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On Finite Models of Regular Identities

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Abstract It is a known result of Austin that there exist nonregular identities with all nontrivial models being infinite. In this note a certain analogue of this result for regular identities is presented and some remarks in this connection are given.

1 Perkins [4] proved that it is undecidable whether an identity (and in consequence, a finite set of identities) has a nontrivial model (i.e., a model of cardinality greater then one). Austin [1] in improving the result of Stein [5] found an identity with infinite models but with no nontrivial finite one. McKenzie [3] proved that it is also undecidable whether an identity (a finite set of identities) has a nontrivial finite model.

All these results are based on properties of nonregular identities. For regular identities (i.e., those with the same variables appearing on both sides, cf. [2]) the model problems mentioned above are trivial. It is known that for each finite set of regular identities the so-called τ-semilattices provide models of arbitrary cardinalities.

More precisely, let τ be a finite type of algebras, i.e., a sequence $\langle n_1, \ldots, n_k \rangle$ of nonnegative integers, and xy a semilattice operation on a set A (a semilattice operation can be defined on every finite set A). For $1 \leqq i \leqq k$ we define an n_i-ary operation on the set A by $f_i = x_1 x_2 \ldots x_{n_i}$. The algebra $\langle A, f_1, .., f_k \rangle$ is then called a *τ-semilattice*. Any τ-semilattice is polynomially equivalent to the corresponding semilattice and clearly is a model for any set of regular identities in type τ. Let us also note that each one-element algebra is a τ-semilattice.

Thus, τ-semilattices can be treated as trivial models for regular identities. Let us now inquire about other models.

We will show that a set of regular identities close to the lattice identities

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has models that are not τ-semilattices, though each such model is infinite (Theorem 1). This is a result analogous to those of Austin [1] and Stein [5], and in consequence leads to a problem for regular identities analogous to that considered by McKenzie [3]. In this case, however, the situation is more complex, since as we will show each single regular identity has a finite model that is not a τ-semilattice (Theorem 2), and therefore the approach of [3] and [4] does not apply here.

Our terminology is standard. If the need arises we recommend that one refer to [6]; some basic notions of graph theory are also assumed to be familiar to the reader.

2 Let $+$ and $\cdot$ be binary function symbols. Consider the following set of identities (we write xy for $x \cdot y$):

$$\Sigma : x + x = x,$$

$$xx = x,$$

$$x + y = y + x,$$

$$xy = yx,$$

$$(x + y)z = (x + z)y.$$

Theorem 1
(a) *Each finite model of Σ is a $\langle 2,2 \rangle$-semilattice*
(b) *There is a model of Σ which is not a $\langle 2,2 \rangle$-semilattice.*

Proof: (a) Let $\mathfrak{A} = \langle A,+,\cdot \rangle$ be a model of Σ. If $x + y = xy$ holds in $\mathfrak{A}$, then by the last identity in Σ the operation $x + y = xy$ is associative and so $\mathfrak{A}$ is a $\langle 2,2 \rangle$-semilattice. We prove that in the opposite case the number of binary polynomials over $\mathfrak{A}$ is infinite, which implies that $\mathfrak{A}$ itself is an infinite model. More precisely, we show that for $n = 0,1,2,\ldots$ the following polynomials

$$s_n(x,y) = x + 2^n y$$

(where $x + ky$ abbreviates $(\ldots((x + y) + y) + \ldots) + y$ with y occurring k-times) are pairwise distinct over $\mathfrak{A}$.

Using the identities of Σ, it is easy to check that

(1) $(s_n(x,y))y = xy$
(2) $(s_n(x,y))x = s_{n-1}(x,y)$.

Indeed, for (1) we have that $(x + ky)y = ((x + (k-1)y) + y)y = (y + (x + (k-1)y))y = (y + y)(x + (k-1)y) = (x + (k-1)y)y = \ldots = xy$.

For (2), $x + 2^n y = ((x + (2^n - 1)y) + y)x = (x + y)(x + (2^n - 1)y) = ((x + (2^n - 2)y) + y)(x + y) = (x + 2y)(x + (2^n - 2)y) = \ldots = (x + 2^{n-1}y)(x + 2^{n-1}y) = s_{n-1}(x,y)$.

Now assume that $s_n(x,y) = s_m(x,y)$ for some $0 \leqq m < n$. Then by (2) $s_{n-m}(x,y) = s_0(x,y) = x + y$; that is, for some $k \geqq 1$, $s_k(x,y)$ is commutative. Using (1) and (2) it follows that $s_{k-1}(x,y) = (s_k(x,y))x = (s_k(y,x))x = yx$, which is commutative as well, and by applying this argument again and again we get that $s_0(x,y) = yx$, that is, $x + y = xy$. This is a contradiction, proving (a).

