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Book Review

Forster, T. E., *Set Theory with a Universal Set: Exploring an Untyped Universe, Oxford Logic Guides* Vol. 20, Oxford University Press, New York, 1992. 152 pages.

1 As the title suggests, this book is an 'exploration' of the set theoretic universe (or universes) under the assumption that there is a set V of all sets. Although several different approaches to V are touched upon, most of the book concerns Quine's theory NF. Indeed the theory NF is, in the reviewer's opinion, by far the most interesting part of the story. The book's coverage of NF in Chapter 2, its exceptional bibliography, and especially its coverage of the Rieger-Bernays permutation model construction described in Chapter 3, makes this book the most complete exposition of NF and its models yet published.

Many readers may be put off by the unconventional notation, and may even be slightly alarmed by the rather casual (and occasionally imprecise) terminology used. The author clearly feels, however, that in a subject as broad as a study of the disparate theories that admit the set V it is the ideas that should take key position, and these will be found in abundance. Thus, to give a familiar example, Cantor's theorem that there is no surjection of a set X onto its powerset $P(X)$ is clearly going to fail for $X = V$ because $P(V) \subseteq V$ but the idea behind the proof can be readily transformed to give the result in NF that there is no surjection from the set of singletons of X into $P(X)$ and its corollary that the cardinality of the set of all singletons is strictly smaller than the cardinality of V itself. (Perhaps this example shows why the often seemingly paranoiac obsession with the 'paradoxes' is of value: often the arguments do seem to be telling us something important about the theories and those sets that they describe.) This book contains a vast number of interesting arguments of varying degrees of difficulty—not only unfamiliar forms of familiar ones—which any reader interested in the subject would do well to assimilate.

This said, the book is not always easy to read: the sequence of definitions and results chosen does not always seem to be ordered in the most natural way, and familiar definitions and concepts are mixed with the unfamiliar ones often without indication of which is which or without preparation for the reader to whom this subject may be quite novel. Within its field, the scope of the book is almost encyclopedic, but at times the reviewer (when trying desperately to find

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such-and-such a result in the book) wondered if 'a random walk' would be a better subtitle; in particular, the inadequate index is a big disappointment. But perhaps I am trying to make the book into something it is not. As Forster says, the book is 'an essay, not a monograph or a textbook', and it is 'a good read' for mathematicians, set theorists and logicians who like seeing ideas in action and have some interest in the subject.

2 Having made the above remarks, I would like to give a more detailed discussion of the book's content, together with some personal reactions to some of the issues it raises. As will be obvious, the selection of topics here is my own. (An excellent—and rather more objective—review by Holmes [9] is to be published in the *Journal of Symbolic Logic*, and this is strongly recommended.) The bulk of the book is about NF and its relatives. The theory NF was devised by Quine as a syntactic modification of Russell and Whitehead's Theory of Simple Types (TST). Quine's objective was to eliminate the inconvenience of having to repeat definitions at each type level. His key notion is that of a *stratified formula*—a formula of the usual (first-order, single sorted) language of set theory which can be made into a formula of TST by assigning type subscripts to all the variables present in it—and he proposed the theory NF, which has the axiom of extensionality and an axiom scheme of existence for each set $\{x:\theta(x)\}$ where $\theta(x)$ is stratified and may possibly contain parameters.

It turns out that NF is quite powerful, and very different from theories in the ZF tradition. NF admits the existence of V, complements of sets, and other rather more curious operations, such as $B(x)=\{y:x\in y\}$, $F(x)=\{y:x\subseteq y\}$, and $b(x)=\{y:y\cap x\neq\varnothing\}$. (This last operation can be thought of as the 'dual' of the powerset operation. 'Duality' in its various guises turns out to be a key theme in NF.) Cardinals and ordinals can be treated in the most natural way (as equivalence classes of sets under equipolence, orderisomorphism) and it turns out that the finite cardinals in NF with their natural addition and multiplication operations gives an interpretation of nth order arithmetic for each standard n just as Frege intended.

This last result, however, rests on the theorem due to Specker that NF refutes the axiom of choice and hence proves the axiom of infinity. It is this theorem that is the main source of doubt that NF may turn out to be inconsistent. (Ironically, 'NF' stands for 'New Foundations'.) The issue here is not that AC is false in this theory—there seems to be no good reason to expect V to be wellorderable—but that Specker's argument seems rather like a lucky fluke, and who is to say there isn't an unlucky fluke waiting to be found?

