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Varying Modal Theories

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Abstract The notion of modal theory is extended by accepting the idea that axioms and language itself vary over a plurality of possible worlds. Inference rules involving different worlds are introduced and completeness is proved by using a notion of 'ugly diagram', which is a graphical means of detecting when a family of modal theories has no model.

Models of modal theories are indexed by a plurality of possible worlds equipped with a binary accessibility relation. It seems natural to extend the notion of modal theory by accepting that axioms, and even language itself, vary over a similar structure.

Here is an argument which supports our point of view, as opposed to already existing work on modal model theory (e.g. [1]). Consider a language L for a modal theory in the usual sense (L is constant). Consider a modal structure M: it varies with the elements of a set I. We may define the "theory of M" as the set of sentences satisfied in the "actual world", but we could as well consider for each $i \in I$ the set T_i of sentences satisfied by M_i in L. A further step consists in the adjunction for each $i \in I$ of constants $\underline{a}_i$ for $a_i \in M(i)$, giving rise to languages $L_i = L \cup \{\underline{a}_i | a_i \in M(i)\}$ varying over the set I of indices.

The aim of this paper is to answer the following preliminary question: when is a family of usual modal theories the theory (in our sense) of a model?

To be specific, we will deal with the system K in the main body of the text but discuss in the last section the extension to other systems.

In the first section, we propose a notion of (K-) theory $(T_i)_{i \in I}$ varying over a structure $\langle I, R \rangle$. Structures and models for these theories are essentially the usual ones (see e.g. [3]), but we note that models validate rules of deduction involving different indices. To take a simple example: if a sentence $\Box\varphi$ is satisfied in i and if iRj, then φ is satisfied in j.

In the second section we describe a notion of consistency. It is clearly necessary but not sufficient to say that for each $i \in I$, T_i is K-consistent; if T_i proves $\Box\varphi$ in K, if iRj and T_j proves $\neg\varphi$ in K, then $T = (T_i)_{i \in I}$ has no model. It is

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shown that T is inconsistent in the sense of the proof theory of Section 1 (i.e., $T \vdash_i \bot$) if and only if T contains (level by level in the sense of K) an "ugly" diagram, typical examples of which being $(iRk,\ iRl,\ jRk)$

$$\begin{array}{ccccc} (i,\Box\varphi) & & (i,\Box\varphi \vee \Box\psi) & & (i,\Box\varphi) \qquad (j,\Box\psi) \\ \downarrow & & \swarrow \quad \searrow & & \searrow \qquad \swarrow \\ (k,\neg\varphi) & & (k,\neg\varphi)\ \ (l,\neg\psi) & & (k,\neg\varphi \vee \neg\psi). \end{array}$$

In the third section, we prove a completeness theorem: the proof remains very close to the completeness proofs in Henkin's style and it could be used as an elementary proof of completeness for usual K-theories (i.e., from our point of view, theories over a one-point set).

1 Languages, structures, and theories over a set with a binary relation Let I be a nonempty set and let R be a binary relation over I.

Definition 1 A language L over $\langle I,R\rangle$ is a family $(L_i)_{i\in I}$ of usual first-order modal languages L_i such that if iRj, then $Op_i^n \subseteq Op_j^n$ and $Rel_i^n \subseteq Rel_j^n$ for all $n \in \omega$. (Op_i^n and Rel_i^n are the sets of symbols of n-ary operations and n-ary relations.)

Terms and formulas of level i are defined as the usual terms and formulas of L_i and denoted by $Term_i$ and $Form_i$ respectively. We assume that the set V of variables is countably infinite and is the same for each $i \in I$: the inclusions $Term_i \subseteq Term_j$ and $Form_i \subseteq Form_j$ are then trivial. In practice, we will consider the (important) case where only the sets of constants Op_i^o vary with i. One could also generalize the given concept of language by allowing I to be a graph and considering for each arrow $a: i \to j$ in I mappings a_{op}^n from Op_i^n to Op_j^n and a_{rel}^n from Rel_i^n to Rel_j^n: the generalization is easy but not motivated at this point of our study.

Definition 2 An L-structure M over $\langle I,R\rangle$ is determined by giving:
(1) for each $i \in I$, an usual L_i-structure $M(i)$; the underlying nonempty set will also be denoted by $M(i)$, and $f^{M(i)}$ and $r^{M(i)}$ denote the interpretations of operation symbols $f \in Op_i^n$ and relation symbols $r \in Rel_i^n$;
(2) for each $i,j \in I$ such that iRj, a mapping of sets M_{ij}: $M(i) \to M(j)$ such that for each $f \in Op_i^n$ and $\bar{a} \in M(i)^n$

$$f^{M(j)} M_{ij}\bar{a} = M_{ij} f^{M(i)} \bar{a}.$$

It is not assumed in general that the transitions are inclusions or that they in some way preserve the relations $r^{M(i)}$.

The interpretation of terms and the satisfaction of formulas in M is the usual one. To be precise, we first define as usual $t(\bar{x})\,[\bar{a}] \in M(i)$ for $t \in Term_i$, $\bar{x}$ a list of variables containing those of t and $\bar{a}$ a matching list of elements of $M(i)$. We then define $M \models_i \varphi(\bar{x})\,[\bar{a}]$ for $\varphi \in Form_i$, $\bar{x}$ a list of variables containing the free variables of φ and $\bar{a}$ a matching list of elements of $M(i)$. Quantifiers are interpreted at the same level

$$(M \models_i \exists y\varphi(\bar{x},y)\,[\bar{a}] \text{ iff } \exists b \in M(i) M \models_i \varphi(\bar{x},y)\,[\bar{a},b])$$

and the inclusions $L_i \subseteq L_j$ and transitions M_{ij} (for iRj) are used to interpret $\Diamond\varphi$ and $\Box\varphi$: e.g., $M \models_i \Box\varphi(\bar{x})[\bar{a}]$ iff $\forall j(iRj \to M \models_j \varphi(\bar{x})\ [M_{ij}\bar{a}])$.

Definition 3 A *theory* T over $\langle I,R\rangle$ in L is a family $(T_i)_{i\in I}$ where for each $i \in I$, T_i is a set of sentences of L_i.

It is well-known from the work of Kripke [2] that even starting from a theory over a one-point set, a model for it is in general over a bigger set. We must therefore extend Definition 2 to allow for that possibility.

