and by I4 $t \leq w$. But t is not overlapped with u and v and hence by I1 $t \leq s$, which is impossible. Suppose now that u does not touch s and consider $\langle t,u\rangle$. We have that $\langle t,u\rangle < t$ but by the consequence from Lemma 1 $\langle t,u\rangle$ does not belong to s. Therefore we have in our defined structure

$$(\alpha)(\alpha < t \rightarrow \alpha < s) \rightarrow t \leq s$$

and hence PI1 holds.

PI2. If two pairs $\{t,s\}$, $\{u,v\}$ of abutting intervals are not equivalent then at least one element from the first pair does not touch some element from the second pair. Suppose that t does not touch u. Then by I3 some interval i abuts both t and u. Now if s does not touch i then we will consider $j = t \nabla i$. It is obvious that $s + j$ (since $t + s$), $t * j$, and $u + j$. Note that s is not overlapped with u, since in the opposite case we would have by I4 that $i \leq (t \nabla s) \nabla u$ and hence $i \leq s$, because i is not overlapped with t and u. In this way we can always find an interval that touches all intervals t,s,u,v and abuts at least one interval from each pair $\{t,s\}$ and $\{u,v\}$. Suppose now that i is such an interval, which abuts t and v, where t is not overlapped with v, and consider some interval r such that $\langle t,s\rangle < r$ and $\langle u,v\rangle < r$. Then by I2 there exists a $k = (t \nabla r) \nabla (r \nabla v)$. Now by I4 we have $i \leq k$ and by I1 $i \leq r$. Therefore i is a minimal interval, containing $\langle t,s\rangle$ and $\langle u,v\rangle$ and hence PI2 holds.

PI3. By I5, for any interval t there exist nonoverlapping intervals u,v such that t abuts both. Consider now points $\langle u,t\rangle$ and $\langle v,t\rangle$. If it were the case that u abutted v, then by I4 we would have that $t \leq (u \nabla v)$, which is impossible. Therefore these points are distinct and just as in the proof for PI2 above we may show that t is a minimal interval, containing these points.

PI4. Suppose that some point $\langle i,j\rangle$ belongs to both t and s. Then by Lemma 1, t touches s and $t \nabla s$ exists. Suppose now that some point $\langle u,v\rangle$ belongs to $t \nabla s$, but does not belong to either t or s. If u does not touch s, then by I3 some interval w abuts both u and s. Consider $r = u \nabla (t \nabla s)$: $w \leq r$ (by I4), but since w is not overlapped with u and s we have that w is a part of t. Therefore $t * u$ (since t overlaps $u \nabla w$ and $(u \nabla w) \nabla t = u \nabla t$). Hence we have that any interval, which touches $t \nabla s$, also touches either t or s. As a consequence we have that both u and v touch either t or s. Suppose now that v does not touch t and u does not touch s. Then we have $u * t$, $s * v$, and some interval k abuts both t and v. By I4 k is a part of $(t \nabla s) \nabla v$ and hence it is a part of s. But on the other hand $k \leq t \nabla u \nabla v$ and therefore it is a part of u. Hence s overlaps u, which is impossible. As a result we have that if $\langle u,v\rangle < t \nabla s$ then either t or s touches both u and v, that is $\langle u,v\rangle < t$ or $\langle u,v\rangle < s$.

PI5. Suppose this it is not true for some points $\alpha = \langle i,j\rangle$, $\beta = \langle t,s\rangle$, and $\gamma = \langle u,v\rangle$. Since $\beta \neq \gamma$, $\{t,s\}$ must contain an interval that does not touch some interval from $\{u,v\}$. Suppose that t does not touch u and r is an interval that abuts t and u (cf. I3). Now $\alpha \neq \beta$, since in the opposite case t is an interval containing α, β but not γ. In the same way $\alpha \neq \gamma$. Consider then intervals $[\alpha\beta]$ and $[\alpha\gamma]$: since $[\alpha\beta]$ must contain γ and $[\alpha\gamma]$ must contain β, we have that these intervals coincide. We will denote this interval by w. By the construction of an interval, bounded by points (from the proof of PI2 above), we have that w abuts some interval, say i, from $\{i,j\}$ and one interval from each pair $\{t,s\}$, $\{u,v\}$, which are not overlapped with i. We must now consider four cases. Let w abut s and v. If s is not overlapped with v, we have that both w and $w \nabla i$ abut v and s and therefore $i \leq w$ (by I4), which is impossible. Hence s overlaps v. Now we have by I4 that $r \leq (u \triangledown v) \triangledown (s \triangledown t)$ (this fusion exists by I2, because $(u \triangledown v)$ overlaps $(s \triangledown t)$) and hence $r \leq (s \triangledown v)$. But on the other hand $r \leq u \triangledown w \triangledown t$ and therefore $r \leq w$. Thus w is overlapped with $v \triangledown s$ and hence it must overlap either v or s — a contradiction. Now let w abut t and v. We have again that t overlaps v and r is a part of both w and v, which is impossible. The case when w abuts s and u is similar, while it is impossible that w abuts t and u, because they are not overlapped. Thus all cases lead to contradiction and therefore PI5 holds.

