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The Strength of the Δ-system Lemma

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Abstract The delta system lemma is not provable in set theory without the axiom of choice nor does it imply the axiom of choice.

1 Introduction A *Δ-system* $\mathcal{G}$ is a collection of sets such that there is a set r with the property that $(\forall A \in \mathcal{G})(\forall B \in \mathcal{G})(A \neq B \Rightarrow A \cap B = r)$. r is called the *root* of $\mathcal{G}$. The *Δ-system lemma* is the statement:

ΔSL *For every uncountable collection $\mathcal{F}$ of finite sets there is an uncountable subcollection $\mathcal{G}$ of $\mathcal{F}$ which forms a Δ-system.*

ΔSL is provable in Zermelo Fraenkel set theory (ZF) with the axiom of choice (AC) as shown by Kunen [3], [4]. We will investigate the strength of ΔSL in ZF (without the axiom of choice). In this theory there are two possible definitions of *X is uncountable*: $|X| \not\leq \aleph_0$ or $\aleph_0 < |X|$. These definitions are equivalent if AC is assumed. In Section 2 below we will use the first definition exclusively. In Section 3 we will investigate the consequences of using the second definition.

We will also refine ΔSL in the following way: $\Delta SL(n)$ will denote, for each positive integer n, the Δ-system lemma for families of n-element sets. We note that $\Delta SL(1)$ is trivially true. Our main goal will be to prove that for any integer $n \geq 2$, $\Delta SL(n)$ is equivalent to ΔSL and also to the conjunction of the two statements:

CU *The union of a countable collection of countable sets is countable.*

and

PC *Every uncountable collection of countable sets has an uncountable subcollection with a choice function.*

2 Using the first definition of uncountable We begin with:

Lemma 2.1 $ZF \vdash (\forall n \in \omega - \{0\})(\Delta SL(n+1) \Rightarrow \Delta SL(n))$.

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Proof: For any X, put $\overline{X} = \{(X,0)\} \cup (X \times \{1\})$. Assume $\Delta SL(n+1)$, and let $\mathcal{F}$ be an uncountable collection of n element sets. Then $\mathcal{F}' = \{X' : X \in \mathcal{F}\}$ is an uncountable collection of $n+1$ element sets and if $\mathcal{H}$ is any uncountable subcollection of $\mathcal{F}'$ which is a Δ-system, then $\mathcal{G} = \{X \in \mathcal{F} : \overline{X} \in \mathcal{H}\}$ is an uncountable Δ-system contained in $\mathcal{F}$. Thus $\Delta SL(n)$ holds.

Theorem 2.2 *For any $n \in \omega$, $n \geq 2$, $\Delta SL(n)$ implies CU.*

Proof: By Lemma 2.1, it suffices to show that $\Delta SL(2)$ implies *CU*. Assume $\Delta SL(2)$ and let $\mathcal{F}$ be a countable collection of countable sets. We may assume that the elements of $\mathcal{F}$ are pairwise disjoint. (If not let $\mathcal{F}' = \{A \times \{A\} : A \in \mathcal{F}\}$ then the countability of $\bigcup\mathcal{F}$ follows from the countability of $\bigcup\mathcal{F}'$.) Assume $\bigcup\mathcal{F}$ is not countable and let

$$\mathcal{G} = \{\{(a,0),(A,1)\} : A \in \mathcal{F} \wedge a \in A\}.$$

$\mathcal{G}$ is uncountable since $\bigcup\mathcal{F}$ is uncountable. Applying $\Delta SL(2)$ to $\mathcal{G}$ gives an uncountable subset $\mathcal{H}$ of $\mathcal{G}$ which is a Δ-system. Suppose that the root of $\mathcal{H}$ is r. If $r \neq \varnothing$ then for some $A \in \mathcal{F}$, $(A,1) \in r$ or $(\exists a \in A)((a,0) \in r)$. But this would mean $\mathcal{H} \subseteq \{\{(a,0),(A,1)\} : a \in A\}$ which implies $\mathcal{H}$ is countable since A is. Therefore $r = \varnothing$. This means that for each $A \in \mathcal{F}$, there is at most one $a \in A$ such that $\{(a,0),(A,1)\}$ is in $\mathcal{H}$ and again $\mathcal{H}$ is countable since $\mathcal{F}$ is.

