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Unifying Some Modifications of the Henkin Construction

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Abstract This paper is a continuation of the work of Leblanc, Roeper, Thau, and Weaver, which modified the Henkin construction to yield various necessary and sufficient conditions for extending a consistent set of sentences in a countable first order language to a maximally consistent and ω-complete set in that language. In this paper the theory of abstract deducibility relations introduced by Goldblatt is extended to provide an abstract setting for these and related results. Modifications of Henkin construction are replaced by Goldblatt's Countable Henkin Principle to yield abstract forms of the ω-completeness theorem, the soundness and completeness of ω-logic, the theorem to the effect that ω-logic is a conservative extension of standard logic for ω-complete sets, and the theorem that all ω-complete sets are ω-consistent. These abstract results specialize to yield the corresponding "concrete" ones.

The main technical innovation (the characterization of the smallest deducibility relation that respects all members of a family of premise–conclusion arguments and extends a given deducibility relation) is motivated by the observation that the deducibility relation determined by ω-logic is the smallest deducibility relation which extends the deducibility relation determined by standard logic and under which every set of sentences respects the ω-rule.

1 Introduction In Henkin [3] the method of constants was introduced to show that every consistent set of first order sentences has a model. The construction of this model is often called the Henkin construction. Part of the construction involves showing that the consistent set can be extended to a maximally consistent set which contains a universal quantification if it contains all of its instances. This extension is accomplished by adding "enough" new individual constants to the nonlogical vocabulary of the language. This addition cannot, in general, be avoided. It is easily shown that, even when the nonlogical vocabulary of the language contains individual constants, there are consistent sets of

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sentences which do not have the desired extension in that language. However, there are cases in which the addition of new constants is unnecessary.

Let $\mathcal{L}_1$ be a countable first order language whose nonlogical vocabulary contains predicate constants of various degrees and infinitely many individual constants $c_0, \ldots, c_n, \ldots$. Let S be a consistent set of sentences in $\mathcal{L}_1$, and consider the following questions: (I) what conditions on S imply that the Henkin construction of a model of S can be carried out "inside" $\mathcal{L}_1$? (i.e., what conditions imply that S has a maximally consistent extension in $\mathcal{L}_1$ which contains a universal quantification in $\mathcal{L}_1$, if it contains each of its instances?); (II) how can the Henkin construction be modified to produce the desired extension?

These questions were discussed in Leblanc et al. [4], where several candidates were considered:

(1) S is ω-complete (i.e., if all instances of $\forall x\varphi(x)$ are provable from S, then $\forall x\varphi(x)$ is provable from S);
(2) S is infinitely extendible (i.e., infinitely many individual constants are foreign to S);
(3) S is consistent in ω-logic; and
(4) S is ω-consistent (i.e., if for all n, $\sim\varphi(c_n)$ is provable from S, then $\exists x\varphi(x)$ is not provable from S).

The following summarizes the relevant results of [4].

Theorem 1 *If S is a consistent set of sentences in $\mathcal{L}_1$, then each of the following implies that S has a maximally consistent ω-complete extension in $\mathcal{L}_1$:*
(1) *S is ω-complete;*
(2) *S is infinitely extendible;*
(3) *S is consistent in ω-logic; and*
(4) *S is ω-consistent, when $\mathcal{L}_1$ does not contain equality and the nonlogical vocabulary of $\mathcal{L}_1$ contains only one predicate and that predicate is unary.*

It was also shown that: (1) consistency in ω-logic is a necessary and sufficient condition for having a maximally consistent and ω-complete extension in $\mathcal{L}_1$; (2) neither ω-completeness nor infinite extendibility are necessary conditions; and (3) ω-consistency is a necessary but not sufficient condition except for $\mathcal{L}_1$ as in Theorem 1 (4).

When $\mathcal{L}_1$ contains equality, parts (1), (2), and (3) of Theorem 1 are consequences of the omitting-types theorem. Let $\Sigma(x) = \{x \neq c_n;$ for all $n\}$. When S is consistent and ω-complete, $\Sigma(x)$ is a nonprincipal type of S. Hence S has a model which omits $\Sigma(x)$. The set of sentences true on this model is maximally consistent and ω-complete. (2) follows from (1) and the observation that infinitely extendible sets are ω-complete. Finally, (3) is also a consequence of (1) (cf. Chang and Keisler [1], p. 81).

Proofs of parts (1), (2), and (3) of Theorem 1 were obtained in [4] by modifying the Henkin construction. In the following, ideas and results of Goldblatt [2] are introduced to provide an abstract setting for these and related results. This approach serves to unify the proofs of [4]. Applications of the Henkin construction and its modifications are replaced by a single abstract "Henkin Principle" which yields various abstract results. These abstract results specialize to yield the completeness theorem as well as parts (1), (2), and (3) of Theorem 1 and related results.

Here, in contrast with the current literature (e.g., [1], pp. 80–81), the omega-concepts (i.e., ω-consistent, ω-complete, ω-rule, and ω-logic) are applied to countable first order languages with countably many constant terms whether or not these terms are numerals. Thus, by 'ω-logic' is meant the result of adding the ω-rule (see Section 2 below) to the deductive system of first order logic, allowing proofs to be infinitely long, and restricting interpretations to those of standard first order logic in which each member of the domain is denoted by a constant term.

