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Relevance and Paraconsistency — A New Approach Part II: The Formal Systems

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Abstract In part I of this paper we introduced what we called "relevance structures". These algebraic structures are based on the idea of relevance domains which are graded according to "degrees of reality" and related (or not) by a certain relevance relation. In the present part we describe the logic RMI which corresponds to these structures, proving it to be sound and strongly complete relative to them. The language of RMI is similar to that of the systems of Anderson and Belnap, but unlike them it is purely intensional: no extensional connective is definable in it, and all its primitive binary connectives have the variable-sharing property. We show that the expressive power of RMI is nevertheless very strong and sufficient for all our needs. In addition, we investigate the main fragments of RMI, as well as its most important extensions. One of these extensions is the system RM (of Dunn and McCall), which is obtained from RMI by adding an axiom to the effect that any two sentences are relevant to each other.

Introduction Our central problems in this work are to find out how a use of inconsistent theories is possible, what kinds of logics can be so used, and what are the possible justifications for it. Henceforth, we shall follow da Costa [11] in calling logics that allow inconsistencies "paraconsistent"; that is to say, a paraconsistent logic is one in which an inconsistency does not necessarily imply everything. Besides their obvious philosophical interest, such logics may also play a practical role (see Nillson [17], p. 408), e.g., we may like to have a computerized system for deriving conclusions which can act efficiently even when it is fed with inconsistent information.

The main difficulty any paraconsistent logic has to overcome is what is

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known as the "Lewis dilemma": We can infer any proposition B from a couple of contradictory sentences, A and $\sim A$, by using only two self-evident and elementary rules of inference. First, we infer $A \vee B$ from A, using weakening. Then we apply the disjunctive syllogism (D.S.) in order to infer B from $A \vee B$ and $\sim A$. It follows that no paraconsistent logic can have an operation of disjunction for which both weakening and D.S. are always valid. The universal validity of at least one must be given up.

Which of these two rules is to be rejected? It seems obvious to me that if Lewis's argument does not apply in concrete situations it is because nobody will try to infer $A \vee B$ from A unless he sees a connection between A and B. In contrast, applications of (D.S.) are frequent and indispensable. Accordingly, it seems preferable to retain (D.S.) while limiting the validity of weakening.

Another major logical problem closely connected with that of paraconsistency is the problem of the implication (or "entailment") connective. It has often been claimed that the "material implication" of classical logic does not represent faithfully our "If . . . then" concept. This is due to the so-called "paradoxes of material implication": $A \supset (B \supset A)$, $B \supset (A \supset A)$, and $\sim A \supset (A \supset B)$. From these classical theorems (together with modus ponens) it follows that if A is true then it is implied by every sentence, while if A is false then every sentence is implied by it. This situation does not correspond to our intuitive concept of "implication". A logic which does have a connective which corresponds to our intuitive implication is a logic with *entailment.*

The two problems are related: in any paraconsistent logic in which M.P. is a rule of inference the "paradox" $\sim A \supset (A \supset B)$ should be rejected. Nevertheless, a solution for one problem is not automatically a solution for the other. Thus $A \supset (B \supset A)$ is a theorem of the da Costa paraconsistent logics of [11]. On the other hand the Lewis argument shows that a system containing a disjunction connective for which both weakening and (D.S.) are among the rules of inference cannot be paraconsistent, even if none of the "paradoxes of implication" is a theorem in it (with respect to its official implication connective).

One should note that the properties of a "proper implication", for example, $\rightarrow$, are neither clear nor agreed upon. Curry [10], for example, takes the deduction theorem as a basic property. Its classical formulation is $T, B \vdash A$ iff $T \vdash B \rightarrow A$. Since $A, B \vdash A$ we get $\vdash A \rightarrow (B \rightarrow A)$. For this reason Curry thought that $A \rightarrow (B \rightarrow A)$ should be valid. Others, among them Church [9], Ackerman [1], and later Anderson and Belnap [2], did not accept the general validity of $A \rightarrow (B \rightarrow A)$, because it then follows that a true A is implied by any other B. Therefore, Church [9] proposed another version of the deduction theorem (the so-called relevant deduction theorem) which $\rightarrow$ should satisfy: $\vdash A_1 \rightarrow (A_2 \rightarrow \ldots \rightarrow (A_n \rightarrow B) \ldots)$ iff there is a proof of B from $A_1, \ldots, A_n$ *which uses all the* A_i*'s*. The minimal system for which such a deduction theorem holds was independently found by Church [9] and Moh [16]. In [2] it is called $R_{\rightarrow}$. In this system, e.g., $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$ is still a logical truth (since C can be deduced from $A \rightarrow B$, $B \rightarrow C$, and A by two applications of M.P.) but $A \rightarrow (B \rightarrow A)$ is not.

Now the term "uses" which appears in the formulation of the relevant deduction theorem is not quite clear. The following inductive definition of "uses" is an obvious candidate: Let $B_1, \ldots, B_n$ be a derivation from $T \cup \{A\}$ (T a theory), then A is *used* in deriving B_i if either B_i is identical with A, or if B_i is obtained (through an application of one of the rules of the system) from previous B_j's ($j < i$) and A is used for deriving at least one of those B_j's. This appears to be satisfactory when $A \notin T$, but becomes perhaps problematic if $A \in T$. The upshot is that one can have satisfactory systems based on $\rightarrow$ and $\sim$ (negation) for which a relevant deduction theorem is valid when "use" is defined in some reasonable way. But attempts to include other connectives lead to extremely complicated competing definitions of "use" which lack intuitive appeal.

Later we shall have more to say about this issue. For the moment, let us assume only the following obvious principle: If $T \cup \{A\} \vdash B$ but $T \nvdash B$ then A is used in every proof of B from $T \cup \{A\}$. Suppose moreover that our language contains in addition to the implication connective, $\rightarrow$, a conjunction connective, $\wedge$, such that $A \wedge B \vdash A$, $A \wedge B \vdash B$, and $A, B \vdash A \wedge B$. If the relevant deduction theorem holds then for any sentences A, B, the sentences $A \wedge B \rightarrow A$, $A \wedge B \rightarrow B$, and $A \rightarrow (B \rightarrow A \wedge B)$ must be theorems. But from this we can deduce $A \rightarrow (B \rightarrow A)$ in $R_{\rightarrow}$. A more complex argument (see [2], pp. 233–234) shows that this remains the case even if we replace $A, B \vdash A \wedge B$ (the *adjunction* rule) by $A \rightarrow B$, $A \rightarrow C \vdash A \rightarrow B \wedge C$ (the *relevant adjunction* rule). Hence the existence of a suitable conjunction, the validity of the relevant deduction theorem, and the nonvalidity of $A \rightarrow (B \rightarrow A)$ are incompatible demands.

The classical and relevant deduction theorems are by no means in conflict. If $\supset$ is an implication connective for which the classical deduction theorem holds (and M.P. is a valid rule of inference) then the intuitive meaning of "$T \vdash A \supset B$" may be: either B is derivable from T alone or there is a proof of B from $T \cup \{A\}$ which uses A. Hence, if a logic contains an implication connective $\rightarrow$ for which the relevant deduction theorem holds, and also a disjunction connective $\vee$ for which $A \vdash A \vee B$, $B \vdash A \vee B$, and $A \rightarrow C$, $B \rightarrow C \vdash (A \vee B) \rightarrow C$ are valid rules of inference, then we can define $A \supset B$ as $(A \rightarrow B) \vee B$. It is then easy to see that M.P. is valid for this $\supset$ and that the classical deduction theorem holds for it. (In particular, every purely implicational theorem of the intuitionistic logic, including $A \supset (B \supset A)$, is true for $\supset$.)

All the systems which we construct in this paper contain a relevant implication $\rightarrow$ such that the relevant deduction theorem holds for their negation-implication fragments (according to the simplest most intuitive definition of "use"). The relevant deduction theorem fails when we add a relevant conjunction, and the above-mentioned observations show why it *must* fail. However, if we define $A \supset B$ as $(A \rightarrow B) \vee B$ (where $B \vee C =_{def} \sim(\sim B \wedge \sim C)$) the classical deduction theorem as well as a strong version of the interpolation theorem hold for $\supset$ (our $\vee$ satisfies the above-mentioned required properties). In view of the discussion of the present section, these seem to be the best possible results concerning the entailment problem.

So far the best known attempts to solve simultaneously the problems of paraconsistency and of relevant implication have been those of the Anderson and Belnap (A & B) school. Their systems are usually known as "relevance logics", and the most important among them are E and R. A detailed presentation of these systems and of the general A & B attitude can be found in [2] and [13].

Essentially, A & B logics are a combination of two components, one relevant and the other extensional. This combination seems to us unfortunate (see the discussion above) and is the source of many paradoxical or unintuitive features of R and E (as we shall show elsewhere). Beyond these paradoxical results, R and E seem to us unsatisfactory also from the following points of view:

(a) They are undecidable (this was recently shown in Urquhart [20]).
(b) They have no simple *intuitive* semantics.[1]
(c) The fundamental concepts underlying the systems R and E (like that of "relevance") have never been clearly defined or explained, and they have neither a semantical nor a syntactical explication.
(d) A major shortcoming of R and E is the total rejection of (D.S.). Dunn admits in [13] that this is the point at which he "loses his audience", which is quite understandable. The arguments of A & B for the a priori rejection of (D.S.) are far from convincing. They admit themselves that there are many cases in practice in which (D.S.) is correctly applied but claim that such applications involve a "relevant" disjunction, not an extensional one. Nevertheless, there is no real counterpart of such a disjunction in their systems. (For a comprehensive discussion of this topic see [8].)

The previous discussion naturally leads us to the following desiderata which, in our view, should be satisfied by any logic L which is to provide an adequate solution to our two problems:

(I) There should be a *proper* provability relation $\vdash_L$ for L. By this we mean that:
  (i) If $A \in T$ then $T \vdash_L A$
  (ii) $T \vdash_L A$ iff there is a finite (possibly empty) subset S of T such that $S \vdash_L A$
  (iii) If $T \cup \{B_1, \ldots, B_n\} \vdash_L A$ and $T \vdash_L B_i$ $(i = 1, \ldots, n)$ then $T \vdash_L A$
  (iv) If $A_1, \ldots, A_n \vdash_L B$ then $A_1', \ldots, A_n' \vdash_L B'$ whenever $A_1', \ldots, A_n', B'$ are obtained from $A_1, \ldots, A_n, B$ respectively by substitutions of formulas for variables.[2]
(II) L should have a negation connective $\sim$ such that p and $\sim\sim p$ are equivalent, but $p, \sim p \nvdash_L q$ if p and q are distinct sentential variables.
(III) L should have a (primitive or defined) implication connective, for which an appropriate version of the deduction theorem holds.
(IV) L should have a (primitive or defined) disjunction connective for which (D.S.) is valid (and hence, by (II), for which weakening is not).
(V) L should be sufficiently simple from a proof-theoretic point of view. (In particular, we want L to be decidable, on a propositional-logic level, and to have a cut-free Gentzen-type formulation.[3])
(VI) L should have a transparent semantics, such that soundness and completeness are obtained.
(VII) The semantics should be based on intuitively clear concepts. It is also desirable that these concepts be expressible in L as much as possible.
(VIII) L should be *syntactically complete*, i.e., $T \vdash_L A$ iff $T_0 \vdash_L A$ for any complete extension T_0 of T (a theory T_0 is complete if for every sentence B either $T \vdash B$ or $T \vdash \sim B$).

We shall prove that the system RMI, which we introduce below, has all the properties in this list (except for the Gentzen-type formulation, to which a separate paper will be devoted). We also show that its language is *purely intensional* (i.e., no nontrivial extensional connectives are definable in it), and that it has a natural multiple-conclusioned version which is strictly stronger in its expressive power than the single-conclusioned version (although it is a conservative extension of it).

Another important feature of our logic is the central role of the relevant disjunction, $+$, with respect to which (D.S.) is true (i.e., $\sim A, A + B$ imply B). Although in our systems $+$ and the relevant implication $\rightarrow$ are definable in terms of each other exactly as they are in the A & B systems (i.e., $A + B = \sim A \rightarrow B$), there are some reasons for choosing $+$ as the primitive, rather than $\rightarrow$:

(1) Its semantic interpretation is more transparent and has nice algebraic properties.
(2) As we have seen above, $\rightarrow$ loses part of its intended meaning in the presence of a conjunction connective (i.e., the relevant deduction theorem fails). This is not true for $+$.
(3) Disjunction plays a crucial role in Lewis's dilemma. It is therefore natural to treat the paraconsistency problem by introducing a relevant disjunction for which (D.S.) holds but weakening does not, and then to proceed to develop and investigate the corresponding system(s).

In addition to (D.S.) and the nonvalidity of weakening, it seems reasonable for us to demand from $+$ the following:

(a) The variable-sharing property (which $\rightarrow$ has in R): $A + B$ should not be a theorem unless A and B share a variable in common.
(b) The truth of at least one of the two disjuncts should be a necessary condition for the truth of $A + B$, or expressed syntactically, we should have: $T, A \vdash_L C$ and $T, B \vdash_L C$ imply $T, A + B \vdash_L C$.
(c) $+$ should be commutative, associative, and idempotent.[4] Syntactically this means, among other things, that we can replace $A + B$ by $B + A$ everywhere in any deduction (or that $\vdash_L (A + B) \leftrightarrow (B + A)$ in case $\leftrightarrow$ is a proper equivalence connective of L).
(d) Inasmuch as our attitude towards disjunction and negation is nonintuitionistic, $\sim A + A$ should be a logical truth. (It seems to me obvious that there is no question about the relevance between the two disjuncts in this case!)
(e) In view of (d), it is reasonable to extend (D.S.) to the *cut* rule: from $A + B$ and $\sim A + C$ to infer $B + C$.

Again, the connective $+$ of RMI has all the properties listed above.

Semantically, RMI corresponds to the algebraic relevance structures which were developed in Avron [7], and it is based on the intuitive ideas which underlie those structures. This includes the idea of Relevance domains, which are graded according to "degrees of reality", and the relevance relation between such domains. For the reader's convenience we end this introduction with a review of the main notions of [7], which we are going to use below.

A *relevance disjunction structure* (r.d.s.) is a structure $\langle D, \leq, \sim, + \rangle$ in which

$\langle D,\leq\rangle$ is a poset, $\sim$ is an involution on this poset and $+$ is an associative, commutative, idempotent, and order-preserving operation on $\langle D,\leq\rangle$ which satisfies the following condition **(D.S.)**: $a \leq b + c \Rightarrow \sim b \leq \sim a + c$. If in addition any two elements a, b of D have a greatest lower bound $a \wedge b$, then $\boldsymbol{D}$ is called a *relevance disjunction lattice* (r.d.l.).