A similar argument gives

Theorem 2.3 *For any $n \in \omega$, $n \geq 2$ $\Delta SL(n)$ implies PC.*

Proof: As in 2.2 it suffices to prove that $\Delta SL(2)$ implies *PC*. Assume $\Delta SL(2)$ and assume that $\mathcal{F}$ is an uncountable family of pairwise disjoint sets. Let

$$\mathcal{G} = \{\{(a,0),(A,1)\} : A \in \mathcal{F} \wedge a \in A\}.$$

$|\mathcal{F}| \not\leq \aleph_0 \Rightarrow |\mathcal{G}| \not\leq \aleph_0$ so $\mathcal{G}$ has an uncountable subset $\mathcal{H}$ which is a Δ-system. As in the proof of 2.2, the root r of $\mathcal{H}$ is empty. Therefore for each $A \in \mathcal{F}$ there is at most one $a \in \mathcal{F}$ such that $\{(a,0),(A,1)\}$ is in $\mathcal{H}$. Therefore

$$\{(A,a) : \{(a,0),(A,1)\} \in \mathcal{H}\}$$

is a choice function for an uncountable subset of $\mathcal{H}$ of $\mathcal{F}$.

Our final result of this section is

Theorem 2.4 *$CU \wedge PC$ implies $(\forall n \in \omega)(n \geq 1 \Rightarrow \Delta SL(n))$.*

Proof: Assume $CU \wedge PC$. The proof of $\Delta SL(n)$ for every $n \geq 1$ is by induction on n. As noted above, the case $n = 1$ is trivial.

Assume $n > 1$, that the theorem is true for all $m < n$ and that $\mathcal{F}$ is an uncountable collection of n element sets. We define:

$$a^1 = \left\{b \in \bigcup\mathcal{F} : (\exists A \in \mathcal{F})(a \in A \wedge b \in A)\right\}$$

for each $a \in \bigcup\mathcal{F}$.

If a^1 is uncountable for some $a \in \bigcup\mathcal{F}$ then, by the countable union theorem, $\{A \in \mathcal{F} : a \in A\}$ is uncountable and therefore,

$$\mathcal{G} = \{A - \{a\} : A \in \mathcal{F} \wedge a \in A\}$$

is an uncountable collection of $n - 1$ element sets. By $\Delta SL(n - 1)$, $\mathcal{G}$ has an uncountable subcollection $\mathcal{H}$ which forms a Δ-system. Then the collection $\{C \cup \{a\} : C \in \mathcal{H}\}$ is an uncountable subcollection of $\mathcal{F}$ which forms a Δ-system.

We therefore may assume a^1 is countable for every $a \in \bigcup\mathcal{F}$. Define (by induction) for each $k \in \omega$, $k > 1$

$$a^k = \bigcup_{b \in a^{k-1}} b^1.$$

Using the countable union theorem and mathematical induction, we see that for each $k \geq 1$ and for each $a \in \bigcup\mathcal{F}$, a^k is countable. It follows (from the countable union theorem) that $[a]$ is countable where (letting $a^0 = \{a\}$)

$$[a] = \bigcup_{k=0}^{\infty} a^k.$$

Therefore $p(a) = \{A \in \mathcal{F} : A \subseteq [a]\}$ is countable since $p(a)$ is a subcollection of the collection of all n-element subsets of $[a]$.