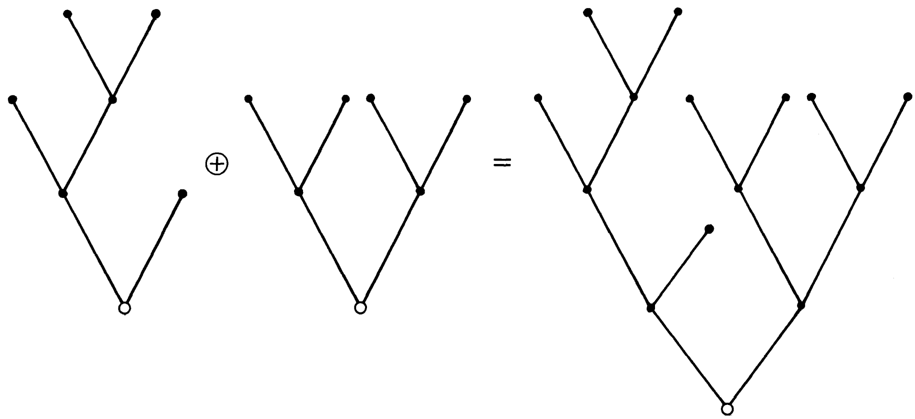

Figure 1.

(b) Let T be the set of the *finite binary trees* on a countable set S with roots as in Figure 1. Isomorphic trees are considered as identical. We define a binary operation $\oplus$ on T as follows: if $t, u \in T$, then $t \oplus u$ is the tree obtained from t and u by adding one more node (intended as the root of $t \oplus u$) and two edges connecting this node with the roots of t and u (see Figure 1). The operation $t \oplus u$ is commutative, but not associative.

Now we define the binary operations $+$ and $\cdot$ on T as follows:

$$x + y = \begin{cases} x, & \text{if } x = y \\ x \oplus y, & \text{otherwise.} \end{cases}$$

Clearly, $x + y$ is idempotent and commutative, but not associative. To define $x \cdot y$, we first introduce some notation. Let

$$y\mathbf{a} = (\dots((y \oplus x_1) \oplus x_2) \oplus \dots) \oplus x_n, \text{ where } \mathbf{a} = \langle x_1, x_2, \dots, x_n \rangle, n \geqq 0$$
$$y\mathbf{a}\bar{\mathbf{a}} = (\dots(((y\mathbf{a}) \oplus x_n) \oplus x_{n-1}) \oplus \dots) \oplus x_1$$
$$y\mathbf{a}z\bar{\mathbf{a}} = (\dots((((y\mathbf{a}) \oplus z) \oplus x_n) \oplus x_{n-1}) \dots) \oplus x_1.$$

By induction on the total number of nodes we define

$$x \cdot y = \begin{cases} y\mathbf{a}, & \text{if } x = y\mathbf{a}\bar{\mathbf{a}} \text{ for some } \mathbf{a} \\ x\mathbf{a}, & \text{if } y = x\mathbf{a}\bar{\mathbf{a}} \text{ for some } \mathbf{a} \\ (y\mathbf{a}) \cdot t, & \text{if } x = y\mathbf{a}t\bar{\mathbf{a}} \text{ for some } t \text{ and } \mathbf{a} \\ (x\mathbf{a}) \cdot t, & \text{if } y = x\mathbf{a}t\bar{\mathbf{a}} \text{ for some } t \text{ and } \mathbf{a} \\ 1, & \text{otherwise.} \end{cases}$$

By 1 we denote here the one-element tree, which is included in T. Note that for $\mathbf{a}$ empty ($n = 0$) we have that $x \cdot x = x$, showing that $x \cdot y$ is idempotent. Clearly, it is also commutative.

Now, let T_0 be the subset of T of all those trees not containing a subtree of the form $x \oplus x$. Then, $x + y$ and $x \cdot y$ are still operations on T_0. We show that

$$(x + y)z = (x + z)y = (y + z)x$$

holds in T_0.

For $x = y = z$ this is trivial.

If $x = y \neq z$, then $(x + y)z = xz$, and $(x + z)y = (x + z)x = xz$ (since $x + z = x\mathbf{a}z\bar{\mathbf{a}}$ with empty $\mathbf{a}$). Similarly, $(y + z)x = xz$.