To balance this, one should note that Quine's main idea has been justified in one important sense: the modified theory NFU which only has the axiom of extensionality for nonempty sets (and allows many urelements, which are identified with the sets with no elements) has been proved consistent by Jensen [10]. (Holmes [9] comments that Forster fails to take advantage of the success of NFU in his discussions of the 'paradoxes'.) In any case, no one has yet found an inconsistency in NF yet; working in NF for some time, one gets a feeling that it is weaker than Specker's result might suggest, and therefore presumably consistent. But this is curious. Why should such a theory resist all attempts to prove it consistent? Either some simple argument is missing, or NF still has some surprises in store for us.

The model theory of NF centers round another argument of Specker's, the so-called 'typical ambiguity' (ambiguity of type-levels), and a related construction, due to Rieger–Bernays, of a 'permutation model'. Specker's argument tells us how to make a many-sorted structure $(M_0, M_1, \dots)$ into a single sorted one—this is not really specific to models of TST and NF at all, and deserves to be better known amongst 'applied' model theorists. Roughly, the trick is to use a compactness argument to build an elementary extension $(M_0^*, M_1^*, \dots)$ with an isomorphism

$$\tau\colon (M_0^*, M_1^*, \dots) \to (M_1^*, M_2^*, \dots).$$

It turns out that all this requires is that the original model satisfies the axiom scheme of typical ambiguity, $\sigma \leftrightarrow \sigma^+$ for all sentences σ, where the $^+$ operation raises type indices by one throughout. The new model can be made into a one-sorted model with domain M_0^* by reinterpreting the relations via τ. For example, in the case of TST and NF we set $x \in_{\mathrm{new}} y$ to hold if and only if $x \in_{\mathrm{old}} \tau(y)$.

The permutation model construction, which is the subject of Forster's Chapter 3, was first invented for ZF but it is particularly useful for constructing new models of NF from old ones. If $(M, \in)$ is a model of set theory and $\sigma\colon M \to M$ is a bijection which preserves sethood in $(M, \in)$ (the key examples are when σ is an automorphism or is an internal set of M) define $x \in^\sigma y$ to hold iff $x \in \sigma(y)$. The resulting structure $(M, \in^\sigma)$ satisfies all stratified sentences true in $(M, \in)$, and hence is a model of NF if the original one was. But unstratified properties can be changed quite significantly: for example one can add or remove 'Quine atoms' (sets of the form $x = \{x\}$) or 'Boffa atoms' (sets of the form $x = \{y : x \in y\}$) almost at will. Another simple but surprising observation here is that for certain σ the new model $(M, \in^\sigma)$ can be regarded as an end extension of the original model.

As one might expect, all the core results alluded to above are treated clearly in Forster's book. His comments on other set theories that admit V also help put NF in context. Essentially, the 'competition' falls into three families (and I am excluding ZF and related theories here for obvious reasons): subsystems of NF, systems of 'positive' set theory, and theories in the style of Church's theory CUS.

Apart from NFU, which is consistent even if the axioms of infinity and choice are added and can be regarded as a serious theory of sets in which to do mathematics, all the known consistency proofs of subsystems of NF are rather weak. NF0 (Forster [4]) and NF_3 (Grishin [8]) are about the most interesting, but they are not really offered as theories in which to 'do' mathematics. The theory KF (Forster and Kaye [6]) was proposed recently as a subtheory of both ZF and NF to which maybe some of the 'pathologies' of NF can be consistently added, but (although it is still early days) no interesting examples have yet surfaced. Even NFU does not seem to be strong enough to reproduce the phenomena (such as the proof of $\neg$AC) that make NF so tantalizing. All of the abovementioned subtheories are discussed in the book under review. Predicative NF (NFP) and its relative NFI (Crabbé [1]) seem to be the main omissions in Forster's book; these at least have been proved consistent, but these theories are still rather weak. In fact, Con(NFI) is a theorem of third-order arithmetic, and Con(NFP) is a theorem of first-order Peano Arithmetic. Intuitionistic NF has also been considered by Dzierzgowski (and briefly by Forster on page 76) but its status is completely unknown. In particular, can Specker's proof of the axiom of infinity be carried out in it?

The 'positive' or 'topological' set theory is associated with Malitz [13], Weydert [14], and Forti and Hinnion [7]. The idea here is a syntactic trick again in which one takes the view that it is negation that is problematic in Frege's comprehension axiom scheme. The resulting set theory turns out to be consistent. Another view of the same family arises from attempting to use sets to approximate classes. This gives the alternative name 'topological set theory' which some authors prefer. The resulting theory (or rather, family of theories) is still rather restrictive in the manipulations that are allowed. Not every set has a complement, for example. Forster mentions these theories in passing, but refers the reader to the references just cited.