Finally, PI6 follows directly from our definition of point.

Note It can be shown that the set of postulates characterizing finite intervals of any linear point structure could be obtained by deleting I5 and adding the following condition:

I5′ $\sim t \leq s \rightarrow (\exists u)(u \leq t \;\&\; \sim(u \circ s) \;\&\; (v)(u \leq v \;\&\; v \leq t \rightarrow v \circ s \vee v = u))$.

Note that I5′ is redundant for intervals that satisfy I5. On the other hand, the set of postulates characterizing convex intervals on linear structures is I0–I4 together with

I6 Any chain of intervals has a fusion.

4 Definitions of points It is well known that there is no definite answer to the question of what we can count as an interval in linear point structures. The general requirement is that intervals must be connected or convex sets; that is, if I is an interval then

$$\alpha \in I \;\&\; \beta \in I \;\&\; \mathrm{b}(\alpha\gamma\beta) \rightarrow \gamma \in I.$$

But in addition to that we may require, for example, that intervals be finite, bounded by points, open, or closed. Just in the same way there is no unique answer to the question of how we may define points in interval structures. There are actually many ways of doing it available in the literature. In this section we will consider most of them and propose some new ones. As will be clear from the following, different definitions may lead to different 'continua'.

Below we will give a general definition of a set of intervals that determines a point. But first we will introduce the following auxiliary definitions:

Definition 5 If F and G are sets of intervals, then
(i) $F \hat{*} G \equiv (t)(s)(t \in F \;\&\; s \in G \rightarrow t * s)$
(ii) $F \hat{\circ} G \equiv (t)(s)(t \in F \;\&\; s \in G \rightarrow t \circ s)$.

Definition 6 A set of intervals is said to be a *point-filter* iff
(i) $F \hat{*} F$
(ii) $(t)(t \in F \rightarrow s \leq t) \rightarrow \sim(\exists u)(u \leq s)$.

In other words, any interval in a point-filter touches any other interval from it, and if some interval is a common part for all its members then it is atomic. Thus, point-filters may have no more than atomic 'intersections'. But they may have no common part at all. Now we will stipulate that any point-filter *determines* some point. But there still remains the possibility of two point-filters determining the same point:

Definition 7 Two point filters *determine the same point* iff $(t)(t \hat{*} F \leftrightarrow t \hat{*} G)$.

Later we will give other identity conditions for points, which will lead to nonstandard point structures.

Now we will distinguish three kinds of point filters, which correspond in fact to different interpretations of the notion of point.

Definition 8

(i) A point-filter is said to be an *atom-point-filter* (apf) iff $(\exists t)(s)(s \in F \to t \leq s)$.
(ii) A point-filter is a *boundary-point-filter* (bpf) iff $\sim(F \hat{\circ} F)$.
(iii) A point-filter is a *limit-point-filter* (lpf) iff $F \hat{\circ} F$ & $\sim(\exists t)(s)(s \in F \to t \leq s)$.

We will say that point-filters of the above kinds determine, respectively, *atom points*, *boundary points*, and *limit points*. It is obvious that any point-filter belongs to one (and only one) of these kinds. It follows from the definition of point-filter that an apf may have only an atomic interval as its common part, whereas any bpf must contain a pair of abutting intervals and hence cannot have a common intersection. We could define lpf simply as a set of intervals, satisfying (iii) above, since this condition implies that it is a point-filter. Note that in our representation theorem (Theorem 3) we in fact used boundary points.

The above types of point-filters roughly correspond to the following understandings of the nature of points:

(i) *Point as an atomic interval.* A distinctive feature of this understanding is that points and intervals are objects of the same kind. It is interesting to note that such an interpretation could be easily incorporated into modern set-theoretical representations of the continuum, since points can be identified with minimal intervals.
(ii) *Point as a boundary.* According to this understanding, which may be ascribed to Aristotle, or even to Pythagoreans and Plato, points are boundaries or ends of intervals;[3] they are obtained in the process of dividing lines and their existence depends on the existence of the intervals they bound. For a long time this understanding was the most common one. Our formal definition stems from Hamblin [9], who gave a corresponding definition for directed linear order (cf. Burgess [5]).
(iii) *Point as a limit.* According to this interpretation a point is an ideal object, which can be obtained as a limit of approximations by an infinite sequence of intervals converging to it. In fact, it is the essence of Whitehead's method of extensive abstraction. Similar approaches have been developed by Russell [15] and Wiener [17] (cf. also van Benthem [1]-[2] and Kamp [10]-[11]).