We may thus assume that x, y, and z are pairwise distinct. According to the definition of $x \cdot y$ we consider four special cases with regard to the term $(x + y)z$.

(1) $z = (x + y)\mathbf{a}\bar{\mathbf{a}} = (x \oplus y)\mathbf{a}\bar{\mathbf{a}}$
(2) $z = (x \oplus y)\mathbf{a}t\bar{\mathbf{a}}$
(3) $x \oplus y = z\mathbf{a}\bar{\mathbf{a}}$ (then either $y = (z \oplus x)\mathbf{b}\bar{\mathbf{b}}$, or dually, $x = (z \oplus y)\mathbf{b}\bar{\mathbf{b}}$)
(4) $x \oplus y = z\mathbf{a}t\bar{\mathbf{a}}$.

In case (1), $(x + y)z = (x \oplus y)\mathbf{a}$, and $(x + z)y = (z + x)y = ((y \oplus x)\mathbf{a}\bar{\mathbf{a}} \oplus x)y = (y\mathbf{b}\bar{\mathbf{b}})y = y\mathbf{b} = (y \oplus x)\mathbf{a}$, as required. The same is true for $(y + z)x$. In case (2) the proof is analogous. In case (3), $(x + y)z = z\mathbf{a}$ $(x + z)y = y(z \oplus x) = ((z \oplus x)\mathbf{b}\bar{\mathbf{b}})(z \oplus x) = (z \oplus x)\mathbf{b} = z\mathbf{a}$ (since, in this case, $\mathbf{b}$ is obtained from $\mathbf{a}$ by deleting the first element x), and also $(y + z)x = (((z \oplus x)\mathbf{b}\bar{\mathbf{b}}) \oplus z)x = (((x \oplus z)\mathbf{b}\bar{\mathbf{b}}) \oplus z)x = (x\mathbf{c}\bar{\mathbf{c}})x = x\mathbf{c} = (z \oplus x)\mathbf{b} = z\mathbf{a}$. The same proof applies for the dual case $x = (z \oplus y)\mathbf{b}\bar{\mathbf{b}}$. Again, in case (4) the proof is analogous. Otherwise, $(x + y)z = 1$. Then, also $(x + z)y = (y + z)x = 1$, because if $(x + z)$ or $(y + z)x \neq 1$, then applying the proof like that above for $y(x + z)$ or $y(y + z)x$, respectively, we obtain that $(x + y)z \neq 1$. Hence, $\langle T_0, +, \cdot \rangle$ is a model of Σ, which completes the proof of the theorem.

3 In view of results of McKenzie [3] and Perkins [4], one can conjecture that the question of whether or not a finite set of regular identities has a nontrivial (not τ-semilattice) model is undecidable. However, our problem is more complicated than those treated of in [3] and [4]. As a matter of fact, their results concern single identities, the undecidability of the question for a finite set of identities being a simple consequence. In contrast to this, for our question we have:

Theorem 2 *Each single regular identity has a nontrivial finite model (not a τ-semilattice).*

Proof: To show this, first note that the problem in question actually concerns the sets of identities with at least one having just a single variable on one side. Indeed, otherwise an algebra $\langle A, F \rangle$ with all the operations equal to a fixed constant provides a suitable model. If a regular identity has a variable on one side, then this is the only variable appearing in this identity. Thus, all that remains to show is that each such identity I has a finite model other than a τ-semilattice. This we show as follows. Suppose that the identity I has a variable x at the right-hand side, and r is the number of the occurrences of x at the left-hand side. If $r = 2$, then putting $f(x) = x$ for all unary operators appearing in I, we have just the identity $xx = x$, and any finite idempotent noncommutative groupoid (with some trivial unary operators added) provides a model as required. If $r = 1$, then the problem is trivial. For $r \geqq 3$, suppose at first that the only function symbol appearing in I is a symbol of a binary operation xy and consider the set of identities

$$\Sigma' = \{I, xy = yx, \ (xy)z = x(yz)\}.$$

Then the identity I can be replaced by just

$$x^r = x$$

and it is clear that any *cyclic group of order* $r - 1$ is a model for Σ, and therefore a model for I itself (by the assumption that $r - 1 > 1$).

Now, if I is an identity in a type τ, then we construct an algebra of type τ from a cyclic group of order $r - 1$ by the same construction as that applied to semilattices in Section 1. The result, which we may call a τ-cyclic group, is clearly a model for I, which completes the proof.

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