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Isomorphic but Not Lower Base-Isomorphic Cylindric Algebras of Finite Dimension

BALÁZS BIRÓ*

Abstract This article deals with Serény's theorem giving sufficient conditions for two cylindric set algebras (*Cs*'s) to be lower base-isomorphic, a cylindric algebra version of Vaught's theorem on the existence of prime models of atomic theories in countable languages. It is proved that Serény's theorem requires all the conditions given in its statement. Here the necessity of the condition of the infinite-dimensionality of the given *Cs*'s is proved via constructing isomorphic but not lower base-isomorphic *Cs*'s of any finite dimension greater than one. A model-theoretical corollary of the above dependence is stated also.

In this paper we will prove that the (cylindric) algebraic version of Vaught's theorem concerning the existence of prime models of atomic theories does not hold for finite dimensional cylindric set algebras, i.e. for algebras corresponding to models for languages with finitely many variable symbols, partly solving a problem posed in [4] and [9]. Let us see this statement in a little more detail. The algebraic version of the Vaught theorem referred to above is the following theorem of Serény [10]: *Any isomorphism between two infinite dimensional countably generated regular and locally finite Cs's with atomic neat n-reducts (for all finite n) is a lower base-isomorphism.* (The notation used in this theorem is defined below, and also in [6] and [7].)

We will show that the condition that the algebras concerned be of infinite dimension cannot be dropped. It is worth adding that we have already proved this fact about all other conditions in this theorem (cf. [2], [3], and [4]). Our treatment is based on [6] and [7]; consequently, we follow the terminology and notation of these volumes except that we denote the full Cs_α with base U by

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$\mathfrak{Sb}(\alpha, U)$. Here we recall only the notions that are connected with our central concepts. Throughout this paper α is an ordinal. Let $\mathfrak{A}$ be a Cs_α with base U and unit element V. Let f be a bijection from U onto a set W. We set $\tilde{f}X = \{y \in {}^\alpha W: f^{-1} \circ y \in X\}$ for any $X \in A$ and call $\tilde{f}$ the *base-isomorphism* induced by f. Let $V' \subseteq V$. We set $rl_{V'}X = X \cap V'$ for any $X \in A$. In this case, $rl_{V'}$ is the *relativization* of $\mathfrak{A}$ to V'. If in addition $rl_{V'}$ is an isomorphism and $V' = {}^\alpha U'$ for a $U' \subseteq U$ then $rl_{V'}$ is called a *strong ext-isomorphism.* $\mathfrak{A}$ is *base-minimal* if it is not strongly ext-isomorphic to any Cs_α except itself. A function g is called a *lower base-isomorphism* if $g = k^{-1} \circ h \circ t$ for some strong ext-isomorphisms k and t and for some base-isomorphism h.

In a little more detail, we will show that the condition that the algebras concerned be infinite-dimensional cannot be dropped in the algebraic version of Vaught's theorem referred to above (proved in [10]) according to which any isomorphism between two countably generated regular locally finite-dimensional Cs's with atomic neat n-reducts for any finite n is a lower base-isomorphism. Consequently, Vaught's theorem mentioned above cannot be extended to languages with finitely many variable symbols.

The results of the present paper also imply that the condition on the characteristic cannot be omitted from Theorem I. 3.6 of [7], according to which any two isomorphic $Cs^{reg} \cap Lf$'s of positive characteristic are base-isomorphic.

First we introduce some special notions and notation, which can for the most part be found in [1], [6], and [7]. Furthermore, we use the notions and results of [1]. It is supposed throughout that $\alpha < \omega$.

If $\mathfrak{A}$ is cylindric set algebra with base U then we write base$(\mathfrak{A}) = U$ (cf. Definition 0.1 of [7]).

If $\alpha < \omega$, for a CA_α $\mathfrak{A}$ we write $\bar{d}$, $\bar{d}^{\mathfrak{A}}$, or $\bar{d}_\alpha$ for $\bar{d}^{\mathfrak{A}}(\alpha \times \alpha) = \Pi\{-d_{\kappa\lambda}: \kappa < \lambda < \alpha\}$. (See Section 1.7 of [6].)

Suppose x and y are elements of a CA_α. Then they are said to be *cylindrically equivalent* if for every $\kappa \in \alpha$, $c_\kappa x = c_\kappa y$ (see [1]).

