Notre Dame Journal of Formal Logic
Volume 31, Number 1, Winter 1990

Linear Logic Displayed

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Abstract "Linear logic" (**LL**; see Girard [6]) was proposed to be of use in computer science, but it can be formulated as a "display logic" (**DL**; see Belnap [2]), which is a kind of Gentzen calculus admitting easy proof of an Elimination Theorem. Thus **LL** is naturally placed within a wider proof-theoretical framework that is known to include relevance, intuitionist, and modal logics, etc., and that permits coherent variations on **LL** itself—including the definition of "punctual logic". In order to accommodate **LL**, two independently useful modifications of **DL** are made. First, **DL** possessed an unmotivated identification of two of its structural constants. This identification is dropped in order to make room in **DL** for the several propositional constants in **LL**. Second, **DL** possessed an unmotivated bias towards connectives that, when they are introduced as consequents, have restrictions put on their antecedents. This bias is abandoned in order to make room in **DL** for a dual pair of modal-like "exponential" connectives of **LL**. The latter modification requires restructuring the proof of the Elimination Theorem for **DL**, rendering it perfectly symmetrical in antecedent and consequent.

1 Introduction To "display" any logic is to exhibit it as a display logic, that is, as the special sort of Gentzen consecution calculus defined in Belnap [2] (**DL**), and thus to place that logic within a certain significant proof-theoretical framework. The aim of this paper is to display "linear logic", which is a logic proposed by Girard [6] in connection with some important computer science considerations.[1] It turns out that the display of linear logic requires some healthy adjustments in the universal features of display logic itself.[2] One set of adjustments is required in order to treat the "exponentiation" connectives of linear logic, and another to treat its four (instead of two) propositional constants while keeping to a single "family" of display logic. After we are done, we will be able to see how displaying linear logic permits a well-organized consideration of some of its essential features and of some of its variants.

Received February 15, 1989

2 Adjustments required by exponentiation The first reason that the methods of **DL** need generalizing is that the **DL** proof of the Elimination Theorem, and the **DL** statement of the conditions sufficient to permit its proof to go through, have a distinct bias towards logics like intuitionism that reserve a special place for connectives that, when they are introduced as consequents, have restrictions put on their antecedents ("consequent-biased connectives"). Note 14 of **DL** observed that its methods could easily be adapted to apply to a logic whose bias was dual to that of intuitionism, but also said that they could not be applied to logics biased in both ways at once. This limitation prevents an immediate application of the techniques of **DL** to the two exponentiation connectives of linear logic, which are absolutely symmetrical between consequent-bias and antecedent-bias: one is biased one way and the other is biased the other way. We overcome this limitation of **DL** by making the universal principles of display logic perfectly symmetrical in antecedent and consequent, as they ought to be. The cost is a certain amount of added proof-theoretical tedium that we try to minimize.

In detail, the **DL** proof of the Elimination Theorem is modified as follows. Instead of continuing to follow Curry [4] by relying on three stages, here we will define just two: the Parametric Stage and the Principal Stage. By the *Principal Stage* we mean exactly the same as Stage 3 in the argument of **DL**, §4.3. To introduce the "Parametric Stage", we first do some naming. The statements named "Stage 1" and "Stage 2" in §4.3 of **DL** contain free *M*, *X*, and *Y* which are there implicitly thought of as universally bound, as usual. Here we explicitly bind *X* and *Y* and leave *M* free, calling the new versions *Cons(1)* and *Cons(2)* respectively. We then define *Cons(1) & Cons(2)*, which will have *M* free, as the conjunction of Cons(1) and Cons(2). Duals are newly introduced as *Ant(1)* and *Ant(2)*, and *Ant(1) & Ant(2)* is defined as their conjunction. Then the *Parametric Stage* is defined as follows: For each formula *M*, either Cons(1) & Cons(2) or Ant(1) & Ant(2). As before, we take the Principal Stage and the Parametric Stage as two quite independent propositions; and as before, it is clear that the Elimination Theorem follows from their conjunction. The argument of Curry rehearsed in §4.3 of **DL** shows how Cons(1) & Cons(2) and the Principal Stage (there called Stages 1–3) together suffice to prove the inductive step of the Elimination Theorem, and a dual argument shows that Ant(1) & Ant(2) and the Principal Stage imply it as well. (These definitions are spelled out below in the Appendix.)