Definition 4 Let L be a language over $\langle I,R\rangle$. Let $\langle J,S\rangle$ be an extension of $\langle I,R\rangle$. The extension $\bar{L}$ of L to $\langle J,S\rangle$ is defined for $j \in J$ by $\overline{Op}_j^n = \bigcup_{i\in I,\, iS^*j} Op_i^n$ and $\overline{Rel}_j^n = \bigcup_{i\in I,\, iS^*j} Rel_i^n$, where S^* is the reflexive and transitive closure of S. An L-structure M extended over $\langle J,S\rangle$ is defined to be an $\bar{L}$-structure.

To axiomatize the theory of L-structures extended over some $\langle J,S\rangle \supseteq \langle I,R\rangle$, we describe rules of deduction for a theory T over I.

By first-order K, we mean here the usual propositional system K (all propositional tautologies, the axiom schema $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, the rule of necessitation) supplemented by the usual rules or axioms of quantification ($\forall x\varphi(x) \to \varphi(t)$ when t is free for x in φ, the rule of generalization), plus also the axiom $x = y \to \Box(x = y)$. We write $T_i \vdash_{\overline{K}} \varphi$ to mean that there exists a finite subset $\{\tau_1, \ldots, \tau_n\}$ of T_i such that $\tau_1 \wedge \ldots \wedge \tau_n \to \varphi$ is a theorem of first-order K. This is a notion which should be sharply distinguished from a more general one, to be denoted $T \vdash_{\bar{i}} \varphi$, which depends upon the whole family $(T_i)_{i\in I}$ and which we define by induction:

Definition 5 The clauses defining $T \vdash_{\bar{i}} \varphi$ are for sentences φ, ψ, χ, $\exists x\alpha(x) \in$ *Form* (L_i):

(a) the two initial clauses:

- (K) $T_i \vdash_{\overline{K}} \varphi \Rightarrow T \vdash_{\bar{i}} \varphi$;
- ($\Box$) $T_i \vdash_{\overline{K}} \Box\varphi,\ iRj,\ T_j \vdash_{\overline{K}} \neg\varphi \Rightarrow T \vdash_{\bar{j}} \bot$;

(b) the four inductive clauses:

- (MP) $T \vdash_{\bar{i}} \varphi,\ T \vdash_{\bar{i}} \varphi \to \psi \Rightarrow T \vdash_{\bar{i}} \psi$;
- ($\bot$) $T \vdash_{\bar{i}} \bot \Rightarrow T \vdash_{\bar{k}} \bot$;
- ($\vee$) $T_i \vdash_{\overline{K}} \varphi \vee \psi,\ T + (i,\varphi) \vdash_{\bar{i}} \chi,\ T + (i,\psi) \vdash_{\bar{i}} \chi \Rightarrow T \vdash_{\bar{i}} \chi$;
- ($\exists$) $T_i \vdash_{\overline{K}} \exists x\alpha(x),\ T + (i,\varphi(c)) \vdash_{\bar{i}} \chi \Rightarrow T \vdash_{\bar{i}} \chi$.

In rules ($\vee$) and ($\exists$), $T + (i,\alpha)$ designates the theory T' given by $T_i' = T_i \cup \{\alpha\}$ and $T_j' = T_j$ for $j \neq i$. In rule ($\exists$), it is assumed that c is a new constant (i.e., c is in no L_i) and the deduction takes place in the language $L' = L + (i,c)$ given by

- $L_j' = L_j$ supplemented by the constant c and this for every j such that iR^*j, and
- $L_k' = L_k$ for every k such that not iR^*k.

The notation $T + (i,\alpha(c))$ will henceforth always presuppose that c is such a constant and that the relevant language is $L + (i,c)$. We also emphasize that φ, ψ, χ and $\exists x\alpha(x)$ are *sentences* of L_i and not formulas in general. We refrain from giving the notion of proof corresponding to $T \vdash_{\bar{i}} \varphi$: proofs are sequences

$(i_1, \varphi_1) \ldots (i_n, \varphi_n)$ satisfying conditions corresponding to the clauses given above. We will also use without mention some obvious properties of $T \vdash_i \varphi$.

To illustrate the use of Definition 5, we give examples of derived rules which are interesting in themselves and will be used in the sequel.

Proposition 1

($\neg$) $T + (i, \neg\varphi) \vdash_i \bot \Rightarrow T \vdash_i \varphi$.

($\Box'$) $T \vdash_i \Box\varphi,\ iRj \Rightarrow T \vdash_j \varphi$.

($\Diamond$) $T \vdash_j \varphi,\ iRj \Rightarrow T \vdash_i \Diamond\varphi$.

($\vee'$) $T \vdash_i \varphi \vee \psi,\ T + (i, \varphi) \vdash_k \chi,\ T + (i, \psi) \vdash_k \chi \Rightarrow T \vdash_k \chi$.

($\exists'$) $T \vdash_i \exists x \varphi(x),\ T + (i, \varphi(c)) \vdash_k \chi \Rightarrow T \vdash_k \chi$.

Proof of ($\neg$): Let $T' \equiv T + (i, \neg\varphi)$. Then $T'_i \vdash_K \bot \to \varphi$, $T' \vdash_i \bot \to \varphi$ by (K), $T' \vdash_i \bot$ by hypothesis and $T' \vdash_i \varphi$ by (MP), i.e.

$$T + (i, \neg\varphi) \vdash_i \varphi. \tag{1}$$

On the other hand,

$$T + (i, \varphi) \vdash_i \varphi \tag{2}$$

and

$$T_i \vdash_K \varphi \vee \neg\varphi. \tag{3}$$

Applying (v) to (1), (2) and (3), we get $T \vdash_i \varphi$, the desired result.

Proof of ($\Box'$): Let $T' \equiv T + (i, \Box\varphi) + (j, \neg\varphi)$. Trivially, $T'_i \vdash_K \Box\varphi$ and $T'_j \vdash_K \neg\varphi$. Then $T' \vdash_j \bot$ by ($\Box$), $T' \vdash_i \bot$ by ($\bot$) and

$$T + (j, \neg\varphi) \vdash_i \neg\Box\varphi \tag{4}$$

by($\neg$). On the other hand, by the hypothesis $T \vdash_i \Box\alpha$,

$$T + (j, \neg\varphi) \vdash_i \Box\varphi. \tag{5}$$

From (4) and (5), we derive $T + (j, \neg\varphi) \vdash_i \bot$ (using $\vdash_K \neg\Box\varphi \to (\Box\varphi \to \bot)$, (K) and MP), $T + (j, \neg\varphi) \vdash_j \bot$ by ($\bot$) and finally $T \vdash_j \varphi$ by ($\neg$) and arguments already used.

Proof of ($\Diamond$): Similar to the proof of ($\Box'$).