Let $n \in \omega$ and, for the time being, denote $\mathfrak{Sb}(\alpha, n)$ by $\mathfrak{A}$ and $\mathfrak{Sb}(\alpha + 1, n + 1)$ by $\mathfrak{B}$. Let $N = \{s \in \bar{d}^{\mathfrak{B}}: n \notin Rg\ s\}$ and for $i \in \alpha + 1$, $Z_i = \{s \in \bar{d}^{\mathfrak{B}}: s_i = n\}$ (cf. the proof of Lemma 2 in [1]).

Finally, for $X \subseteq \bar{d}^{\mathfrak{A}}$ define $X^+ \subseteq \bar{d}^{\mathfrak{B}}$ as follows:

$$X^+ = \{s \in N: \alpha 1 s \in X\} \cup \{s \in Z_\alpha: \alpha 1 s \notin X\} \cup \{\{s \in Z_i: s^i_{s(\alpha)} \in X\}: i \in \alpha\}.$$

Here

(1) $s^i_{s(\alpha)}$

is the function differing from s in place i where its value is $s(\alpha)$ (cf. 3.1.1 in [6]).

Now we state some lemmas:

Lemma 1 *If $X \subseteq \bar{d}$ then $\bar{d} \sim X^+ = (\bar{d} \sim X)^+$.*

Let $1 \le i \in \omega$. Define a series $\langle X^i_\alpha: 2 \le \alpha < \omega\rangle$ of elements of $\mathfrak{Sb}(\alpha, \omega)$ by recursion on α such that $X^i_\alpha \in \mathfrak{Sb}(\alpha, \alpha + i)$. Let

$$X^i_2 = \{\langle k, m\rangle \in {}^2(i + 2): k + 1 \equiv m (\text{mod } i + 2)\}$$

(see Figure 1). For $\alpha \ge 2$ let $X^i_{\alpha+1} = (X^i_\alpha)^+$. We then have the following lemma.

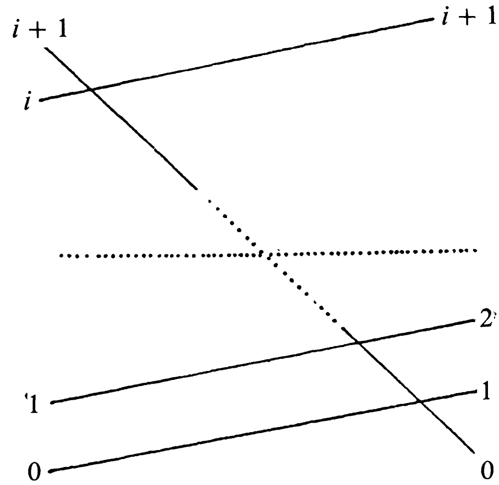

Figure 1

Lemma 2
(i) *If* $i \geq 1$ *and* $\alpha \geq 2$ *then* $0 \subset X^i_\alpha \subset \bar{d}_\alpha$;
(ii) *Both* X^i_α *and* $d \sim X^i_\alpha$ *are cylindrically equivalent to* $\bar{d}$.

Proof: (i) can be proved by induction on α using Lemma 1. (ii) can be seen in the same way as Lemma 2 of [1], using Lemma 1 of the present paper.

If $\mathfrak{A}$ is a cylindric algebra then $At\,\mathfrak{A}$ denotes the set of atoms of $\mathfrak{A}$.

Lemma 3 *Assume that* $\mathfrak{A} \subseteq \mathfrak{C} \in CA_\alpha$, $|A| < \omega$, $v \in At\,\mathfrak{A}$, x *and* $y \in C$, $x + y = v$, $x \cdot y = 0$, *and* $x \neq 0 \neq y$ *hold in* $\mathfrak{C}$ *and both* x *and* y *are cylindrically equivalent to* v. *Let* $\mathfrak{B} = \mathfrak{Sg}^{\mathfrak{C}}(\{x,y\} \cup A)$. *Under these hypotheses*
(i) $At\,\mathfrak{B} = At\,\mathfrak{A} \sim \{v] \cup \{x,y\}$
(ii) $B = \mathfrak{Sg}^{\mathfrak{B}\restriction\mathfrak{C}}(A \cup \{x,y\})$.