Now we need to ask what it takes to prove the Parametric Stage. The point is that sometimes we cannot prove Cons(1) & Cons(2) for each formula *M* of a given display logic, nor can we prove Ant(1) & Ant(2) for each *M*; but we *can* prove that for each *M* at least one of Cons(1) & Cons(2) or Ant(1) & Ant(2) holds.

The condition under which the proof of Cons(1) & Cons(2) as given in §4.3 of **DL** goes through for a given *M* is easy, but we make it just a little easier by giving it a name. Let us say that a formula *M*, relative to a display logic, is *Cons-regular* (see Curry [4]) provided that: (1) when *M* is a parametric *consequent* part you can substitute an *arbitrary* structure, and (2) when *M* is a parametric *antecedent* part you can substitute any structure that *could* be on the antecedent side of the turnstile when *M* has just been introduced as principal in the consequent.

Somewhat more rigorously, if less transparently, a formula *M* is Cons-regular if (1) every rule is closed under replacing *consequent* parametric M by *arbitrary* X, and (2) every rule is closed under replacing *antecedent* parametric M by any X such that $X \vdash M$ is the conclusion of an inference with M *principal*. And most tediously, even if most rigorously, M is Cons-regular under the following two conditions: (1) if M occurs as a consequent parameter of an inference of the logic falling under a certain rule, then the same rule also sanctions the inference obtained by replacing each member of the congruence class of M (in the sense of §4.1 of **DL**) with an arbitrary structure X; and (2) if M occurs as an antecedent parameter of an inference of the logic falling under a certain rule, then the same rule also sanctions the inference obtained by replacing each member of the congruence class of M with any structure X such that $X \vdash M$ is the conclusion of an inference of the logic in which M is *not* parametric.

Observe that to determine if M is Cons-regular requires checking every rule. But this is easy to do, for, as explained under the verification of C6 in §4.2 of **DL**, as long as any rule is stated with the help of structure-variables (instead of formula-variables) for its parameters, and as long as there are no extra "side-conditions" or provisos added to the statement of the rule, it cannot possibly prevent the Cons-regularity of any M.

It is easy to check that the Cons-regularity of M suffices for Cons(1) & Cons(2) holding for M; details were given in the proof of Stage 1 in §4.2 of **DL**. *Ant-regular* is defined in exactly the dual way. So the Ant-regularity of M suffices for Ant(1) & Ant(2) holding for M.

We say that a formula is *regular* if it is either Cons-regular or Ant-regular. The *asymmetric* requirements C6 and C7 of §4.2 of **DL** are now replaced by this perfectly balanced

Regularity condition *Every formula must be regular.*

In accord with the above discussion, if a rule is closed under substitution of structures for any of its parametric constituents, as is the case for "most" rules, that rule cannot be a threat to regularity. Only rules that are *not* closed under substitution of structures for parametric constituent formulas can cause worry.

The above adjustments to the universal principles of display logic will allow us to treat of the exponential connectives of linear logic, as we see below.

3 Adjustments for propositional constants Linear logic involves four propositional constants, two on the side of truth and two on the side of falsehood, with negation making a definability-pairing between them. What appears to prevent **DL** from representing these four propositional constants in a single family is the fact that **DL** entered for each of the families it considered a pair of structural postulates, $\mathbf{I}^*+$ and $\mathbf{I}^*-$, that identify an antecedent occurrence of **I** with that of $\mathbf{I}^*$, and also a consequent occurrence of **I** with that of $\mathbf{I}^*$, thus permitting only two propositional constants per family. All that is needed, however, is to withhold for linear logic the postulates $\mathbf{I}^*+$ and $\mathbf{I}^*-$, which in any event are not organic to display logic (the matter is discussed a little in Note 4 of **DL**). Then we shall have **I** and $\mathbf{I}^*$ in antecedent position representing the two propositional constants on the side of truth, and **I** and $\mathbf{I}^*$ in consequent position representing the two propositional constants on the side of falsehood.