The most important property of these structures is that every r.d.s. D has a unique subset $\boldsymbol{T_D}$ (the truth subset of D) such that for every a, b in D, $a \leq b$ iff $a \to b \in T_D$, where $a \to b = \sim a + b$. So we have:

$$T_D = \{a \mid \sim a \leq a\} = \{a \mid \sim a + a \leq a\}$$
$$= \{a \mid \sim a + a = a\} = \{a \mid \exists c \sim c + c \leq a\}.$$

The *relevance domain* of an element a in an r.d.s. D is the set $|a| = \{c \mid \sim c + c = \sim a + a\}$. $|a|$ is closed under $\sim$ and $+$, and unless it is degenerate (i.e., a singleton) it forms a Boolean algebra relative to $\langle +,\sim,\leq\rangle$. In case D is also an r.d.l. $|a|$ is closed also under $\wedge$ and is a Boolean algebra relative to $\langle +,\wedge,\sim,\leq\rangle$. Every relevance domain $|a|$ has a unique representative in T_D, namely, $\sim a + a$. Hence the partial order $\leq$ on T_D induces a partial order (the "grading" relation) on the set of relevance domains. This makes this set an upper semilattice in which degenerate domains are necessarily minimal.

Two elements a, b of an r.d.s. D are called *relevant* to each other (notation: aRb) if $(\sim a + a) + (\sim b + b) \in T_D$. The relation R is reflexive and symmetric but not necessarily transitive. We have that if $|a| = |b|$ then aRc iff bRc and that if $a \not R b$ then $a + b \notin T_D$. In an r.d.l. $a \wedge b \in T_D$ if $a \in T_D$, $b \in T_D$, and aRb.

We call an r.d.s. D *full* if for every $a \in D$, either $a \in T_D$ or $\sim a \in T_D$. A full r.d.s. D is necessarily an r.d.l., and its relevance domains form a tree under $\leq$ (where by a *tree* we mean here an upper semilattice in which the subsets $\{x \mid x \geq a\}$ are totally ordered). Each nondegenerate relevance domain is then a two-valued Boolean algebra. Moreover, if $|a| < |b|$ then $a + b = b$ while if $a \not R b$ then $a + b = a \wedge b$ and $\sim(a + b) \in T_D$.

The above properties completely characterize full r.d.l.'s. To make this statement precise, proceed as follows: Let $\langle T,\leq\rangle$ be a tree, and let t,f,I be three objects referred to below as the basic truth values. Order them by $<$ so that f $<$ I $<$ t and define $\sim$t = f, $\sim$f = t, $\sim$I = I. Define next a structure $\langle D,\leq,\sim,+\rangle$ as follows: (1) $D \subseteq T \times \{\mathrm{t,f,I}\}$ and for every $a \in T$ either $\{(a,\mathrm{t}),(a,\mathrm{f})\} \subseteq D$ or $(a,\mathrm{I}) \in D$ but not both; (2) If $(a,\mathrm{I}) \in D$ then a is minimal in T; (3) $(a,v_1) \leq (b,v_2)$ iff either $a = b$ and $v_1 \leq v_2$ or $a \leq b$ and $v_2 = \mathrm{t}$ or $b \leq a$ and $v_1 = \mathrm{f}$; (4) $\sim(a,v) = (a,\sim v)$; (5) $(a,v_1) + (b,v_2)$ is $(a,\sup_{\leq}\{v_1,v_2\})$ if $a = b$, (b,v_2) if $a < b$, (a,v_1) if $b < a$ and $(\sup_{\leq}\{a,b\},\mathrm{f})$ otherwise. The resulting structure is a full r.d.s. in which $(a,v) \in T_D$ if $v \in \{\mathrm{t,I}\}$. Conversely, every full r.d.s. D is isomorphic to a unique structure which is based on T_D according to the above method.

A particularly important r.d.s. is $\boldsymbol{A}_\omega$ which is obtained by the above construction if we start with a two-leveled, infinite tree and make all its minimal elements degenerate. We denote the elements of A_ω by T,F,I_1,I_2,I_3,.... Obviously $\sim\mathrm{T} = \mathrm{F}$, $\sim\mathrm{F} = \mathrm{T}$, $\sim\mathrm{I}_n = \mathrm{I}_n$, $\mathrm{F} < \mathrm{I}_n < \mathrm{T}$, $\mathrm{T} + x = x + \mathrm{T} = \mathrm{T}$, $x + \mathrm{F} = \mathrm{F} + x = \mathrm{F}$ if $x \neq \mathrm{T}$, $\mathrm{I}_i + \mathrm{I}_i = \mathrm{I}_i$ while $\mathrm{I}_i + \mathrm{I}_j = \mathrm{F}$ if $i \neq j$. A_ω has among r.d.s.'s the same role the two-valued Boolean algebra has among Boolean algebras, i.e., it is a polynomially free r.d.s. A more complex structure, called in Avron [7] *the canonical structure*, is a polynomially free r.d.l. This canonical structure differs from the Sugihara matrix by having, for each $n > 0$, a denumerable set of values I_n^k so that $\sim\mathrm{I}_n^k = \mathrm{I}_n^k$ and $-m < \mathrm{I}_n^k < m$ iff $m \geq n$.

The canonical structure M is a *prime* r.d.l. This means that $a \vee b \in T_M$ iff $a \in T_n$ or $b \in T_M$ (and so $\vee$ behaves like an extensional "or" relative to M). Every prime r.d.l. is necessarily full.

A The system **RMI**$_{\rightleftarrows}$ In this section we introduce and investigate the minimal formal system which meets the demands we have set forth in the introduction. Its language contains exactly those connectives that were explicitly required. It has been convenient to use $+$ (the relevant disjunction) as a basic operation of the algebraic structures of [7]. Historically, however, relevant logic has been approached mostly from its syntactical aspect, and there $\rightarrow$ has been taken as primitive. In order to relate our formal systems to those that have been investigated by Anderson–Belnap et al., we start by presenting a system based on $\rightarrow$ and $\sim$. Then we shall present an equivalent formulation in terms of $+$ and $\sim$.

A.1 The system RMI$_{\rightleftarrows}$ (first formulation)

Primitive connectives $\sim, \rightarrow$

Defined Connectives $A + B =_{def} \sim A \rightarrow B$

$A \circ B =_{def} \sim(A \rightarrow \sim B)$

Axioms IM.1 $A \rightarrow (A \rightarrow A)$

IM.2 $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$

IM.3 $A \rightarrow (B \rightarrow C) \rightarrow (B \rightarrow (A \rightarrow C)$

IM.4 $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$

N1 $\sim\sim A \rightarrow A$

N2 $(A \rightarrow \sim B) \rightarrow (B \rightarrow \sim A)$

Rule of Inference $A, A \rightarrow B \vdash B.$

A.2 Historical background If we replace the so-called mingle axiom IM.1 by the weaker IM.1′, $A \rightarrow A$, we get the system $R_{\rightleftarrows}$ proposed by A & B. In this system, as well as in its subsystem $R_{\rightarrow}$ (whose axioms are IM.1′ and IM2–IM4) the following form of the relevant deduction theorem holds: $\vdash A_1 \rightarrow (A_2 \rightarrow \ldots \rightarrow (A_n \rightarrow B) \ldots)$ iff there is a proof of B from $A_1, \ldots, A_n$ which uses all members of the *multiset* $[A_1, \ldots, A_n]$. In such a multiset a wff may occur more than once and the definition of "uses" requires that each occurrence be used separately at least once in the proof.

Dunn and McCall sought systems satisfying a stronger form of the relevant deduction theorem, obtained by replacing "uses all members of the multiset $[A_1, \ldots, A_n]$" by "uses all members of the set $\{A_1, \ldots, A_n\}$" (we agree with them that this indeed is the more natural notion). Now, from the one-member sequence (A) we can deduce (by definition) A, hence by the first form of the relevant deduction theorem $A \rightarrow A$ should be valid. But since $\{A\} = \{A, A\}$ this is also a proof of A from $\{A,A\}$ and so by the strong relevant deduction theorem $A \to (A \to A)$ should be valid. This has led Dunn and McCall to replace IM.1′ by the stronger IM.1. The system obtained from $R_{\to}$ in this way is $RMO_{\to}$, and they have shown that in every extension of it which is closed under substitutions and which has M.P. for $\to$ as the sole rule of inference the strong relevant deduction theorem holds (this includes $RMI_{\overset{\sim}{\to}}$).

By the relevant deduction theorem $RMI_{\overset{\sim}{\to}}$ clearly satisfies demands (I)–(III) set forth in the introduction. We now show that the defined connective $+$ of $RMI_{\overset{\sim}{\to}}$ satisfies condition (IV), including all the subdemands (a)–(e) appearing in that section. For this purpose it would be convenient to give $RMI_{\overset{\sim}{\to}}$ a new formulation, $RMI_{\overset{\sim}{+}}$, in which $+$ is taken as the primitive connective (instead of $\to$) and which mirrors the subdemands (c)–(e). The equivalence of $RMI_{\overset{\sim}{+}}$ and $RMI_{\overset{\sim}{\to}}$ makes it clear that every system with a proper relevant disjunction should be an extension of $RMI_{\overset{\sim}{\to}}$.

A.3 The system $RMI_{\overset{\sim}{+}}$ (second formulation of $RMI_{\overset{\sim}{\to}}$)

Primitive Connectives $\sim, +$
Defined Connectives $A \to B =_{def} \sim A + B$
$A \circ B =_{def} \sim(\sim A + \sim B)$
Axioms $\sim A + A$ (excluded middle)
Rules
(1) The structural rule **(S)**: $\phi(A_1,\ldots,A_n) \vdash \psi(A_1,\ldots,A_n)$ whenever $\phi(A_1,\ldots,A_n)$ and $\psi(A_1,\ldots,A_n)$ are two sentences obtained from $A_1,\ldots,A_n$ using only the connective $+$ and in which all the A_i's occur. (Example: $((A_1 + A_2) + (A_2 + A_3)) + (A_3 + (A_2 + A_1)))$)
(2) The cut rule **(cut)**:

$$A + B, \sim A + C \vdash B + C.$$

A.4 Notes
(1) We can decompose (S) into the following six elementary rules:

$$\frac{A}{A+A}, \quad \frac{A+A}{A}, \quad \frac{C+(A+A)}{C+A}, \quad \frac{C+(A+B)}{C+(B+A)},$$

$$\frac{D+(A+(B+C))}{D+((A+B)+C)}, \quad \frac{D+((A+B)+C)}{D+(A+(B+C))}.$$

The proof of this fact starts by showing that $B + A$ follows from $A + B$ using $\sim A + A$ and cut. Other details are easy.
(2) From the relevant disjunction point of view what distinguishes $RMI_{\overset{\sim}{\to}}$ from the weaker $R_{\overset{\sim}{\to}}$ is the idempotency of $+$: The rule $\dfrac{A}{A+A}$ is not valid in $R_{\overset{\sim}{\to}}$, only its converse is. In fact, the mingle axiom IM.1 is equivalent to the schema $A \to (A + A)$.

A.5 Theorem *The two formulations of $RMI_{\overset{\sim}{\to}}$ are strongly equivalent (i.e., $T \vdash_{RMI_{\overset{\sim}{\to}}} A$ iff $T \vdash_{RMI_{\overset{\sim}{+}}} A$).*

Proof: To prove that $RMI_{\overset{\sim}{\to}}$ includes $RMI_{\overset{\sim}{+}}$ we show that $RMI_{\overset{\sim}{\to}}$ is closed under substitutions of equivalences, i.e., that $A \to B,\ B \to A \vdash_{RMI_{\overset{\sim}{\to}}} \phi(A) \to \phi(B)$ (where $\phi(B)$ is obtained from $\phi(A)$ by replacing some occurrences of A by B). Then we show that in $RMI_{\overset{\sim}{\to}}$ $A + B$ is equivalent to $B + A$, $A + (B + C)$ to $(A + B) + C$, and $A + A$ to A and that $\vdash_{RMI_{\overset{\sim}{\to}}} (A + B) \to ((\sim A + C) \to (B + C))$, (except for the provability of $A \to (A + A)$ these are all properties of $R_{\overset{\sim}{\to}}$ too). From this the implication $T \vdash_{RMI_{\overset{\sim}{+}}} A \Rightarrow T \vdash_{RMI_{\overset{\sim}{\to}}} A$ follows immediately, where T is any set of wff's.

For the converse we show that the relevant deduction theorem (for $\to$) of $RMI_{\overset{\sim}{\to}}$ holds in $RML_{\overset{\sim}{+}}$. The proof is by a straightforward induction on lengths of proofs.[5] Using this, IM.1–IM.4 and N2 follow easily. N1 is the result of applying cut to the axioms: $\sim\sim\sim A + \sim\sim A$ and $\sim A + A$. It remains to show that any $RMI_{\overset{\sim}{+}}$ theory is closed under M.P. for $\to$, i.e., $A,\ \sim A + B \vdash_{RMI_{\overset{\sim}{+}}} B$. To obtain this, infer from A (by (S)) $A + A$. A cut of this and of $\sim A + B$ then yields $A + B$. Another cut, this time of $A + B$ and $\sim A + B$, gives $B + B$ and then we infer B.

We turn next to completeness theorems.

A.6 Theorem *$RMI_{\overset{\sim}{\to}}$ is strongly sound and complete relative to r.d.s.'es (i.e., $T \vdash_{RMI_{\overset{\sim}{\to}}} A$ iff $v(A) \in T_D$ whenever D is an r.d.s. and v is a valuation in D such that $v(B) \in T_D$ for every $B \in T$).*

Proof: For the soundness part we show that if T is a theory in the language of $RMI_{\overset{\sim}{\to}}$, $T \vdash_{RMI_{\overset{\sim}{\to}}} A$, $\boldsymbol{D}$ is an r.d.s., and $(\boldsymbol{D}, v)$ is a model of T, then A is true in $(\boldsymbol{D}, v)$ (i.e., $v(A) \in T_D$). We prove this by induction on the length of a given proof in $RMI_{\overset{\sim}{+}}$ of A from T. (The facts we need concerning T_D are reviewed in the introduction above and proved in I.8 of [7].) If $A \in T$ then A is true in $(\boldsymbol{D}, v)$. If A is an axiom of $RMI_{\overset{\sim}{+}}$ (i.e., $A = \sim B + B$ for some B) then $v(A) = \sim v(B) + v(B)$, which belongs to T_D. If A results from B by (S) then $v(A) = v(B)$ and we apply the induction hypothesis to B. Finally, if A follows from two previous theorems by a cut, then in view of the induction hypothesis it suffices to show that if $a + b \in T_D$ and $\sim a + c \in T_D$ then $b + c \in T_D$. But if $a + b \in T_D$ and $\sim a + c \in T_D$ then $\sim a \le b$, $a \le c$ and so $\sim a + a \le b + c$. Hence $b + c \in T_D$.