Proof: This lemma is a special case of Lemma 3 of [1].

Now, for $i \in \omega \sim 1$ and $\alpha \in \omega \sim 2$ define $\mathfrak{A}^i_\alpha \in Cs_\alpha$ as follows: Let $\mathfrak{A}^i_\alpha = \mathfrak{Sg}^{\mathfrak{Sb}(\alpha,\alpha+i)}(\{X^i_\alpha\})$, the Cs_α with base $\alpha + i$ generated by X^i_α. Sometimes $\mathfrak{A}(i,\alpha)$ will be written instead of $\mathfrak{A}^i_\alpha$.

We will need the following lemmas.

Lemma 4 $At\,\mathfrak{A}^i_\alpha = \{\Pi\{d_R$: *R is an equivalence class of* $\Sigma\}\cdot\bar{d}({}^2\alpha \sim \Sigma)$: Σ *is an equivalence relation on* α *such that it is not the identity relation of* $\alpha\} \cup \{X^i_\alpha, \bar{d} \sim X^i_\alpha\}$.

Proof: This lemma follows easily from Lemmas 2 and 3.

Lemma 5 *If* $\alpha \geq 2$ *and* $i \geq 1$, *then* $\mathfrak{A}^i_\alpha$ *is base-minimal.*

Proof: We prove the statement by induction on α. For $\alpha = 2$ the lemma can be easily seen to be true. Now suppose the lemma is true for α. We are going to prove it for $\alpha + 1$. Suppose $W \subseteq \alpha + 1 + i$ is such that

(2) $\langle Y \cap {}^{\alpha+1}W$: $Y \in A^i_{\alpha+1}\rangle$

is an isomorphism on $\mathfrak{A}^i_{\alpha+1}$. For the sake of brevity denote ${}^{\alpha+1}W$ by V, the relativization of $\mathfrak{A}^i_{\alpha+1}$ to V by $\mathfrak{R}$, and X^i_α *by* X. First, we claim that

(3) $\alpha + i \in W.$

In fact, suppose the contrary, i.e.

(4) $\alpha + i \notin W.$

Since $|\text{base}\,(\mathfrak{A}^i_{\alpha+1})| > \alpha + 1$, $\mathfrak{A}^i_{\alpha+1} \models \Pi\{-d_{jk}: j < k < \alpha + 1\} \neq 0$; hence, by (2),

(5) $\mathfrak{R} \models \Pi\{-d_{jk}: j < k < \alpha + 1\} \neq 0.$

Let

(6) $q \in \cap\{V \sim D^{[V]}_{jk}: j < k < \alpha + 1\}.$

We have

$$\begin{aligned} q \in c^{\mathfrak{R}}_\alpha \bar{d}_{\alpha+1} &= c^{\mathfrak{A}(i,\alpha+1)}_\alpha \bar{d}_{\alpha+1} \cap V && \text{by (6)} \\ &= c^{\mathfrak{A}(i,\alpha+1)}_\alpha (\bar{d} \sim X^i_{\alpha+1}) \cap V && \text{by Lemma 2 (ii)} \\ &= c^{\mathfrak{R}}_\alpha (\bar{d} \sim (x^+ \cap V)). \end{aligned}$$

Hence there is a $u \in W$ such that

(7) $q^\alpha_u \in \bar{d}_{\alpha+1} \sim X^+$

(cf. (1)). By (4) and (6)

(8) $q^\alpha_u \in N.$

So, by (7), $\alpha 1 q \in X$ thus $q^\alpha_{\alpha+1} \in X^+$. Consequently $q \in c^{\mathfrak{A}(i,\alpha+i)}_\alpha X^+ \cap V = c^{\mathfrak{R}}_\alpha X^+$, so there is a $t \in W$ such that $q^\alpha_t \in X^+$. However, by (7), (8), and the definition of X^+ the only possible choice for t is $\alpha + i$, hence $\alpha + i \in W$. This contradiction proves (3). Set

$$\begin{aligned} \mathfrak{A} &= \mathfrak{A}^i_\alpha, \\ W' &= W \sim \{\alpha + i\}, \\ V' &= {}^\alpha W'. \end{aligned}$$