We also claim that $\mathbf{F} = \{p(a) : a \in \mathcal{F}\}$ is a partition of $\mathcal{F}$ and further that

$$p(a) \neq p(b) \Rightarrow (\forall A \in p(a))(\forall B \in p(b))(A \cap B = \emptyset).$$

We leave the proof of this claim to the reader.

It follows that the collection $\mathbf{F}$ is uncountable since $\mathcal{F}$ is. By *PC*, $\mathbf{F}$ has an uncountable subcollection $\mathbf{E}$ with a choice function f. The set $\{f(p(a)) : p(a) \in \mathbf{E}\}$ is therefore an uncountable subcollection of $\mathcal{F}$ which forms a Δ-system with root $\emptyset$.

Combining the theorems above gives us:

Corollary 2.5 *For each $n \in \omega$, $n \geq 2$, the following are equivalent*:
(1) $\Delta SL(n)$
(2) $CU \wedge PC$
(3) ΔSL.

Proof: All that remains to be shown is that $(\forall n \in \omega)(\Delta SL(n)) \Rightarrow \Delta SL$ since for each $n \in \omega$, $n \geq 2$, $\Delta SL(n)$ implies $CU \wedge PC$ which implies $(\forall n \in \omega)(\Delta SL(n))$. But this follows easily from the countable union theorem, which implies that for every uncountable collection $\mathcal{F}$ of finite sets there is an $n \in \omega$ and an uncountable subcollection all of whose elements have cardinality n.

3 Comparing the two definitions of uncountable Up to this point the meaning of *X is uncountable* has been $|X| \nleq \aleph_0$. Sets which are uncountable in this sense are clearly *Dedekind infinite sets* and the assertion that our two definitions of *uncountable* are equivalent is equivalent to the assertion $W_{\aleph_0}$ that every infinite set is Dedekind infinite (this is the notation of Jech [2].) The assertion $W_{\aleph_0}$ has been studied extensively. See for example Howard and Yorke [1], [2], Spišiak and Vojtáš [6], and Truss [7].

In this section we will use $\Delta SL(\not\leq \aleph_0)$ and $\Delta SL(\not\leq \aleph_0, n)$ for the statements ΔSL and $\Delta SL(n)$ from the previous section and we will use $\Delta SL(> \aleph_0)$ and $\Delta SL(> \aleph_0, n)$ for the corresponding statements using the second definition of uncountable.

Theorem 3.1 *For all $n \in \omega$, $n \geq 2$, $\Delta SL(> \aleph_0, n) \Rightarrow W_{\aleph_0}$.*

Proof: Assume $\Delta SL(> \aleph_0, n)$ for some $n \geq 2$. Then by an easy argument similar to the one in the proof of 2.1, $\Delta SL(> \aleph_0, 2)$ holds. Now we argue that $\Delta SL(> \aleph_0, 2)$ implies that every Dedekind finite set is finite. Assume that A is a Dedekind finite set which is not finite. We may also assume that $A \cap \omega = \emptyset$. Let $\mathcal{F} = \{\{k, a\} : k \in \omega \wedge a \in A\}$. Then clearly $|\mathcal{F}| \geq \aleph_0$. If $|\mathcal{F}| = \aleph_0$ then $|A| \leq \aleph_0$ contradicting our assumptions so we have $|\mathcal{F}| > \aleph_0$. Applying $\Delta SL(> \aleph_0, 2)$ to $\mathcal{F}$ gives a subcollection $\mathcal{G}$ of $\mathcal{F}$ such that $|\mathcal{G}| > \aleph_0$ and $\mathcal{G}$ is a Δ-system. If r is the root of $\mathcal{G}$ then either $|r| = 0$ or $|r| = 1$. If $|r| = 0$ then

$$|A| \geq |\{a \in A : (\exists x \in \mathcal{G})(a \in x)\}| = |\mathcal{G}| > \aleph_0.$$

On the other hand if $r = \{t\}$ then there are two possibilities, either $t \in A$ or $t \in \omega$. If $t \in A$ then $|\mathcal{G}| \leq |\{\{k, t\} : k \in \omega\}| = \aleph_0$ which is impossible. If $t \in \omega$ then $|A| \geq |\{\{t, a\} : a \in A\}| \geq |\mathcal{G}| > \aleph_0$ a contradiction.

