Notre Dame Journal of Formal Logic
Volume 33, Number 3, Summer 1992

A Sequent- or Tableau-style System for Lewis's Counterfactual Logic VC

IAN PHILIP GENT

Abstract In a 1983 paper, de Swart gave sequent based proof systems for two counterfactual logics: Stalnaker's **VCS** and Lewis's **VC.** In this paper I demonstrate that de Swart's system for **VC** is incorrect by giving a counterexample. This counterexample does not affect de Swart's system for **VCS.** Then I give a new sequent- or tableau-style proof system for **VC** together with soundness and completeness proofs. The system I give is closely modeled on de Swart's.

1 Introduction Lewis [2] presented a counterfactual logic **VC**. de Swart [1] presented first a sequent-based proof system for Stalnaker's counterfactual logic **VCS**, together with soundness and completeness proofs, and then a proof system for **VC**. Unfortunately, the soundness and completeness proofs for **VC** were only sketched. In this paper I show that de Swart's system for **VC** is incorrect, in that there is a **VC**-valid formula which the system reports to be invalid. This paper concentrates exclusively on **VC**; de Swart's work on **VCS** is not affected by the counterexample to **VC**.

In the rest of this section, I very briefly introduce Lewis's logic **VC**. In Section 2 I describe de Swart's system for **VC** and in Section 3 I give a counterexample to this system. In Section 4 I give a new proof system for **VC**, for which I give soundness and completeness proofs in Sections 5 and 6 respectively. This system cannot exactly be described as either a tableau or a Gentzen sequent system but is closely related to both. It differs from tableau systems in that nodes in a derivation tree are labeled with sets of formulas rather than single formulas, but it differs from Gentzen sequent systems in that I do not use the sequent arrow, preferring the semantic notion of signed formulas.

The language of **VC** contains standard propositional connectives $\wedge$, $\vee$, $\neg$, $\supset$, the propositional constants $\top$ and $\bot$, and the extra connectives $\leq$ and $\Box\!\!\rightarrow$. $\mathcal{A} \leq \mathcal{B}$ is read as "$\mathcal{A}$ is at least as possible as $\mathcal{B}$". The connective $\Box\!\!\rightarrow$ is used for

Received November 26, 1990; revised May 29, 1991

counterfactual implication; $\mathcal{A} \Box\!\!\rightarrow \mathcal{B}$ is read as "If $\mathcal{A}$ were the case, then $\mathcal{B}$ would be the case". The following definition gives the semantics of **VC**.

Lewis defines $\Box\!\!\rightarrow$ and $\leq$ in terms of each other. In view of this interdefinedness, it is necessary to consider only one of the connectives. In this paper $\leq$ is used.

Definition 1.1 A *model for* **VC** is a quadruple $\langle \mathbf{I}, \mathbf{R}, \leq, [\![\ ]\!] \rangle$ which satisfies:

(1) **I** is a nonempty set of possible worlds.
(2) **R** is a binary relation on **I**, representing the mutual accessibility relation of possible worlds.
(3) $\leq$ is a three place relation on **I**, such that for each $i \in \mathbf{I}$ there is a binary relation $\leq_i$ on **I**. Furthermore, each $\leq_i$ must be transitive and connected on $\{j \mid j \in \mathbf{I} \text{ and } i\mathbf{R}j\}$. (The latter requirement is that if $i\mathbf{R}j$ and $i\mathbf{R}k$ then either $j \leq_i k$ or $k \leq_i j$ or both must be true.)
(4) $[\![\ ]\!]$ assigns to each formula $\mathcal{A}$ of **VC** a subset $[\![\mathcal{A}]\!]$ of **I**, representing the set of worlds where $\mathcal{A}$ is true. $[\![\ ]\!]$ must satisfy the following requirements:
 (4.1) $[\![\mathcal{A} \wedge \mathcal{B}]\!] = [\![\mathcal{A}]\!] \cap [\![\mathcal{B}]\!]$
 (4.2) $[\![\mathcal{A} \vee \mathcal{B}]\!] = [\![\mathcal{A}]\!] \cup [\![\mathcal{B}]\!]$
 (4.3) $[\![\neg\mathcal{A}]\!] = \mathbf{I} - [\![\mathcal{A}]\!]$
 (4.4) $[\![\mathcal{A} \supset \mathcal{B}]\!] = (\mathbf{I} - [\![\mathcal{A}]\!]) \cup [\![\mathcal{B}]\!]$
 (4.5) $[\![\top]\!] = \mathbf{I}$
 (4.6) $[\![\bot]\!] = \varnothing$
 (4.7) $[\![\mathcal{A} \leq \mathcal{B}]\!] = \{i \mid i \in \mathbf{I}$ and for all $j \in [\![\mathcal{B}]\!]$ such that $i\mathbf{R}j$ there is some $k \in [\![\mathcal{A}]\!]$ such that $k \leq_i j\}$.
(5) (The Centering Assumption) **R** is reflexive on **I**; and if $i\mathbf{R}j$ and $i \neq j$ then $\neg j \leq_i i$ (and so by the connectivity of $\leq_i$ and reflexivity of **R**, and in an obvious notation, $i <_i j$).