Leblanc et al. [4] uses 'term-consistent' and 'term-complete' in the way that 'ω-consistent' and 'ω-complete' are used here. There is no terminology used in that paper in the way that 'ω-rule' and 'ω-logic' are used here, although 'infinite instantial induction' is used in a footnote in the way that 'ω-rule' is used here.

2 Goldblatt [2] extracts the essence of the Henkin construction and presents it in the form of abstract principles. In this section the preliminary definitions are reviewed and the relevant principles are stated.

Let $\mathcal{L}$ be a language such that (1) if φ is a sentence in $\mathcal{L}$, then $\sim\varphi$ is also a sentence in $\mathcal{L}$; and (2) $\mathcal{L}$ contains the sentence $\perp$ which is thought of as a contradictory sentence. For convenience, $\mathcal{L}$ will denote the set of sentences in $\mathcal{L}$. $\mathcal{P}(\mathcal{L})$ denotes the power set of $\mathcal{L}$.

Let $\vdash$ be a subset of the cartesian product $\mathcal{P}(L) \times \mathcal{L}$, $\vdash$ is called a *deducibility relation* provided $\vdash$ satisfies the following:

D1 if $S \vdash \varphi$ and $S \subseteq S'$, then $S' \vdash \varphi$;
D2 if $\varphi \in S$, then $S \vdash \varphi$;
D3 if $S \cup \{\varphi\} \vdash \psi$ and $S \vdash \varphi$, then $S \vdash \psi$; and
D4 $S \vdash \varphi$ iff $S \cup \{\sim\varphi\} \vdash \perp$.

$\vdash$ *is a finitary deducibility relation* provided $\vdash$ is a deducibility relation which satisfies

D5 if $S \vdash \varphi$, then there is S', a finite subset of S, such that $S' \vdash \varphi$.

In [2] the following condition is used in place of D3:

D3′ if $S \vdash \varphi$ and $S \cup \{\varphi\} \vdash \perp$, then $S \vdash \perp$.

Clearly, D3 implies D3′. Moreover, if $\vdash$ satisfies D1, D4, and D3′, then $\vdash$ also satisfies D3. To understand this suppose that $\vdash$ satisfies D1, D4, and D3′. To show that $\vdash$ satisfies D3, suppose that $S \cup \{\varphi\} \vdash \psi$ and $S \vdash \varphi$. To show that $S \vdash \psi$. Since $S \vdash \varphi$, by D1 $S \cup \{\sim\psi\} \vdash \varphi$; further, since $S \cup \{\varphi\} \vdash \psi$, by D4 $S \cup \{\sim\psi\} \cup \{\phi\} \vdash \perp$; hence by D3′ $S \cup \{\sim\psi\} \vdash \perp$; and by D4 $S \vdash \psi$.

Given $S \subseteq \mathcal{L}$, $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$, S is $\vdash$*-consistent* provided $S \nvdash \perp$; S is *finitely* $\vdash$*-consistent* provided all finite subsets of S are $\vdash$-consistent; S is *maximally finitely* $\vdash$*-consistent* provided S is finitely $\vdash$-consistent and no proper extension of S is finitely $\vdash$-consistent; finally, S is *maximally* $\vdash$*-consistent* provided S is $\vdash$-consistent and no proper extension of S is $\vdash$-consistent.

Notice that if $\vdash$ is any subset of $\mathcal{P}(\mathcal{L}) \times \mathcal{L}$, then finitely $\vdash$-consistent sets are closed under unions of chains. Thus, by Zorn's Lemma, we have:

Lemma 1 [Abstract Lindenbaum Principle: cf. [2]] *If* $\vdash\, \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ *and S is finitely* $\vdash$*-consistent, then S has a maximally finitely* $\vdash$*-consistent extension.*

Notice that if $\vdash$ is a deducibility relation and S is finitely $\vdash$-consistent, then for all φ in $\mathcal{L}$, either $S \cup \{\varphi\}$ or $S \cup \{\sim\varphi\}$ is finitely $\vdash$-consistent. Otherwise there are S' and S'' finite subsets of S such that $S' \cup \{\varphi\} \vdash \perp$ and $S'' \cup \{\sim\varphi\} \vdash \perp$. Hence by D1, $S' \cup S'' \cup \{\varphi\} \vdash \perp$ and $S' \cup S'' \cup \{\sim\varphi\} \vdash \perp$. Thus by D4, $S' \cup S'' \vdash \varphi$, and by D3, $S' \cup S'' \vdash \perp$. But $S' \cup S''$ is a finite subset of S, a contradiction. Therefore, if S is maximally finitely $\vdash$-consistent, then for all φ in $\mathcal{L}$, either $\varphi \in S$ or $\sim\varphi \in S$. Otherwise, as $S \subset S \cup \{\varphi\}$ and $S \subset S \cup \{\sim\varphi\}$, S is not maximally $\vdash$-consistent.