The last family of theories are interpreted directly in ZF plus global choice (Church, Sheridan, Mitchell). These are discussed in Chapter 4 of Forster's book. Again, they have the advantage that they come equipped with consistency proofs, but these theories only allow the kinds of manipulations of big sets that were built in at the outset, and they tell us little or nothing new about V and similar sets, nor do they say anything new about the small wellfounded sets since these are preserved in the construction.

3 Perhaps this is a good point to take stock and to discuss what one hopes to gain from a theory of sets that admits V as a set. (Quine, in particular, and certain other *NFistes* have been criticized as not having a precise enough conception of their proposed universe of sets; and without some philosophical or practical motivation this 'Cinderella of logic' (page 2) might turn out to be an ugly sister after all.)

There is no doubt that 'large' objects have a role to play in mathematics. At a by-now familiar, but nevertheless esoteric level, large cardinals (a rather different sort of large object from sets such as V) are a powerful addition to set theory in the ZF tradition; but even at a more mundane level the rectangular box we all drew around Venn diagrams at school to signify the universe is occasionally useful (my students sometimes use one!). Actually, this is not so far removed from the common practice in model theory of working within a 'big model' which is always a bit bigger than the last model you thought of.

My favorite motivation for set theories with V is that these theories might turn out to be a powerful way of describing these situations with consequences for the small everyday objects in mathematics. I don't propose any particular theory or any particular conception of set, but the NF approach seems to be about the most potent on offer at present. No one (yet) has found a new principle in NF that gives us new results about small objects, but it seems that many mathematical ideas are expressed much more naturally in NF. I have already mentioned the definitions of 'cardinal' and 'ordinal'; here is another taken from pages 39–40 of Forster's book. As is well-known, the wellfounded sets are the sets on which one can do $\in$-induction; evidently this is the same as saying that x is wellfounded iff

$$\forall y(P(y) \subseteq y \rightarrow x \in y),$$

just as ω is the least set containing 0 and closed under successor. But working within ZF (without foundation) this memorable definition must be converted to the equivalent but unsuggestive

$$\forall y(x \in y \rightarrow \exists z \in y(z \cap y = \emptyset)).$$

Nor is this the only case when a definition by recursion is more natural with large sets around.

Taking this a bit further, a set theorist with this in mind and mainly interested in NF as a new way of describing the familiar 'small' sets will be particularly interested in the various species of small set that can exist. Of these the most important genera are the wellfounded sets and the Cantorian sets (sets x which are in 1-1 correspondence with $\{\{y\} : y \in x\}$). Cantorian sets are the ones for which Cantor's proof of $|x| < |P(x)|$ goes through, and large sets such as V are not Cantorian in NF, as we have seen. The most important sub-genus within the Cantorian sets is the class of strongly Cantorian sets, those for which $\{(y, \{y\}) : y \in x\}$ is a set. The strongly Cantorian sets actually satisfy a weak separation principle for Δ_0 formulas (curiously this result is omitted from the book under review, but see Forster [2], page 24), and it is the wellfounded strongly Cantorian sets which would appear to form the most natural submodel of ZF-like sets. A natural project would be to examine the effect of reflection principles for this submodel, but this has not been carried through.

All this said, Forster's main motivation for the set theories on offer seems to be the old conception of sets as extensions of predicates. To disallow the set V of all sets, he says, is to deny the predicate of 'being identical to oneself'. I am not at all taken by this argument: it does not even indicate that there may be problems associated with the predicate 'being a nonmember of oneself'. If the sets-as-predicates argument is to be taken seriously, I feel it would be much better used as justification for the (almost unknown) set theory proposed by Krajíček [11], [12] based on modal logic. Krajíček's axioms are the modal logic **S4+BF+LP**, where BF is Barcan's formula

$$\forall x \Box \varphi(x) \rightarrow \Box \forall x \varphi(x)$$

and LP is 'Leibniz's principle'

$$(x = y \rightarrow \Box x = y) \wedge (x \neq y \rightarrow \Box x \neq y),$$

together with the set abstraction scheme

$$\exists x \forall y((\Box(y \in x) \leftrightarrow \varphi(y)) \wedge (\Box(y \notin x) \leftrightarrow \Box\neg\varphi(y)))$$

for all formulas φ—which may have other free-variables (except x itself of course). Krajíček showed that this theory is strong enough to interpret arithmetic. It turns out, though, that the usual axiom of extensionality is inconsistent with these axioms—but whether this is a problem within this modal setting is debatable—nor has a consistency result been obtained for Krajíček's theory.