Proof of ($\vee'$): Let $T_1 \equiv T + (i, \varphi \vee \psi)$, $T_2 \equiv T_1 + (k, \neg\chi)$, $T' = T_2 + (i, \varphi)$ and $T'' = T_2 + (i, \psi)$. Then $T' \vdash_k \neg\chi$ (since T' contains T_2), $T' \vdash_k \chi$ (since T' contains $T + (i, \varphi)$ and $T + (i, \varphi) \vdash_k \chi$), from which we derive $T' \vdash_k \bot$ and $T' \vdash_i \bot$ by ($\bot$), i.e.,

$$T_2 + (i, \varphi) \vdash_i \bot. \tag{6}$$

Similarly, the consideration of T'' gives

$$T_2 + (i, \psi) \vdash_i \bot. \tag{7}$$

Since we also have

$$(T_2)_i \vdash_K \varphi \vee \psi, \tag{8}$$

we can apply (v) to (6), (7) and (8) to obtain: $T_2 \vdash_i \bot$, hence $T_2 \vdash_k \bot$ by ($\bot$), $T_1 \vdash_k \chi$ by ($\neg$) and finally $T \vdash_k \chi$ by transitivity of consequence and the hypothesis $T \vdash_i \varphi \vee \psi$.

Proof of ($\exists'$): Similar to the proof of (v$'$), letting $T_1 \equiv T + (i, \exists x\varphi(x))$, $T_2 \equiv T_1 + (k, \neg\chi)$ and $T' \equiv T_2 + (i, \varphi(c))$.

We now turn to the soundness theorem.

Definition 6 Let L be a language over $\langle I, R\rangle$, let T be an L-theory over I and M be an L-structure extended over $\langle J, S\rangle \supseteq \langle I, R\rangle$. M is a model of T (in symbols $M \models T$) if for every $i \in I$ and every sentence $\tau \in T_i$, $M \models_i \tau$. Let φ be a sentence of L_i; φ is a semantic consequence of T at level i (in symbols $T \models_i \varphi$) if for every model M of T, $M \models_i \varphi$.

Theorem 2 (Soundness) *If* $T \vdash_i \varphi$, *then* $T \models_i \varphi$.

Proof: Let M be a model of T. We prove by induction on the proof of φ that $M \models_i \varphi$. The soundness of (K) is a well-known fact. The soundness of ($\Box$) is an immediate consequence of the definition of satisfaction for $\Box\varphi$. Similarly for (MP) and implication. Rule ($\bot$) is sound because for every i, $M \not\models_i \bot$. For rule (v), since $T_i \models_K \varphi \vee \psi$, $M \models_i \varphi \vee \psi$, hence $M \models_i \varphi$ or $M \models_i \psi$; in the first case, $M \models_i T + (i, \varphi)$, hence by induction $M \models_i \chi$; the second case is analogous. For rule ($\exists$), since $T_i \models_K \exists x\varphi(x)$, $M \models_i \exists x\varphi(x)$ and for some element $a \in M(i)$, $M \models_i \varphi(x)[a]$. Add a new constant c to every L_j with iR^*j and interpret c in M at level i by a and at level j with $iRj_1\, Rj_2R \ldots Rj_{n-1}\, Rj$ by $a_j = M_{j_{n-1}j} \ldots M_{j_1j_2}\, M_{ij_1}(a)$. This turns M into a structure M^+ for $L^+ = L + (i, c)$ and the relation between M and M^+ is such that

(a) for every j with iR^*j, every formula $\psi(\bar{y}, c)$ of L_j^+ and every $\bar{b}$ in $M(j)$,

$$M^+ \models_j \psi(\bar{y}, c)[\bar{b}] \text{ iff } M \models_j \psi(\bar{y}, x)[\bar{b}, a_j]$$

and

(b) for every k such that not iR^*k, every formula $\psi(\bar{y})$ of L_k^+ and every $\bar{b}$ in $M(k)$,

$$M^+ \models_k \psi(\bar{y})[\bar{b}] \text{ iff } M \Vdash_k \psi(\bar{y})[\bar{b}].$$

Using this, it is clear that M^+ is a model of $T + (i, \varphi(c))$; hence by induction, $M^+ \models_i \psi$ and $M \models_i \psi$ again by the relation between M and M^+.

2 Nice theories

The notion of consequence described in the preceding section mixes indices. Rules ($\Box'$) and ($\Diamond$) of Proposition 1 are typical examples of this phenomenon and they generally suffice to detect most inconsistencies.

As a first example, consider the situation represented by

$$(i, \Box\alpha \vee \Box\beta)$$
$$\swarrow \quad \searrow$$
$$(j, \neg\alpha \vee \Box\gamma) \qquad (k, \neg\beta \wedge \Box\delta)$$
$$\searrow \quad \swarrow$$
$$(l, \neg\gamma \vee \neg\delta)$$

meaning that $iRjRl$, $iRkRl$ and that, at each level, one has the displayed sentences, e.g., $T_i \vdash_K \Box\alpha \vee \Box\beta$. Rule ($\Box'$) applied to k and l and rule ($\Diamond$) applied to i and k will easily yield the contradiction.

But in the following example, more complex situations are suggested and rule (v) seems to be necessary:

$$
\begin{array}{ccc}
 & (i, \Box\alpha \vee \Box\beta) & \\
 & \swarrow \quad \searrow & \\
(j, \neg\alpha \vee \Box\gamma) & & (k, \neg\beta \vee \Box\delta) \\
 & \searrow \quad \swarrow & \\
 & (l, (\neg\gamma \wedge \neg\delta) \vee (\Box\varphi \wedge \Box\psi)) & \\
 & \swarrow \quad \searrow & \\
(m, \Box\chi \vee \neg\varphi) & & (n, \Box\omega \vee \neg\psi) \\
 & \searrow \quad \swarrow & \\
 & (p, \neg\chi \vee \neg\omega). &
\end{array}
$$

To obtain a contradiction, we may proceed (informally) as follows. In $T + (i, \Box\alpha)$ we have successively: α in j, $\Box\gamma$ in j, γ in l, not $(\neg\gamma \wedge \neg\delta)$ in l, $\Box\varphi \wedge \Box\psi$ in l, φ in m and ψ in n, $\Box\chi$ in m and $\Box\omega$ in n, χ and ω in p, $\bot$ in p, $\bot$ in i. Similarly, in $T + (i, \Box\beta)$ we have $\bot$ in i and may conclude by applying rule (v).

To clarify this somewhat involved combinatorics of deductions, we propose a "graphic" view of it.