4 Linear logic Let us now see how these adjustments to **DL** permit the display of linear logic.

Girard's linear logic is a single-family display logic, with numerous formula-connectives. The following notation is that of Girard [6], except that we take advantage of his proposal in Note 2 regarding symbols for *par* and *with*: interdefinable "multiplicative" **1** and $\bot$, and interdefinable "additive" $\top$ and **0**, are 0-place; $A^\bot$ (linear negation) is 1-place; "multiplicative" interdefinable $A \otimes B$, $A \sqcup B$, and $A \multimap B$ (times, par, and linear implication) are 2-place, and "additive" interdefinable $A \sqcap B$ and $A \oplus B$ (with and plus) are also 2-place. There are also the interdefinable multiplicative 1-place "exponentials" $!A$ and $?A$ (of course and why not), and interdefinable additive quantifiers $\bigwedge xA$ and $\bigvee xA$. In all cases the interdefinability is via the linear negation, $A^\bot$, in a familiar De Morgan-like way. We use A, B, and M as formula-variables.

To make a display logic of linear logic, we introduce a single family of structural connectives: **I** (0-place), $*$ (1-place), and $\circ$ (2-place). That is, as for *any* display logic, a *structure* is defined as **I** or any formula, or the result of applying $*$ to one structure, or $\circ$ to two structures. (A chief part of the point of the current enterprise is that in displaying linear logic we introduce no *ad-hocery*, but only the standard means available in display logic. Otherwise the clarity and ease with which we display linear logic would not count as evidence in favor of the power of display logic.) We use W, X, Y, and Z as structure-variables. A *consecution* always has the form $X \vdash Y$, where X and Y are structures. We can define the "positive" and "negative" parts of a structure in the obvious way (flipping for $*$), and define "antecedent" parts of $X \vdash Y$ as positive parts of X or negative parts of Y, and dually for "consequent" parts of $X \vdash Y$.

Certain postulates are common to every family of every display logic, which also hold for linear logic: The identity schema $A \vdash A$ for each variable A (§3.1), and the eight display equivalences of §3.2 that permit the display of any antecedent part of $X \vdash Y$ as the antecedent (standing alone) of a display-equivalent consecution, and dually for any consequent part. For example, $X \circ Y \vdash Z$ is display-equivalent to $X \vdash Y^* \circ Z$, which displays X as the antecedent standing alone.

As structural rules for the display logic version of linear logic, we need from §3.4 the rules **I**+ and **I**− that make **I** in *antecedent* position an identity; the rule **CI** for commutativity of $\circ$, and the rule B which in context guarantees associativity. (Linear logic distinguishes itself as "linear" by not postulating either weakening, **KI**, or contraction, **WI**.) We also need to add a rule that makes **I** in *consequent* position a "bottom":

$$\frac{X \vdash \mathbf{I}}{X \vdash Y}\ (\vdash\!\mathbf{I} - \mathrm{K}).$$

From this we have the rule ($\mathbf{I}^*$ − K), from $\mathbf{I}^* \vdash Y$ to infer $X \vdash Y$, which makes $\mathbf{I}^*$ in antecedent position (not an identity but) a "top". This rule is a variant on the rule (**I** − K) of §3.4 of **DL**.

As connective-governing postulates common to nearly every display logic, we also have for linear logic the standard postulates from §3.3 for the multiplicative connectives, though the change in notation will make things a little confusing[3]:

DL rule from §3.3	Linear logic connective governed	Comment
&	$\otimes$, multiplicative *times*	meet-like
$\vee$	$\sqcup$, multiplicative *par*	join-like
$\rightarrow$	$\multimap$, linear implication	conditional-like
$\sim A$	$A^{\perp}$, linear negation	negation-like
t	**1**, multiplicative identity for *times*	verum-like
f	**0**, additive identity for *plus* (see below)	falsum-like

For falsum-like $\perp$ and verum-like $\top$ in linear logic we need to add the obvious rules that make $\top$ and **0** interdefinable by negation, and **1** and $\perp$ interdefinable in the same way; which is to say, the rule for $\perp$ copies the **DL** rules for **f**, but with **I*** in place of **I**; and the rule for $\top$ copies the **DL** rules for **t**, but with **I*** in place of **I**.

Furthermore, for the "additive" connectives $\sqcap$ (Girard's meet-like *with*) and $\oplus$ (Girard's join-like *plus*) linear logic postulates the rules that in display logic are called "structure-free", since they involve no structural connectives (§6.5 of **DL**). And Girard's universal-like and existential-like quantifiers $\bigwedge x$ and $\bigvee x$ are also added, with structure-free rules as described in §6.2 for $\forall x$ and $\exists x$ respectively.