Further, if $\vdash$ is a deducibility relation and S is maximally $\vdash$-consistent, then for all $\varphi \in \mathcal{L}$, (1) either $\varphi \in S$ or $\sim\varphi \in S$; (2) it is not the case that $\varphi \in S$ and $\sim\varphi \in S$; and (3) if $S \vdash \varphi$, then $\varphi \in S$. To understand this suppose that $\vdash$ is a deducibility relation and S is maximally $\vdash$-consistent. Let $\varphi \in \mathcal{L}$. Suppose that $\varphi \notin S$ and $\sim\varphi \notin S$. Thus $S \cup \{\varphi\}$ and $S \cup \{\sim\varphi\}$ are extensions of S. Since S is maximally $\vdash$-consistent, $S \cup \{\varphi\} \vdash \perp$ and $S \cup \{\sim\varphi\} \vdash \perp$. Thus by D4, $S \vdash \varphi$; and by D3, $S \vdash \perp$; a contradiction. Now suppose that $\varphi \in S$ and $\sim\varphi \in S$. By D2, $\{\varphi\} \vdash \varphi$. Hence by D4, $\{\varphi, \sim\varphi\} \vdash \perp$. Therefore by D1, $S \vdash \perp$; a contradiction. Finally, suppose that $S \vdash \varphi$ and $\varphi \notin S$. By (1) above, $\sim\varphi \in S$; and by D4, $S \cup \{\sim\varphi\} \vdash \perp$, a contradiction.

Members (S', φ) of $\mathcal{P}(\mathcal{L}) \times \mathcal{L}$, are called *premise-conclusion arguments* (or p–c arguments). S' is called the *premises of* (S', φ) and φ is called the *conclusion of* (S', φ). Intuitively, p–c arguments are thought of as inferences and sets of p–c arguments are thought of as rules of inference. Let $(S', \varphi) \in \mathcal{P}(\mathcal{L}) \times \mathcal{L}$, $\vdash\, \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$, S *respects* (S', φ) (in $\vdash$) provided if for all $\psi \in S'$, $S \vdash \psi$, then $S \vdash \varphi$; S is *closed under* (S', φ) provided if $S' \subseteq S$, then $\varphi \in S'$; and S *decides* (S', φ) provided either $\varphi \in S$ or there is ψ in S' such that $\sim\psi \in S$. Let $I \subseteq \mathcal{P}(L) \times \mathcal{L}$, S *respects* I (in $\vdash$) provided S respects every member of I (in $\vdash$); S is *closed under* I provided S is closed under every member of I; and S *decides* I iff S decides every member if I.

Goldblatt extracted the following principle from the Henkin construction.

Theorem 2 [The Countable Abstract Henkin Principle: cf. [2]] *If* $\vdash$ *is a deducibility relation, I is a countable set of p–c arguments, S is* $\vdash$*-consistent and each finite extension of S respects I* (*in* $\vdash$), *then S has a finitely* $\vdash$*-consistent extension which decides I.*

3 In this section attention is focused on ω-completeness. It is immediate from the definitions of the last section that any set which extends a set which decides I also decides I and that any finitely $\vdash$-consistent set which decides I is closed under I (cf. Lemma 1, [2], p. 38). The major result of this section is an abstract form of what some authors have called the ω-completeness theorem (cf. Vaught [5], p. 27).

Corollary 1 *If $\vdash$ is a finitary deducibility relation, I is countable, S is $\vdash$-consistent, and each finite extension of S respects I (in $\vdash$), then S has a maximally $\vdash$-consistent extension which is closed under I.*

Proof: By Theorem 2, S has a finitely $\vdash$-consistent extension, S', which decides I. By Lemma 1, there is S'' such that $S' \subseteq S''$ and S'' is maximally finitely $\vdash$-consistent. Hence S'' also decides I; and S'' is closed under I. Finally, since $\vdash$ is finitary, S'' is maximally $\vdash$-consistent.

Now let $\vdash^1$ be the finitary deducibility relation on $\mathcal{L}_1$ provided by first order logic. Let $I(\omega)$ denote $\{(\{\psi(c_n)\colon \text{for all } n\}, \forall x\psi(x))\colon \text{for all } \psi(x) \text{ a formula in } \mathcal{L}_1)\}$. Intuitively, $I(\omega)$ is the ω-rule or the rule of infinite instantial induction (see [4], note 6). Since $\mathcal{L}_1$ is countable, $I(\omega)$ is also countable. Notice that S is ω-complete iff S respects $I(\omega)$. Since any finite extension of a set which respects $I(\omega)$ also respects $I(\omega)$ (see [4], Lemma 8), the following is immediate from Corollary 1.

Corollary 2 *If S is $\vdash^1$-consistent and respects $I(\omega)$, then S has a maximally $\vdash^1$-consistent extension which is closed under $I(\omega)$.*

Since every infinitely extendible set is ω-complete (see [4], remarks following the proof of Theorem 7), it follows from the above that if S is $\vdash^1$-consistent and infinitely extendible, then S has a maximally $\vdash^1$-consistent extension which is closed under $I(\omega)$. Further, even when $\mathcal{L}_1$ must be extended by introducing infinitely many new individual constants, S is infinitely extendible (hence ω-complete) in the larger language. Hence Corollary 1 specializes to cover both the standard Henkin construction, when new constants must be added, and the case when S is ω-complete in $\mathcal{L}_1$. Furthermore, Corollary 1 also specializes to yield the countable omitting types theorem (see [2], p. 40).