To get strong completeness, suppose that $T \nvdash_{RMI_{\overset{\sim}{\to}}} \phi$. We construct the Lindenbaum algebra of T in the usual manner: First we define $C \sim_T B$ iff both $C \to B$ and $B \to C$ are theorems of T. $\sim_T$ is a congruence relation. Let D be the set of equivalence classes of $\sim_T$. We define in D: $[C] \le [B]$ iff $T \vdash_{RMI_{\overset{\sim}{\to}}} C \to B$, $\sim[C] = [\sim C]$, and $[C] + [D] = [C + D]$. It is not difficult to see that $\sim$, $+$, and $\le$ are well-defined and that $\boldsymbol{D} = \langle D, \le, \sim, + \rangle$ is an r.d.s. Moreover, in D, $[A] \in T_D \Leftrightarrow \sim[A] \le [A] \Leftrightarrow T \vdash_{RMI_{\overset{\sim}{\to}}} \sim A \to A \Leftrightarrow T \vdash A + A \Leftrightarrow T \vdash_{RMI_{\overset{\sim}{\to}}} A$. By defining $v(C) = C$ we get therefore a model $(\boldsymbol{D}, v)$ of T in which ϕ is not true.

A.7 Corollary *$T \vdash_{RMI_{\overset{\sim}{\to}}} A \to B$ iff $v(A) \le v(B)$ in every model $(\boldsymbol{D}, v)$ of T where $\boldsymbol{D}$ is an r.d.s.*

A.8 Theorem *$RMI_{\overset{\sim}{\to}}$ is strongly complete for full r.d.s.'s.*

Proof: The soundness part follows from Theorem A.6.

For the converse, suppose that $T \nvdash_{RMI_{\overset{\sim}{\to}}} \phi$. By Theorem A.6 there exists an r.d.s. $\boldsymbol{D}$ and a valuation v in D such that $(\boldsymbol{D}, v)$ is a model of T and $v(\phi) \notin$

T_D. II.20 of [7] then entails that there exists a full r.d.s. $\boldsymbol{M}$ and a homomorphism h of $\boldsymbol{D}$ on $\boldsymbol{M}$ such that $h(v(\phi)) \notin T_M$. On the other hand $h(v(A)) \in T_M$ for every $A \in T$, since $a \in T_D \Rightarrow h(a) \in T_M$. Obviously $h \circ v$ is a valuation in $\boldsymbol{M}$, and $(\boldsymbol{M}, h \circ v)$ is a model of T which is not a model of ϕ.

Definition We call a model $(\boldsymbol{M}, v)$, in which M is a full r.d.s., a *full model.*

A.9 Corollaries[6]

(1) *$RMI_{\overrightarrow{=}}$ is syntactically complete (i.e., every $RMI_{\overrightarrow{=}}$-theory is the intersection of the sets of theorems of all its complete extensions—see the introduction, demand (VIII))*

(2) $P, {\sim}P \nvdash_{RMI_{\overrightarrow{=}}} Q$

(3) *If $T, A \vdash_{RMI_{\overrightarrow{=}}} C$ and $T, B \vdash_{RMI_{\overrightarrow{=}}} C$ then $T, A + B \vdash_{RMI_{\overrightarrow{=}}} C$.*

Proof: (1) Let T be an $RMI_{\overrightarrow{=}}$-theory and suppose that $T \nvdash_{RMI_{\overrightarrow{=}}} A$. By Theorem A.8 there exists a full r.d.s. $\boldsymbol{M}$ and a valuation v such that $(\boldsymbol{M}, v)$ is a model of T and $v(A) \notin T_M$. Let $T' = \{B \mid v(B) \in T_M\}$. Then $T \subseteq T'$, T' is complete, every theorem of T belongs to T', and $T' \nvdash_{RMI_{\overrightarrow{=}}} A$. Hence A is not in the intersection of the complete extensions of T.

(2) Assume in A_ω that $v(P) = \mathrm{I}_1$ and $v(Q) = \mathrm{F}$. Then $(\boldsymbol{A}_\omega, v)$ is a model of $\{{\sim}P, P\}$ which is not a model of Q.

(3) Suppose that $(\boldsymbol{M}, v)$ is a full model of $T \cup \{A + B\}$. Then either A or B is true in $(\boldsymbol{M}, v)$. Since both $T, A \vdash_{RMI_{\overrightarrow{=}}} C$ and $T, B \vdash_{RMI_{\overrightarrow{=}}} C$, it follows that C is true in $(\boldsymbol{M}, v)$. Hence C is true in any full model of $T \cup \{A + B\}$ and so $T, A + B \vdash_{RMI_{\overrightarrow{=}}} C$ (by Theorem A.8).

A.10 Theorem *$RMI_{\overrightarrow{=}}$ is finitely strongly complete relative to finite full r.d.s.'s.*

Proof: Suppose that $B_1, \ldots, B_n \nvdash A$. We show that there exists a finite full model $(\boldsymbol{M}, v)$ of $B_1, \ldots, B_n$ which is not a model of A. By Theorem A.8 there is a full model, say $(\boldsymbol{M}', v')$, for $B_1, \ldots, B_n$ which is not a model of A. Let $P_1, \ldots, P_k$ be all the sentential variables occurring in $\{B_1, \ldots, B_n, A\}$. Let $\boldsymbol{M}$ be the sub-full r.d.s. of $\boldsymbol{M}'$ generated by $\{v(P_1) \ldots v(P_k)\}$. By II.25 of [7] $\boldsymbol{M}$ is a finite full r.d.s. Let v be the restriction of v' to M. Obviously, $(\boldsymbol{M}, v)$ is a model of the kind requested.

A.11 Theorem[7] *$\boldsymbol{A}_\omega$ is a characteristic matrix for $RMI_{\overrightarrow{=}}$.*

Proof: By Theorem A.6 $\vdash_{RMI_{\overrightarrow{=}}} A$ iff $v(A) \in T_D$ whenever $\boldsymbol{D}$ is an r.d.s. and v is a valuation in D. Now every sentence A of the language of $RMI_{\overrightarrow{=}}$ defines in a natural way an algebraic expression in the $\{{\sim}, +\}$-language. Our present theorem is therefore a direct consequence of the fact that A_ω is a polynomially free r.d.s. (see [7], I.12).

A.12 Corollaries

(a) *Let A be a sentence with at most n propositional variables. Then $\vdash_{RMI_{\overrightarrow{=}}} A$ iff A is valid in $\boldsymbol{A}_n$ (the finite sub-r.d.s. of A_ω with n neutral values)*

(b) *$RMI_{\overrightarrow{=}}$ is decidable*

(c) *$RMI_{\overrightarrow{=}}$ has the variable-sharing property for $+$ and $\rightarrow$, i.e., $A + B$ or $A \rightarrow B$ are provable only if A and B share a variable*

(d) D.S. *for $+$ is valid in $RMI_{\overrightarrow{=}}$ but weakening is not. Moreover, the provability of A or of B or even both does not guarantee the provability of $A + B$.*

Definition Let A be a sentence. Then $\boldsymbol{Pv}(A)$ is the set of propositional variables occurring in A.
(e) *If* $\boldsymbol{Pv}(B) \subseteq \boldsymbol{Pv}(A)$, *then* $A \vdash_{RMI_{\eqsim}} A + B$ *and* $A \vdash_{RMI_{\eqsim}} B \to A$. *(This is the* weak weakening *rule)*
(f) *If* $\boldsymbol{Pv}(B) \subseteq \boldsymbol{Pv}(A)$, *then* $\vdash_{RMI_{\eqsim}} A \to (B \to B)$ *and* $\vdash_{RMI_{\eqsim}} ((A \to B) \to A) \to A$
(g) *If* $\vdash_{RMI_{\eqsim}} A$ *and* $\vdash_{RMI_{\eqsim}} B$ *then* $\vdash_{RMI_{\eqsim}} A + B$ *iff* A *and* B *share a variable.*

We turn now to the expressive power of $RMI_{\eqsim}$. We shall show that the intuitive ideas of relevance and of grading from [7] can be formalized in it.

A.13 Definition Let A, B be sentences in the language of $RMI_{\eqsim}$.
(a) $R^+(A,B) =_{def} ({\sim}A + A) + ({\sim}B + B)$
(b) $|A| \le |B| =_{def} (A \to A) \to (B \to B) (= {\sim}({\sim}A + A) + ({\sim}B + B))$.

A.14 Proposition *Let* $\boldsymbol{D}$ *be an r.d.s. and* v *a valuation in* D. *Then:*
(a) $R^+(A,B)$ *is true in* $(\boldsymbol{D}, v)$ *iff* $v(A) R v(B)$
(b) $|A| \le |B|$ *is true in* $(\boldsymbol{D}, v)$ *iff* $|v(A)| \le |v(B)|$.

Proof: The proof is easy (compare to I.19, I.21 of [7]!).

A.15 Proposition
(1) $\vdash_{RMI_{\eqsim}} R^+(A,A)$
(2) $\vdash_{RMI_{\eqsim}} R^+(A,B) \to R^+(B,A)$
(3) $\vdash_{RMI_{\eqsim}} (A + B) \to R^+(A,B)$, $\vdash_{RMI_{\eqsim}} (A \to B) \to R^+(A,B)$
(4) $\vdash_{RMI_{\eqsim}} R^+(A,B) \to R^+({\sim}A,B)$
(5) $\vdash_{RMI_{\eqsim}} R^+(A,B) \to R^+(A,B + C)$, $\vdash_{RMI_{\eqsim}} R^+(A,B) \to R^+(A,C + B)$; *in particular,* $\vdash_{RMI_{\eqsim}} R^+(A,A + B)$ *and* $\vdash_{RMI_{\eqsim}} R^+(B,A + B)$
(6) *If* $\boldsymbol{Pv}(A) \cap \boldsymbol{Pv}(B) \ne \varnothing$ *then* $\vdash_{RMI_{\eqsim}} R^+(A,B)$.

Proof: (6) is a corollary of Corollary A.12(g). The rest can be shown by using A_ω.

A.16 Proposition
(1) $\vdash_{RMI_{\eqsim}} |A| \le |A|$
(2) $\vdash_{RMI_{\eqsim}} |A| \le |B| \to |{\sim}A| \le |B|$
(3) $\vdash_{RMI_{\eqsim}} |A| \le |A + B|$, $\vdash_{RMI_{\eqsim}} |B| \le |A + B|$
(4) $\vdash_{RMI_{\eqsim}} |A| \le |C| \to (|B| \le |C| \to |A + B| \le |C|)$
$\vdash_{RMI_{\eqsim}} |A| \le |C| \to (|B| \le |C| \to |A \to B| \le |C|)$
(5) $\perp_{RMI_{\eqsim}} |A| \le |B| \to (|B| \le |C| \to |A| \le |C|)$
(6) *if* $\boldsymbol{Pv}(B) \subseteq \boldsymbol{Pv}(A)$ *then* $\vdash_{RMI_{\eqsim}} |B| \le |A|$.

Proof: Use A_ω (note: (3) is an immediate corollary of (6)).

Two other relations which we would like to express in the language of $RMI_{\eqsim}$ are the equivalence between propositions and the relation of belonging to the same relevance domain. For this we need first some formal properties of $\circ$:

A.17 Lemma
(1) $\vdash_{RMI_{\eqsim}} ((A \circ B) \to C) \to (A \to (B \to C))$
$\vdash_{RMI_{\eqsim}} (A \to (B \to C)) \to ((A \circ B) \to C)$
(2) $\vdash_{RMI_{\eqsim}} A \to (B \to A \circ B)$
(3) *If* $\boldsymbol{Pv}(A) \subseteq \boldsymbol{Pv}(B)$ *then* $\vdash_{RMI_{\eqsim}} A \circ B \to A$ *and* $\vdash_{RMI_{\eqsim}} B \circ A \to A$

(4) $\vdash_{RMI_{\cong}} A \circ A \to A$, $\vdash_{RMI_{\cong}} A \to A \circ A$
(5) $\vdash_{RMI_{\cong}} A \circ B \to B \circ A$
(6) $\vdash_{RMI_{\cong}} A \circ (B \circ C) \to (A \circ B) \circ C$, $\vdash_{RMI_{\cong}} (A \circ B) \circ C \to A \circ (B \circ C)$
(7) $\vdash_{RMI_{\cong}} \sim R^{+}(A,B) \to A \circ B$.

Note (1)-(2), (4b)-(6) of the above lemma are true also for $R_{\cong}$. (4a), (3), and (7) are not. (7) is the main reason we cannot consider $\circ$ to be a "relevant conjunction".

A.18 Definition
(1) $A \leftrightarrow B =_{def} (A \to B) \circ (B \to A)$
(2) $(|A| = |B|) =_{def} (|A| \leq |B|) \circ (|B| \leq |A|)$.

A.19 Proposition
(1) $\vdash_{RMI_{\cong}} (A \leftrightarrow B) \to (A \to B)$, $\vdash_{RMI_{\cong}} (A \leftrightarrow B) \to (B \to A)$
(2) $\vdash_{RMI_{\cong}} (A \to B) \to ((B \to A) \to (A \leftrightarrow B))$
(3) $\vdash_{RMI_{\cong}} |A| = |B| \to |A| \leq |B|$, $\vdash_{RMI_{\cong}} |A| = |B| \to |B| \leq |A|$
(4) $\vdash_{RMI_{\cong}} |A| \leq |B| \to (|B| \leq |A| \to |A| = |B|)$
(5) *If* $\boldsymbol{D}$ *is an r.d.s. and* v *is a valuation in* D *then* $|A| = |B|$ *is true in* $(\boldsymbol{D}, v)$ *iff* $|v(A)| = |v(B)|$ *in* $\boldsymbol{D}$.

Proof: (1)-(4) follow from Lemma A.17. (5) is obvious.

Note A.19(1)-(4) are true also for $R_{\cong}$, although A.17(3) in general is not.

We end this section with an example of the difference between weak and finitely strong completeness.

A.20 Proposition *$RMI_{\cong}$ is not finitely strongly complete relative to $\boldsymbol{A}_{\omega}$ (recall that this means that there are sentences $A_1, \ldots, A_n, B$ such that $A_1, \ldots, A_n \nvdash_{RMI_{\cong}} B$ but $v(B) \in T_{A_{\omega}}$ whenever $v(A_1), v(A_2), \ldots, v(A_n) \in T_{A_{\omega}}$).*

Proof: Let $|A| = |p| \leq |q|$, $B = |q| \leq |p| \to p$ (p, q are different atomic variables). It is not hard to see that the necessary and sufficient conditions for a full model $(\boldsymbol{M}, v)$ to be a model of A and B but not of p are that $|v(p)| < |v(q)|$ and $v(p) \notin T_M$. This can happen in the Sugihara matrix but not in $\boldsymbol{A}_{\omega}$. Hence $A, B \nvdash_{RMI_{\cong}} p$, but this is not reflected in $\boldsymbol{A}_{\omega}$.

B The system $RMI_{\min}$ In the previous section we saw that $RMI_{\cong}$ is, to a great extent, an optimal relevance logic. However, from one aspect it is less than satisfactory: its language is not rich enough. An especially important classical connective, which has no counterpart in $RMI_{\cong}$, is conjunction. The connective $\circ$ cannot be taken seriously as a candidate since (among other things) $A \circ B$ may be true even if both A and B are false ($\circ$ is appropriately called "cotenability" in Routley and Meyer [18]).