The connective $\Box\!\!\rightarrow$ can be defined in terms of $\leq$ by

$$\mathcal{A} \Box\!\!\rightarrow \mathcal{B} \equiv_{df} (\bot \leq \mathcal{A}) \vee \neg((\mathcal{A} \wedge \neg\mathcal{B}) \leq (\mathcal{A} \wedge \mathcal{B}))$$

and its semantics are given by

$$[\![\mathcal{A} \Box\!\!\rightarrow \mathcal{B}]\!] = \{i \mid i \in \mathbf{I} \text{ and if there is some } j \in [\![\mathcal{A}]\!] \text{ such that } i\mathbf{R}j \text{ then there is some } k \in [\![\mathcal{A} \wedge \mathcal{B}]\!] \text{ such that there is no } l \in [\![\mathcal{A} \wedge \neg\mathcal{B}]\!] \text{ such that } i\mathbf{R}l \text{ and } l \leq_i k\}.$$

Definition 1.2 A formula $\mathcal{A}$ is **VC**-*valid*, written "$\vDash_{\mathbf{VC}} \mathcal{A}$", if and only if, in every model $\langle \mathbf{I}, \mathbf{R}, \leq, [\![\ ]\!] \rangle$ for **VC**, $[\![\mathcal{A}]\!] = \mathbf{I}$.

2 de Swart's system for VC

Definition 2.1 A *signed formula* is any formula of the form $[\mathcal{A}]^+$ or $[\mathcal{A}]^-$ where $\mathcal{A}$ is a formula of **VC**. The reader may prefer to read $[\mathcal{A}]^+$ as T$\mathcal{A}$ and $[\mathcal{A}]^-$ as F$\mathcal{A}$.

Definition 2.2 A *sequent* is a set of signed formulas.[1] Below, I will sometimes write "$\Sigma, [\mathcal{A}]^+$" to mean "$\Sigma \cup \{[\mathcal{A}]^+\}$", and similarly with more than one signed formula.

Each rule that is defined below consists of one sequent above one or more other sequents derived from it.

Definition 2.3 A *derivation* of a finite *sequent* Σ is a finite schema of sequents such that

(a) Σ is the highest sequent in the schema.
(b) If a sequent in the schema has any sequents immediately below it, they are the sequents derived from Σ by applying one of the rules.
(c) If a sequent has no sequents below it, then $[\bot]^+ \in \Sigma$ or $[\top]^- \in \Sigma$ or for some formula $\mathcal{B}$, $[\mathcal{B}]^+ \in \Sigma$ and $[\mathcal{B}]^- \in \Sigma$.

Definition 2.4 A *derivation* of a *formula* $\mathcal{A}$ of **VC** is a derivation of the sequent $\{[\mathcal{A}]^-\}$. The rules are as follows:

T$\wedge$	$\Sigma, [\mathcal{B} \wedge \mathcal{C}]^+$ $\Sigma, [\mathcal{B}]^+, [\mathcal{C}]^+$	F$\wedge$	$\Sigma, [\mathcal{B} \wedge \mathcal{C}]^-$ $\Sigma, [\mathcal{B}]^- \mid \Sigma, [\mathcal{C}]^-$
T$\vee$	$\Sigma, [\mathcal{B} \vee \mathcal{C}]^+$ $\Sigma, [\mathcal{B}]^+ \mid \Sigma, [\mathcal{C}]^+$	F$\vee$	$\Sigma, [\mathcal{B} \vee \mathcal{C}]^-$ $\Sigma, [\mathcal{B}]^-, [\mathcal{C}]^-$
T$\supset$	$\Sigma, [\mathcal{B} \supset \mathcal{C}]^+$ $\Sigma, [\mathcal{B}]^- \mid \Sigma, [\mathcal{C}]^+$	F$\supset$	$\Sigma, [\mathcal{B} \supset \mathcal{C}]^-$ $\Sigma, [\mathcal{B}]^+, [\mathcal{C}]^-$
T$\neg$	$\Sigma, [\neg\mathcal{B}]^+$ $\Sigma, [\mathcal{B}]^-$	F$\neg$	$\Sigma, [\neg\mathcal{B}]^-$ $\Sigma, [\mathcal{B}]^+$
T$\leq$	$\Sigma, [\mathcal{B} \leq \mathcal{C}]^+$ $\Sigma, [\mathcal{B} \leq \mathcal{C}]^+, [\mathcal{B}]^+ \mid \Sigma, [\mathcal{B} \leq \mathcal{C}]^+, [\mathcal{C}]^-$	F$\leq$	$\Sigma, [\mathcal{B} \leq \mathcal{C}]^-$ $\Sigma, [\mathcal{B} \leq \mathcal{C}]^-, [\mathcal{B}]^-$