4 In this section attention is focused on ω-logic. To this end the theory of deducibility relations is developed further. The idea here is motivated by the observation that the deducibility relation provided by ω-logic ($\vdash_\omega$) is the smallest deducibility relation which extends $\vdash^1$ and under which each subset of $\mathcal{L}_1$ respects $I(\omega)$.

Lemma 2 *If $\vdash$ is a deducibility relation, $I \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$, then there is a unique deducibility relation which is the smallest deducibility relation which extends $\vdash$ and under which each subset of $\mathcal{L}$ respects I.*

Proof: Let $\vdash$ be a deducibility relation, $I \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$. Let $\Delta = \{\vdash'\colon \vdash \subseteq \vdash', \vdash'$ is a deducibility relation and every subset of $\mathcal{L}$ respects I (in $\vdash'$)$\}$. Notice that $\mathcal{P}(\mathcal{L}) \times \mathcal{L} \in \Delta$. It is easily verified that $\bigcap\Delta \in \Delta$. Hence $\bigcap\Delta$ is the smallest deducibility relation extending $\vdash$ under which each subset of $\mathcal{L}$ respects I.

$\vdash_I$ denotes the smallest such deducibility relation. Intuitively, $\vdash_I$ is the deducibility relation which results from adding the rule I to the logic which determines $\vdash$. Notice that $\vdash_I$ need not be finitary even if $\vdash$ is. It will be shown below that if the premise of each member of I is finite and $\vdash$ is finitary, then $\vdash_I$ is also finitary.

The following abstract principle specializes to yield part (3) of Theorem 1.

Theorem 3 *If $\vdash$ is a finitary deducibility relation, I is countable and S is $\vdash_I$-consistent, S has a maximally $\vdash$-consistent extension which is closed under I.*

Proof: Let $\vdash$ be a finitary deducibility relation and I be a countable subset of $\mathcal{P}(\mathcal{L}) \times \mathcal{L}$. Suppose that S is $\vdash_I$-consistent. Then by Theorem 2 and Lemma 1, there is S'' such that $S \subseteq S''$, S'' is maximally finitely $\vdash_I$-consistent and closed under I. Further, for φ in $\mathcal{L}$ either $\varphi \in S''$ or $\sim\varphi \in S''$.

Since $\vdash \subset \vdash_I$, S'' is finitely $\vdash$-consistent. As $\vdash$ is finitary, S'' is $\vdash$-consistent. It remains to show that S'' is maximally $\vdash$-consistent. Suppose otherwise. Then there is φ in $\mathcal{L}$ such that $S'' \subset S'' \cup \{\varphi\}$ and $S'' \cup \{\varphi\}$ is $\vdash$-consistent. Thus $\varphi \notin S''$, and $\sim\varphi \in S''$. By D2, $S'' \cup \{\varphi\} \vdash \varphi$. Thus by D4, $S'' \cup \{\varphi\} \cup \{\sim\varphi\} \vdash \bot$. But $S'' \cup \{\varphi\} \cup \{\sim\varphi\} = S'' \cup \{\varphi\}$. Hence $S'' \cup \{\varphi\}$ is $\vdash$-inconsistent, a contradiction.

The above and the observation (which follows from the soundness of ω-logic) that $\vdash_\omega = \vdash^1_{I(\omega)}$ yields the following:

Corollary 3 *If S is a subset of $\mathcal{L}_1$ which is $\vdash_\omega$-consistent, then S has a maximally $\vdash^1$-consistent extension which is closed under $I(\omega)$.*

It was mentioned above that $\vdash_\omega$-consistency is a necessary and sufficient condition for having a maximally $\vdash^1$-consistent extension which is closed under $I(\omega)$. There is an abstract principle which specializes to yield this result as well:

Theorem 4 *If $\vdash$ is a finitary deducibility relation and I is countable, then the following are equivalent:*
(1) *S is $\vdash_I$-consistent; and*
(2) *S has a maximally $\vdash$-consistent extension which is closed under I.*

This theorem is proved in the next section. Before proceeding, some of the consequences of this result are established.

Corollary 4 *If $\vdash$ is a finitary deducibility relation, I is countable, S is $\vdash$-consistent and all finite extensions of S respect I (in $\vdash$), then for all φ in $\mathcal{L}$, $S \vdash \varphi$ iff $S \vdash_I \varphi$.*

Proof: Let $\vdash$, I, and S be as above. Let φ be in $\mathcal{L}$. Suppose $S \vdash \varphi$. Since by construction $\vdash \subseteq \vdash_I$, $S \vdash_I \varphi$. Suppose that $S \vdash_I \varphi$. To show that $S \vdash \varphi$. Suppose otherwise. By D4, $S \cup \{\sim\varphi\}$ is $\vdash$-consistent. By hypothesis, all finite extensions of $S \cup \{\sim\varphi\}$ respect I (in $\vdash$). Hence by Corollary 1, $S \cup \{\sim\varphi\}$ has a maximally $\vdash$-consistent extension which is closed under I. Thus by Theorem 4, $S \cup \{\sim\varphi\} \nvdash_I \bot$. But by supposition and D4, $S \cup \{\sim\varphi\} \vdash_I \bot$, a contradiction.