We are now going to add a "relevant conjunction" connective $\wedge$ to the language of $RMI_{\cong}$. We shall see that we get by this quite a rich language, which is, nevertheless, purely intensional.

Note In what follows we use without proofs various properties of $RMI_{\cong}$. The reader may reconstruct the proofs by using A_{ω}.

B.1 The system $RMI_{\min}$

Basic connectives $\sim, \rightarrow, \wedge$
Defined connectives $A + B = \sim A \rightarrow B$
$A \vee B = \sim(\sim A \wedge \sim B)$
Axioms **IM.1–1M.4, N1–N2** (see A.1)
(C1) $A \wedge B \rightarrow A$
(C2) $A \wedge B \rightarrow B$
Rules (1) $A, A \rightarrow B \vdash B$ (M.P.)
(2) $A \rightarrow B, A \rightarrow C \vdash A \rightarrow (B \wedge C)$ (re. adj.)[8]

Explanation That any (relevant) conjunction operator should satisfy (C1) and (C2) is generally agreed. On the other hand the adjunction rule $A, B \vdash A \wedge B$ cannot be justified from the point of view of relevance logic. For even if we assume both A and B to be true, relevance considerations should determine whether or not it makes sense to accept their conjunction. The intuition which bars constructions such as: "If the moon is made of cheese then $2 + 2 = 4$", should also bar constructions such as: "The moon is not made of cheese and $2 + 2 = 4$". On the other hand the truth of $A \rightarrow B$ and of $A \rightarrow C$ should imply the truth of $A \rightarrow (B \wedge C)$, because the common antecedent A guarantees the necessary relevance. Thus, instead of adjunction we have relevant adjunction. The resulting system $RMI_{\min}$ satisfies the minimal requirements concerning relevant conjunction.

It turns out that we have a natural interpretation for $\wedge$ in the structures we considered in [7], without having to extend these structures. In any r.d.l. interpret $\wedge$ as the g.l.b. (greatest lower bound). Using '$\wedge$' both for the syntactic symbol and for its interpretation we put: $a \wedge b =$ g.l.b. of a and b. Then, as we shall prove, we get strong completeness of $RMI_{\min}$ with respect to this semantics. If however we limit ourselves only to r.d.l.'s which are full we get a stronger system, denoted by *RMI*, which has important advantages over $RMI_{\min}$ (recall that as far as $\sim, +$ are concerned full r.d.l.'s and r.d.l.'s yield the same syntactic systems). All natural strengthenings of *RMI* are obtainable by restricting the class of r.d.l.'s which is to serve as its semantics; this line of investigation is pursued in the last section.

B.2 Proposition
(1) $\vdash_{RMI_{\min}} A \rightarrow A \vee B$
(2) $\vdash_{RMI_{\min}} B \rightarrow A \vee B$
(3) $A \rightarrow C, B \rightarrow C \vdash_{RMI_{\min}} (A \vee B) \rightarrow C$
(4) $\vdash_{RMI_{\min}} (A + B) \rightarrow A \vee B$
(5) $\vdash_{RMI_{\min}} (\sim A \vee A) \leftrightarrow (\sim A + A)$.

Proof: Use (C1), (C2), re. adj., contraposition ($\vdash A \rightarrow \sim\sim A$, $\vdash \sim\sim A \rightarrow A$), and the following theorems of $RMI_{\approx}$:
(1) $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A + B) \rightarrow C))$
(2) $A \rightarrow \sim A + A$.

B.3 Theorem *$RMI_{\min}$ is strongly complete relative to r.d.l.'s, where the operation on r.d.l.'s which corresponds to the connective $\wedge$ is the g.l.b. which we denote in [7] by the same symbol (recall that this means that $T \vdash_{RMI_{\min}} A$ iff A is true in every model $(\boldsymbol{D}, v)$ of T in which D is an r.d.l.).*

Proof: The soundness part follows immediately from the fact that in an r.d.s. D, $a \leq b$ iff $a \rightarrow b \in T_D$, and the definition of a g.l.b.

For the converse, let T be an $RMI_{\min}$-theory (i.e., a set of sentences in the language of $RMI_{\min}$). Construct the Lindenbaum algebra $\boldsymbol{D}$ of T as in the proof of Theorem A.6. Then D is an r.d.s., and it is easy to see that $[A \wedge B]$ is the g.l.b. of $[A]$ and $[B]$. Hence D is an r.d.l. Defining $v(A) = [A]$ we get a model $(\boldsymbol{D}, v)$ such that $\boldsymbol{D}$ is an r.d.l. and in which exactly the theorems of T are true.

B.4 Corollaries
(1) $\sim P, P \nvdash_{RMI_{\min}} Q$
(2) *$RMI_{\min}$ is a conservative extension of $RMI_{\rightleftarrows}$*
(3) *$RMI_{\min}$ has the variable-sharing property for $+, \rightarrow$ and $\wedge$; i.e., if either $A + B$, $A \rightarrow B$, or $A \wedge B$ is provable then A and B share a variable.*

Proof: (2) follows from the completeness of $RMI_{\rightleftarrows}$ relative to full r.d.s.'s (which are r.d.l.'s by II.4. of [7]). The proofs of (1) and of (3) in the case of $+$ and $\rightarrow$ are exactly as in Corollaries A.9(2) and A.12. The variable-sharing property of $\wedge$ is proved similarly (by using $\boldsymbol{A}_2$).

In $RMI_{\min}$ we can give an alternative formal definition of the relevance relation.

B.5 Definition $R^{\wedge}(A,B) =_{def} \sim(\sim A \wedge A) \wedge \sim(\sim B \wedge B)$.

Note Since $\vdash_{RMI_{\min}} \sim(\sim A \wedge A) \leftrightarrow (\sim A + A)$ (by Proposition B.2(5)), $R^{\wedge}(A,B)$ is equivalent to $(\sim A + A) \wedge (\sim B + B)$.

B.6 Proposition
(1) $\vdash_{RMI_{\min}} R^{\wedge}(A,B) \rightarrow R^{+}(A,B)$
(2) $R^{+}(A,B) \vdash_{RMI_{\min}} R^{\wedge}(A,B)$
(3) $A, B, R(A,B) \vdash_{RMI_{\min}} A \wedge B$
(4) $R(A,B) \vdash_{RMI_{\min}} R(A \wedge C, B \wedge C)$
(5) $R(A,B) \vdash_{RMI_{\min}} R(A \vee C, B \vee C)$
(6) $\vdash_{RMI_{\min}} R(A \wedge B, A)$, $\vdash_{RMI_{\min}} R(A \wedge B, B)$, $\vdash_{RMI_{\min}} R(A, A \vee B)$, $\vdash_{RMI_{\min}} R(B, A \vee B)$.

Note By (1) and (2) it does not matter if $R(A,B)$ in (3)–(6) means either $R^{+}(A,B)$ or $R^{\wedge}(A,B)$.

Proof: (1) This follows from the note to Definition B.5 and the following theorem of $RMI_{\min}$: $A \wedge B \rightarrow A + B$. This theorem is derived using (C1), (C2) and the fact that $\vdash_{RMI_{\rightleftarrows}} (C \rightarrow A) \rightarrow ((C \rightarrow B) \rightarrow (C \rightarrow (A + B)))$.

(2) This follows from the completeness theorem B.3, Proposition A.14(a), and the fact that if $a, b \in T_D$ and aRb then $a \wedge b \in T_D$ (see [7], I.25).

(3) Similar to (2).

(4) We show that $R^{+}(A,B) \vdash_{RMI_{\min}} R^{+}(A \wedge C, B \wedge C)$. We start from the equivalence of $R^{+}(A,B)$ and $A \rightarrow (B \rightarrow (\sim B \rightarrow A))$ in $RMI_{\rightleftarrows}$. From this equivalence follows:

$$\text{(i)}\quad R^{+}(A,B) \vdash_{RMI_{\min}} A \wedge C \rightarrow (B \wedge C \rightarrow (\sim B \rightarrow A)).$$

Since $\vdash_{RMI_{\rightleftarrows}} C \to (C \to (\sim C \to C))$ we have also:

$$\text{(ii)}\ \vdash_{RMI_{\min}} A \wedge C \to (B \wedge C \to (\sim C \to C)).$$

Since $\vdash_{RMI_{\min}} (A \wedge C) \to (A + C)$ (by (1)) and since $\vdash_{RMI_{\rightleftarrows}} (A + C) \to (C \to (\sim C \to A))$ we have that $\vdash_{RMI_{\min}} (A \wedge C) \to (C \to (\sim C \to A))$. Hence:

$$\text{(iii)}\ \vdash_{RMI_{\min}} (A \wedge C) \to (B \wedge C \to (\sim C \to A)).$$

Similarly:

$$\text{(iv)}\ \vdash_{RMI_{\min}} (A \wedge C) \to (B \wedge C \to (\sim B \to C)).$$

Using the equivalence of $A \to (B \to C)$ and $(A \circ B) \to C$ in $RMI_{\rightleftarrows}$, re. adj., and contraposition we can derive from (i) and (iv) that

$$\text{(v)}\ R^+(A,B) \vdash_{RMI_{\min}} A \wedge C \to (B \wedge C \to (\sim (A \wedge C) \to B)).$$

Similarly we can derive from (ii) and (iii):

$$\text{(vi)}\ \vdash_{RMI_{\min}} A \wedge C \to (B \wedge C \to (\sim (A \wedge C) \to C)).$$

From (v) and (vi) follows finally, by re. adj., that $R^+(A,B) \vdash_{RMI_{\min}} A \wedge C \to (B \wedge C \to (\sim (A \wedge C) \to B \wedge C)))$. This is equivalent to what we wish to prove.

(5) This follows from (4) and $R^+(A,B) \vdash_{RMI_{\rightleftarrows}} R^+(\sim A, \sim B)$.

(6) Immediate from $A \to B \vdash_{RMI_{\rightleftarrows}} R^+(A,B)$.

In the following lemma we gather together some other proof-theoretical properties of $RMI_{\min}$ that we shall need later.

B.7 Lemma

(1) $B \to C \vdash_{RMI_{\min}} A \wedge B \to A \wedge C$
$B \to C \vdash_{RMI_{\min}} A \vee B \to A \vee C$

(2) (i) $\vdash_{RMI_{\min}} (A \to B) \wedge (A \to C) \to (A \to (B \wedge C))$
(ii) $\vdash_{RMI_{\min}} (A \to C) \wedge (B \to C) \to ((A \vee B) \to C))$

(3) $\vdash_{RMI_{\min}} (A \wedge (A \to B)) \to B.$

Proof: Since $A \to (B \to C)$ is equivalent in $RMI_{\rightleftarrows}$ to $A \circ B \to C$ we have that:

$$\vdash_{\mathrm{RMI}_{\min}} [((A \to B) \wedge (A \to C)) \circ A] \to B \vdash_{RMI_{\min}} [((A \to B) \wedge (A \to C)) \circ A] \to C.$$

From this (2)(i) follows by re. adj. and the same equivalence. We leave the other parts to the reader.

The results of this section up to this point show that $RMI_{\min}$ satisfies many of the demands listed in the introduction. Nevertheless $RMI_{\min}$ is still unsatisfactory because of its following shortcomings:

B.8 Proposition

(1) *$RMI_{\min}$ is not syntactically complete (i.e., its set of theorems is not the intersection of all its complete theories)*

(2) *The relevant deduction theorem fails in $RMI_{\min}$*

(3) *From $T,A \vdash_{RMI_{\min}} C$ and $T,B \vdash_{RMI_{\min}} C$ it does not follow in general that $T,A + B \vdash_{RMI_{\min}} C$.*

Proof: (1) We show that the following sentence is not provable in $RMI_{\min}$, although it is provable in every complete $RMI_{\min}$-theory:

(*) $\quad A \wedge (B \vee {\sim}B) \rightarrow ((A \wedge B) \vee (A \wedge {\sim}B))$.

The second part of this claim follows from the fact that $B \vdash_{RMI_{\vec{\approx}}} B \leftrightarrow ({\sim}B + B)$ and from Proposition B.2(5), since the two facts entail that if $T \vdash_{RMI_{\min}} B$ then (*) is equivalent in T to $A \wedge B \rightarrow (A \wedge B) \vee (A \wedge {\sim}B)$, which is an instance of (C1). Similarly, if $T \vdash_{RMI} {\sim}B$ then (*) is equivalent in T to $(A \wedge {\sim}B) \rightarrow ((A \wedge B) \vee (A \wedge {\sim}B))$. Hence (*) is provable if T is complete.

For the first part it suffices to give an example of an r.d.l. in which (*) is not valid. For this take any Boolean algebra B containing an element b distinct from 1_B and 0_B. Obtain B' by adding to B an element I and by defining $\mathrm{I} + \mathrm{I} = \mathrm{I}$, ${\sim}\mathrm{I} = \mathrm{I}$, and $a + \mathrm{I} = \mathrm{I} + a = a$ for $a \in B$. It can easily be seen that B' is an r.d.l. (we have that $\mathrm{I} \wedge 1_B = \mathrm{I} \wedge \mathrm{I} = \mathrm{I}$ and $\mathrm{I} \wedge a = 0_B$ otherwise), and that by letting $v(p) = \mathrm{I}$, $v(q) = b$ we get a countermodel to (*). (In fact, $v(p \wedge (q \vee {\sim}q) \rightarrow ((p \wedge q) \vee (p \wedge {\sim}q))) = 0_B$ while $T_{B'} = \{1_B, \mathrm{I}\}$.)

(2) Since $\vdash_{RMI_{\vec{\approx}}} A \rightarrow A$ we have, by re. adj., that $A \rightarrow B \vdash_{RMI_{\min}} A \rightarrow (A \wedge B)$. On the other hand $\nvdash_{RMI_{\min}} A \rightarrow (A \wedge B)$. Had the relevant deduction theorem been valid we should have had that $\vdash (A \rightarrow B) \rightarrow (A \rightarrow A \wedge B)$. However, this sentence is not a theorem of $RMI_{\min}$, as is seen by defining $v(A) = \mathrm{I}$, $v(B) = 1_B$ in the substructure $\{1_B, 0_B, \mathrm{I}\}$ of B' from the proof of (1). (This substructure is isomorphic to the finite Sugihara matrix M_3.) (See 29.3.4 of [2].)

(3) Suppose for a contradiction that $RMI_{\min}$ had this property. Then in particular we should have that if $T, A \vdash_{RMI_{\min}} C$ and $T, {\sim}A \vdash_{RMI_{\min}} C$ then $T, {\sim}A + A \vdash_{RMI_{\min}} C$, i.e., that $T \vdash_{RMI_{\min}} C$. This property is easily seen to be equivalent to syntactical completeness, contradicting (1).

Note The relevant deduction theorem fails also for the full system *RMI* to be presented below and with the same counterexample. This counterexample works, in fact, for every other known system containing conjunction in which $A \rightarrow (B \rightarrow A)$ fails (including all the systems discussed by A & B). However, in *RMI* we have the classical deduction theorem for another implication connective, $\supset$ (discussed already in the introduction), which we are now going to introduce.