There is one additional rule that is considerably more complicated. Its general name is F $\leq (m, n)$. It applies to a set of m formulas of the form $[\mathcal{A} \leq \mathcal{D}]^-$ and n formulas of the form $[\mathcal{U} \leq \mathcal{V}]^+$. It is only applicable if $m \geq 1$, but n may be 0.

de Swart [1] gives only special cases of this rule, and the reader is left to infer the general case. Fortunately, the counterexample in Section 3 relies only on the special cases that were explicitly given. The following special cases of the rule F $\leq (m, n)$ will be enough for me.

F $\leq$ (1,0) A sequent $\Sigma, [\mathcal{A} \leq \mathcal{D}]^-$ is derivable if the following sequent is derivable:

$$[\mathcal{A}]^-, [\mathcal{D}]^+.$$

F $\leq$ (1,1) A sequent $\Sigma, [\mathcal{A} \leq \mathcal{D}]^-, [\mathcal{U}_1 \leq \mathcal{V}_1]^+$ is derivable if both of the following sequents are derivable:

$$[\mathcal{A}]^-, [\mathcal{D}]^+, [\mathcal{V}_1]^-$$

$$[\mathcal{A}]^-, [\mathcal{U}_1]^+.$$

$F \le (1,2)$ A sequent $\Sigma, [\mathfrak{A} \le \mathfrak{D}]^-, [\mathfrak{U}_1 \le \mathfrak{V}_1]^+, [\mathfrak{U}_2 \le \mathfrak{V}_2]^+$ is derivable if all the following three sequents are derivable:

$$[\mathfrak{A}]^-, [\mathfrak{D}]^+, [\mathfrak{V}_1]^-, [\mathfrak{V}_2]^-$$
$$[\mathfrak{A}]^-, [\mathfrak{U}_1]^+, [\mathfrak{V}_2]^-$$
$$[\mathfrak{A}]^-, [\mathfrak{V}_1]^-, [\mathfrak{U}_2]^+$$

and at least one of the following two sequents is derivable.

$$[\mathfrak{A}]^-, [\mathfrak{U}_1]^+, [\mathfrak{V}_2]^+$$
$$[\mathfrak{A}]^-, [\mathfrak{U}_2]^+, [\mathfrak{V}_1]^+.$$

$F \le (1,3)$ A sequent $\Sigma, [\mathfrak{A} \le \mathfrak{D}]^-, [\mathfrak{U}_1 \le \mathfrak{V}_1]^+, [\mathfrak{U}_2 \le \mathfrak{V}_2]^+, [\mathfrak{U}_3 \le \mathfrak{V}_3]^+$ is derivable if all the following four sequents are derivable:

$$[\mathfrak{A}]^-, [\mathfrak{D}]^+, [\mathfrak{V}_1]^-, [\mathfrak{V}_2]^-, [\mathfrak{V}_3]^-$$
$$[\mathfrak{A}]^-, [\mathfrak{U}_1]^+, [\mathfrak{V}_2]^-, [\mathfrak{V}_3]^-$$
$$[\mathfrak{A}]^-, [\mathfrak{V}_1]^-, [\mathfrak{U}_2]^+, [\mathfrak{V}_3]^-$$
$$[\mathfrak{A}]^-, [\mathfrak{V}_1]^-, [\mathfrak{V}_2]^-, [\mathfrak{U}_3]^+$$

and if each of the three sequents is derivable in at least one of the following six sets of sequents.