Although point-filters of different kinds cannot coincide, they may determine the same point. Thus, a boundary point may coincide with some limit point. But atom points can coincide with neither boundary points nor limit points. Many problems arise if we mix points of different kinds. Thus, for any atomic interval there exist two boundary points, which bound it. Now if we treat any boundary point as an atom point, it will acquire its own boundaries and we will achieve a quick reproduction of points in the vicinity of our source atom (cf. Kamp [11]).[4]

For any atomic interval t the sets $\{t\}$ and $\{s : t \leq s\}$ are, respectively, minimal and maximal apf's, and for any apf there is a maximal and a minimal apf that determine the same point. Two maximal or minimal apf's determine the same point if and only if they coincide.

Any set of intervals satisfying the condition

(*) $(t)(t \in F \leftrightarrow t \mathbin{\hat{*}} F)\ \&\ {\sim}(F \mathbin{\hat{\circ}} F)$

is a maximal bpf. On the other hand minimal bpf's are simply pairs of abutting intervals. Here we also have that for any bpf there are a maximal and a minimal bpf that determine the same (boundary) point. Two maximal bpf's determine the same point only if they coincide, whereas two minimal bpf's $\{t,s\}$ and $\{u,v\}$ determine the same point if and only if both t and s touch u and v (see the proof of Theorem 3).

The sets of intervals satisfying

(**) $(t)(t \in F \leftrightarrow t \mathbin{\hat{\circ}} F)\ \&\ {\sim}(\exists s)(u)(u \in F \rightarrow s \leq u)$

are exactly maximal lpf's, and for any lpf there is a maximal lpf determining the same point. But as can easily be shown there are no minimal lpf's.

In view of the above facts we may identify atom points either with atomic intervals or with maximal sets of intervals, which have some common part, whereas boundary points could be defined as maximal bpf's, that is as sets of intervals satisfying (*) above. In this way we may avoid the use of "higher-order" definitions of points as certain sets of sets of intervals. Note also that boundary points could also be defined as sets of intervals, satisfying the formula:

$$(t)(t \in F \leftrightarrow t \mathbin{\hat{*}} F\ \&\ {\sim}(t \mathbin{\hat{\circ}} F)).$$

These sets correspond to maximal sets of intervals that have some common boundary point.

Unfortunately, the situation with limit points is more complicated. Two different maximal lpf's may determine the same point, and this happens just when this point is also determined by some bpf. Suppose that $\{t,s\}$ is a pair of abutting intervals and define the following two sets:

$$Ft = \{u : u \circ t\ \&\ u * s\} \qquad \text{and} \qquad Fs = \{u : u \circ s\ \&\ u * t\}.$$

Both these sets are point-filters, and in the case when no atomic interval from t abuts s and no atom from s abuts t (this obviously holds, for example, when there are no atomic intervals) both are maximal lpf's, determining the same point, which is also determined by the bpf $\{t,s\}$. Therefore limit points cannot be simply defined as maximal lpf's (contrary to Russell, Wiener, Kamp, and van Benthem) at least if we want to obtain an ordinary linear point structure (but see below).

Nonatomic structures We will say that a linear interval structure is *nonatomic* if it satisfies

I7 $(t)(\exists s)(s < t)$.

This structure characterizes intervals in dense linear point structures. Since in this case there are no apf's, the definition of point-filters can now be simplified:

Definition 6′ A set of intervals F is a *point-filter* iff
(i) $F \hat{*} F$
(ii) $t \hat{*} F \; \& \; s \hat{*} F \rightarrow t * s$.

It is easy to show that (ii) above is equivalent to the condition that elements from F do not contain a common part. Now, two point-filters F, G will determine the same point if and only if $F \hat{*} G$. Note also that lpf's may now be defined simply as point-filters, satisfying the condition $F \hat{\circ} F$. Sets of intervals satisfying

(*) $(t)(t \in F \leftrightarrow t \hat{*} F)$

are now exactly maximal point-filters, and for any point-filter there is obviously a maximal point-filter determining the same point. Moreover, two maximal point-filters determine the same point if and only if they coincide. Thus we may identify points in nonatomic structures with maximal point-filters, that is with sets of intervals, satisfying (*) above.