B.9 Definition $\quad A \supset B =_{def} (A \rightarrow B) \vee B$.

Explanation The connective $\vee$ is the De Morgan dual of $\wedge$, and since the r.d.l.'s are (nondistributive) De Morgan lattices it is interpreted in every r.d.l. as the l.u.b.

Now, if aRb in a full r.d.l. $\boldsymbol{M}$ then $a \vee b \in T_M$ iff either $a \in T_M$ or $b \in T_M$, so in this case $\vee$ behaves like an extensional disjunction. But, since $\vdash_{RMI_{\vec{\approx}}} R^+(A, A \rightarrow B)$, $A \rightarrow B$ and B have relevant values in any r.d.s. Hence in full r.d.l.'s the disjunction on the right hand side of the last definition behaves extensionally. The intuitive meaning of $A \supset B$ is, therefore, really: "either B is true or A relevantly entails B", where the "either or" can in this particular case be interpreted extensionally.

The following lemma shows that $\supset$ has in $RMI_{\min}$ the following essential property of an implication:

B.10 Lemma $A, A \supset B \vdash_{RMI_{\min}} B.$

Proof: Since $A \vdash_{RMI_{\vec{=}}} (A \to B) \to B$ and $\vdash_{RMI_{\vec{=}}} B \to B$ we have by Proposition B.2(3) that $A \vdash_{RMI_{\min}} (A \supset B) \to B$.

Our task in the next section will be to extend $RMI_{\min}$ to a system free from the shortcomings of $RMI_{\min}$. The way to do so is almost dictated to us by the following:

B.11 Proposition *A sentence A is a theorem of every complete $RMI_{\min}$-theory iff it is valid in every full r.d.s.*

Proof: One direction is trivial, since the set of sentences which are true in a given full model $(\boldsymbol{M}, v)$ is a complete $RMI_{\min}$-theory.

For the converse, suppose T is a complete $RMI_{\min}$-theory and that $T \nvdash_{RMI_{\min}} A$. The Lindenbaum algebra of T is then an r.d.l. in which A is not valid (see the proof of Theorem B.3.). This r.d.l. is full since T is complete. Hence A is not valid in every full r.d.s.

Proposition B.11 means that every syntactically complete extension of $RMI_{\min}$ must include the set of sentences which are valid in every full r.d.l. We next introduce, therefore, the system which corresponds to the semantics of full r.d.l.'s.

C Formulations and properties of RMI

C.1 The system RMI (first formulation) This is the system obtained from $RMI_{\min}$ by the addition of the following axiom-schema:

(RD) $R^{\wedge}(B, C) \supset [A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)]$.

The reason for adding this axiom is given by the following theorem:

C.2 Completeness Theorem *RMI is strongly complete relative to full r.d.l.'s.*

Proof: For the soundness part it is enough to use the tree-based representation of full r.d.l.'s (see the Introduction or [7], II.8, II.13) in order to show that (RD) is valid in every full r.d.l. (This is a bit tedious. Later we shall see that a sentence of the form $A \supset B$ is valid in every full r.d.s. iff every model $(\boldsymbol{M}, v)$ of A is also a model of B. Using this the validity of (RD) is immediate, recalling that in a full r.d.l. if bRc then either $b \vee c = b$ or $b \vee c = c$.) For the completeness part it suffices to show that if $T \nvdash_{RMI} A$ then there exists a prime theory T_0 (i.e., a theory T_0 such that $T_0 \vdash C \vee D$ iff either $T \vdash C$ or $T \vdash D$) such that $T \subseteq T_0$ and $T_0 \nvdash_{RMI} A$. In $RMI_{\min}$ the Lindenbaum algebra of such a T_0 determines a prime (hence full) r.d.l. $\boldsymbol{M}$ and a valuation v in M such that exactly the theorems of T_0 are true in $(\boldsymbol{M}, v)$. Such an $(\boldsymbol{M}, v)$ is therefore a full model of T in which A is not true.

The existence of such a T_0 is an immediate consequence of the following:

Lemma $T, B \vdash_{RMI} A,\ T, C \vdash_{RMI} A \Rightarrow T, B \vee C \vdash_{RMI} A.$

Proof: As in [2] (p. 301), to prove the lemma it suffices to have:

(*) $T, B \vdash_{RMI} C \Rightarrow T, B \vee D \vdash_{RMI} C \vee D.$

For then we infer

$$T, B \vdash_{RMI} A \Rightarrow T, B \vee C \vdash_{RMI} A \vee C$$
$$T, C \vdash_{RMI} A \Rightarrow T, C \vee A \vdash_{RMI} A \vee A$$

and so, since $A \vee A \vdash_{RMI} A$, we get $T, B \vee C \vdash_{RMI} A$.

The proof in [2] of (*) (with $\vdash_{RMI}$ replaced by $\vdash_E$ or $\vdash_R$) is by induction on the length of the proof of C from $T \cup \{B\}$. The two nontrivial steps in this induction correspond to the inference rules. In order to reproduce this proof in *RMI* we only need to show that:

(a) $D \vee A, D \vee (A \rightarrow B) \vdash_{RMI} D \vee B$
(b) $D \vee (A \rightarrow B), D \vee (A \rightarrow C) \vdash_{RMI} D \vee (A \rightarrow B \wedge C)$.

Now, the proof of (a), e.g., in [2] makes an essential use of the full adjunction rule (from A and B to infer $A \wedge B$) and of the distribution axiom, both of which are not valid in *RMI*. In the present case, however, adjunction and distribution are both available wherever needed. We illustrate this claim in the proof of (b) (the proof of (a) is similar). We start with the following lemma:

Lemma $R^+(B, C) \vdash_{RMI} (A \vee B) \wedge (A \vee C) \rightarrow (A \vee (B \wedge C))$.

Proof: Indeed, $R^\wedge(\sim B, \sim C)$ follows from $R^+(B, C)$ by Propositions A.15(4) and B.6(2). Applying (RD) and Lemma B.10 we therefore get:

$$R^+(B, C) \vdash_{RMI} \sim A \wedge (\sim B \vee \sim C) \rightarrow ((\sim A \wedge \sim B) \vee (\sim A \wedge \sim C)).$$

From this the lemma follows by contraposing and using the definition of $\vee$.

The proof of (b) is now straightforward: By Propositions B.6(5), B.6(3), and the fact that $\vdash_{RMI_{\overrightarrow{\sim}}} R^+(A \rightarrow B, A \rightarrow C)$, we have that:

$$D \vee (A \rightarrow B), D \vee (A \rightarrow C) \vdash_{RMI} (D \vee (A \rightarrow B)) \wedge (D \vee (A \rightarrow C)).$$

From this (b) follows by using the lemma and Lemma B.7(1)–(2). (In the proof of (a) we must use Lemma B.7(3) instead of B.7(2).)

By inspecting the proof of the last theorem we see that we have in fact proved that *RMI* is also strongly complete (see Proposition A.20) relative to prime r.d.l.'s. This can be sharpened as follows:

C.3 Theorem *If T is an RMI theory and $T \nvdash_{RMI} A$ then there is a model $(\boldsymbol{M}, v)$ of T such that $\boldsymbol{M}$ is a prime r.d.l., $v(A) \notin T_M$, and $v(A)$ is minimal among the normal domains of M.*

Proof: This follows easily from Theorem C.2 and II.29 of [7].

C.4 Definition We shall call a model $(\boldsymbol{M}, v)$ in which $\boldsymbol{M}$ is a prime r.d.l. a *prime model.*

C.5 Theorem *RMI is finitely strongly complete relative to finite full (prime) r.d.l.'s.*

Proof: This follows from Theorems C.2 and C.3 in exactly the same way in which Theorem A.10 follows from Theorem A.8.

C.6 Theorem *The canonical full r.d.l. is characteristic for RMI. Moreover, RMI is finitely strongly complete (see Proposition A.20) relative to it.*

Proof: The first part is immediate from the fact that the canonical structure is a polynomially free full r.d.s. (II.32 of [7]). The second part is a consequence of Theorem C.5 and the fact that every finite prime r.d.l. is embeddable in the canonical full r.d.l.

We give next an alternative formulation of *RMI*, which is more similar to the usual formulations of *R* and *RM*, and in which one can clearly see the differences between *RMI* and those two systems.

C.7 RMI (second formulation)

Primitive connectives $\sim, \rightarrow, \wedge, \vee$

Axioms
(1) $A \rightarrow (A \rightarrow A)$
(2) $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
(3) $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$
(4) $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$
(5) $\sim\sim A \rightarrow A$
(6) $(A \rightarrow \sim B) \rightarrow (B \rightarrow \sim A)$
(7) $A \wedge B \rightarrow A$
(8) $A \wedge B \rightarrow B$
(9) $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$
(10) $A \rightarrow A \vee B$
(11) $B \rightarrow A \vee B$
(12) $(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C)$

Rules of inference
(I) $A, A \rightarrow B \vdash B$
(II) $R^{+}(B,C), B, C \vdash B \wedge C$
(III) $R^{+}(B,C) \vdash A \wedge (B \vee C) \rightarrow ((A \wedge B) \vee (A \wedge C))$.

Here $+$ and R^{+} are defined in terms of $\sim$ and $\rightarrow$ as before. (Thus $R^{+}(A,B)$ can be written simply as $\sim(A \rightarrow A) \rightarrow (B \rightarrow B)$.)

C.8 Proposition *The two formulations of RMI given above are strongly equivalent (i.e., $T \vdash_{RMI} A$ according to the first iff $T \vdash_{RMI} A$ according to the second).*

Proof: Denote temporarily the first formulation by $RMI_{(I)}$ and the second by $RMI_{(II)}$. From Proposition B.2(1)–(3), Proposition B.6(3), Lemma B.7(2), Lemma B.10, and (RD) it follows that $T \vdash_{RMI_{(II)}} A \Rightarrow T \vdash_{RMI_{(I)}} A$.

For the converse we note first that, since $\vdash_{RMI_{\rightrightarrows}} R^{+}(A \rightarrow B, A \rightarrow C)$, we can use the second rule of $RMI_{(II)}$ and then its ninth axiom to derive $A \rightarrow (B \wedge C)$ from $A \rightarrow B$ and $A \rightarrow C$. Hence we can easily transfer the proof of Theorem C.3 into a completeness proof of $RMI_{(II)}$ relative to the same semantics. From this the desired equivalence immediately follows.

C.9 Corollaries

(1) *By adding either $R^{+}(A,B)$ or $R^{\wedge}(A,B)$ to RMI we get RM (see 29.3.4 of [2]).*

(2) *By dropping the third rule from the second formulation of RMI we obtain a new formulation of* $RMI_{\min}$. *(This follows from the proof of Proposition C.8.)*

We turn now to proof-theoretical properties of *RMI*.

C.10 Theorem *RMI is syntactically complete and it is the minimal extension of* $RMI_{\min}$ *having this property.*

Proof: In the proof of Theorem C.2 we showed that if $T \nvdash_{RMI} A$ then there exists a prime theory T_0 such that $T \subseteq T_0$ and $T \nvdash_{RMI} A$. But since $\vdash_{RMI} {\sim}A \vee A$ (Proposition B.2(5)), every prime *RMI*-theory is also complete. Hence *RMI* is syntactically complete. The second part of the theorem follows from Proposition B.11 and Theorem C.2.

C.11 Theorem *RMI is a strongly conservative extension of* $RMI_{\overset{\sim}{\to}}$.

Proof: This is an immediate consequence of the strong completeness of both relative to full r.d.l.'s.

C.12 Theorem *RMI is decidable. Moreover, the provability relation it induces between finite sets of sentences and sentences is decidable too.*

Proof: Immediate from Theorem C.5.

C.13 Theorem
(1) *RMI has the variable-sharing property for* $\rightarrow$, $+$ *and* $\wedge$
(2) ${\sim}P, P \nvdash_{RMI} Q$
(3) *If* $T, A \vdash_{RMI} C$ *and* $T, A \vdash_{RMI} C$ *then* $T, A \vee B \vdash_{RMI} C$ *and* $T, A + B \vdash_{RMI} C$.

Proof: The proof is easy by using full r.d.l.'s and the strong completeness theorem.

C.14 On the connective $\vee$ ***in RMI*** This connective may be properly called "semi-extensional". In fact, it is no less "extensional" than the corresponding connectives of E and R, since like them it has all the positive properties of classical disjunction: It is associative, commutative, and idempotent, and the following basic rules are valid for it: (i) $A \vdash A \vee B$, (ii) $B \vdash A \vee B$, (iii) If $T, A \vdash C$ and $T, B \vdash C$ then $T, A \vee B \vdash C$. Moreover, $\vee$ behaves exactly as an extensional disjunction relative to the class of prime r.d.l.'s, which suffices for a strong characterization of *RMI*. (By extensional behavior we mean that $A \vee B$ is true in a prime model iff either A is true there or B is.)

On the other hand, with respect to negation $\vee$ loses much of its classical character. Except for the law of excluded middle (${\sim}A \vee A$) almost no classical rule concerning $\vee$ and $\sim$ is true for it. Thus, for example, ${\sim}A, {\sim}B \nvdash_{RMI} {\sim}(A \vee B)$ (this rule is equivalent to the adjunction rule of R).[9]

Here are some nice properties of $RMI_{\overset{\sim}{\to}}$ which *RMI* lacks: *RMI* lacks the Scroggs property which $RMI_{\overset{\sim}{\to}}$ has; $\boldsymbol{A}_\omega$ is not characteristic for *RMI* ($A \vee (A \rightarrow B)$, e.g., is valid in it but not in the canonical r.d.s.); Corollaries A.12(e)–(f), Propositions A.15(6) and A.16(6) are not true for *RMI*. (Thus, e.g., $\nvdash_{RMI} R(A \vee B, A \vee C)$ although $A \vee B$, $A \vee C$ share a variable.) A particularly important property of $RMI_{\overset{\sim}{\to}}$ which *RMI* lacks is the deduction theorem for $\rightarrow$ (see the note after Proposition B.8). The next section deals with a sort of substitute which we have for *RMI*.

D The deduction and interpolation theorems In the previous section we showed that *RMI* satisfies all our demands (I)–(VIII) from the Introduction (including properties (a)–(e) of $+$) except perhaps two: the validity of an appropriate version of the deduction theorem and the existence of a cut-free Gentzen-type formulation for *RMI*. We shall now show that the connective $\supset$, defined in Definition B.9, behaves "almost" like the classical implication. We have already seen that M.P. is valid for it (Lemma B.10). We next show:

D.1 The Deduction Theorem for $\supset$ *$T,A \vdash_{RMI} B$ iff $T \vdash_{RMI} A \supset B$.*[10]

Proof: The "if" part follows from Lemma B.10.