Let $\langle I, R\rangle$ be a nonempty set with a binary relation, and let L be a language over $\langle I, R\rangle$.

A *labeled diagram* D (over I) is determined by giving for each $i \in I$ a sentence e_i of L_i, the *label* of i in D, in such a way that e_i be Tr ("true") except for a finite number of indices.

We often identify two diagrams if their labels are equivalent in first-order K.

If D and D' are two diagrams with labels (e_i^D) and $(e_i^{D'})$ and if $k \in I$, we define the *wedge in k* $D'' = D \underset{k}{\vee} D'$ by the labels:

$$
\begin{aligned}
e_k^{D''} &= e_k^D \vee e_k^{D'}, \\
e_i^{D''} &= e_i^D \wedge e_i^{D'}, \qquad \text{for } i \neq k.
\end{aligned}
$$

Let $k \in I$ and let c be a constant of L_k. Let $D \equiv D(c)$ be a diagram. The labels e_i^D of D in i may be written $[c/x]e_i^D(x)$ where c does not occur in $e_i^D(x)$. We define the $\exists$-*wedge in k* $D' = \exists_k x D(x)$ by the labels

$$
\begin{aligned}
e_k^{D'} &\equiv \exists x e_k^D(x) \\
e_i^{D'} &\equiv \forall x e_i^D(x) \qquad \text{for } i \neq k.
\end{aligned}
$$

Let T be a theory over $\langle I, R\rangle$ in L. We say that T *contains* diagram D if for every $i \in I$, $T_i \vdash_K e_i^D$. (The interest of this concept is that it works level by level and that only a finite number of indices are really concerned.)

We now define a graphic version of inconsistency:

Definition 7 Ugly diagrams are defined inductively by the following clauses:
(1) for every $i \in I$, the diagram

$$e_i^D \equiv \bot$$
$$e_k^D \equiv Tr \qquad \text{for } k \neq i$$

is ugly;

(2) for every $i,j \in I$ with iRj, $i \neq j$, the diagram

$$e_i^D \equiv \Box\varphi$$
$$e_j^D \equiv \neg\varphi$$
$$e_k^D \equiv Tr \qquad \text{for } k \neq i,j$$

is ugly;

(3) for every $i \in I$ with iRi, the diagram

$$e_i^D \equiv \Box\varphi \wedge \neg\varphi$$
$$e_k^D \equiv Tr \qquad \text{for } k \neq i$$

is ugly;

(4) if D_1 and D_2 are ugly, then for every $i \in I$, $D_1 \vee_i D_2$ is ugly;

(5) if $i \in I$ and $D(c)$ is in the language $L + (i,c)$ and is ugly, then $\exists_i x\, D(x)$ is ugly in the language L.

A theory T is *ugly* if it contains an ugly diagram, *nice* otherwise.

Note that in (5) c occurs at most in the labels e_j^D for which iR^*j. As for the notation $T + (i,\alpha(c))$, $D(c)$ and $\exists_i x\, D(x)$ will appear only in contexts where c is a new constant and $D(c)$ is in the language $L + (i,c)$.

Since the indices for which the label is Tr play no significant role, we will often omit them, thus denoting by $(i,\bot)$ diagrams of type (1), by

$$\begin{array}{c}(i,\Box\varphi)\\ \downarrow \\ (j,\neg\varphi)\end{array}$$

diagrams of type (2) and by $(i,\Box\varphi \wedge \neg\varphi)$ diagrams of type (3). Type (3) has been separated from type (2) for technical reasons only: we prefer to handle diagrams labeled by formulas rather than by sets of formulas.

The definition of ugly diagram allows us to characterize the notion of consequence.

Theorem 3 *Let T be a theory over $\langle I,R\rangle$ in L, let $i \in I$ and φ be a sentence of L_i. Then $T \vdash_i \varphi$ iff $T + (i,\neg\varphi)$ is ugly.*

Proof: (A) We prove first by induction on the form of D that if $T + (i,\neg\varphi)$ contains the ugly diagram D, then $T \vdash_i \varphi$.

Case 1. Suppose D is $(k,\bot)$. If $k \neq i$, then $T_k \vdash_{\mathrm{K}} \bot$, $T \vdash_k \bot$ by (K), $T \vdash_i \bot$ by ($\bot$) and $T \vdash_i \varphi$ by "e falso" in K and (MP). If $k = i$, then $T_i \cup \{\neg\varphi\} \vdash_{\mathrm{K}} \bot$ and $T \vdash_i \varphi$ follows easily.

Case 2. Suppose D is the elementary diagram

$$\begin{array}{c}(k,\Box\alpha)\\ \downarrow \\ (l,\neg\alpha)\end{array}$$

with kRl, $k \neq l$.

Case 2.1. If $i \notin \{k,l\}$, then $T_k \vdash_{K} \Box\alpha$ and $T_l \vdash_{K} \neg\alpha$, $T \vdash_{l} \bot$ by ($\Box$), $T \vdash_{i} \bot$ by ($\bot$) and $T \vdash_{i} \varphi$ by "e falso" in K and (MP).

Case 2.2. If $i = k$, then $T_i \cup \{\neg\varphi\} \vdash_{K} \Box\alpha$ and $T \vdash_{K} \neg\alpha$. From the first we get $T_i \vdash_{K} \varphi \vee \Box\alpha$ and from the second $T \vdash_{i} \Diamond\neg\alpha$ by ($\Diamond$). From this, $T \vdash_{i} \varphi$ will easily follow.

Case 2.3. If $i = l$, then $T_k \vdash_{K} \Box\alpha$ and $T_i \cup \{\neg\varphi\} \vdash_{K} \neg\alpha$. From the first we get $T \vdash_{i} \alpha$ by ($\Box'$) and from the second $T_i \vdash_{K} \varphi \vee \neg\alpha$. From this, $T \vdash_{i} \varphi$ will easily follow.

Case 3. Suppose D is the elementary diagram $(k, \Box\alpha \wedge \neg\alpha)$ with kRk.

Case, 3.1. If $i \neq k$, then $T_k \vdash_{K} \Box\alpha \wedge \neg\alpha$, $T_k \vdash_{K} \Box\alpha$ and $T_k \vdash_{K} \neg\alpha$, $T \vdash_{k} \bot$ by ($\Box$), $T \vdash_{i} \bot$ by ($\bot$) and $T \vdash_{i} \varphi$ by "e falso" in K and (MP).