Concerning other sources of axioms for large sets, there is a beautiful discussion in Section 1.2.1 concerning possible alternatives for foundation and $\in$-induction in a theory that admits nonwellfounded sets. Consider a set x, and a sequence $x \ni x_1 \ni x_2 \ni \cdots$. If x is wellfounded this must terminate. For nonwellfounded sets Forster proposes a game G_x starting with $x_0 = x$ and played by two players I and II, I playing $x_i \in x_{i-1}$ for odd i and II playing $x_{i+1} \in x_i$. I (or II) loses if (s)he can't play. Let I also denote the collection of sets for which I has a winning strategy, and similarly for II. Then $V \in \mathrm{I}$ and $\varnothing \in \mathrm{II}$, $\mathrm{II} = P(\mathrm{I})$, and dually $\mathrm{I} = b(\mathrm{II})$. This suggests a hierarchy

$$I_0 = \{V\},\ \mathrm{I}_{\alpha+1} = b(\mathrm{II}_\alpha),\ \mathrm{II}_0 = \{\varnothing\},\ \mathrm{II}_{\alpha+1} = P(\mathrm{I}_\alpha).$$

Forster suggests that 'empty sets ought to be sufficiently dense in the transitive closure of [each set] x for one player or the other to be able to force a win', and derives from this $V = \mathrm{I} \cup \mathrm{II}$ and a principle of 'pseudoinduction'—induction on the 'pseudorank' α of a set in I_α or II_α. Even more important is the game-theoretic discussion of extensionality and the game $G_{x=y}$. These ideas deserve to be explored further. In fact it is surprising that the hierarchy above is not explored further in Forster's Chapter 4 on interpretations which seems to be set up just for this sort of thing.

4 So what does an untyped universe look like? One aspect that has already been mentioned is the duality between large sets and small. This was seen at its clearest in the discussion of the classes I and II above, but appears throughout. A model of NF is, amongst other things, a particularly interesting Boolean algebra with V and $\varnothing$ (Forster notates this Λ) as **1** and **0**. Some authors have proposed axioms of duality, for example the existence of isomorphisms $(V, \in) \to (V, \notin)$ (see Forster [3] for example), but these ideas have to be treated with care: after all, what evidence have we that the universe is really like this?

Mathematics in NF looks very much like the usual mathematics, at least for small objects, but the theory is rather disappointing for larger sets. Cardinal arithmetic which starts in a promising way with the discussion of Cantor's paradox, Specker's result on the axiom of infinity and Hartog's Lemma rather fizzles out. It seems that NF is too weak to prove much of interest, or that the field of discussion is too big to expect our favorite theorems to transfer over. But there are still problems: NF even seems too weak to prove the existence of infinitely many alephs, apparently. The major stumbling block in transferring ZF-style theorems over is that inductions can only be justified in NF for stratified formulas—a seemingly very serious limitation.

On the other hand, one of the most intriguing aspects of NF is whether it can have ω-standard models. This has been exploited as a source of new axioms. For example, the T operation on cardinals, $T(|x|) = |\{\{y\} : y \in x\}|$ raises the type of its argument. Thus $\forall n T(n) = n$ (where n ranges over finite cardinals) is not stratified and cannot be proved by induction. On the other hand, T is an automorphism of the finite cardinals with $+$, $\cdot$, etc., so if T is not the identity then there are nonstandard finite cardinals. The statement $\forall n T(n) = n$ is called the axiom of counting (AxCount) and is rather powerful; in fact NF + AxCount proves Con(NF). Another example is given by the tree $\tau(\alpha)$ of a cardinal α at the heart of Specker's proof of $\neg$AC. This tree is defined to be the least set τ of cardinals containing α such that if $2^\beta \in \tau$ then $\beta \in \tau$. It is partially ordered by the transitive closure of $\beta < \gamma \Leftrightarrow_{\mathrm{def}} 2^\beta = \gamma$. It turns out that these trees are always wellfounded, and, if the axiom of counting holds, the tree on $|V|$ is of infinite rank. On the other hand, the existence of cardinals whose trees are of infinite rank is not consistent with ZFC and it is an open problem if it is consistent with ZF.

Perhaps I should conclude with a more concrete picture of a model of NF (due to Boffa) with which set-theorists in the 'ZF school' can experiment. Let $(M, \in)$ be a model of ZF (or just Z) and let it have an automorphism J, and sets a, b with $b = J(a)$ and $|P(a)| = |b|$. Then the collection of $x \in a$ with $x \in_{\mathrm{new}} y$ iff $x \in fJ(y)$ is a model of NF, where $f \in M$ is a bijection $b \to P(a)$. In particular, the original model $(M, \in)$ cannot satisfy AC. This still seems the most reasonable possibility of constructing models of NF, but getting an automorphism to send an object to something exactly the same size as its power set is a tall order.

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