For the "only if" part, suppose that $T,A \vdash_{RMI} B$ but $T \nvdash_{RMI} A \supset B$. By Theorem C.3 we then find a prime model $(\boldsymbol{M}, v)$ of T in which $A \supset B$ is not true (i.e., $\mathrm{val}(v(A \supset B) = \mathrm{f})$ and $|v(A \supset B)|$ is minimal among the normal domains. Hence $\mathrm{val}(v(B)) = \mathrm{f}$, $\mathrm{val}(v(A \to B)) = \mathrm{f}$ and $v(A), v(A \to B)$ are normal. Now, since in every r.d.s. $|v(A \to B)| \geq |v(B)|$, $v(A \supset B) = v(B)$ in the present case (see [7], II.14). Hence $|v(B)|$ is minimal in the normal domains. Since $T,A \vdash_{RMI} B$, $\mathrm{val}(v(A)) = \mathrm{f}$ too. Hence $v(A)$ is normal and so $|v(A)| \geq |v(B)|$. This, in turn, implies that $\mathrm{val}(v(A \to B)) = \mathrm{t}$, which is a contradiction.

D.2 A Generalization *$T, A_1, \ldots, A_n \vdash_{RMI} B$ iff $T \vdash_{RMI} B \vee \bigvee\!\!\!\bigvee_{1 \leq i \leq n} (A_i \to B) \vee \bigvee\!\!\!\bigvee_{1 \leq i < j \leq n} (A_i \to (A_j \to B)) \vee \bigvee\!\!\!\bigvee_{1 \leq i < j < k \leq n} (A_i \to (A_j \to (A_k \to B))) \vee \bigvee\!\!\!\bigvee \ldots \vee (A_1 \to (A_2 \to \ldots \to (A_n \to B) \ldots))$, where $\bigvee\!\!\!\bigvee_{1 \leq i < j \leq n} A_i \to (A_j \to B)$ etc. are defined in the obvious way.*

Proof: By induction on n, using Theorem D.1 and:
(i) $\vdash_{RMI} R^+(A \to B, C \to B)$
(ii) $\vdash_{RMI} R^+(A \to B, B)$
(iii) $R^+(A,B) \vdash_{RMI} (C \to (A \vee B)) \leftrightarrow ((C \to A) \vee (C \to B))$.

D.3 A Deduction Theorem for Complete Theories *Let T be a complete theory. Then $T,A \vdash_{RMI} B$ iff either $T \vdash_{RMI} B$ or $T \vdash_{RMI} A \to B$.*

Proof: One direction is trivial. For the other, assume that $T,A \vdash_{RMI} B$, where T is a complete theory. By Theorem D.1 $T \vdash_{RMI} B \vee (A \to B)$. Now, since $\vdash_{RMI_{\overset{\sim}{\to}}} {\sim}[(A \to B) \to B] \to [B \to (A \to B)]$, the completeness of T implies that either $T \vdash_{RMI} (A \to B) \to B$ or $T \vdash_{RMI} B \to (A \to B)$. Since $A \to C, B \to C \vdash_{RMI} (A \vee B) \to C$, we have in the first case that $T \vdash_{RMI} B$ and in the second that $T \vdash_{RMI} A \to B$.

The connective $\supset$ lacks, of course, the variable-sharing property, since $\vdash_{RMI} A \supset (B \supset B)$. However, only theorems of this sort can violate the variable-sharing principle in the case of $\supset$:

D.4 Theorem *If $\vdash_{RMI} A \supset B$ then either $\vdash_{RMI} B$ or A and B share a variable.*

Proof: Suppose that $\nvdash_{RMI} B$ and there is no variable common to A and B. By Theorem C.6 we can find a valuation in the canonical r.d.l. for which $v(B)$ is not designated. Since A and B share no variable we may assume that v assigns to all the variables of A the same neutral value. Hence $v(A)$ is neutral and so designated, while $v(B)$ is not designated. It follows that $v(A \supset B)$ is not designated, contradicting $\vdash_{RMI} A \supset B$.

In view of Theorem D.4, the validity in *RMI* of the following strong version of Craig's Interpolation Theorem might be expected:

D.5 The Interpolation Theorem $\vdash_{RMI} A \supset B$ *if either* $\vdash_{RMI} B$ *or there is an interpolant C, containing only atomic variables common to A and B such that* $\vdash_{RMI} A \supset C$ *and* $\vdash_{RMI} C \supset B$. *C has the form of a* $\vee$*-disjunction of* $\wedge$*-conjunctions of sentences in the language of* $RMI_{\rightleftarrows}$.

Note Classically there is also a third possibility: The provability of $\sim A$. In *RMI*, in contrast, that $\vdash_{RMI} \sim A$ does not entail that $\vdash_{RMI} A \supset B$, since $\sim A, A \nvdash_{RMI} B$.

Proof: The "if" part is obvious. For the converse, it is enough by Theorem D.4 to show that an interpolant C as above exists whenever A and B share a variable. Let, accordingly, $p_1, \ldots, p_n$ be all the atomic variables common to A and B $(n \geq 1)$, and let $p_{n+1}, \ldots, p_{n+m}$ be the other variables of A. We construct an interpolant C as follows: To every finite prime model $(\boldsymbol{M}, v)$ of A we shall assign a sentence $C(\boldsymbol{M}, v)$ with the following properties: (i) all its atomic variables are from $\{p_1, \ldots, p_n\}$; (ii) it is a conjunction of sentences in the language of $RMI_{\rightleftarrows}$; (iii) it is true in $(\boldsymbol{M}, v)$; (iv) every finite prime model $(\boldsymbol{M}', v')$ of $C(\boldsymbol{M}, v)$ (where v' is not defined for $p_{n+1}, \ldots, p_{n+m}$) may be extended to a finite prime model $(\boldsymbol{M}'' v'')$ of A (by this we mean that $\boldsymbol{M}'$ is a sub-r.d.l. of $\boldsymbol{M}''$ and that $v''(\varphi) = v'(\varphi)$ whenever $v'(\varphi)$ is defined). Once $C(\boldsymbol{M}, v)$ has been produced for any such $(\boldsymbol{M}, v)$ we take C to be the $\vee$-disjunction of all the $C(\boldsymbol{M}, v)$'s. (The number of the $C(\boldsymbol{M}, v)$'s will be finite since up to isomorphism there is essentially only a finite number of different finite prime models $(\boldsymbol{M}, v)$ of A (ignoring superfluous elements of M) and $C(\boldsymbol{M}, v)$ will be identical to $C(\boldsymbol{M}', v')$ whenever $(\boldsymbol{M}, v)$ and $(\boldsymbol{M}', v')$ are essentially the same in this sense.)

By property (iii) of the $C(\boldsymbol{M}, v)$'s, at least one disjunct of C will be true in any model of A. By the deduction theorem it will follow therefore that $\vdash_{RMI} A \supset C$. On the other hand, from $\vdash_{RMI} A \supset B$, property (iv) of the $C(\boldsymbol{M}, v)$'s, the deduction theorem, and the fact that if $D_1 \vdash_{RMI} B$, $D_2 \vdash_{RMI} B$ then $D_1 \vee D_2 \vdash_{RMI} B$, it follows that $\vdash_{RMI} C(\boldsymbol{M}, v) \supset B$ and so that $\vdash_{RMI} C \supset B$.

It therefore remains, given a finite prime model $(\boldsymbol{M}, v)$ of A, to construct an appropriate $C(\boldsymbol{M}, v)$. We use for this the following observation:

(*) Let $\psi_1, \ldots, \psi_t$ be all the sentences of the form $p_i + p_j$, where $1 \leq i \leq j \leq n + m$. Using II.25 of [7] and the fact that in full r.d.l.'s $|a| \vee |b| = |a + b|$, it is easy to see that if $(\boldsymbol{M}, v)$ is a prime model of A and $(\boldsymbol{M}', v')$ is another prime model satisfying:

(1) If $|v(\psi_i)| \geq |v(A)| (1 \leq i \leq t)$ then $v'(\psi_i) \in T_{M'} \Leftrightarrow v(\psi_i) \in T_M$
(2) If $|v(\psi_i)| \geq |v(A)|$ then $|v(\psi_i)| \leq |v(\psi_j)|$ or $|v(\psi_j)| \leq |v(\psi_i)|$ iff $|v'(\psi_i)| \leq |v'(\psi_j)|$ or $|v'(\psi_j)| \leq |v'(\psi_i)|$ respectively $(1 \leq i,j \leq t)$

then $(\boldsymbol{M}', v')$ is also a model of A. (Note: $p_i + p_j$ is equivalent to p_i when $i = j$.)

(*) can easily be strengthened as follows:

(**) If $(\boldsymbol{M}, v)$ is a prime model of A then so is any other prime model $(\boldsymbol{M}', v')$ satisfying:

(1′) If $|v(\psi_i)| \geq |v(A)|$ then if ψ_i is true in $(\boldsymbol{M}, v)$ it is true also in $(\boldsymbol{M}', v')$ (similarly with $\sim\psi_i$)
(2′) If $|v(\psi_i)| \geq |v(A)|$ then $|\psi_i| \leq |\psi_j|$ is true in $(\boldsymbol{M}', v')$ if it is true in $(\boldsymbol{M}, v)$ (similarly with $|\psi_j| \leq |\psi_i|$, $\sim(|\psi_i| \leq |\psi_j|)$, and $\sim(|\psi_j| \leq |\psi_i|)$).

Note The difference between (*) and (**) lies in the fact that the truth of (say) $\sim(|\psi_i| \leq |\psi_j|)$ in $(\boldsymbol{M}', v')$ does not guarantee that $|v'(\psi_i)| \not\leq |v'(\psi_j)|$, since there is also the possibility that $v'(\psi_i) = v'(\psi_j)$ and both are neutral. It is easy to see however that in this case (*) holds with respect to $(\boldsymbol{M}', v')$ and another prime model of A, $(\boldsymbol{M}'', v'')$, which results from $(\boldsymbol{M}, v)$ by replacing some subsets of M of the form $\{x \in M: |x| \leq |a|\}$ by the single element $(|a|,\mathrm{I})$ (we use here the tree-representation of full r.d.l.'s), and defining $v''(p) = (|a|,\mathrm{I})$ if $v(p)$ was replaced by $(|a|,\mathrm{I})$, $v''(p) = v(p)$ otherwise. (As in the proof of Theorem C.3 such an $(\boldsymbol{M}'', v'')$ is a model of at least any sentence that was true in the original $(\boldsymbol{M}, v)$.) Hence (**) does follow from (*).

From (**) we can obtain without difficulties:

(***) Let $(\boldsymbol{M}, v)$ be a model of A, and let $(\boldsymbol{M}', v')$ be any model such that (1′) and (2′) of (**) hold whenever $v'(\psi_i)$ is defined. Then $(\boldsymbol{M}', v')$ can be extended to a model of A (with the meaning of "extension" we explained above).

The construction of $C(\boldsymbol{M}, v)$ for a given finite prime model $(\boldsymbol{M}, v)$ of A is now straightforward: Let $\psi_1, \ldots, \psi_t$ be as above and suppose that $\psi_1, \ldots, \psi_s$ $(s \leq t)$ are all the sentences among the ψ_i's which contain only variables from $\{p_1, \ldots, p_n\}$. If there is no $1 \leq i \leq s$ such that $|v(\psi_i)| \geq |v(A)|$ we take $C(\boldsymbol{M}, v)$ to be $p_1 \to p_1$ (it is easy to see that (***) and $\vdash_{RMI} A \supset B$ entail that $\vdash_{RMI} B$ in this case). Otherwise we take $C(\boldsymbol{M}, v)$ to be the conjunction of all the sentences of the following forms which are *true* in $(\boldsymbol{M}, v)$:

(1) ψ_i, if $1 \leq i \leq s$ and $|v(\psi_i)| \geq |v(A)|$
(2) $\sim\psi_i$, if $1 \leq i \leq s$ and $|v(\psi_i)| \geq |v(A)|$
(3) $|\psi_i| \leq |\psi_j|$, if $1 \leq i, j \leq s$, and either $|v(\psi_i)| \geq |v(A)|$ or $|v(\psi_j)| \geq |v(A)|$
(4) $\sim(|\psi_i| \leq |\psi_j|)$ if $1 \leq i, j \leq s$, and either $|v(\psi_i)| \geq |v(A)|$ or $|v(\psi_j)| \geq |v(A)|$.

Obviously this $C(\boldsymbol{M}, v)$ has properties (i) and (ii) required above. By (***) it also has property (iv). Finally, we have that $|v(\varphi)| \geq |v(A)|$ for every conjunct ψ of $C(\boldsymbol{M}, v)$. Hence *all the conjuncts are relevant to each other* in $(\boldsymbol{M}, v)$, and since they are also true there (by definition) so is their conjunction $C(\boldsymbol{M},v)$. This concludes the proof of Theorem D.5.

Notes

(1) By substituting $B = A$ in the above theorem it follows that given a sentence A we can find another sentence C, which is a $\vee$-disjunction of $\wedge$-conjunctions of sentences in the language of $RMI_{\rightleftarrows}$, such that $\vdash_{RMI} A \supset C$ and $\vdash_{RMI} C \supset A$. One should remember, however, that this "normal form" is *not equivalent* to A since $\sim A \supset \sim C$ may not be a theorem of *RMI* (we have, e.g., that $\vdash_{RMI} A \supset B$ and $\vdash_{RMI} B \supset A$ whenever $\vdash_{RMI} A$ and $\vdash_{RMI} B$).

(2) The method of proof of Theorem D.5 can be used (in a simpler form) for deriving an interpolation theorem for $\rightarrow$ in $RMI_{\rightleftarrows}$. An easier proof of this was given in Avron [3].

In the formulations of *RMI* given above the primitive connectives were essentially the purely relevant ones: $\rightarrow$ (or $+$), $\wedge$, and $\sim$. The following theorem shows (among other things) that we can take as primitives the semi-extensional connectives of *RMI*: $\sim$, $\vee$, and $\supset$ ($\sim$ can be considered as relevant as well as an extensional connective; see below).

For the following theorem recall that we call A and B equivalent in *RMI* if $\vdash_{RMI} A \leftrightarrow B$, where $A \leftrightarrow B$ is defined either as $(A \rightarrow B) \circ (B \rightarrow A)$ or $(A \rightarrow B) \wedge (B \rightarrow A)$ (the two formulations are equivalent in *RMI*). The equivalence of A and B is also equivalent to their having the same value in every model. We have also: $A \leftrightarrow B \vdash_{RMI} \varphi(A) \leftrightarrow \varphi(B)$.

D.6 Theorem

(1) *$A \rightarrow B$ is equivalent in RMI to $(A \supset B) \wedge (\sim B \supset \sim A)$ (thus it is definable in terms of $\sim$, $\vee$, and $\supset$)*

(2) *For all*

$$A_1, A_2 \in \{R^+(A,B), R^\wedge(A,B), A \supset (B \supset (A \wedge B)),$$
$$\sim A \supset (\sim B \supset \sim (A \vee B))\}$$

we have $A_1 \vdash_{RMI} A_2$

(3) *$A \rightarrow B$ holds in a full model $(\boldsymbol{M},v)$ if $A \supset B$, $\sim B \supset \sim A$, and $R(A,B)$ hold there. Hence $T \vdash_{RMI} A \rightarrow B$ iff $T \vdash_{RMI} A \supset B$, $T \vdash_{RMI} \sim B \supset \sim A$, and $T \vdash_{RMI} \sim A \supset (\sim B \supset \sim (A \vee B))$*

(4) *$A \leftrightarrow B$ holds in a full model $(\boldsymbol{M},v)$ iff $A \supset B$, $B \supset A$, $\sim A \supset \sim B$, $\sim B \supset \sim A$,* and *$R(A,B)$ hold there.*

Proof: We leave the easy details to the reader.