In nonatomic linear structures any boundary point coincides with some limit point. However it is still possible that some limit points do not coincide with any boundary points (we have in this respect a clear and grounded correspondence with rational vs. real points on a real line). Corresponding 'pure' lpf's could be characterized as point-filters, satisfying

$$(t)(t \hat{*} F \rightarrow t \hat{\circ} F),$$

and pure limit points could be identified with sets of intervals, satisfying

$$F \hat{\circ} F \; \& \; (t)(t \hat{*} F \rightarrow t \in F).$$

The set of points determined by all point-filters forms a complete dense linear order, and it coincides with the set of all boundary points if and only if the following condition holds:

I6′ Any bounded chain of intervals has a fusion.

5 Nonstandard continua In the history of the philosophical analysis of time we can find attempts to develop alternative temporal ontologies. Most of them were inspired by the desire to avoid Zeno's paradoxes and to resolve problems concerning the description of change (see Kretzmann [13]). Our frameworks allow us to describe some of these attempts.

Discrete continuum Suppose that an open linear interval structure in which I6′ holds satisfies also the following *discreteness axiom*:

I8 $t + s \rightarrow (\exists u)(\mathrm{At}(u) \; \& \; u \leq t \; \& \; u + s)$

where $\mathrm{At}(u)$ means that u is an atomic interval. It is obviously an atomic structure. Moreover, in this structure there are no lpf's and hence the corresponding point structure could be represented as an alternated sequence of atom points and boundary points that join them. Such a structure was proposed by Joannes Damascius (see Sorabji [16]) as an answer to Zeno's paradoxes. It reappears in some works of the fourteenth-century atomists (see Zoubov [18]) and in works of Giordano Bruno, who distinguished two kinds of indivisibles—minima and termini. According to Bruno, minima are indivisible magnitudes proper, whereas termini are their boundaries.

Leibniz' continuum This continuum could be roughly described as a result of 'splitting' all points of an ordinary continuum on pairs of points, which will serve as primary boundaries of corresponding abutting intervals. The origins of such a construction can be found in Aristotle's distinction between continuity and contiguity (cf. *Physics* 227a6–23). This structure was proposed by Leibniz in his dialogue "Pacidius Philalethi" (1676) as a representation of the structure of time. And it has reappeared recently as a (perhaps undesirable) by-product of the Russell-Wiener construction of points (see van Benthem [3]).

In order to give a formal description of it we will define *half-point-filters* (hpf's) as sets of pairwise overlapping intervals, which satisfy the following condition:

$$(*) \quad (t)(s)(t \mathbin{\hat{\circ}} F \,\&\, s \mathbin{\hat{\circ}} F \rightarrow s \circ t).$$

Note that any hpf is a point-filter. We will consider hpf's as determining *half-points*, and give for them the following identity condition:

Two hpf's F and G determine the same half-point iff $F \mathbin{\hat{\circ}} G$.

In view of (*) above it is an equivalence relation. Now, maximal sets of pairwise overlapped intervals are exactly maximal hpf's and two maximal hpf's determine the same half-point if and only if they coincide. Thus, we may identify half-points with maximal sets of pairwise overlapped intervals. For any pair of abutting intervals in a nonatomic linear structure there are exactly two maximal hpf's that determine the same point as this pair of intervals. Hence in complete linear structures to any point two half-points will correspond.

We may also define the notion of a *'generalized' point* (g-point, for short), which will comprise in some sense 'ordinary' points and half-points. In order to do this we simply provide another identity condition for point-filters:

Two point-filters F and G determine the same g-point
iff $(t)(t \mathbin{\hat{\circ}} F \leftrightarrow t \mathbin{\hat{\circ}} G)$.

This equivalence relation is stronger than the identity relation for points (cf. Definition 7 above), and hence point-filters that determine the same g-point also determine the same point. In addition, point-filters that are not hpf's determine the same point if and only if they determine the same g-point. Thus, to any 'standard' point will correspond three g-points, two of which are at the same time half-points, while the third g-point could be interpreted as the 'union' of these half-points in the following sense: the union of corresponding hpf's is a point-filter that determines the third g-point. Although this structure looks strange, it also has precedents in the history of the philosophical analysis of time (see Knuuttila and Incerilehtinen [12] and Brentano's views in Chisholm [7]).

NOTES

1. The theory, described below, is in fact a special case of a general mereological theory, suggested in Bochman [4].

2. In accordance with the approach proposed in this paper we would personally prefer the following formulation of Theorem 2: *The set of axioms* P1–P5 *characterizes point structures generated by open linear* PI-*structures.*" But for historical reasons we know primarily what it is to be "a linear point structure" whereas point-interval structures are known only in a derived way.

3. Such an understanding is reflected also in Euclid's *Elements*, where points are characterized as ends of lines.

4. In Sextus Empiricus (*Adversus Mathematicos*, IX, 282–293) we find the following puzzle: If a line consists of points, then divisions of it would lead either to the emergence of some new points or to divisions of the points themselves. It seems that here we have the same mixture of different kinds of points.

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