The following is immediate from Corollary 4 and the observation that finite extensions of sets which respect $I(\omega)$ (in $\vdash^1$) also respect $I(\omega)$.

Corollary 5 *If $S \subseteq \mathcal{L}(1)$, S is $\vdash^1$-consistent and respects $I(\omega)$ (in $\vdash^1$) then for all φ in $\mathcal{L}_1$, $S \vdash^1 \varphi$ iff $S \vdash_\omega \varphi$.*

Notice that Corollary 4 and Theorem 3 imply Theorem 4. Suppose that S has a maximally $\vdash$-consistent extension which is closed under I. Let S' be such an extension. Notice that all finite extensions of S' respect I. Thus by Corollary 4, S' is $\vdash_I$-consistent. Thus, as $S \subseteq S'$, S is $\vdash_I$-consistent by D1. In the next section

Theorem 4 is proved by showing that the conclusion of Corollary 4 holds when $\vdash$ is a deducibility relation and S is maximally $\vdash$-consistent and closed under I.

5 This section is devoted to proving Theorem 4. It suffices to establish the following: if $\vdash$ is a deducibility relation and S has a maximally $\vdash$-consistent extension which is closed under I, then S is $\vdash_I$-consistent. This result is obtained by giving a more detailed characterization of $\vdash_I$. In particular, $\vdash_I$ will be characterized as the union of a chain of deducibility relations.

Lemma 3 *If β is an infinite limit ordinal and $\{\vdash_\alpha : \alpha < \beta\}$ is a chain of deducibility relations (i.e., if $\alpha < \alpha'$, then $\vdash_\alpha \subseteq \vdash \alpha'$), then $\cup\{\vdash_\alpha : \alpha < \beta\}$ is a deducibility relation.*

Proof: Let $\vdash_\beta = \cup\{\vdash_\alpha : \alpha < \beta\}$. Let $(S, \varphi) \in \mathcal{P}(\mathcal{L}) \times \mathcal{L}$. Notice that $S \vdash_\beta \varphi$ iff there is $\alpha < \beta$ such that $S \vdash_\alpha \varphi$. It is easily verified that $\vdash_\beta$ satisfies D1–D4.

Notice that it also follows that the union of a chain of finitary deducibility relations is also a finitary deducibility relation.

Lemma 4 *If $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$, then there is unique $\vdash'$ such that* (1) $\vdash \subseteq \vdash'$; (2) *$\vdash'$ is a deducibility relation; and* (3) *if $\vdash''$ is any deducibility relation such that $\vdash \subseteq \vdash''$, then $\vdash' \subseteq \vdash''$.*

Proof: Notice that the uniqueness of $\vdash'$ is immediate. The existence of $\vdash'$ can be established by a proof analogous to that of Lemma 2. However, because a more detailed characterization is needed below, a different proof is given here.

Let $\vdash$ be any subset of $\mathcal{P}(\mathcal{L}) \times \mathcal{L}$.

Let:

$\mathrm{D1}(\vdash) = \{(S, \varphi) : \text{there is } S' \subseteq S,\ S' \vdash \varphi\}$;
$\mathrm{D2}(\vdash) = \{(S, \varphi) : \varphi \in S\}$
$\mathrm{D3}(\vdash) = \{(S, \varphi) : S \cup \{\psi\} \vdash \varphi \text{ and } S \vdash \psi\}$
$\mathrm{D4}(\vdash) = \{(S, \varphi) : S \cup \{\sim\varphi\} \vdash \perp\} \cup \{(S \cup \{\sim\varphi\}, \perp) : S \vdash \varphi\}$.

Define $\{\vdash^n : n \geq 0\}$ as follows:

(1) $\vdash^0 = \vdash$
(2) $\vdash^{n+1} = \vdash^n \cup \mathrm{D1}(\vdash^n) \cup \mathrm{D2}\ (\vdash^n) \cup \mathrm{D3}\ (\vdash^n) \cup \mathrm{D4}\ (\vdash^n)$.

Notice that $\vdash^n \subseteq \vdash^{n+1}$. Let $\vdash' = \cup\{\vdash n : n > 0\}$. It is easily verified that $\vdash'$ is a deducibility relation. Let $\vdash''$ be any deducibility relation such that $\vdash \subseteq \vdash''$. It is easily verified by mathematical induction that $\vdash^n \subseteq \vdash''$ for each n. Thus, $\vdash' \subseteq \vdash''$.

Let $\vdash$ be a deducibility relation, $I \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$, and let κ be the least regular cardinal such that for all $(S', \varphi) \in I$ the cardinality of $S' < \kappa$. Define $\{\vdash_\alpha : \alpha < \kappa\}$ as follows:

(1) $\vdash_0 = \vdash$;
(2) $\vdash_\alpha = \cup\{\vdash_\lambda : \lambda < \alpha\}$ when α is an infinite limit ordinal; and
(3) $\vdash_{\alpha+1}$ is the least deducibility relation extending $\vdash_\alpha \cup \{(S, \varphi)$: there is (S', φ) in I and $S \vdash_\alpha \psi$, for all $\psi \in S'\}$. Finally, let $\vdash_\kappa = \cup\{\vdash_\alpha : \alpha < \kappa\}$. By Lemma 3 and Lemma 4, $\vdash_\kappa$ is a deducibility relation.