Corollary 3.2 *For any $n \in \omega$, $n \geq 2$, the following are equivalent*:
(1) $\Delta SL(> \aleph_0, n)$
(2) $CU \wedge PC \wedge W_{\aleph_0}$
(3) $\Delta SL(> \aleph_0)$.

Proof: Under the assumption $W_{\aleph_0}$, $\Delta SL(\not\leq \aleph_0, k)$ and $\Delta SL(> \aleph_0, k)$ are equivalent for every $k \in \omega$, $k \geq 2$, as are $\Delta SL(\not\leq \aleph_0)$ and $\Delta SL(> \aleph_0)$. Since $\Delta SL(> \aleph_0, n)$ implies $W_{\aleph_0}$, the corollary follows from 2.5.

We note that as a consequence of the corollary, $\Delta SL(> \aleph_0)$ implies $\Delta SL(\not\leq \aleph_0)$.

Finally, we show that $\Delta SL(\not\leq \aleph_0) \Rightarrow \Delta SL(> \aleph_0)$ is not a theorem of *ZF*. We will do this by showing that $\Delta SL(\not\leq \aleph_0)$ is true and $\Delta SL(> \aleph_0)$ is false in the ordered Mostowski model ([2], p. 49). This will give $\nvdash_{ZFU} \Delta SL(\not\leq \aleph_0) \Rightarrow \Delta SL(> \aleph_0)$ where *ZFU* is *ZF* weakened to permit the existence of atoms. We will then appeal to the transfer results of Pincus [5] to obtain our desired result $\nvdash_{ZF} \Delta SL(\not\leq \aleph_0) \Rightarrow \Delta SL(> \aleph_0)$.

We begin with a brief description of the construction of the ordered Mostowski model. Let M' be a model of $ZFU + AC$ with a countable set A of atoms and a linear ordering $<$ of A with order type that of the rational numbers. Let G be the group of all order preserving permutations of A and for each finite subset E of A, let $\text{fix}(E) = \{\phi \in G : (\forall a \in E)(\phi(a) = a)\}$. Let Γ be the filter of subgroups of G generated by $\{\text{fix}(E) : E \subseteq A \wedge E \text{ finite}\}$. The ordered Mostowski model M is the permutation model determined by Γ, that is

$$M = \{x \in M' : (\forall y \in \{x\} \cup \text{trcl}(x))(\exists H \in \Gamma)(\forall \phi \in H)(\phi(y) = y)\}.$$

In this formula, $\text{trcl}(x)$ denotes the transitive closure of x. If $x \in M$ and E is a finite subset of A, we say *E is a support of x* if $(\forall \phi \in \text{fix}(E))(\phi(x) = x)$. We will make use of the following facts about M:

(4) Every $x \in M$ has a least support, $\text{supp}(x)$, which satisfies $(\forall \phi \in G)(\phi(x) = x \Leftrightarrow \phi \in \text{fix}(\text{supp}(x))$.
(5) $W_{\aleph_0}$ is false in M because in M, $|A| \not\leq \aleph_0$ and $|A| \not\geq \aleph_0$.
(6) If $x \in M$ is finite, then $\text{supp}(x) = \bigcup_{t \in x} \text{supp}(t)$.

Theorem 3.3 *$\Delta SL(> \aleph_0)$ is false in M.*

Proof: This follows from 3.1 and (5).

Theorem 3.4 *$\Delta SL(\not\leq \aleph_0)$ is true in M.*

Proof: By 2.4, it suffices to prove $\Delta SL(\not\leq \aleph_0, 2)$. Suppose $\mathcal{F} \in M$ is a collection of 2 element sets such that, in M, $|\mathcal{F}| \not\leq \aleph_0$.