Case 3.2. If $i = k$, then $T_i \cup \{\neg\varphi\} \vdash_{K} \Box\alpha \wedge \neg\alpha$. (1)

Let $T' = T + (i, \neg\varphi)$; then $T' \vdash_{i} \Box\alpha$, $T' \vdash_{i} \alpha$ by ($\Box'$), $T' \vdash_{i} \bot$ by (1) and arguments already used, $T \vdash_{i} \varphi$ by ($\neg$).

Case 4. Suppose $T + (i, \neg\varphi)$ contains $D' \underset{k}{\vee} D''$ where D' and D'' are ugly.

Case 4.1. If $k \neq i$, we have

$$\text{in } k : T_k \vdash_{K} e_k^{D'} \vee e_k^{D''} \tag{2}$$

$$\text{in } i : T_i \cup \{\neg\varphi\} \vdash_{K} e_i^{D'} \wedge e_i^{D''}$$

$$\text{in } l \notin \{i,k\} : T_l \vdash_{K} e_l^{D'} \wedge e_l^{D''}.$$

Consider the theories $T' = T + (k, e_k^{D'}) + (i, \neg\varphi)$ and $T'' = T + (k, e_k^{D''}) + (i, \neg\varphi)$. Clearly T' contains D', hence by inductive hypothesis

$$T + (k, e_k^{D'}) \vdash_{i} \varphi. \tag{3}$$

Similarly, T'' contains D'' and

$$T + (k, e_k^{D''}) \vdash_{i} \varphi. \tag{4}$$

It suffices to apply ($\vee'$) to (2), (3) and (4) to obtain $T \vdash_{i} \varphi$, the desired result.

Case 4.2. If $k = i$, we have

$$\text{in } k = i : T_k \cup \{\neg\varphi\} \vdash_{i} e_k^{D'} \vee e_k^{D''}, \text{ i.e. } T_k \vdash_{K} \varphi \vee e_k^{D'} \vee e_k^{D''} \tag{5}$$

$$\text{in } l \neq k : T_l \vdash_{K} e_l^{D'} \wedge e_l^{D''}.$$

Consider the theories $T' = T + (k, e_k^{D'}) + (k, \neg\varphi)$ and $T'' = T + (k, e_k^{D''}) + (k, \neg\varphi)$. Clearly T' contains D', hence by inductive hypothesis:

$$T + (k, e_k^{D'}) \vdash_{k} \varphi. \tag{6}$$

Similarly, T'' contains D'' and

$$T + (k, e_k^{D''}) \vdash_{k} \varphi. \tag{7}$$

Finally,

$$T + (k, \varphi) \vdash_{k} \varphi, \tag{8}$$

and it suffices to apply an immediate extension of (v′) to (5), (6), (7) and (8) to obtain $T \vdash_{\bar{K}} \varphi$, the desired result.

Case 5. Suppose $T + (i, \neg\varphi)$ contains $\exists_k x D(x)$ for some ugly $D(c)$ in $L + (k, c)$.

Case 5.1. If $k \neq i$, we have:

$$\text{in } k: T_k \vdash_{\mathrm{K}} \exists x e_k^D(x) \tag{9}$$

$$\text{in } i: T_i \cup \{\neg\varphi\} \vdash_{\mathrm{K}} \forall x e_i^D(x)$$

$$\text{in } l \notin \{i,k\}: T_l \vdash_{\mathrm{K}} \forall x e_l^D(x).$$

Consider in $L + (k,c)$ the theory $T' = T + (k, e_k^D(c)) + (i, \neg\varphi)$. Clearly T' contains $D(c)$, hence by the inductive hypothesis

$$T + (k, e_k^D(c)) \vdash_{\bar{i}} \varphi. \tag{10}$$

It remains to apply (∃′) to (9) and (10) to obtain $T \vdash_{\bar{i}} \varphi$, the desired result.

Case 5.2. If $k = i$, we have:

$$\text{in } k = i: T_k \cup \{\neg\varphi\} \vdash_{\mathrm{K}} \exists x e_k^D(x), \text{ i.e. } T_k \vdash_{\mathrm{K}} \varphi \vee \exists x e_k^D(x) \tag{11}$$

$$\text{in } l \neq k: T_l \vdash_{\mathrm{K}} \forall x e_l^D(x).$$

Consider in $L + (k,c)$ the theory $T' = T + (k, e_k^D(c)) + (i, \neg\varphi)$. Clearly T' contains $D(c)$, hence by the inductive hypothesis

$$T + (k, e_k^D(c)) \vdash_{\bar{k}} \varphi. \tag{12}$$

On the other hand

$$T + (k, \varphi) \vdash_{\bar{k}} \varphi. \tag{13}$$

From (11), (12) and (13) we conclude $T \vdash_{\bar{k}} \varphi$ by an easily established derived rule:

$$\text{if } T_k \vdash_{\mathrm{K}} \varphi \vee \exists x \psi(x),\ T + (k, \psi(c)) \vdash_{\bar{k}} \chi \text{ and } T + (k, \varphi) \vdash_{\bar{k}} \chi \text{ then } T \vdash_{\bar{k}} \chi.$$

(B) We now show how to associate inductively with every proof $T \vdash_{\bar{i}} \varphi$ an ugly diagram D contained in $T + (i, \neg\varphi)$.

(Rule K) Suppose $T_i \vdash_{\mathrm{K}} \varphi$ and we conclude $T \vdash_{\bar{i}} \varphi$. To this use of (K) we associate the ugly diagram $(i, \bot)$ which is clearly contained in $T + (i, \neg\varphi)$.

(Rule □) Suppose $T_i \vdash_{\mathrm{K}} \Box\varphi$, iRj, $T_j \vdash_{\mathrm{K}} \neg\varphi$ and we conclude $T \vdash_{\bar{j}} \bot$. If $i \neq j$, we associate to this use of (□) the ugly diagram

$$\begin{array}{c}(i, \Box\varphi)\\ \downarrow \\ (j, \neg\varphi)\end{array}$$

which is clearly contained in $T + (j, \neg\bot)$. If $i = j$, we associate the ugly diagram $(i, \Box\varphi \wedge \neg\varphi)$ which is also contained in $T + (j, \neg\bot)$, i.e. in $T + (i, \neg\bot)$.

(Rule MP) Suppose $T \vdash_{\bar{i}} \varphi$, $T \vdash_{\bar{i}} \varphi \to \psi$ and we conclude $T \vdash_{\bar{i}} \chi$. The inductive hypothesis is that there exist ugly diagrams D' and D'' such that $T + (i, \neg\varphi)$ contains D' and $T + (i, \neg(\varphi \to \psi))$ contains D''. It is easy to show by reasoning in K only that $T + (i, \neg\psi)$ contains $D \equiv D' \vee_i D''$.