All of the above components of linear logic, then, are assembled from existing ingredients of display logic. What is *not* envisaged in **DL** are any connectives like the exponentials ! and ? of linear logic; but they fit smoothly into the apparatus now that we have made the appropriate symmetrizing adjustments. For convenience in stating the rules for these connectives, we say that a structure Y is *exponentially restricted* if whenever Y contains a formula B, if B is a positive part of Y, then B has the form $!A$, and if B is a negative part of Y, then B has the form $?A$. In other words, no formulas except $!A$'s are positive parts of Y, and no formulas except $?A$'s are negative parts of Y. Furthermore, Y must not contain **I** as a negative part. And we define *dual exponentially restricted* dually. Then we postulate the following rules:

$$\frac{Y \vdash A}{Y \vdash !A} \qquad \frac{A \vdash X}{!A \vdash X}$$

$$\frac{X \vdash A}{X \vdash ?A} \qquad \frac{A \vdash Z}{?A \vdash Z}$$

where X is unrestricted, but where Y must be exponentially restricted and Z must be dual exponentially restricted. (We would not need to define "dual exponentially restricted" were we content to write Z as Y^*.)

There are also two additional structural rules, which are stated with the help of the concept of exponential restriction, namely, restricted forms of weakening, K, and of contraction, **WI**:

$$\frac{X \vdash Z}{X \circ Y \vdash Z}\ (\mathrm{K}^{\mathrm{exp}}) \qquad \frac{Y \circ Y \vdash X}{Y \vdash X}\ (\mathbf{WI}^{\mathrm{exp}})$$

provided Y is exponentially restricted. These rules strengthen those of Girard, and work even in circumstances in which Girard's do not work, although in the special context of the other rules of linear logic Girard's rules are equivalent to these. What is crucial is that we are using structure-variables in our statement where Girard uses formula-variables. When **CI** and B are present this makes no difference, but when they are absent it gives us control that we would otherwise not have.

We will now try to clarify the above by way of summary. We display linear logic via the following axioms and rules. To save space, we will use $\Rightarrow$ between the premises and conclusion of a rule ($\Leftrightarrow$ for reversible rules).

Axioms $A \vdash A$, for each variable A.

Display equivalences Consecutions in the same group are equivalent (mutually inferable).
(1) $X \circ Y \vdash Z \Leftrightarrow X \vdash Y^* \circ Z$
(2) $X \vdash Y \circ Z \Leftrightarrow X \circ Y^* \vdash Z \Leftrightarrow X \vdash Z \circ Y$
(3) $X \vdash Y \Leftrightarrow Y^* \vdash X^* \Leftrightarrow X^{**} \vdash Y.$

Structural rules
(**I**+) $X \vdash Y \Rightarrow \mathbf{I} \circ X \vdash Y$
(**I**−) $\mathbf{I} \circ X \vdash Y \Rightarrow X \vdash Y$
(**CI**) $X \circ Y \vdash Z \Rightarrow Y \circ X \vdash Z$
(B) $W \circ (X \circ Y) \vdash Z \Rightarrow (W \circ X) \circ Y \vdash Z$
($\vdash \mathbf{I} - K$) $X \vdash \mathbf{I} \Rightarrow X \vdash Y$
($\mathrm{K}^{\mathrm{exp}}$) $X \vdash Z \Rightarrow X \circ Y \vdash Z$, provided Y is exponentially restricted
($\mathrm{WI}^{\mathrm{exp}}$) $Y \circ Y \vdash X \Rightarrow Y \vdash X$, provided Y is exponentially restricted.