We first handle the case where $(\forall x \in \mathcal{F})(\text{supp}(x) \subseteq \text{supp}(\mathcal{F}))$. In this case $\mathcal{F}$ is well-orderable in M since $\text{supp}(\mathcal{F})$ is a support for any well-ordering of $\mathcal{F}$. This together with $(|\mathcal{F}| \not\leq \aleph_0)^M$ implies that $|\mathcal{F}| \geq \aleph_1$ in M. Therefore $|\mathcal{F}| \geq \aleph_1$ in M'. Hence, in M' (since M' satisfies AC) $\mathcal{F}$ has a subcollection $\mathcal{G}$ such that $\mathcal{G}$ forms a Δ-system and $|\mathcal{G}| = \aleph_1$. But $\text{supp}(\mathcal{F})$ is a support for $\mathcal{G}$ and for each element of $\mathcal{G}$, therefore $\text{supp}(\mathcal{F})$ is a support for a bijection between $\mathcal{G}$ and $\aleph_1$. It follows that $\mathcal{G} \in M$, $|\mathcal{G}| = \aleph_1$ in M and $\mathcal{G}$ is a Δ-system in M.

On the other hand if there is some $x \in \mathcal{F}$ such that $\text{supp}(x) \not\subseteq \text{supp}(\mathcal{F})$, suppose that $x = \{t_1, t_2\}$ and that $\text{supp}(\mathcal{F}) \cup \text{supp}(x) = \{a_1, a_2, \ldots, a_n\}$ where $a_1 < \cdots < a_n$. Fix a j such that $a_j \in \text{supp}(x) - \text{supp}(\mathcal{F})$. We will assume that $1 < j < n$. The proof can easily be modified to handle the cases $j = 1$ and $j = n$. Let $E = (\text{supp}(\mathcal{F}) \cup \text{supp}(x)) - \{a_j\}$.

By (6) there are three possibilities:

case 1. $a_j \in \text{supp}(t_1) - \text{supp}(t_2)$.
case 2. $a_j \in \text{supp}(t_2) - \text{supp}(t_1)$.
case 3. $a_j \in \text{supp}(t_1) \cap \text{supp}(t_2)$.

In case 1 we construct a subcollection $\mathcal{G}$ of $\mathcal{F}$ such that
(i) $\mathcal{G}$ has support $\subseteq E$
(ii) $|\mathcal{G}| = |\{a : a_{j-1} < a < a_{j+1}\}|$ in M.
(iii) $\mathcal{G}$ is a Δ-system with root $\{t_2\}$.

This will suffice since it follows from (ii) and (5) that $|\mathcal{G}| \not\leq \aleph_0$ in M.

$\mathcal{G}$ is defined by

$$\mathcal{G} = \{\phi(x) : \phi \in \text{fix}(E)\}. \tag{7}$$

We first note that $\mathcal{G} \subseteq \mathcal{F}$ since for any $\phi \in \text{fix}(E)$, $\phi(\mathcal{F}) = \mathcal{F}$ and therefore $\phi(x) \in \mathcal{F}$. Part (i) is clear. For (ii) we claim that the set $f = \{(\phi(x), \phi(a_j)) : \phi \in \text{fix}(E)\}$ is a one-to-one function from $\mathcal{G}$ onto $\{a : a_{j-1} < a < a_{j+1}\}$ with support E. It is clear that f has domain $\mathcal{G}$ and that f is onto $\{a : a_{j-1} < a < a_{j+1}\}$. The relation f is one-to-one, for suppose that $\phi, \psi \in \text{fix}(E)$ and $\phi(a_j) = \psi(a_j)$ then $\psi^{-1}\phi(a_j) = a_j$ so that $\psi^{-1}\phi \in \text{fix}(E \cup \{a_j\})$. It follows that $\psi^{-1}\phi(x) = x$, hence $\phi(x) = \psi(x)$. Similarly f is a function since for $\phi, \psi \in \text{fix}(E)$, $\phi(x) = \psi(x) \Rightarrow \psi^{-1}\phi(x) = x$ which by (4) implies $\psi^{-1}\phi(a_j) = a_j$. Hence $\phi(a_j) = \psi(a_j)$.