(Rule ⊥) Suppose $T \vdash_{\bar{i}} \bot$ and we conclude $T \vdash_{\bar{K}} \bot$. By inductive hypothesis, there exists an ugly diagram D such that $T + (i, \neg\bot)$ contains D. The same D is obviously contained $T + (k, \neg\bot)$.

(Rule ∨) Suppose $T_i \vdash_{\bar{K}} \varphi \vee \psi$, $T + (i, \varphi) \vdash_{\bar{i}} \chi$ and $T + (i, \psi) \vdash_{\bar{i}} \chi$ and we conclude $T \vdash_{\bar{i}} \chi$. By inductive hypothesis, there are ugly diagrams D' and D'' such that $T + (i, \varphi) + (i, \neg\chi)$ contains D' and $T + (i, \psi) + (i, \neg\chi)$ contains D''. It is easy to show that $T + (i, \neg\chi)$ contains $D \equiv D' \underset{i}{\vee} D''$.

(Rule ∃) Suppose $T_i \vdash_{\bar{K}} \exists x\, \varphi(x)$, $T + (i, \varphi(c)) \vdash_{\bar{i}} \psi$ and we conclude $T \vdash_{\bar{i}} \psi$. By inductive hypothesis $T + (i, \varphi(c)) + (i, \neg\psi)$, which is in $L + (i, c)$ contains some ugly $D(c)$ in the same language. Using the theorem on constants in K it is easy to show that $T + (i, \neg\psi)$ contains the ugly $D = \exists_i x\, D(x)$.

Theorem 3 shows that ugly diagrams are really a substitute for "contradiction". We illustrate this by giving the proof of a deduction theorem.

Corollary 4 *If $T + (i, \varphi) \vdash_{\bar{i}} \psi$, then $T \vdash_{\bar{i}} \varphi \to \psi$.*

Proof: If $T + (i, \varphi) \vdash_{\bar{i}} \psi$, then, by Theorem 3, $T + (i, \varphi) + (i, \neg\psi)$ contains an ugly D, $T + (i, \neg(\varphi \to \psi))$ contains D, and $T \vdash_{\bar{i}} \varphi \to \psi$ by Theorem 3.

3 Completeness

The soundness theorem (Theorem 2) and the characterization of consequence contained in Theorem 3 give:

Proposition 5 *If T has a model, then T is nice.*

Proof: If T is ugly, then by Theorem 3, T contains some ugly diagram D, $T + (i, \neg\bot)$ contains D, $T \vdash_{\bar{i}} \bot$, $T \models_{\bar{i}} \bot$ by Theorem 2, hence finally T has no model.

We now proceed to prove the converse. Let L be a language over $\langle I, R\rangle$ and let T be a theory over $\langle I, R\rangle$ in L. Suppose T is nice and find a model M of it extended over some $\langle J, S\rangle \supseteq \langle I, R\rangle$. We may proceed very classically in Henkin's style with three kinds of elementary steps: (a) adjunction of φ or $\neg\varphi$ to maximalize, (b) adjunction of constants to enrich for ∃, (c) adjunction of new worlds to enrich for ◇. We first deal with these steps separately, then show (d) that each nice theory is contained in a maximal nice, ∃-rich, ◇-rich theory, (e) for which it is easy to construct a model.

(a) *Adjunction of φ or $\neg\varphi$.*

Lemma 6 *Let T be a nice theory over $\langle I, R\rangle$ in L. Let $i \in I$ and φ be a sentence of L_i. Then $T + (i, \varphi)$ or $T + (i, \neg\varphi)$ is nice.*

Proof: If both $T' = T + (i, \varphi)$ and $T'' = T + (i, \neg\varphi)$ are ugly, they contain ugly diagrams D' and D'' respectively. Then clearly T will contain the ugly diagram $D' \underset{i}{\vee} D''$.

(b) *Adjunction of constants.*

Lemma 7 *Let T be a nice theory over $\langle I, R\rangle$ in L. Let $i \in I$, let $\exists x\, \varphi(x)$ be a sentence of L_i and assume $T_i \vdash_{\bar{K}} \exists x \varphi(x)$. Introduce in L_i and in every L_j with iR^*j a new constant c, thus forming the language $L' \equiv L + (i, c)$. Consider in L' the new theory $T' \equiv T + (i, \varphi(c))$. The claim is that T' is nice.*

Proof: Otherwise T' contains an ugly diagram $D(c)$. Then T contains $\exists_i xD(x)$ which is ugly. This is easily seen as follows. In $k \neq i$, $T_k = T'_k \vdash_{\overline{K}} e_k^D(c)$ and since c is new, $T_k \vdash_{\overline{K}} \forall x e_k^D(x)$. In i, $T'_i \equiv T_i \cup \{\varphi(c)\} \vdash_{\overline{K}} e_i^D(c)$, hence $T_i \vdash_{\overline{K}} \varphi(c) \to e_i^D(c)$, $T_i \vdash_{\overline{K}} \forall x(\varphi(x) \to e_i^D(x))$ since c is new, and by the hypothesis $T_i \vdash_{\overline{K}} \exists x\varphi(x)$ we get $T_i \vdash_{\overline{K}} \exists x e_i^D(x)$.

(c) *Adjunction of new worlds.* We need a new construction on diagrams. Let D be a diagram in I' and i^- be a "minimal" element of I' having a unique "predecessor" $i \in I'$ in the sense that iRi^- and for all $j \in I'$, not i^-Rj and $jRi^- \Rightarrow j = i^-$. Define over $I = I' - \{i^-\}$ a diagram D^* by:

$$e_i^{D^*} \equiv e_i^D \wedge \Diamond e_{i^-}^D$$
$$e_j^{D^*} \equiv e_j^D \text{ for } j \neq i.$$

Lemma 8 *If D is ugly (in I'), then D^* contains an ugly diagram (in I).*

Proof: By induction on the form of D. The case $(k,\bot)$ is trivial.
Consider the case

$$\begin{array}{c}(k,\Box\alpha)\\ \downarrow\\ (l,\neg\alpha),\end{array}$$

with kRl, $k \neq l$. If $l \neq i^-$, then D^* is D. If $l = i^-$, then $k = i$ and D^* is $(i,\Box\alpha \wedge \Diamond\neg\alpha)$ which contains $(i,\bot)$.

Consider the case $(k,\Box\alpha \wedge \neg\alpha)$ with kRk. The case $k = i^-$ is excluded, and for $k \neq i^-$, $D^* = D$.