Connective rules (right and left rules for each connective, separated by semicolons)
(**1**) $\mathbf{I} \vdash \mathbf{1}$; $\mathbf{I} \vdash X \Rightarrow \mathbf{1} \vdash X$
($\bot$) $X \vdash \mathbf{I}^* \Rightarrow X \vdash \bot$; $\bot \vdash \mathbf{I}^*$
($\top$) $\mathbf{I}^* \vdash \top$; $\mathbf{I}^* \vdash X \Rightarrow \top \vdash X$
(**0**) $X \vdash \mathbf{I} \Rightarrow X \vdash \mathbf{0}$; $\mathbf{0} \vdash \mathbf{I}$
($\otimes$) $X \vdash A$ and $Y \vdash B \Rightarrow X \circ Y \vdash A \otimes B$; $A \circ B \vdash X \Rightarrow A \otimes B \vdash X$
($\sqcup$) $X \vdash A \circ B \Rightarrow X \vdash A \sqcup B$; $A \vdash X$ and $B \vdash Y \Rightarrow A \sqcup B \vdash X \circ Y$
($\multimap$) $X \circ A \vdash B \Rightarrow X \vdash A \multimap B$; $X \vdash A$ and $B \vdash Y \Rightarrow A \multimap B \vdash X^* \circ Y$
($^\perp$) $X \vdash A^* \Rightarrow X \vdash A^\perp$; $A^* \vdash X \Rightarrow A^\perp \vdash X$
($\sqcap$) $X \vdash A$ and $X \vdash B \Rightarrow X \vdash A \sqcap B$; $A \vdash X \Rightarrow A \sqcap B \vdash X$; $B \vdash X \Rightarrow A \sqcap B \vdash X$
($\oplus$) $X \vdash A \Rightarrow X \vdash A \oplus B$; $X \vdash B \Rightarrow X \vdash A \oplus B$; $A \vdash X$ and $B \vdash X \Rightarrow A \oplus B \vdash X$
($\bigwedge$) $X \vdash Aa \Rightarrow X \vdash \bigwedge xAx$, provided a does not occur in the conclusion; $Aa \vdash X \Rightarrow \bigwedge xAx \vdash X$
($\bigvee$) $X \vdash Aa \Rightarrow X \vdash \bigvee xAx$; $Aa \vdash X \Rightarrow \bigvee xAx \vdash X$, provided a does not occur in the conclusion
(!) $Y \vdash A \Rightarrow Y \vdash !A$, provided Y is exponentially restricted; $A \vdash X \Rightarrow !A \vdash X$
(?) $X \vdash A \Rightarrow X \vdash ?A$; $A \vdash Z \Rightarrow ?A \vdash Z$, provided Z is dual exponentially restricted.

5 Elimination Theorem Is the Elimination Theorem provable for linear logic so formulated? The existing argument of **DL** will not work, but if we ignore quantifiers for the moment, those of the above rules involving (dually) exponentially restricted structures are the only restricted rules for linear logic; so it is easy to see that every formula is regular. In fact, $!A$ is Cons-regular, $?A$ is Ant-regular, and all other formulas, atomic or complex, are both Cons- and Ant-regular (*bi-regular*, as we might say).

Since this is the distinctive feature of our treatment (over and above the considerations of **DL**) it is worthwhile to go into the matter a little; we argue for the regularity of a couple of sample formulas. First, $A \otimes B$. This formula is regular because it cannot occur at all as a parameter in one of the exponentially restricted structures. Therefore, an arbitrary structure can be substituted for $A \otimes B$ wherever it is a parameter; that is, it is bi-regular. Second, $!A$. Observe first, however, that $!A$ is *not* Ant-regular (hence not bi-regular); for example, $!A \vdash B \Rightarrow\ !A \vdash\ !B$ is an instance of the rule $\vdash!$ in which $!A$ is an antecedent parameter, but $!A$ cannot be replaced by an arbitrary structure, X, without violating the exponential restriction on $\vdash!$. But $!A$ *is* Cons-regular and therefore regular. In the first place, if it occurs parametrically as part of an unrestricted structure in an inference, it is obvious that it can be replaced with a structure X *ad lib.* Secondly, suppose it occurs parametrically as part of an exponentially restricted structure, Y, in an inference. All such structures occur as antecedent parts in our rules (there are only three rules to check), so the occurrence of $!A$ is positive or negative in Y according as it is antecedent or consequent in the consecutions forming the premise and conclusion of the inference. It cannot be a negative part of Y (all such parts of exponentially restricted Y have the form $?B$ by definition of exponential restriction) and so must be a positive part of Y and thus an antecedent part of the premise/conclusion consecutions. But in this circumstance the definition of Cons-regularity requires replacement not by arbitrary structures but only by structures X that can appear on the left in the rule $\vdash!$, which is to say, structures that are themselves exponentially restricted. And it is easy to see that replacing a positive formula of an exponentially restricted structure by an exponentially restricted structure generates another exponentially restricted structure, so that the replacement of $!A$ by X will still fall under the same rule. Thirdly, if $!A$ occurs parametrically in a dual exponentially restricted structure, a dual argument suffices, including the tongue-twisting but obvious remark that the replacement of a negative part of a dual exponentially restricted structure by an exponentially restricted structure results in a dual exponentially restricted structure.