Notice that κ will never be any larger than the cardinal successor of the cardinality of $\mathcal{L}$. Further, if each member of I has only finitely many premises, then $\kappa = \aleph_0$. When $S \vdash_\kappa \varphi$, then there is a least ordinal $\alpha < \kappa$ such that $S \vdash_\alpha \varphi$. When $\alpha > 0$, α is a successor ordinal. α is called the *rank* of (S, φ). It is shown below that given I there is $\{\vdash_\alpha : \alpha < \kappa\}$ such that $\vdash_\kappa = \vdash_I$ and that when S is maximally $\vdash$-consistent and closed under I, then for all φ if $S \vdash_I \varphi$, then the rank of (S, φ) is zero.

Lemma 5 *If $\vdash$ is a deducibility relation and $I \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$, then $\vdash_I = \vdash_\kappa$.*

Proof: Given $\vdash$, a deducibility relation, and $I \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$, there is κ a regular cardinal such that $\vdash \subseteq \vdash_\kappa$ and $\vdash_\kappa$ is a deducibility relation. Let $S \subseteq \mathcal{L}$. To show that S respects I in $\vdash_\kappa$. Let $(S', \varphi) \in I$. Suppose that for all $\psi \in S'$, $S \vdash_\kappa \psi$. To show that $S \vdash_\kappa \varphi$. For each $\psi \in S'$, there is $\alpha(\psi) < \kappa$ such that $S \vdash_{\alpha(\psi)} \psi$. Since the cardinality of $S' < \kappa$ and κ is regular, there is $\alpha < \kappa$ such that $\alpha(\psi) \leq \alpha$, for all $\psi \in S'$. Thus $\vdash_{\alpha(\psi)} \subseteq \vdash_\alpha$, and for all $\psi \in S'$, $S \vdash_\alpha \psi$. Thus, by construction $S \vdash_{\alpha+1} \varphi$; and $S \vdash_\kappa \varphi$. Thus $\vdash_I \subseteq \vdash_\kappa$. It is easily verified by transfinite induction on κ that $\vdash_\alpha \subseteq \vdash_I$ for all $\alpha < \kappa$. Thus $\vdash_\kappa \subseteq \vdash_I$.

When for all (S', φ) in I, S' is finite, it follows from the above that $\kappa = \aleph_0$ and that $\vdash_I$ is a finitary deducibility relation if $\vdash$ is.

Lemma 6 *If $\vdash$ is a deducibility relation, $I \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ and S is maximally $\vdash$-consistent and closed under I, then for all φ, $S \vdash \varphi$ iff $S \vdash_I \varphi$.*

Proof: Suppose $\vdash$ is a deducibility relation, $I \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ and S is maximally $\vdash$-consistent and closed under I. By Lemma 5, $\vdash_\kappa = \vdash_I$. Thus, it suffices to show that for all $\alpha < \kappa$, $S \vdash \varphi$ iff $S \vdash_\alpha \varphi$. Proceed by transfinite induction on κ. Since $\vdash \subseteq \vdash_\alpha$, if $S \vdash \varphi$, then $S \vdash_\alpha \varphi$. Let $T = \{\alpha : \alpha < \kappa$ and for all φ, if $S \vdash_\alpha \varphi$, then $S \vdash \varphi\}$.

Since $\vdash_0 = \vdash$, by construction $0 \in T$. Let α be an ordinal $<\kappa$. Suppose for all $\beta < \alpha$ that $\beta \in T$. To show that $\alpha \in T$. Suppose that α is a limit ordinal. By construction, $\vdash_\alpha = \bigcup\{\vdash_\beta : \beta < \alpha\}$. Suppose that $S \vdash_\alpha \varphi$. Then there is $\beta < \alpha$ such that $S \vdash_\beta \varphi$. Since $\beta \in T$, $S \vdash \varphi$.

Suppose that α is a successor ordinal. Thus $\alpha = \beta + 1$, where $\beta < \alpha$. By construction $\vdash_{\beta+1}$ is the smallest deducibility relation extending $\vdash_\beta^0 = \vdash_\beta \cup \{(S', \varphi)$: there is (S'', φ) in I such that for all $\psi \in S''$, $S' \vdash_\beta \psi\}$.

Let $\vdash_\beta^{n+1}$ denote $\vdash_\beta^n \cup \mathrm{D1}(\vdash_\beta^n) \cup \mathrm{D2}(\vdash_\beta^n) \cup \mathrm{D3}(\vdash_\beta^n) \cup \mathrm{D4}(\vdash_\beta^n)$. By the proof of Lemma 4, $\vdash_{\beta+1} = \bigcup\{\vdash_\beta^n : n \geq 0\}$. Thus $S \vdash_{\beta+1} \varphi$ iff there is $n \geq 0$ such that $S \vdash_\beta^n \varphi$. We proceed by showing that for all n, $S \vdash_\beta \varphi$ iff $S \vdash_\beta^n \varphi$. Since $\vdash_\beta \subset \vdash_\beta^n$, it suffices to show that if $S \vdash_\beta^n \varphi$, then $S \vdash_\beta \varphi$.