For the case $D = D' \underset{k}{\vee} D''$, one verifies that in each of the cases $k = i$ and $k \notin \{i,i^-\}$, D^* contains $D'^* \underset{k}{\vee} D''^*$, and that for $k = i^-$, D^* contains $D'^* \underset{i}{\vee} D''^*$.

Consider finally the case $\exists_k xD(x)$ with $D(c)$ ugly in $L + (k,c)$. If $k = i^-$, the condition that $D(c)$ is in $L + (i^-,c)$ shows that c has no occurrence in the labels of $D(c)$ other than $e_{i^-}^D(c)$; in fact, as is easily shown by a trivial induction on diagrams, c does not occur either in $e_{i^-}^D(c)$. Consequently, $\exists_{i^-} xD(x)$ is (K-equivalent to) $D(c)$ and $(\exists_{i^-}xD(x))^*$ is $D(c)^*$ which contains an ugly diagram by the induction hypothesis.

For the case $k = i$, we denote by $D_0(c)$ the ugly diagram contained in $D^*(c)$ which is given by the induction hypothesis; we denote by $D_1(c)$ the diagram which has the same labels as $D^*(c)$ except that in i, $e_i^{D_1}(c) \equiv e_i^D(c) \wedge \Diamond\forall x e_{i^-}^D(x)$; then clearly, $D_1(c)$ contains $D^*(c)$ (using $\Diamond\forall x e_{i^-}^D(x) \to \Diamond e_{i^-}^D(c)$); hence $D_1(c)$ contains $D_0(c)$, $\exists_i xD_1(x)$ contains $\exists_i xD_0(x)$ which is ugly by the inductive definition of ugly diagram; it remains to observe that $\exists_i xD_1(x)$ is (K-equivalent to) $(\exists xD(x))^*$: in i, the label of the first is $\exists x e_i^{D_1}(x) \equiv \exists x(e_i^D(x) \wedge \Diamond\forall x e_{i^-}^D(x))$, while the label of the second is $\exists x e_i^D(x) \wedge \Diamond\forall x e_{i^-}^D(x))$; in $k \neq i$, the labels clearly coincide. The case $k \notin \{i,i^-\}$ is handled by observing that $(\exists_k xD(x))^*$ contains $\exists_k xD^*(x)$: in i, the label of the first is $\forall x e_i^D(x) \wedge \Diamond\forall x e_{i^-}^D(x)$ (1), while the label of the second is $\forall x(e_i^D(x) \wedge \Diamond e_{i^-}^D(x))$ (2). And (1) implies (2) because $\Diamond\forall x\varphi(x) \to \forall x\Diamond\varphi(x)$ is a theorem of K; in $l \neq i$, the labels coincide.

Lemma 9 *Let T be a nice theory over $\langle I,R\rangle$ in L. Let $i \in I$, φ be a sentence of L_i and assume $T_i \vdash_{\mathrm{K}} \Diamond\varphi$. Add to I a new element i^- with the only condition iRi^-, thus giving a new set $I' = I \cup \{i^-\}$ and a new relation $R' = R \cup \{\langle i,i^-\rangle\}$ extending $\langle I,R\rangle$. Extend L to L' over $\langle I',R'\rangle$ by letting $L_{i^-} = L_i$. Extend T to T' by letting $T'_{i^-} = \{\varphi\}$. The claim is that T' is nice.*

Proof: Otherwise T' contains some ugly diagram D (over I'). We show that T contains D^*. Since $T'_{i^-} = \{\varphi\}$ and T' contains D, $\{\varphi\} \vdash_{\mathrm{K}} e^D_{i^-}$, hence successively, $\vdash_{\mathrm{K}} \varphi \to e^D_{i^-}$, $\vdash_{\mathrm{K}} \Box(\varphi \to e^D_{i^-})$, $\vdash_{\mathrm{K}} \Diamond\varphi \to \Diamond e^D_{i^-}$, $T_i \vdash_{\mathrm{K}} \Diamond e^D_{i^-}$ (since $T_i \vdash_{\mathrm{K}} \Diamond\varphi$), $T_i \vdash_{\mathrm{K}} e^{D^*}_i$ (since $T_i \vdash_{\mathrm{K}} e^D_i$). For $k \neq i$, it is trivial that $T_k \vdash_{\mathrm{K}} e^{D^*}_k$ since $T_k = T'_k$, $e^{D^*}_k = e^D_k$ and $T'_k \vdash_{\mathrm{K}} e^D_k$.

(d) *Maximalization and enrichment.* The adaptation of the usual definitions of maximality, richness, etc. . . . is easy:

Definition 8 Let L' be a language over $\langle I',R'\rangle$. By $|L'|$, we mean the cardinal $\Sigma_{i\in I'}|Form(L_i)|$. Let T' be a theory over I' in L'.

(1) T' is *maximal* if T' is maximal for inclusion, among nice theories over L'.
(2) T' is *$\exists$-rich* if for every $i \in I'$, every sentence $\exists x\varphi(x) \in L'_i$, $T'_i \vdash_{\mathrm{K}} \exists x\varphi(x)$ implies $T'_i \vdash_{\mathrm{K}} \varphi(c)$ for some constant c of L'_i.
(3) T' is *$\Diamond$-rich* if for every $i \in I'$, every sentence $\Diamond\varphi \in L'_i$, $T'_i \vdash_{\mathrm{K}} \Diamond\varphi$ implies $T'_j \vdash_{\mathrm{K}} \varphi$ for some $j \in I'$ with iRj.

Lemma 10 *Let T be a nice theory over $\langle I,R\rangle$ in L. There exist an extension $\langle I',R'\rangle$ of $\langle I,R\rangle$, a language L' over $\langle I',R'\rangle$ such that L' restricted to $\langle I,R\rangle$ is an enrichment of L by constants and $|L'| \leq |L|$, and there exists a theory T' over $\langle I',R'\rangle$ in L' such that T' restricted to $\langle I,R\rangle$ contains T and T' is maximal, $\exists$-rich, and $\Diamond$-rich.*

Proof: Enumerate all pairs $(i_\xi,\varphi_\xi)_{\xi<\alpha}$ with $i_\xi \in I$ and φ_ξ a sentence of L_i. Construct a chain $(I_\xi,R_\xi,L_\xi,T_\xi)_{\xi<\alpha}$, starting with (I,R,L,T) and define $(I_{\xi+1}, R_{\xi+1}, L_{\xi+1}, T_{\xi+1})$ by considering (i_ξ,φ_ξ). To avoid lengthy definitions, we describe only $T_{\xi+1}$. If $T_\xi + (i_\xi,\neg\varphi_\xi)$ is nice, take $T_{\xi+1} = T_\xi + (i_\xi,\neg\varphi_\xi)$. If $T_\xi + (i_\xi,\neg\varphi_\xi)$ is not nice, then

$$T^* = T_\xi + (i_\xi,\varphi_\xi)$$

is nice (Lemma 6). If φ_ξ is of the form $\exists x\varphi(x)$, make the construction of Lemma 7 and take

$$T_{\xi+1} = (T^* + (i_\xi,\varphi(c)).$$

If φ_ξ is of the form $\Diamond\varphi$, make the construction of Lemma 9 and take

$$T_{\xi+1} = T^* + (i_\xi^-,\varphi).$$

In other cases, take $T_{\xi+1} = T^*$.