We will prove that

(9) $rl^{\mathfrak{A}}_V$

is an isomorphism. To prove this it is enough to show that $Y \cap V' \neq 0$ whenever $Y \in At\ \mathfrak{A}$. By Lemma 4 if $Y \in At\ \mathfrak{A}$ then

(10) Y is an atom of the minimal subalgebra of $\mathfrak{A}$,
(11) $Y = X$, or
(12) $Y = \bar{d} \sim X.$

By (5) $|W| \geq \alpha + 1$, so $|W'| \geq \alpha$. Hence, as it is easy to see, if Y is an atom of the minimal subalgebra of $\mathfrak{A}$ then $Y \cap V' \neq 0$. This disposes of case (10). For case (11), by (3) and (5) take a $q \in V \cap \bar{d}_{\alpha+1}$ such that $q_\alpha = \alpha + i$. In this case, by Lemma 2 and (2),

$$q \in c^{\mathfrak{A}(i,\alpha+i)}_{(\alpha)} (\bar{d}_{\alpha+1} \sim X^+) \cap \bar{d}_{\alpha+1} \cap V = c^{\mathfrak{R}}_{(\alpha)} (\bar{d}_{\alpha+1} \sim X^+) \cap \bar{d}_{\alpha+1}.$$

Hence there exist $q'_0, \ldots, q'_{\alpha-1} \in W'$ such that the series $q' \stackrel{d}{=} \langle q'_0, \ldots, q'_{\alpha-1}, \alpha + i\rangle \in (\bar{d}_{\alpha+1} \sim X^+) \cap V$, thus $\alpha 1 q' \in X \cap V'$. Case (12) can be treated similarly. By (9) and the induction hypothesis we have that $W' = \alpha + i$. This fact and (3) prove that $W = \alpha + i + 1$.

Now we have our main theorem.

Theorem 6 *For any $\alpha \in \omega \sim 2$ there exist infinitely many pairwise isomorphic, and finite (hence atomic and countable) but not lower base-isomorphic Cs_α^{reg}'s.*

Proof: We claim that for $i \in \omega \sim 1$ the Cs_α's $\mathfrak{A}_\alpha^i$ satisfy the requirements of the Theorem. In fact, $\mathfrak{A}_\alpha^i$'s are trivially finite.

By Lemma 2(ii) $\Delta X_\alpha^i = \Delta \bar{d}_\alpha = \alpha$, hence X_α^i is regular. Thus, by Corollary 3.1.54 of [6], $\mathfrak{A}_\alpha^i$ is regular.

The fact that the algebras $\mathfrak{A}_\alpha^i$ are isomorphic can be proved easily by Lemma 3 by induction on the complexity of terms.

If $i \neq j$, $\mathfrak{A}_\alpha^i$ and $\mathfrak{A}_\alpha^j$ cannot be lower base-isomorphic, since $|\text{base } (\mathfrak{A}_\alpha^i)| \neq |\text{base } (\mathfrak{A}_\alpha^j)|$, and they are base-minimal by Lemma 5.

Corollary 7 *The condition that the algebras concerned be of infinite dimension cannot be dropped in the algebraic version of Vaught's theorem (Theorem 1 of [10]), according to which any isomorphism between two isomorphic countably generated regular locally finite Cs's of infinite dimension with atomic neat n-reducts for any finite n is a lower base-isomorphism.*

Corollary 8 *The Vaught theorem (Theorem 2.3.4 of [5] or Theorem 27.10 of [8]) on prime models of atomic theories in (ordinary) languages of power ω cannot be extended to languages with finitely many, but more than one, variable symbols.*

Proof: By Corollary 7 herein and Section 4.3 of [6]. See also Theorem 6 of the present paper.

Corollary 9 *The condition for the cardinalities cannot be dropped from Theorem I.3.6 of [7] which states the following: If $\mathfrak{A}$ and $\mathfrak{B} \in Cs_\alpha^{reg} \cap Lf_\alpha$, $\mathfrak{A} \cong \mathfrak{B}$, and $|base\ (\mathfrak{A})| \cap |base\ (\mathfrak{B})| < \alpha \cap \omega$ then $\mathfrak{A}$ and $\mathfrak{B}$ are base-isomorphic.*

Proof: The proof is similar to those for Theorem 6 and Corollary 7.

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Mathematical Institute of the
Hungarian Academy of Sciences
Box 127 H-1364
Budapest, Hungary