We proceed by mathematical induction. Let $T' = \{n : n \geq 0$ and for all $S'' \subseteq S$, if $S'' \vdash_\beta^n \varphi$, then $S \vdash_\beta \varphi\}$. First we show that $0 \in T'$. Let $S'' \subseteq S$. Suppose that $S'' \vdash_\beta^0 \varphi$. Either $S'' \vdash_\beta \varphi$ or there is $(S', \varphi) \in I$ such that $S'' \vdash_\beta \psi$, for all $\psi \in S'$. Suppose $S'' \vdash_\beta \varphi$. Since $\vdash_\beta$ is a deducibility relation, by D1 $S \vdash_\beta \varphi$. Suppose there is $(S', \varphi) \in I$ such that for all $\psi \in S'$, $S'' \vdash_\beta \varphi$. By D1 $S \vdash_\beta \psi$, for all $\psi \in S'$. Since $\beta \in T$ and S is maximally $\vdash$-consistent, $S' \subseteq S$. And as S is closed under I, $\varphi \in S$. Thus by D2, $S \vdash_\beta \varphi$.

Suppose that $n \in T'$. To show that $n + 1 \in T'$, let $S'' \subseteq S$. Suppose that $S'' \vdash_\beta^{n+1} \varphi$. By construction, $\vdash_\beta^{n+1} = \vdash_\beta^n \cup \mathrm{D1}(\vdash_\beta^n) \cup \mathrm{D2}(\vdash_\beta^n) \cup \mathrm{D3}(\vdash_\beta^n) \cup \mathrm{D4}(\vdash_\beta^n)$. There are five cases to consider.

Case 1: Suppose that $S'' \vdash_\beta^n \varphi$. Since $n \in T'$, $S \vdash_\beta \varphi$.

Case 2: Suppose that $(S'', \varphi) \in D1(\vdash_\beta^n)$. Thus there is $S''' \subseteq S''$ such that $S'' \vdash_\beta^n \varphi$. Since $n \in T'$ and $S''' \subseteq S$, $S \vdash_\beta \varphi$.

Case 3: Suppose that $(S'', \varphi) \in D2(\vdash_\beta^n)$. Thus $\varphi \in S''$. By D2, $S'' \vdash_\beta \varphi$; and by D1, $S \vdash_\beta \varphi$.

Case 4: Suppose that $(S'', \varphi) \in D3(\vdash_\beta^n)$. Thus there is ψ such that $S'' \cup \{\psi\} \vdash_\beta^n \varphi$ and $S'' \vdash_\beta^n \varphi$. Since $n \in T'$, and $\beta \in T$, $S \vdash \varphi$. As S is maximally $\vdash$-consistent, $\psi \in S$ and $S'' \cup \{\psi\} \subseteq S$. Since $n \in T'$, $S \vdash_\beta \varphi$.

Case 5: Suppose that $(S'', \varphi) \in D4(\vdash_\beta^n)$. Thus either $S'' \cup \{\sim\varphi\} \vdash_\beta^n \perp$ or φ is $\perp$, and there is ψ such that $S'' = S''' \cup \{\sim\psi\}$ and $S''' \vdash_\beta^n \psi$. Suppose that $S'' \cup \{\sim\varphi\} \vdash_\beta^n \perp$. To show that $S \vdash_\beta \varphi$. Suppose that $S \nvdash_\beta \varphi$. Since $\vdash \subseteq \vdash_\beta$, $S \nvdash \varphi$. By D2, $\varphi \notin S$. Since S is maximally $\vdash$-consistent, $\sim\varphi \in S$. Therefore $S'' \cup \{\sim\varphi\} \subseteq S$ and, as $n \in T'$, $S \vdash_\beta \perp$. Since $\beta \in T$, $S \vdash \perp$, a contradiction. Finally, we claim that it is not the case that φ is $\perp$ and there is ψ such that $S'' = S''' \cup \{\sim\psi\}$ and $S''' \vdash_\beta^n \psi$. Suppose otherwise. Notice that $S''' \subseteq S$ and $\sim\psi \in S$. Since $n \in T'$, and $\beta \in T$, $S \vdash \psi$. Since S is maximally $\vdash$-consistent, $\psi \in S$. Hence, $\{\psi, \sim\psi\} \subseteq S$, a contradiction.

Thus for all $n \geq 0$, $S \vdash_\beta \varphi$ iff $S \vdash_\beta^n \varphi$; and $\beta + 1 = \alpha \in T$. Therefore for all $\alpha < \kappa$, all φ in $\mathcal{L}$ if $S \vdash_\alpha \varphi$, then $S \vdash \varphi$.

It remains to show the following:

Lemma 7 *If $\vdash$ is a deducibility relation, $I \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ and S has a maximally $\vdash$-consistent extension which is closed under I, then S is $\vdash_I$-consistent.*

Proof: Let S'' be a maximally $\vdash$-consistent extension of S which is closed under I. By Lemma 6, for all φ in $\mathcal{L}$, $S'' \vdash \varphi$ iff $S'' \vdash_I \varphi$. Thus, since S'' is $\vdash$-consistent, $S'' \nvdash_I \perp$. Since $\vdash_I$ is a deducibility relation, by D1, $S \nvdash_I \perp$. Thus, S is $\vdash_I$-consistent.