For limit steps, we take unions. With this construction, we obtain $(I^{(1)}, R^{(1)}, L^{(1)}, T^{(1)})$ which satisfy (1), (2), and (3) of Definition 8 but only for $i \in I$, sentences of L_i, and provability in T_i. It suffices to perform the construction ω

times to obtain the result. Of course, in limit steps, we use the finite character of ugly diagrams. The cardinality result $|L'| \leq |L|$ follows from the construction.

(e) *Completeness.*

Theorem 11 *Let T be a theory over $\langle I,R\rangle$ in L. If T is nice, then T has a model over some $\langle I',R'\rangle$ extending $\langle I,R\rangle$ with $\Sigma_{i\in I'}|M_i| \leq |L|$.*

Proof: By Lemma 10, it suffices to prove that if T' is maximal, $\exists$-rich, $\Diamond$-rich over $\langle I',R'\rangle$ in L', then T' has a model M over that same I'. Here is the definition of M: $M(i)$ is the quotient of the set of closed terms of L_i by the equivalence $t \sim_i t'$ iff $(t = t') \in T_i$. Interpret the functional symbol f in $M(i)$ by $f^{M(i)}(\bar{t}/\sim_i) = (f\bar{t})/\sim_i$ and the relation symbol r by $r^{M(i)} = \{\bar{t}/\sim_i \,|\, r\bar{t} \in T_i\}$. Define M_{ij} by $M_{ij}(t/\sim_i) = t/\sim_j$. All these definitions make sense and determine a model because we admitted in first-order K the axiom $x = y \to \Box(x = y)$. It is easy to show by the usual inductions that $M \models_i \varphi[\bar{t}/\sim_i]$ iff $\varphi(\bar{t}) \in T_i$, hence the result.

5 Extension to other logics In this section, we present some comments on the proof and on the possibilities of extension. We will confine ourselves to the simplest propositional cases.

1. For the system K itself, it is interesting to remark that if one starts from a binary relation R that is irreflexive or asymmetric or intransitive or is a tree, the construction preserves those properties and the model obtained satisfies them immediately: it is an advantage of the method that it is not necessary to "unravel" the model.

2. The scheme (t) $\Box\varphi \to \varphi$. This corresponds to the semantic condition that R is reflexive. It is easy here to revise the construction replacing everywhere K by Kt. The most natural way to do it is to transform the relation R over I into its reflexive closure $\bar{R} = R \cup \{\langle i,i\rangle | i \in I\}$. The key point is to prove that if T is nice over $\langle I,R\rangle$, then so is T over $\langle I,\bar{R}\rangle$: just look at the new relations iRi. One can also use that fact but apply it only at the end of the construction: the construction gives a maximal $\Diamond$-rich T over some $\langle I,R\rangle$; the same T is maximal $\Diamond$-rich over $\langle I,\bar{R}\rangle$. One could finally also keep the construction as it is: it gives a model M over some $\langle I,R\rangle$; that same model is also a structure $\bar{M}$ over $\langle I,\bar{R}\rangle$ with the property that $M \models_i \varphi$ iff $\bar{M} \models_i \varphi$, the inductive proof of which uses $M \models_i \Box\psi \to \psi$. Whatever the method, we obtain a model over a reflexive R and if the relation one starts with is asymmetric or intransitive, it remains so at the end of the construction.

3. The schemes $\Box\varphi \to \Box\Box\varphi$ and $\varphi \to \Box\Diamond\varphi$. These correspond respectively to the semantic conditions that R is transitive and R is symmetric. One can adapt the foregoing observations replacing R by its transitive or its symmetric closure.

4. The scheme $\Diamond T$. This corresponds to the semantic condition that R is serial: $\forall i \exists j\, iRj$. The construction will immediately ensure this, since for every i, $T_i \vdash \Diamond T$, a point i^- is added in the "adjunction of new worlds".

5. The scheme $\Diamond\Box\varphi \to \Box\Diamond\varphi$. This corresponds to the semantic condition: $\forall i \forall j \forall k (iRj \wedge iRk \to \exists l\,(jRl \wedge kRl))$. It seems much harder here to adapt the construction step by step. We can however obtain the result that every theory T has a model satisfying that condition as follows. The construction gives a T over

$\langle I,R\rangle$ where $\langle I,R\rangle$ is a substructure of the canonical structure $\langle I_c,R_c\rangle$ (iR_cj iff $\{\varphi | \Box\varphi \in T_i\} \subseteq T_j$): if iRj and $\Box\varphi \in T_i$ then $\varphi \in T_j$, for otherwise $\neg\varphi \in T_j$ and T is ugly. By the usual completeness theorem one has a model $\bar{M}$ over $\langle I_c,R_c\rangle$ satisfying the semantic condition and such that:

$$\bar{M} \models_{\bar{k}} \varphi \text{ iff } \varphi \in T_k, \text{ for every } k \in I_c.$$

By our unmodified completeness theorem there is an M with

$$M \models_{\bar{i}} \varphi \text{ iff } \varphi \in T_i, \text{ for every } i \in I.$$

Consequently:

$$\bar{M} \models_{\bar{i}} \varphi \text{ iff } M \models_{\bar{i}} \varphi, \text{ for every } i \in I,$$

and $\bar{M}$ is a model of the original theory satisfying the semantic condition.

6. For other schemes such as $\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$, the indirect argument of the foregoing point seems even more necessary, since the semantic condition corresponding to it is not preserved by union of chains.

Of course, in cases 5 and 6 the result is interesting only from the point of view of the original problem: it gives no new insight into completeness proofs for those extensions of K; one could also say that our proof exhibits conditions ("niceness") for a family $(T_i)_{i\in I}$ of theories being embeddable in the canonical model.

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