In view of the deduction theorem for $\supset$ and the properties of $\vee$ listed in C.14 we have that if we take $\sim$, $\vee$, and $\supset$ to be the primitive connectives of *RMI* then the "positive" fragment of *RMI* (i.e., that $\{\supset,\vee\}$-fragment) is as least as strong as the corresponding intuitionistic fragment. Actually we have more:

D.7 Proposition *The set of theorems of RMI in the $\{\supset,\vee\}$-language is identical to the corresponding fragment of the system LC of Dumment (which is stronger than the intuitionistic fragment, see Dummett [12]). An axiomatization of this fragment can be therefore given as follows:*

Axioms (1) $A \supset (B \supset A)$
(2) $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
(3) $((A \supset B) \supset C) \supset ((B \supset A) \supset C) \supset C$
(4) $A \supset A \vee B$
(5) $B \supset A \vee B$
(6) $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$.

Rule of inference $A, A \supset B \vdash B$ (M.P. for $\supset$).

Proof: The validity of Axioms (1), (2), and (4)–(6) (which are the intuitionistic axioms) follows from the deduction theorem and C.14. Adding the third axiom is equivalent to adding $(A \supset B) \vee (B \supset A)$, and the validity of it in full r.d.l.'s can easily be checked.

For the converse we note that in Avron [4] it was shown that even the corresponding fragment of the stronger *RM* is identical with that of *LC* (this was essentially proved already in Dunn and Meyer [14]).

D.8 Note Using Theorem D.6 and Proposition D.7 it is not difficult to give a quite perspicuous axiomatization of *RMI* in the $\{\sim, \vee, \supset\}$-language with M.P. for $\supset$ as the single rule of inference. Note that $A \supset \sim\sim A$, $\sim\sim A \supset A$, $\sim A \vee A$, $A \supset (\sim B \supset \sim(A \supset B))$, $\sim(A \supset B) \supset \sim B$, $\sim(A \vee B) \supset \sim A$, and $\sim(A \vee B) \supset \sim B$ are valid while $\sim A \supset (A \supset B)$, $(A \supset B) \supset (\sim B \supset \sim A)$, $\sim A \supset (\sim B \supset \sim(A \vee B))$, and $\sim A \supset ((A \vee B) \supset B)$ are not. In this formulation *RMI* looks like a paraconsistent logic rather then a relevance logic.

We end this section with some independence results concerning definability in *RMI*. Let us say that an n-connective # of *RMI* (primitive or defined in *RMI*-language) is definable in *RMI* in terms of a given set of other connectives of *RMI* if there exists a formula $C(p_1, \ldots, p_n)$, formulated in terms of the given set of connectives, such that

$$\vdash_{RMI} \#(p_1, \ldots, p_n) \leftrightarrow C(p_1, \ldots, p_n).$$

D.9 Proposition
(1) $\vee$ *and* $\wedge$ *are not definable in terms of* $\sim$, $\rightarrow$, *and* $\supset$
(2) $\supset$ *is not definable in terms of* $\sim$ *and* $\rightarrow$
(3) $+$, $\rightarrow$, $\wedge$, *and* $\vee$ *are not definable in terms of* $\sim$ *and* $\supset$
(4) $+$, $\rightarrow$, *and* $\supset$ *are not definable in terms of* $\sim$ *and* $\wedge$.

Proof: In [4] we have shown (1), (2), and (4) to be true even for the stronger *RM*. To show (3) note that if $A(p,q)$ is a sentence in the language $\{p, q, \sim, \supset\}$ then in every full model $(\boldsymbol{M}, v)$ we have that $|v(A)| = |v(p)|$ or $|v(A)| = |v(q)|$, $p + q$, $p \wedge q$, $p \rightarrow q$, and $p \vee q$ all lack this property.

E A multiple-conclusioned formulation of RMI In this section we give *RMI* a new kind of presentation. We use what in Shoesmith and Smiley [19] is called a "multiple-conclusioned logic", and is also called in Gabbay [15] a "Scott system". This version of *RMI* is in fact stronger than the single-conclusioned ones, in the sense that it allows us to express (and prove) logical facts concerning full r.d.l.'s which *cannot be expressed* in the former versions of *RMI*. On the other hand the new system is a conservative extension of *RMI* (this means that if *T* is an *RMI*-theory and A a sentence in the language of *RMI* then A follows from T according to the new system iff $T \vdash_{RMI} A$). The Gentzen-type formulation of *RMI* and the normal-form techniques connected with it correspond directly to the system of the present section, and not to the system we have investigated so far.

By a "Scott system" we understand as in [15] a binary relation $|\vdash$ between finite sets of sentences, closed under substitutions, and satisfying the following conditions:

(a) $A \mid\vdash A$ (i.e., $\{A\} \mid\vdash \{A\}$)
(b) If $\Gamma \mid\vdash \Delta$, $\Gamma \subset \Gamma'$, and $\Delta \subset \Delta'$ then $\Gamma' \mid\vdash \Delta'$
(c) (cut): If $\Gamma_1 \mid\vdash \Delta_1 \cup \{A\}$ and $\Gamma_2 \cup \{A\} \mid\vdash \Delta_2$ then $\Gamma_1 \cup \Gamma_2 \mid\vdash \Delta_1 \cup \Delta_2$.

(The intuitive interpretation of $\Gamma \mid\vdash \Delta$ is: in each case in which all the sentences of Γ are true at least one of the sentences of Δ is true.)

E.1 Definition (a) The relation $|\vdash_{RMI}$ is the minimal Scott system whose formulas are constructed in the language based on $\{\sim,\rightarrow,\wedge\}$ and which satisfies:

(1) $|\vdash_{RMI} A$ for every axiom of $RMI_{\min}$
(2) $|\vdash_{RMI} A, \sim A$
(3) $A, A \rightarrow B \mid\vdash_{RMI} B$
(4) $A \rightarrow B, A \rightarrow C \mid\vdash_{RMI} A \rightarrow (B \wedge C)$.

(b) If T_1 and T_2 are two sets of sentences then $T_1 \mid\vdash_{RMI} T_2$ iff there exist finite sets $\Gamma_1 \subset T_1$ and $\Gamma_2 \subset T_2$ such that $\Gamma_1 \mid\vdash_{RMI} \Gamma_2$.

Note What is essentially added here to $RMI_{\min}$ is the condition $|\vdash_{RMI} A, \sim A$. Without this the transition to a multiple-conclusioned system would be a conservative extension of $RMI_{\min}$, since by a theorem of Scott (see [15], pp. 7–8), if $\vdash$ is an ordinary (single-conclusioned) provability relation (see (I) of the introduction) and we define $\Gamma \mid\vdash \Delta$ iff $\Gamma \vdash A$ for some $A \in \Delta$, then $|\vdash$ is the minimal Scott relation extending $\vdash$ and $\Gamma \mid\vdash A$ iff $\Gamma \vdash A$.

E.2 A Completeness Theorem *Let T_1 and T_2 be sets of sentences. Then $T_1 \mid\vdash_{RMI} T_2$ iff every full model $(\mathbf{M}, v)$ of T_1 is also a model of at least one of the sentences of T_2.*

Proof: The soundness direction is by now trivial.

For the converse assume that $T_1 \mid\not\vdash_{RMI} T_2$. Let T_0 be a maximal extension of T_1 such that $T_0 \mid\not\vdash_{RMI} T_2$. We show first that if $\Delta = \{A_1, \ldots, A_n\}$ is a finite set of sentences then $T_0 \mid\vdash_{RMI} \Delta$ iff $A_i \in T_0$ for some i. Evidently $A_i \in T_0$ implies $T_0 \mid\vdash \Delta$. Assume, for a contradiction, that $T_0 \mid\vdash_{RMI} \Delta$ and $A_i \notin T_0$ for all i. By the maximality of T_0 this means that there exist finite $\Gamma_i \subseteq T_0$ and $\Delta_i \subset T_2$ such that $A_i, \Gamma_i \mid\vdash_{RMI} \Delta_i$. Using n cuts we obtain that $T_0 \mid\vdash_{RMI} \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_n$. (For example, in case $n = 2$ our assumptions imply that there are subsets $\Gamma_0, \Gamma_1, \Gamma_2$ of T_0 and subsets Δ_1, Δ_2 of T_2 such that $\Gamma_0 \mid\vdash_{RMI} A_1, A_2$; $\Gamma_1, A_1 \mid\vdash_{RMI} \Delta_1$; $\Gamma_2, A_2 \mid\vdash_{RMI} \Delta_2$. A cut of the first two sequents yields $\Gamma_0, \Gamma_1 \mid\vdash_{RMI} A_2, \Delta_1$ and a cut of this sequent and of $\Gamma_2, A_2 \mid\vdash_{RMI} \Delta_2$ yields $\Gamma_0, \Gamma_1, \Gamma_2 \mid\vdash_{RMI} \Delta_1, \Delta_2$. Since $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \subseteq T_0$, this means that $T_0 \mid\vdash \Delta_1 \cup \Delta_2$.) This contradicts our assumption that $T_0 \mid\not\vdash_{RMI} T_2$.

Since $|\vdash_{RMI} A, \sim A$ it follows that T_0 is a complete $RMI_{\min}$ theory. We have also: $T_0 \vdash_{RMI} B$ iff $B \in T_0$. Such a theory determines, as usual (see the proof of Proposition B.11), a full model $(\boldsymbol{M}, v)$ in which exactly the elements of T_0 are true. In particular, all the sentences of T_1 are true in $(\boldsymbol{M}, v)$ while all the sentences of T_2 are not (for evidently $T_0 \cap T_2 = \emptyset$).

E.3 Corollary $\vdash_{RMI} A$ *iff* $|\vdash_{RMI} A$.

Note Various Scott systems for various logics were already developed and used in the past. In all these past cases their use was not essential but only (sometimes) technically convenient. By this we mean that in all those Scott systems a connective $\vee$ was available such that $T \mid\vdash A_1, \ldots, A_n$ iff $T \mid\vdash A_1 \vee A_2 \vee \ldots \vee A_n$. Thus, facts expressible in the multiple-conclusioned system could have been translated into the single-conclusioned counterpart. In the case of *RMI* things are essentially different:

E.4 Theorem *No formula* $\varphi(A,B)$ *in the language of RMI has the property that* $T \mid\vdash_{RMI} A,B$ *iff* $T \vdash_{RMI} \varphi(A,B)$.

Proof: Suppose for a contradiction that φ is such a formula. Let $A = p \supset q$ and $B = q \supset p$. $A \vee B$ is easily seen to be valid in every prime model and so $\vdash_{RMI} A \vee B$. It follows that if T is a prime theory then either $T \vdash_{RMI} A$ or $T \vdash_{RMI} B$. Hence if T is prime then $T \mid\vdash_{RMI} A,B$; and so $T \vdash_{RMI} \varphi(A,B)$. By the proof of Theorem C.2 this implies that $\vdash_{RMI} \varphi(A,B)$ and so, by our assumption on φ, that $\mid\vdash_{RMI} A,B$. Now consider a full model in which p, q, and pRq are false; then both A and B are false (although $A \vee B$ holds!), which is a contradiction.

In view of the last theorem it is clear that for the multiple-conclusioned system (in contrast to the single-conclusioned one) there is an essential difference between prime models and others:

E.5 Proposition *Let* $\mid\vdash_{RMIP}$ *be the Scott system obtained by replacing (2) of Definition E.1 by (2)′:* $A \vee B \mid\vdash_{RMIP} A,B$. *Then:*

(a) $T_1 \mid\vdash_{RMIP} T_2$ *if every prime model of* T_1 *is a model of at least one of the sentences of* T_2

(b) $T \mid\vdash_{RMIP} A_1, \ldots, A_n$ *iff* $T \mid\vdash_{RMIP} A_1 \vee A_2 \vee \ldots \vee A_n$ *iff* $T \vdash_{RMI} A_1 \vee A_2 \vee \ldots \vee A_n$

(c) $T \mid\vdash_{RMIP} A$ *iff* $T \mid\vdash_{RMI} A$ *iff* $T \vdash_{RMI} A$

(d) *If* $T_1 \mid\vdash_{RMI} T_2$ *then* $T_1 \mid\vdash_{RMIP} T_2$, *but not vice versa:* $\mid\vdash_{RMIP} A \supset B, B \supset A$ *but* $\mid\not\vdash_{RMI} A \supset B, B \supset A$.

Proof: We leave the proof to the reader.

F Extensional connectives The meaning of "extensional connective" in classical logic is that the truth or falsity of a compound formed by applying the connective depends only on the truth values of the various components, each of these being either 'true' or 'false'. This may indicate how to extend the notion for nonclassical logics, but one should not expect there to be a definition covering all cases by which connectives can be meaningfully classified into "extensional" and "nonextensional". Suppose that the logic has a semantics based on the notions of model and satisfaction, such that a strong completeness theorem holds. Then, with respect to this given semantics, one could define a connective (say #) to be extensional if in any model the satisfaction or nonsatisfaction of $\#(A_1,\ldots,A_n)$ is determined once we know which of the A_i's are satisfied. However, the same logic may have a different semantics (satisfying strong completeness) so that a connective may be "extensional" in one but "nonextensional" in another. In the case of classical logic, the standard two-valued semantics is the obvious natural choice, for it underlies the very intuitions that the logic expresses. But in other cases there may not be a clearly preferred semantics. *RMI* is strongly complete with respect to the semantics of full r.d.l.'s as well as with respect to the narrower semantics of prime r.d.l.'s. The connective $\vee$ is nonextensional in the first but extensional in the second.

In the case of a multiple-conclusioned logic, the above-mentioned semantic definition of extensionality can be put in an equivalent syntactic form so that it no longer depends on the particular semantics. The definition as given in Gabbay [15] is:

F.1 Definition Let $|\vdash$ be a Scott consequence relation and let $\#(p_1,\ldots,p_n)$ be a formula based on the propositional variables $p_1,\ldots,p_n$. We say that $\#(p_1,\ldots,p_n)$ is *extensional* (or *classical*) if for every sequence $\bar{a} \in \{0,1\}^n$ the following holds:

Let $\Gamma_{\bar{a}} = \{p_i | a_i = 1\}$, $\Delta_{\bar{a}} = \{p_i | a_i = 0\}$; then either $\Gamma_{\bar{a}} \mid\vdash \Delta_{\bar{a}},\#(p_1,\ldots,p_n)$ or $\Gamma_{\bar{a}},\#(p_1,\ldots,p_n) \mid\vdash \Delta_{\bar{a}}$ (but not both, which should be the case unless the consequence relation is trivial). If f: $\{0,1\}^n \to \{0,1\}$ then $\#(p_1,\ldots,p_n)$ is extensional with truth table f if $\Gamma_{\bar{a}} \mid\vdash \Delta_{\bar{a}}$, $\#(p_1,\ldots,p_n)$ whenever $f(\bar{a}) = 1$, and $\Gamma_{\bar{a}}$, $\#(p_1,\ldots,p_n) \mid\vdash \Delta_{\bar{a}}$ whenever $f(\bar{a}) = 0$.