Theorem 4 is immediate from Lemma 7 and Theorem 3. The following is also a consequence of Lemma 6.

Theorem 5 *If $\vdash$ is a deducibility relation, then*
(1) *S is maximally $\vdash_I$-consistent iff S is maximally $\vdash$-consistent and closed under I; and*
(2) *if $\vdash$ is finitary and I countable, S is $\vdash_I$-consistent iff it has a maximally $\vdash_I$-consistent extension.*

Proof: Suppose that $\vdash$ is a deducibility relation.

(1) Suppose that S is maximally $\vdash$-consistent and closed under I. By Lemma 6, S is $\vdash_I$-consistent. Suppose that $\varphi \notin S$. To show that $S \cup \{\varphi\}$ is not $\vdash_I$-consistent. Since S is maximally $\vdash$-consistent, $\sim\varphi \in S$. Hence $\{\varphi, \sim\varphi\} \subseteq S \cup \{\varphi\}$ and $S \cup \{\phi\}$ is not $\vdash_I$-consistent. Suppose that S is maximally $\vdash_I$-consistent. Since $\vdash \subseteq \vdash_I$, S is $\vdash$-consistent. Further, if $\varphi \notin S$, then $S \cup \{\varphi\}$ is not $\vdash$-consistent. Hence S is maximally $\vdash$-consistent. Let $(S', \varphi) \in I$. Suppose that $S' \subseteq S$. By D2, $S \vdash_I \psi$ for all $\psi \in S'$. Hence $S \vdash_I \varphi$; and since S is maximally $\vdash_I$-consistent, $\varphi \in S$. Hence S is closed under I.

(2) Suppose that $\vdash$ is finitary and I is countable. Suppose that S is $\vdash_I$-con-

sistent. By Theorem 4, S has a maximally $\vdash$-consistent extension which is closed under I. By (1), this extension is maximally $\vdash_I$-consistent. Clearly, if S has a maximally $\vdash_I$-consistent extension, S is itself $\vdash_I$-consistent.

Theorem 5 provides a characterization of the maximally $\vdash_I$-consistent sets when $\vdash$ is a deducibility relation and a characterization of the $\vdash_I$-consistent sets when $\vdash$ is a finitary and I is countable. Theorem 5 specializes to yield that S is $\vdash_\omega$-consistent iff S has a maximally $\vdash_\omega$-consistent extension.

The above setting also provides an abstract version of the fact that all consistent and ω-complete sets are ω-consistent. Let $\vdash$ be a deducibility relation and let $I \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$. S is *I-consistent in* $\vdash$ iff for all (S', φ) in I, if $S \vdash \psi$ for all ψ in S', then $S \nvdash \sim\varphi$. Notice that if $S \subseteq \mathcal{L}(1)$, S is $I(\omega)$-consistent in $\vdash^1$ iff S is ω-consistent. Suppose that S, a subset of $\mathcal{L}$, is $\vdash$-consistent and respects I. Then S is I-consistent in $\vdash$. Suppose otherwise. Then there is (S', φ) in I such that $S \vdash \psi$ for all $\psi \in S'$ and $S \vdash \sim\varphi$. As S respects I, $S \vdash \varphi$. Thus by D4, $S \cup \{\sim\varphi\} \vdash \perp$. Hence by D3, $S \vdash \perp$, and S is not $\vdash$-consistent; a contradiction. Thus we have:

Corollary 6 *If $\vdash$ is a deducibility relation, $I \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ and $S \subseteq L$, then if S respects I and is $\vdash$-consistent, then S is I-consistent in $\vdash$.*

It is immediate from the above that all consistent and ω-complete sets are ω-consistent.

Goldblatt [2] used Theorem 2 and Lemma 1 to provide alternative proofs of the completeness and omitting-types theorems for $\mathcal{L}_1$ and countable fragments of $\mathcal{L}_{\infty\omega}$. Thus it follows from earlier remarks that Theorem 2 and Lemma 1 yield proofs of parts (1), (2), and (3) of Theorem 1, when $\mathcal{L}_1$ contains equality. It has been shown above that Theorem 2 and Lemma 1 imply, by purely combinatorial arguments, abstract forms of the ω-completeness theorem (Corollary 1), the soundness and completeness of ω-logic (Theorem 4), and the theorem to the effect that ω-logic is a conservative extension of standard logic for ω-complete sets (Corollary 4). These abstract results yield alternative proofs of the corresponding "concrete" ones. In the case of Theorem 4, the corresponding result is immediate from the soundness of ω-logic. In the other two cases, the corresponding results follow from the observation that any finite extension of an ω-complete set is ω-complete.

The novelty of the approach taken here is twofold: (1) it unifies the proofs of Leblanc et al. in [4] by replacing various modifications of the Henkin construction by a single abstract construction; and (2) it isolates that feature of ω-complete sets which is essential to these results.

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