$$[\mathfrak{A}]^-, [\mathfrak{U}_1]^+, [\mathfrak{V}_2]^+ \mid [\mathfrak{A}]^-, [\mathfrak{U}_1]^+, [\mathfrak{V}_3]^+ \mid [\mathfrak{A}]^-, [\mathfrak{U}_2]^+, [\mathfrak{V}_3]^+$$
$$[\mathfrak{A}]^-, [\mathfrak{U}_1]^+, [\mathfrak{V}_2]^+ \mid [\mathfrak{A}]^-, [\mathfrak{U}_1]^+, [\mathfrak{V}_3]^+ \mid [\mathfrak{A}]^-, [\mathfrak{U}_3]^+, [\mathfrak{V}_2]^+$$
$$[\mathfrak{A}]^-, [\mathfrak{U}_2]^+, [\mathfrak{V}_1]^+ \mid [\mathfrak{A}]^-, [\mathfrak{U}_2]^+, [\mathfrak{V}_3]^+ \mid [\mathfrak{A}]^-, [\mathfrak{U}_1]^+, [\mathfrak{V}_3]^+$$
$$[\mathfrak{A}]^-, [\mathfrak{U}_2]^+, [\mathfrak{V}_1]^+ \mid [\mathfrak{A}]^-, [\mathfrak{U}_2]^+, [\mathfrak{V}_3]^+ \mid [\mathfrak{A}]^-, [\mathfrak{U}_3]^+, [\mathfrak{V}_1]^+$$
$$[\mathfrak{A}]^-, [\mathfrak{U}_3]^+, [\mathfrak{V}_1]^+ \mid [\mathfrak{A}]^-, [\mathfrak{U}_3]^+, [\mathfrak{V}_2]^+ \mid [\mathfrak{A}]^-, [\mathfrak{U}_1]^+, [\mathfrak{V}_2]^+$$
$$[\mathfrak{A}]^-, [\mathfrak{U}_3]^+, [\mathfrak{V}_1]^+ \mid [\mathfrak{A}]^-, [\mathfrak{U}_3]^+, [\mathfrak{V}_2]^+ \mid [\mathfrak{A}]^-, [\mathfrak{U}_2]^+, [\mathfrak{V}_1]^+.$$

Note that in the rule $F \le (m, n)$, the sequent Σ disappears in the derived sequents. This occurs because the derived sequents can be seen as referring to different possible worlds, and so many statements about the original world become irrelevant.

Remark 2.5 Note that, due to the above definition of the six sets of three sequents, $F \le (1,3)$ can lead to a derivation only if:

for any pair (j, k) with $1 \le j < k \le 3$, either $\{[\mathfrak{A}]^-, [\mathfrak{U}_j]^+, [\mathfrak{V}_k]^+\}$
or $\{[\mathfrak{A}]^-, [\mathfrak{U}_k]^+, [\mathfrak{V}_j]^+\}$ is derivable.

This remark will be used in the proof of Theorem 3.2 in the next section.

3 A counterexample to de Swart's system Consider the formula of **VC**

(1) $$(\mathfrak{A} \le \mathfrak{C} \wedge \mathfrak{C} \le \mathfrak{D} \wedge \mathfrak{D} \le (\neg\mathfrak{D} \wedge \mathfrak{B})) \supset (\mathfrak{A} \le \mathfrak{B}).$$

Theorem 3.1 (1) *is* **VC**-*valid.*

Proof: It is necessary to prove that $[\![(1)]\!] = \mathbf{I}$ in any model $\langle \mathbf{I}, \mathbf{R}, \leq, [\![\ ]\!] \rangle$ for **VC**. So from the semantics of **VC**, it is necessary to prove

$$(\mathbf{I} - ([\![\mathcal{A} \leq \mathcal{C}]\!] \cap [\![\mathcal{C} \leq \mathcal{D}]\!] \cap [\![\mathcal{D} \leq (\neg\mathcal{D} \wedge \mathcal{B})]\!])) \cup [\![\mathcal{A} \leq \mathcal{B}]\!] = \mathbf{I}.$$

To do this, it is sufficient to prove that for any $i \in \mathbf{I}$, if $i \in ([\![\mathcal{A} \leq \mathcal{C}]\!] \cap [\![\mathcal{C} \leq \mathcal{D}]\!] \cap [\![\mathcal{D} \leq (\neg\mathcal{D} \wedge \mathcal{B})]\!])$, then $i \in [\![\mathcal{A} \leq \mathcal{B}]\!]$.

Consider any $i_0 \in ([\![\mathcal{A} \leq \mathcal{C}]\!] \cap [\![\mathcal{C} \leq \mathcal{D}]\!] \cap [\![\mathcal{D} \leq (\neg\mathcal{D} \wedge \mathcal{B})]\!])$. Choose any $i_1 \in \mathbf{I}$ such that $i_0\mathbf{R}i_1$ and $i_1 \in [\![\mathcal{B}]\!]$ (if there is no such i_1 then $i_0 \in [\![\mathcal{A} \leq \mathcal{B}]\!]$ is trivially true). Now either $i_1 \in [\![\mathcal{D}]\!]$ or $i_1 \in [\![\neg\mathcal{D}]\!]$. In either case, I will prove that there must be some $i' \in \mathbf{I}$ such that $i_0\mathbf{R}i'$, $i' \leq_{i_0} i_1$ and $i' \in [\![\mathcal{A}]\!]$. Since the choice of i_1 was arbitrary, I will have established $i_0 \in [\![\mathcal{A} \leq \mathcal{B}]\!]$.