(iii) follows since $\text{supp}(t_2) \subseteq E$ therefore for each $\phi \in \text{fix}(E)$, ϕ fixes t_2.

Case 2 is similar to case 1. In case 3 we construct a subcollection $\mathcal{G}$ of $\mathcal{F}$ satisfying (i), (ii), and

(iii′) $\mathcal{G}$ is a Δ-system with root $\varnothing$.

As in case 1 we define $\mathcal{G}$ by (7). The proofs of (i) and (ii) are identical to the case 1 proofs. For (iii′) we first note that for $\phi \in \text{fix}(E)$ with $\phi(x) \neq x$ we can conclude by (4) that $\phi(a_j) \neq a_j$ and therefore by (4) that

(8) $$\phi(t_1) \neq t_1 \wedge \phi(t_2) \neq t_2.$$

It follows from (4) that $\phi(\text{supp}(z)) = \text{supp}(\phi(z))$ for all $z \in M$ and therefore

(9) $$\phi(t_1) \neq t_2 \wedge \phi(t_2) \neq t_1.$$

Combining (8) and (9) gives us $(\forall\phi \in \text{fix}(E))(\phi(x) \neq x \Rightarrow \phi(x) \cap x = \varnothing)$. It follows that the elements of $\mathcal{G}$ are pairwise disjoint. This completes the proof of 3.4.

The proof that the independence results can be transferred to *ZF* will require the following lemma.

Lemma 3.5 *For any ordinal α, if $\mathcal{A}$ is a collection of sets such that $|\mathcal{A}| \leq \aleph_{\alpha+1}$ and $(\forall x \in \mathcal{A})(|x| \leq \aleph_\alpha)$ then $|\bigcup\mathcal{A}| \not\geq \aleph_{\alpha+2}$.*

Proof: Assume the hypotheses and that $|\bigcup\mathcal{A}| \geq \aleph_{\alpha+2}$. Let $Z \subseteq \bigcup\mathcal{A}$ have cardinality $\aleph_{\alpha+2}$, then $\mathcal{A}' = \{x \cap Z : x \in \mathcal{A}\}$ satisfies $|\mathcal{A}'| \leq \aleph_{\alpha+1}$ and $(\forall x \in \mathcal{A}')(|x| \leq \aleph_\alpha)$ and $|\bigcup\mathcal{A}'| = \aleph_{\alpha+2}$. Let $\lhd$ be a well-ordering of $\bigcup\mathcal{A}'$. For each $x \in \mathcal{A}'$, $\lhd \restriction x$ is an ordering of type $< \aleph_{\alpha+1}$. From these well-orderings together with a well-ordering of $\mathcal{A}'$ of type $\leq \aleph_{\alpha+1}$ it is easy to construct a one-to-one function from $\bigcup\mathcal{A}'$ into $\aleph_{\alpha+1} \times \aleph_{\alpha+1}$. But $|\aleph_{\alpha+1} \times \aleph_{\alpha+1}| = \aleph_{\alpha+1}$ (see [4], p. 293). This contradiction completes the proof of the lemma.