So the Elimination Theorem is provable for quantifier-free linear logic with exponentials.

As for the quantifier rules of linear logic, they are structure-free rules, like those for $\sqcap$ and $\oplus$, and accordingly are of little special interest. It is true that the restrictions on the rule $\vdash\wedge$ and its dual $\vee\vdash$ prevent the regularity of any formula whatever (for example, an arbitrary structure cannot be substituted for a parametric formula in an instance of $\vdash\wedge$ and still be an instance of the same rule), but as Gentzen [5] showed in an exactly similar situation, given proofs of $X \vdash M$ and $M \vdash Y$ one can always find by substitution a pair of proofs such that relative to those proofs there is enough regularity. (Perhaps we should define this more general concept, but it is not so clear that it is worth it if quantifiers are the only example.) So quantifiers are no bar to the Elimination Theorem for all of linear logic.

With the help of the Elimination Theorem, it is no more than "axiom-chopping" to show that $\mathbf{I} \vdash A$ holds in the display logic formulation if and only if A does in the Hilbert formulation, and that $A \multimap B$ holds in the Hilbert formulation if and only if $A \vdash B$ holds in the display version. They are equivalent.

6 Variants of linear logic The known formulations of linear logic seem to *depend on* the associativity and commutativity of $\otimes$, which is in certain respects a defect. The display logic formulation of linear logic does not depend on them, and it therefore permits us to consider variants of linear logic that are nonassociative or noncommutative. The reason is that our treatment of the various connectives and—especially—our argument for the Elimination Theorem, even with respect to exponentials, do not at all depend on the postulation by linear logic of either associativity or commutativity. We can drop these postulates and still have a coherent concept of all the connectives, including exponentials, or add other postulates, or subtract some and add others; the rationality of the concepts carried by the various connectives, as expressed in the rules given above, is completely independent of which structural rules are postulated for a given family, and of which connectives are present. This is a typical strength of display logic. The upshot is that we may look to the desired applications in computer science as the sole determiner of whether we make our logic associative or commutative or whatever.

There is also no bar to adding further exponentially restricted structural properties if applications should suggest the wisdom of so doing. For instance, if we are treating of a calculus weaker than linear logic for which associativity and commutativity are not postulated, we can add exponentially restricted rules of associativity or commutativity; all formulas would continue to be regular, and the Elimination Theorem would remain provable. Indeed, we could just add the exponentially restricted structural rules we liked, since none are needed for, and none stand in the way of, the Elimination Theorem.

To make the point, let us define "punctual logic" as the one that goes beyond linearity by postulating *no* structural rules (except doubtless **I**+ and **I**− for the multiplicative family), neither associativity nor commutativity nor weakening nor contraction. Punctual logic is still "strong enough" in the sense of §II.3 of Girard [6], since if we give the missing structural properties back to the exponentials we can still translate intuitionist logic. That is, the postulation of commutativity and associativity for the multiplicative family doubtless has a point, but not the point of making the logic strong enough to contain intuitionism. The proof, resting on Girard's work, is routine.

(One keeps wondering whether these techniques suggest a "multiplicative quantifier" that is related to $\otimes$ in the way that additive $\wedge$ is related to additive $\sqcap$. Nothing has yet emerged, however.)

Linear logic postulates only one family, $(\mathbf{I},{}^*,\circ)$, a fact that may be important for its intended applications; but as a logical matter it needs to be noted that the coherence of the concept of exponentiation, or of other features of linear logic, does not depend on this adherence to a single family. For example, one could add the Boolean family (§5.1 of **DL**), for which one postulates "all" structural rules. If one did so, one would immediately obtain distribution for the "additive" connectives $A \sqcap B$ and $A \oplus B$, without further postulation concerning them, as mentioned in §6.5 of **DL**. One would then have to choose how to define "exponentially restricted", a choice that should be guided by the requirement of regularity as herein defined. Any such choice that renders all formulas regular will permit the proof of an Elimination Theorem and hence will be coherent (but of course not necessarily of use in computer science applications).