If $i_1 \in [\![\mathcal{D}]\!]$, then, since $i_0 \in [\![\mathcal{C} \leq \mathcal{D}]\!]$, there is some $i_2 \in \mathbf{I}$ such that $i_0\mathbf{R}i_2, i_2 \in [\![\mathcal{C}]\!]$ and $i_2 \leq_{i_0} i_1$. Then since $i_0 \in [\![\mathcal{A} \leq \mathcal{C}]\!]$, there is some $i' \in \mathbf{I}$ such that $i_0\mathbf{R}i', i' \in [\![\mathcal{A}]\!]$ and $i' \leq_{i_0} i_2$. Since $\leq_{i_0}$ is transitive, $i' \leq_{i_0} i_1$.

If $i_1 \in [\![\neg\mathcal{D}]\!]$, then $i_1 \in [\![\neg\mathcal{D} \wedge \mathcal{B}]\!]$. Since $i_0 \in [\![\mathcal{D} \leq (\neg\mathcal{D} \wedge \mathcal{B})]\!]$, there is some $i_2 \in \mathbf{I}$ such that $i_0\mathbf{R}i_2, i_2 \in [\![\mathcal{D}]\!]$ and $i_2 \leq_{i_0} i_1$. Then by a similar chain to that in the last paragraph we can establish that there is some $i' \in \mathbf{I}$ with the required conditions.

Theorem 3.2 (1) *is not derivable in de Swart's system.*

Proof: A derivation for (1) is a derivation for

$$\{[(\mathcal{A} \leq \mathcal{C} \wedge \mathcal{C} \leq \mathcal{D} \wedge \mathcal{D} \leq (\neg\mathcal{D} \wedge \mathcal{B})) \supset (\mathcal{A} \leq \mathcal{B})]^-\}. \tag{2}$$

In searching a derivation, we find one sequent to which only $F \leq (m, n)$ applies and which does not contain $[\mathcal{E}]^-$ and $[\mathcal{E}]^+$ for any $\mathcal{E}$. This sequent is derived as follows. (Each sequent is annotated by the rule used to produce it, and "RHS" indicates that the right hand sequent of the two possible new sequents was chosen.)

$\{[(\mathcal{A} \leq \mathcal{C} \wedge \mathcal{C} \leq \mathcal{D} \wedge \mathcal{D} \leq (\neg\mathcal{D} \wedge \mathcal{B})) \supset (\mathcal{A} \leq \mathcal{B})]^-\}$ (2)

$\{[\mathcal{A} \leq \mathcal{C} \wedge \mathcal{C} \leq \mathcal{D} \wedge \mathcal{D} \leq (\neg\mathcal{D} \wedge \mathcal{B})]^+, [\mathcal{A} \leq \mathcal{B}]^-\}$ F$\supset$

$\{[\mathcal{A} \leq \mathcal{C}]^+, [\mathcal{C} \leq \mathcal{D}]^+, [\mathcal{D} \leq (\neg\mathcal{D} \wedge \mathcal{B})]^+, [\mathcal{A} \leq \mathcal{B}]^-\}$ T$\wedge$ ×2

$\{[\mathcal{A} \leq \mathcal{C}]^+, [\mathcal{C} \leq \mathcal{D}]^+, [\mathcal{D} \leq (\neg\mathcal{D} \wedge \mathcal{B})]^+, [\mathcal{A} \leq \mathcal{B}]^-, [\mathcal{A}]^-\}$ F$\leq$

$\{[\mathcal{A} \leq \mathcal{C}]^+, [\mathcal{C} \leq \mathcal{D}]^+, [\mathcal{D} \leq (\neg\mathcal{D} \wedge \mathcal{B})]^+, [\mathcal{A} \leq \mathcal{B}]^-, [\mathcal{A}]^-, [\mathcal{C}]^-, [\mathcal{D}]^-, [(\neg\mathcal{D} \wedge \mathcal{B})]^-\}$ T$\leq$ (RHS) ×3

$\{[\mathcal{A} \leq \mathcal{C}]^+, [\mathcal{C} \leq \mathcal{D}]^+, [\mathcal{D} \leq (\neg\mathcal{D} \wedge \mathcal{B})]^+, [\mathcal{A} \leq \mathcal{B}]^-, [\mathcal{A}]^-, [\mathcal{C}]^-, [\mathcal{D}]^-, [\mathcal{B}]^-\}$ F$\wedge$ (RHS)

To substantiate Theorem 3.2 it remains only to show, by a case analysis, that no possible application of an $F \leq (m, n)$ rule to the last sequent above leads to a derivation.