Now we note that

(10) $$(\forall Z)(|Z| \not\geq \aleph_3 \rightarrow (\forall\mathcal{F} \in \mathcal{P}\mathcal{P}(Z))\,[((|\mathcal{F}| \not\leq \aleph_0) \wedge (\forall t \in \mathcal{F})(|t| = 2)) \rightarrow (\exists\mathcal{G} \in \mathcal{P}\mathcal{P}(Z))(\mathcal{G} \subseteq \mathcal{F} \wedge (|\mathcal{G}| \not\leq \aleph_0) \wedge \mathcal{G} \text{ is a } \Delta\text{-system})])$$

(which is $\Delta SL(\not\leq \aleph_0)$ restricted to families $\mathcal{F}$ such that $|\bigcup\mathcal{F}| \not\geq \aleph_3$) is injectively boundable in the sense of [5] since $|Z| \not\geq \aleph_3$ and $|Z|_- \leq \aleph_2$ are equivalent (see [5].) Since $\neg\Delta SL(> \aleph_0)$ is boundable (in the sense of [5]), we can use the transfer results of [5] to obtain a model of *ZF* in which (10) is true and $\Delta SL(> \aleph_0)$ is false. Therefore to complete our argument it will be sufficient to prove the following lemma.

Lemma 3.6 $ZF \vdash (10) \rightarrow \Delta SL(\not\leq \aleph_0)$.

Proof: Let $\mathcal{F}$ be a collection of pairs such that $|\bigcup\mathcal{F}| \geq \aleph_3$. Let $X \subseteq \bigcup\mathcal{F}$ satisfy $|X| = \aleph_2$ and let $\mathcal{G} = \{A \in \mathcal{F} : A \cap X \neq \varnothing\}$. For each $a \in X$ let $\mathcal{F}_a = \{A \in \mathcal{G} : (\exists b)(A = \{a,b\} \wedge b \notin X)\}$. If for some $a \in X$, $|\mathcal{F}_a| \not\leq \aleph_0$ then $\mathcal{F}_a$ is an uncountable Δ-system (with root $\{a\}$) and we are done. We may therefore assume that for every $x \in X$, $|\mathcal{F}_a| \leq \aleph_0$. It follows that for every $a \in X$, $|\bigcup\mathcal{F}_a| \leq \aleph_0$. By 3.5 $|\bigcup\{\bigcup\mathcal{F}_a : a \in X\}| \not\geq \aleph_3$. If we let $\mathcal{F}' = \{A \in \mathcal{F} : A \subseteq X\}$, then $\mathcal{G} = \mathcal{F}' \; U \; (\bigcup\{\mathcal{F}_a : a \in X\})$ so $|\bigcup\mathcal{G}| \not\geq \aleph_3$. Also by 3.5 $|\mathcal{G}| \not\leq \aleph_0$. Therefore by (10), $\mathcal{G}$ has an uncountable subcollection which forms a Δ-system. This completes the proof of our independence result which we state as:

Theorem 3.7 $ZF \nvdash \Delta SL(\not\leq \aleph_0) \Rightarrow \Delta SL(> \aleph_0)$.

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REFERENCES

[1] Howard, P. and M. Yorke, "Definitions of finite," *Fundamenta Mathematicae*, vol. 133 (1989), pp. 169–177.

[2] Jech, T., *The Axiom of Choice*, Studies in Logic and the Foundations of Mathematics, vol. 75, North-Holland, Amsterdam, 1973.

[3] Kunen, K., "Combinatorics," pp. 371–401 in *Handbook of Mathematical Logic*, edited by J. Barwise, North-Holland, Amsterdam, 1977.

[4] Kunen, K., *Set Theory: An Introduction to Independence Proofs*, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland, Amsterdam, 1980.

[5] Pincus, D., "Zermelo-Fraenkel consistency results by Fraenkel-Mostowski methods," *The Journal of Symbolic Logic*, vol. 37 (1972), pp. 721–743.

[6] Spišiak, L. and P. Vojtáš, "Dependencies between definitions of finiteness," *Czechoslovak Mathematical Journal*, vol. 38 (1988), pp. 389–397.

[7] Truss, J., "Classes of Dedekind finite cardinals," *Fundamenta Mathematicae*, vol. 84 (1974), pp. 188–208.

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