In the light of Rezus [9], it is perhaps useful to observe that there is already present in linear logic an **S5**-type of modality, with $\Box A$ definable by $\top \multimap A$, as can be seen from the point of view of display logic by examining the structural rules used as the means for defining **S5** in §5.6 of **DL**. The details of the transfer from **DL** are, however, a little confusing, because *there* we: (1) identified **I** and $\mathbf{I}^*$ in similar positions and (2) stated the properties indifferently in terms of **I** or $\mathbf{I}^*$. But *here*, having (1′) failed to identify **I** and $\mathbf{I}^*$ in similar positions, (2′) we have in each case to decide whether *here* it is **I** or $\mathbf{I}^*$ that is to have the properties *there* indifferently attributed to **I** or $\mathbf{I}^*$. Having made the transfer properly, however, it is doubtless better to restate uniformly the properties in terms of **I** in consequent position, keeping in mind for this purpose the equivalent definition of $\Box A$ as $\mathbf{0} \sqcup A$. For example, the key Brouwersche postulate of §3.4 of **DL** is the inference from $X \circ Y \vdash \mathbf{I}$ to $Y \circ X \vdash \mathbf{I}$, which is *here* just a special case of commutativity. In any event, given the following rules it is easy to see that linear logic distinguishes no more than six "modalities" in the sense of Parry [8], the same ones distinguished by **S5**: $\Box A$, A, $\Diamond A = (\Box(A^{\perp}))^{\perp}$, and the negations of these.

$$\frac{X \vdash \mathbf{I} \circ A}{X \vdash \Box A} \qquad \frac{A \vdash X}{\Box A \vdash \mathbf{I} \circ X}$$

$$\frac{X \vdash A}{(\mathbf{I} \circ X^*)^* \vdash \Diamond A} \qquad \frac{X^* \circ A \vdash \mathbf{I}}{\Diamond A \vdash X}.$$

For example, here is a proof that $\Diamond\Box A$ implies $\Box A$; it illustrates the precision with which the rules interrelate:

$A \vdash A$	Identity
$\Box A \vdash \mathbf{I} \circ A$	$(\Box)$
$\Box A \circ A^* \vdash \mathbf{I}$	Display equivalences
$\Box A \circ A^* \vdash (\mathbf{I}^* \circ \mathbf{I}^*)^*$	$\vdash \mathbf{I} -$ K
$\Box A \circ (\mathbf{I}^* \circ \mathbf{I}^*) \vdash A$	Display equivalence
$(\Box A \circ \mathbf{I}^*) \circ \mathbf{I}^* \vdash A$	B$\vdash$
$\Box A \circ \mathbf{I}^* \vdash \mathbf{I} \circ A$	Display equivalence
$\Box A \circ \mathbf{I}^* \vdash \Box A$	$(\Box)$
$\Box A \circ \Box A^* \vdash \mathbf{I}$	Display equivalences
$\Box A^* \circ \Box A \vdash \mathbf{I}$	Brouwersche; here just CI$\vdash$
$\Diamond\Box A \vdash \Box A$	$(\Diamond)$

Observe that adding these rules causes almost no extra work in verifying that the Elimination Theorem goes through; it is obvious to the eye that all formulas remain regular. Note in particular that these rules have no "side-conditions" or provisos on their parameters, even though the connectives introduced are modal in character.

How do we know that $\Box A$ stays modal and does not collapse, and in particular that the six modalities do not collapse? Meyer [7] shows that the so-defined $\Box A$ together with other connectives is not just **S5**-like but constitutes *precisely* **S5** when the context is not linear logic but its proper supersystem (except for exponentials), the calculus **R** of relevant implication. And the Elimination Theorem tells us that exponentials constitute a conservative extension of linear logic.

How do we know that $\Box A$ does not yield as much as **S5** in the context of linear logic (instead of **R**)? We know this because of the absence of contraction. And this leads us to wonder if it would be interesting to add a limited form of contraction to make the $\Box A$ part of linear logic even more **S5**-like, say by postulating the inference from $X \circ X \vdash \mathbf{I} \circ Y$ to $X \vdash \mathbf{I} \circ Y$, so that we could prove $\Box A \multimap (\Box A \multimap B) \vdash \Box A \multimap B$. That would appear to be following the lead provided by Girard's treatment of exponentiation; perhaps. Etc.

I have not mentioned semantics. Beyond those of Girard [6] and Avron [1] and works cited there, permit me to call your attention to Routley et al. [11] as (among other things) a sourcebook on the semantics of the family of weaker systems to which linear logic belongs.

Appendix The definitions leading to a reshaping of the **DL** proof of the Elimination Theorem are as follows. The words "parametric" and "principal" are to be understood as in §4.1—and especially Definition 4.1—of **DL**.