$F \leq (1,0)$ applied to $[\mathcal{A} \leq \mathcal{B}]^-$. This leads to the single sequent $\{[\mathcal{A}]^-, [\mathcal{B}]^+\}$, which is not derivable.

$F \leq (1,1)$ applied to $[\mathcal{A} \leq \mathcal{B}]^-, [\mathcal{A} \leq \mathcal{C}]^+$. One of the derived sequents is $\{[\mathcal{A}]^-, [\mathcal{B}]^+, [\mathcal{C}]^-\}$ which is not derivable.

$F \leq (1,1)$ applied to $[\mathcal{A} \leq \mathcal{B}]^-, [\mathcal{C} \leq \mathcal{D}]^+$. One of the derived sequents is $\{[\mathcal{A}]^-, [\mathcal{B}]^+, [\mathcal{D}]^-\}$ which is not derivable.

$F \leq (1,1)$ applied to $[\mathcal{A} \leq \mathcal{B}]^-, [\mathcal{D} \leq (\neg\mathcal{D} \wedge \mathcal{B})]^+$. One of the derived sequents is $\{[\mathcal{A}]^-, [\mathcal{B}]^+, [(\neg\mathcal{D} \wedge \mathcal{B})]^-\}$ which is not derivable.

$F \leq (1,2)$ applied to $[\mathcal{A} \leq \mathcal{B}]^-, [\mathcal{A} \leq \mathcal{C}]^+, [\mathcal{C} \leq \mathcal{D}]^+$. One of the derived sequents is $\{[\mathcal{A}]^-, [\mathcal{B}]^+, [\mathcal{C}]^-, [\mathcal{D}]^-\}$ which is not derivable.

$F \leq (1,2)$ applied to $[\mathcal{A} \leq \mathcal{B}]^-, [\mathcal{A} \leq \mathcal{C}]^+, [\mathcal{D} \leq (\neg\mathcal{D} \wedge \mathcal{B})]^+$. One of the derived sequents is $\{[\mathcal{A}]^-, [\mathcal{B}]^+, [\mathcal{C}]^-, [(\neg\mathcal{D} \wedge \mathcal{B})]^-\}$ which is not derivable.

$F \leq (1,2)$ applied to $[\mathcal{A} \leq \mathcal{B}]^-, [\mathcal{C} \leq \mathcal{D}]^+, [\mathcal{D} \leq (\neg\mathcal{D} \wedge \mathcal{B})]^+$. One of the derived sequents is $\{[\mathcal{A}]^-, [\mathcal{C}]^+, [(\neg\mathcal{D} \wedge \mathcal{B})]^-\}$ which is not derivable.

$F \leq (1,3)$ applied to $[\mathcal{A} \leq \mathcal{B}]^-, [\mathcal{A} \leq \mathcal{C}]^+, [\mathcal{C} \leq \mathcal{D}]^+, [\mathcal{D} \leq (\neg\mathcal{D} \wedge \mathcal{B})]^+$. To see that $F \leq (1,3)$ fails to lead to a derivation, recall Remark 2.5 and apply it to $[\mathcal{C} \leq \mathcal{D}]^+$ and $[\mathcal{D} \leq (\neg\mathcal{D} \wedge \mathcal{B})]^+$. Then the relevant sequents are $\{[\mathcal{A}]^-, [\mathcal{C}]^+, [(\neg\mathcal{D} \wedge \mathcal{B})]^+\}$ and $\{[\mathcal{A}]^-, [\mathcal{D}]^+, [\mathcal{D}]^+\}$, neither of which is derivable.

Together Theorems 3.1 and 3.2 establish that de Swart's system for **VC** is incorrect.[2]

4 A sequent based proof system for VC In this section I describe a proof system for **VC** similar to de Swart's. In Section 5 I prove the soundness theorem for this system, and in Section 6 I prove the completeness theorem.

All the definitions of Section 2 apply equally to this system, and all the rules are the same except for $F \leq (m,n)$. The definition of $F \leq (m,n)$ is given below.

Definition 4.1

$$F \leq (m,n) : \Sigma, [\mathcal{A}_1 \leq \mathcal{D}_1]^-, \ldots, [\mathcal{A}_m \leq \mathcal{D}_m]^-, [\mathcal{U}_1 \leq \mathcal{V}_1]^+, \ldots, [\mathcal{U}_n \leq \mathcal{V}_n]^+$$
$$\Sigma_1 | \Sigma_2 | \cdots | \Sigma_m | (*)$$

where

$$\Sigma_i = \{[\mathcal{A}_1]^-, \ldots, [\mathcal{A}_m]^-, [\mathcal{D}_i]^+, [\mathcal{V}_1]^-, \ldots, [\mathcal{V}_n]^-\} \text{ for } 1 \leq i \leq m$$

and where $(*)$ is the following special condition, which only applies if $n \geq 1$.