Now suppose that we have a semantics based on a class of models and a satisfaction notion such that for all sets of formulas Γ,Δ:

$\Gamma \mid\vdash \Delta$ iff every model satisfying all the formulas of Γ satisfies also some formulas of Δ.

Then $\#(p_1,\ldots,p_n)$ is extensional with truth table f iff in every model the truth value of $\#(p_1,\ldots,p_n)$ is f (truth value $(p_1),\ldots,$ truth value (p_n)), where the truth value is, by definition, 1 if the formula is satisfied, 0 if not. (Note that this concept of semantics for a Scott system is "stronger" or "richer" than the usual concept of semantics for single-conclusioned systems.)

We have seen that there is more than one way of extending *RMI* to a multiple-conclusioned system. The following theorem is true with respect to the extension which we proposed and is based on the semantics of the class of all full r.d.l.'s.

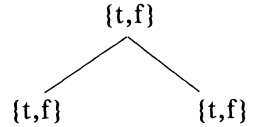
F.2 Theorem *Of the sixteen possible binary truth functions only the following three can be represented by extensional connectives in the multiple-conclusioned extension of RMI:*

$$f_1(a_1,a_2) \equiv a_1;\ f_2(a_1,a_2) \equiv a_2;\ f_3(a_1,a_2) \equiv 1.$$

Proof: Let $\varphi = \varphi(p_1,p_2)$ be a formula containing only the propositional variables p_1,p_2. Consider the class of all full models (M,v) such that $v(p) \not R v(q)$. We claim that exactly one of the following possibilities holds:

(a) In all such models $\mathrm{val}(v(\varphi)) = \mathrm{t}$ and $|v(\varphi)| = |v(p_1)| \vee |v(p_2)|$
(b) In all such models $\mathrm{val}(v(\varphi)) = \mathrm{f}$ and $|v(\varphi)| = |v(p_1)| \vee |v(p_2)|$
(c) In all such models $|v(\varphi)| = |v(p_1)|$, and $\mathrm{val}(v(\varphi))$ is determined via some fixed function $g: \{\mathrm{t,I,f}\} \to \{\mathrm{t,I,f}\}$ by $\mathrm{val}(v(p_1))$. Moreover, $g(\mathrm{I}) = \mathrm{I}$.
(d) As in (c) but with p_1 replaced by p_2.

Assume for the moment the claim: If φ is extensional with truth table f, then in case (c), e.g., $f(a_1, a_2)$ depends only on a_1. This follows immediately from considering those models $(\boldsymbol{M}, v)$ which belong to the class for which the claim was made and in which $\boldsymbol{M}$ is the full r.d.s.:



Also, since I corresponds to the standard value 1 and in case (c) $g(\mathrm{I}) = \mathrm{I}$ we must have that $f(1, \ ) = 1$. Hence case (c) gives rise either to f_1 or to f_3. Similarly case (d) gives rise either to f_2 or f_3. It is also easily seen that in case (a) f must be f_3. Finally, in case (b) f can only be the constant function 0. But in a model in which $v(p_1) = v(p_2)$ and $\mathrm{val}(v(p_1)) = \mathrm{I}$ we must have that $\mathrm{val}(v(\varphi)) = \mathrm{I}$ (true in general). Hence we must have that $f(1,1) = 1$. Thus in case (b) φ cannot be extensional.

Proof of the claim: We use an induction on the length of φ:

(1) If $\varphi(p,q) = p$ or $\varphi(p,q) = q$ then φ has property (c) or (d) respectively.
(2) If $\varphi = {\sim}\psi$ then φ has property (a) (property (b)) if ψ has property (b) (property (a)), and φ has property (c) (or (d)) iff ψ has the same property.
(3) If $\varphi = \psi_1 + \psi_2$ then if either ψ_1 or ψ_2 has property (a) so does φ. If both lack (a) but one of them has (b) so does φ. If both have (c) or both have (d) so does φ. If one has (c) while the other has (d) then φ has property (b).
(4) If $\varphi = \psi_1 \wedge \psi_2$ then if ψ_1 has property (a) then φ has the same property that ψ_2 has. Similarly if ψ_2 has (a). If one of ψ_1, ψ_2 has property (b) so does φ. Other cases are similar to the case of $+$.

Note "Classical connectives" with truth functions f_1, f_2, or f_3 are definable in every Scott system having "logical truths" (i.e., sentences A such that $|\vdash A$).

F.3 By relaxing somewhat the requirements concerning extensionality we get a richer class of "extensional connectives". Call a *generalized truth table* a function $g: \{\mathrm{t,I,f}\}^n \to \{0,1\}$ such that $g(\mathrm{I,I,\ldots,I}) = 1$. Call $\#(p_1, \ldots, p_n)$ *weakly extensional* with generalized table g if in any model $(\boldsymbol{M}, v)$ we have $h(\mathrm{val}(\#(p_1, \ldots, p_n))) = g(\mathrm{val}(v(p_1)), \ldots, \mathrm{val}(v(p_n)))$, where $h(\mathrm{t}) = h(\mathrm{I}) = 1$ and $h(\mathrm{f}) = 0$. Then we have:

F.4 Theorem *Of the* 512 *possible two-argument generalized truth tables there are seven which can be represented by weakly extensional formulas. The seven formulas are:*

(1) $p \to p$ (or $(p \to q) \to (p \to q)$)
(2) p (or $((q \to q) \supset p)$)
(3) $\sim p$
(4) $\sim (p \to p)$
(5) q
(6) $\sim q$
(7) $\sim (q \to q)$.

The proof is by the same sort of analysis we used in the proof of Theorem F.2 and is left to the reader.

Note For sentences with a single propositional variable there is no difference between *RMI* and classical logic, and all four possible (generalized) truth functions have corresponding connectives in *RMI*. Theorem F.4 shows that essentially nothing new is added while considering binary truth functions. Moreover, it also shows that negation is the only nontrivial weakly extensional connective which is definable in *RMI*.

G Extensions of RMI In this section we briefly investigate various extensions of *RMI*. An extension, we recall, is obtained by adding new axiom schemes and, possibly, new derivation rules. A key result in this investigation is the following:

G.1 Theorem *Let X be an extension of RMI. Let $P =$ the class of all full r.d.l.'s $\boldsymbol{M}$ such that for every A, if $\vdash_X A$ then A holds in $\boldsymbol{M}$ under all valuations. Then X is weakly complete relative to P (i.e., $\vdash_X A$ iff A is valid in all elements of P). Moreover, if X is obtained from RMI by adding only axiom schemes then X is strongly complete relative to P. The same holds if we restrict ourselves to prime r.d.l.'s.*

Proof: Without loss in generality we may assume that $\vdash_X A$ iff $A \in X$. Let $B \notin X$. By the proof of Theorem C.2 we can find a prime theory T_0 such that $X \subseteq T_0$ but $T_0 \nvdash_{RMI} B$. As usual, the Lindenbaum algebra of T_0 is a prime r.d.l. $\boldsymbol{M}$ in which $[A] \in T_M$ if $T_0 \vdash_{RMI} A$. Since X is closed under substitutions, this entails that every theorem of X is valid in $\boldsymbol{M}$. B, however, is not valid (use, as usual, the canonical valuation $v(A) = [A]$). From this the first part of the theorem follows at once. For the second part assume that $T \nvdash_X B$. If X is obtained by adding only axiom schemes then $T \vdash_X B$ is equivalent to $T \cup X \vdash_{RMI} B$. Extend $T \cup X$ to a prime T_0 and argue as before.

G.2 Corollary *RM is obtained from RMI by each of the following methods: (This strengthens a result of Avron* [5]):
(1) *Adding the scheme $R^+(A,B)$*
(2) *Adding the scheme $R^\wedge(A,B)$*
(3) *Strengthening* re. adj. *in the second formulation of RMI to the full adjunction rule: From A and B infer $A \wedge B$*
(4) *Adding the disjunctive syllogism* (D.S.) *for $\vee$ as an extra rule of inference: $\sim A, A \vee B \vdash B$.*

Proof: That we get *RM* by adding either $R^+(A,B)$ or $R^\wedge(A,B)$ to *RMI* was already proved in Corollary C.9. From this (3) also follows immediately.

That *RM* is closed under (D.S.) for $\vee$ is a theorem of Meyer and Dunn (see [2]). Suppose we add this rule to *RMI*. Let A and B be any two sentences, and define $C = \sim(A \to A)$, $D = \sim(B \to B) \vee R^{\wedge}(A,B)$. Then $\vdash_{RMI} \sim C, \vdash_{RMI} C \vee D$ (by De Morgan laws). Two applications of (D.S.) yield $R^{\wedge}(A,B)$. Hence (4) follows from (2).

G.3 Corollary[11] *RM is strongly complete relative to Sugihara chains.*

Proof: $R^{+}(A,B)$ is valid exactly in full r.d.l.'s in which every two elements are relevant, i.e. (see [7], II.7.C), full r.d.l.'s that are totally ordered by $\leq$. These are exactly the Sugihara chains. Hence the theorem follows from Theorem G.1 and Corollary G.2(1).

Note In view of our results the most natural formulation of *RM* is *RMI* (first formulation) + $R^{+}(A,B)$. Since (D) (the distribution axiom) holds in *RM* we may simplify the formulation by replacing (RD) (relevant distribution) with (D). What we get is simply $RMI^{1}_{\rightleftharpoons} \cup R_{fde}$[12] (the first-degree-entailments fragment of R and E). This formulation of *RM* is then conservative with respect to its implication-negation fragment.

G.4 Discussion Corollary G.2 determines the exact place of *RM* among relevance logics. Since in *RM* every two sentences are relevant one can describe it as being obtained from *RMI* by giving up the relevance idea while keeping the notion of grading as expressed in the Sugihara chains. (Note that it does not matter whether we choose to violate the variable-sharing principle for $+$ or for $\wedge$ for this!).

Finally, we review other results on extensions of *RMI*. The proofs are not difficult, and most of them were essentially given in [5].

G.5 Definition RMI_D is the system obtained from *RMI* by strengthening (RD) to (D) (the distribution axiom).

G.6 Theorem

(1) *RMI_D is strongly complete for the set of full r.d.l.'s which have at most one pair of elements a, b such that $a \not R b$. Such full r.d.l.'s are either linear (i.e., Sugihara chains) or they result from a chain of normal domains by adding two neutral minimal elements at the end of the chain. (In both cases the r.d.l.'s are prime.)*

(2) *RMI_D has the variable-sharing property for both $\wedge$ and $\to$. Its first-degree-entailment fragment is exactly R_{fde}, while its negation-implication is $RMI^{(2)}_{\rightleftharpoons}$ (see Avron* [3]*).*

Discussion The difference between *AS*, the characteristic matrix of RMI_D, and the Sugihara matrix, which is characteristic of *RM*, is that *AS* has two "zeroes" instead of one. From a semantical point of view the effect of adding the distribution axiom to *RMI* is therefore similar to that of adding to it (adj.) or (D.S.) for $\vee$. It is remarkable that in R and also E there is an intimate connection between (D) and (D.S.) for $\vee$: (D) plays a key role in the proof of the admissibility of (D.S.) in R and E ([2], p. 313).

G.7 Theorem *If we replace the axiom* (RD) *of RMI by* $[R^{\wedge}(A,B) \vee R^{\wedge}(A,C) \vee R^{\wedge}(B,C)] \supset [A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)]$ *we get a system which is strongly complete for full r.d.l.'s in which every abnormal domain is < every normal one. This system is a conservative extension of* $RMI_{\overrightarrow{\sim}}$ *(but not of* R_{fde}*).*

G.8 Theorem *By adding either of:* (i) $A \vee (A \rightarrow B)$ (ii) $((A \supset B) \supset A) \supset A$ (iii) $\sim (A \supset B) \supset A$ *to RMI we get a system which is strongly complete relative to r.d.l.'s which have a single normal domain.* A_ω *is a characteristic matrix for this system. Moreover, this system has the same Scroggs property that* $RMI_{\overrightarrow{\sim}}$ *has (see* [3]*), it is conservative with respect to* $RMI_{\overrightarrow{\sim}}$*, and includes every other extension of RMI which has this property. Its* $\{\cup,\supset\}$*-fragment is identical to the corresponding classical one.*

NOTES

1. An interesting Kripke-style semantics *was* developed by Routley and Meyer in [18]. Their structures are very complex, though, and in my opinion they are difficult to use while shedding no real new light on the systems. Few people would claim that they are a faithful explication of the basic ideas underlying R and E.

2. A relation $\vdash_L$ satisfying (i)–(iii) is called in Gabbay [15] "a Tarski consequence relation", while a relation satisfying (i)–(iv) is called there a "Tarski system".

3. No such formulations for the systems R and E are known. Avron [6] provides one for RM.

4. From the syntactic point of view idempotency is what distinguished our + from the corresponding connective of R. To us, at least, it seems unquestionably a property any disjunction should possess. Our semantical results heavily depend on it (see [7]).

5. $R_{\overrightarrow{\sim}}$ and $RMI_{\overrightarrow{\sim}}$ have convenient Gentzen-type calculi in which every such claim can easily be proved. See [2] and [3].

6. These are all true for $R_{\overrightarrow{\sim}}$ also. It is not difficult to give constructive proofs in both cases.

7. This theorem was first proved in [3], using another method. The corollaries that follow were also shown there and are included here for the sake of completeness. In [3] it is shown also that $RMI_{\overrightarrow{\sim}}$ has a strong "Scroggs" property: It has no finite characteristic matrix, but every proper extension of it is characterized by some A_n. Similar results hold for its pure implicational fragment.

8. In R, E, and RM we have as primitive the adjunction rule: $A, B \vdash A \wedge B$. The relevant adjunction rule of $RMI_{\min}$ is taken as primitive also in R_{fde}—the "first-degree-entailments" fragment of R and E (see [2], [13]).

9. Another difference concerning $\vee$ between R, E, and RM on the one hand, and RMI on the other, is with respect to the disjunctive syllogism for $\vee$. R, E, and RM are closed under this rule, RMI is not (and there is no reason why it should be).

10. This theorem, as well as the interpolation theorem for $\supset$ (see below), was first proved in [4] in the case of RM.

11. This is a result of Dunn's which extends Meyer's theorem that the Sugihara matrix is characteristic for *RM*. Both results are in fact instances of Theorem G.1.

12. See [3] for the meaning of $RMI^{\frac{1}{\rightarrow}}$ and [2], Chapter III for the meaning of R_{fde}.

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