Cons(1) = Stage 1 as a property of M. For all X and Y: if (Hyp C1a) $X \vdash M$ is derivable; and if (Hyp C1b) for all X', if there is a derivation of $X' \vdash M$ ending in an inference in which displayed M is *not* parametric (i.e., ending in an inference in which displayed M is principal), then $X' \vdash Y$ is derivable; then $X \vdash Y$ is derivable.

Cons(2) = Stage 2 as a property of M. For all X and Y: if (Hyp C2a) $M \vdash Y$ is derivable; if (Hyp C2b) for all Y', if there is a derivation of $M \vdash Y'$ ending in an inference in which displayed M is *not* parametric (i.e., ending in an inference in which displayed M is principal) then $X \vdash Y'$ is derivable; and if (Hyp C2c) $X \vdash M$ is the conclusion of some inference in which M is not parametric (i.e., in which M is principal); then $X \vdash Y$ is derivable.

Ant(1) = Dual stage 1 as a property of M. For every X and Y: if (Hyp A1a) $M \vdash Y$ is derivable; and if (Hyp A1b) for all Y', if there is a derivation of $M \vdash Y'$ ending in an inference in which displayed M is *not* parametric (i.e., ending in an inference in which displayed M is principal), then $X \vdash Y'$ is derivable; then $X \vdash Y$ is derivable.

Ant(2) = Dual stage 2 as a property of M. For every X and Y: if (Hyp A2a) $X \vdash M$ is derivable; if (Hyp A2b) for all X', if there is a derivation of $X' \vdash M$

ending in an inference in which displayed M is *not* parametric (i.e., ending in an inference in which displayed M is principal) then $X' \vdash Y$ is derivable; and if (Hyp A2c) $M \vdash Y$ is the conclusion of some inference in which M is not parametric (i.e., in which M is principal); then $X \vdash Y$ is derivable.

Cons(1) & Cons(2) = Stage 1 and Stage 2 conjointly, as a property of M: both Cons(1) and Cons(2).

Ant(1) & Ant(2) = Dual stage 1 and Dual stage 2 conjointly, as a property of M: both Ant(1) and Ant(2).

Parametric Stage For all M, either Cons(1) & Cons(2) or Ant(1) & Ant(2).

Principal Stage Assume that for each of $X \vdash M$ and $M \vdash Y$ there are derivations ending in inferences in which the respective displayed M's are *not* parametric, and that for all X', Y' and proper subformulas M' of M, $X' \vdash Y'$ is derivable if $X' \vdash M'$ and $M' \vdash Y'$ are. Then $X \vdash Y$ is derivable.

It is elementary by induction on the complexity of formulas that if the Parametric and Principal Stages hold for a calculus, then so does the Elimination Theorem: if $X \vdash M$ and $M \vdash Y$ are both provable, then so is $X \vdash Y$.

NOTES

1. "Linear logic is the first attempt to solve the problem of parallelism *at the logical level*" ([6], p. 3). And though the pre-echo is striking, it does seem unlikely that Bosanquet [3] was a genuine anticipation: "What makes Inference linear is respect for the independence of its terms" ([3], p. 20).
2. We presuppose what we must: that the reader is familiar with both linear logic (Girard [6]) and display logic (Belnap [2]), which we refer to as **DL**. The following were quite helpful for understanding linear logic: Avron [1], Rezus [9] and [10], a lecture by Girard that was hosted by Carnegie-Mellon University in the fall of 1987, and blackboarding with Andrea Asperti shortly thereafter. Avron [1] in particular is outstanding in relating Girard's work to the mainstream of studies in nonclassical logic. Thus, one could take Avron [1] and Belnap [2] as sufficient background for this paper. It needs to be added that, alas, the background is essential—to summarize 100 pages of Girard or 42 of Belnap would be a waste of trees.
3. In this study the notation for connectives is a particular headache, partly because there are so many connectives with which to deal, partly because Girard [6] employs totally nonstandard notation for standard purposes, and partly because **DL** reluctantly goes its own way over notation for connectives (since its purposes require notation for indefinitely many families of connectives). Avron [1] introduces some sanity, which we recommend, but we nevertheless choose to employ the connective notation of Girard [6] in order to make it indubitably clear that it is linear logic that is being presented as a display logic.

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