$(*)$ There is a sequence $i_1, i_2, \ldots, i_n$ which is a permutation of $1, 2, \ldots, n$ and is such that each of the following sequents is derivable.[3]

$$\{[\mathcal{A}_1]^-, \ldots, [\mathcal{A}_m]^-, [\mathcal{U}_{i_1}]^+\}$$
$$\{[\mathcal{A}_1]^-, \ldots, [\mathcal{A}_m]^-, [\mathcal{U}_{i_2}]^+, [\mathcal{V}_{i_1}]^-\}$$
$$\{[\mathcal{A}_1]^-, \ldots, [\mathcal{A}_m]^-, [\mathcal{U}_{i_3}]^+, [\mathcal{V}_{i_1}]^-, [\mathcal{V}_{i_2}]^-\}$$
$$\ldots$$
$$\{[\mathcal{A}_1]^-, \ldots, [\mathcal{A}_m]^-, [\mathcal{U}_{i_n}]^+, [\mathcal{V}_{i_1}]^-, [\mathcal{V}_{i_2}]^-, \ldots, [\mathcal{V}_{i_{n-1}}]^-\}.$$

In the rest of this paper, references to the proof system for **VC** refer to this system and not de Swart's.

5 Soundness

Theorem 5.1 (Soundness Theorem) *For any formula $\mathcal{A}$ of* **VC**, *if there is a derivation for $\mathcal{A}$ then $\vDash_{\mathbf{VC}} \mathcal{A}$.*

Proof: If there is a derivation of $\mathcal{A}$, there is a derivation with $[\mathcal{A}]^-$ as its uppermost sequent. So it suffices to show that for any sequent $\{[\mathcal{B}_1]^+, [\mathcal{B}_2]^+, \ldots, [\mathcal{B}_m]^+, [\mathcal{C}_1]^-, \ldots, [\mathcal{C}_n]^-\}$ in the derivation of $[\mathcal{A}]^-$,

(3) $$\vDash_{\mathbf{VC}} (\mathcal{B}_1 \wedge \mathcal{B}_2 \wedge \cdots \wedge \mathcal{B}_m) \supset (\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_n).$$

This will be done by induction, on the size of the derivation.

Induction base: A derivation of minimal size is one in which the original sequent contains $[\bot]^+, [\top]^-$, or both $[\mathcal{B}]^+$ and $[\mathcal{B}]^-$ for some formula *B*.

In the first case, we need $\vDash_{\mathbf{VC}} (\bot \wedge \mathcal{B}_1 \cdots \wedge \mathcal{B}_m) \supset (\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_n)$, which is certainly satisfied by **VC**. In the second and third cases, we need $\vDash_{\mathbf{VC}} (\mathcal{B}_1 \wedge \mathcal{B}_2 \wedge \cdots \wedge \mathcal{B}_m) \supset (\top \vee \mathcal{C}_1 \vee \cdots \vee \mathcal{C}_n)$ and $\vDash_{\mathbf{VC}} (\mathcal{B} \wedge \mathcal{B}_1 \wedge \cdots \wedge \mathcal{B}_m) \supset (\mathcal{B} \vee \mathcal{C}_1 \vee \cdots \vee \mathcal{C}_n)$ respectively, both of which are satisfied by **VC**.

Induction step: For each rule, we have to show that if (3) holds for all the sequents the rule leads to, then (3) holds for the original sequent. I will give an example of doing this for one of the propositional rules, which are straightforward, and then I shall do this for the rules T≤, F≤, and F ≤ (m, n). This will complete the proof of the soundness theorem.

Induction Step for the rule T⊃: The induction hypothesis is that, in any model $\langle \mathbf{I}, \mathbf{R}, \leq, [\![\ ]\!] \rangle$,

(4) $$[\![(\mathcal{B}_1 \wedge \cdots \wedge \mathcal{B}_m) \supset (\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_n \vee \mathcal{D})]\!] = \mathbf{I}$$

and

(5) $$[\![(\mathcal{B}_1 \wedge \cdots \wedge \mathcal{B}_m \wedge \mathcal{E}) \supset (\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_n)]\!] = \mathbf{I}.$$

We have to show that

$$[\![(\mathcal{B}_1 \wedge \cdots \wedge \mathcal{B}_m \wedge (\mathcal{D} \supset \mathcal{E})) \supset (\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_n)]\!] = \mathbf{I}.$$

Consider any $i \in \mathbf{I}$. It is sufficient to consider the case where

(6) $$i \in [\![\mathcal{B}_1]\!], i \in [\![\mathcal{B}_2]\!], \ldots, i \in [\![\mathcal{B}_m]\!]$$

(7) $$i \in [\![\mathcal{D} \supset \mathcal{E}]\!]$$

and to show

(8) $$i \in [\![\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_n]\!].$$

Now (7) means that either $i \notin [\![\mathcal{D}]\!]$ or $i \in [\![\mathcal{E}]\!]$. If $i \notin [\![\mathcal{D}]\!]$ then, from (4) and (6), we have (8) as required. If $i \in [\![\mathcal{E}]\!]$ then, from (5) and (6), we have (8) as required.

Induction Step for the Rule T≤: The induction hypothesis is that, in any model $\langle \mathbf{I}, \mathbf{R}, \leq, [\![\ ]\!] \rangle$,

(9) $$[\![(\mathcal{B}_1 \wedge \cdots \wedge \mathcal{B}_m \wedge (\mathcal{U} \leq \mathcal{V}) \wedge \mathcal{U}) \supset (\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_n)]\!] = \mathbf{I}$$

and

(10) $$[\![(\mathcal{B}_1 \wedge \cdots \wedge \mathcal{B}_m \wedge (\mathcal{U} \leq \mathcal{V})) \supset (\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_n \vee \mathcal{V})]\!] = \mathbf{I}.$$

We need to show that

$$[\![(\mathcal{B}_1 \wedge \cdots \wedge \mathcal{B}_m \wedge (\mathcal{U} \leq \mathcal{V})) \supset (\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_n)]\!] = \mathbf{I}.$$

Consider any $i \in \mathbf{I}$. It is sufficient to consider only the case where

(11) $$i \in [\![\mathcal{B}_1]\!], i \in [\![\mathcal{B}_2]\!], \ldots, i \in [\![\mathcal{B}_m]\!]$$

(12) $$i \in [\![\mathcal{U} \leq \mathcal{V}]\!]$$

and to show

(13) $$i \in [\![\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_n]\!].$$

Suppose $i \notin [\![\mathcal{V}]\!]$. Then from (10), (11), and (12), we get (13) as required.
Suppose $i \in [\![\mathcal{V}]\!]$. From (12) and the semantics of **VC**, we have

(14) There is some $k \in [\![\mathcal{U}]\!]$ such that $i\mathbf{R}k$ and $k \leq_i i$.

But by the Centering Assumption, if $k \leq_i i$ then $k = i$, so from (14) $i \in [\![\mathcal{U}]\!]$. Using (9), (11), and (12), we get (13) as required.

Induction step for F$\leq$: The induction hypothesis is

(15) $$[\![(\mathcal{B}_1 \wedge \cdots \wedge \mathcal{B}_m) \supset (\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_n \vee (\mathcal{A} \leq \mathcal{B}) \vee \mathcal{A})]\!] = \mathbf{I}.$$

We need to show that

$$[\![(\mathcal{B}_1 \wedge \cdots \wedge \mathcal{B}_m) \supset (\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_n \vee (\mathcal{A} \leq \mathcal{B}))]\!] = \mathbf{I}.$$

Consider any $i \in \mathbf{I}$. It is sufficient to consider only the case where

(16) $$i \in [\![\mathcal{B}_1]\!], i \in [\![\mathcal{B}_2]\!], \ldots, i \in [\![\mathcal{B}_m]\!]$$

(17) $$i \notin [\![\mathcal{C}_1]\!], i \notin [\![\mathcal{C}_2]\!], \ldots, i \notin [\![\mathcal{C}_n]\!]$$

and to show

(18) $$i \in [\![\mathcal{A} \leq \mathcal{B}]\!].$$

Suppose $i \notin [\![\mathcal{A}]\!]$. Then by (15), (16), (17), we have (18) as required.
Suppose $i \in [\![\mathcal{A}]\!]$. Now, by the Centering Assumption, if $j \in \mathbf{I}$ and $i\mathbf{R}j$ then $i \leq_i j$. So in particular, if $j \in [\![\mathcal{B}]\!]$ and $i\mathbf{R}j$ then $i \leq_i j$. This gives us (18) as required.

Induction step for F $\leq (m, n)$, $m \geq 1$, $n \geq 0$: The induction hypothesis is that, for any model $\langle \mathbf{I}, \mathbf{R}, \leq, [\![\ ]\!] \rangle$,

(19) $$[\![\mathcal{D}_j \supset (\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_m \vee \mathcal{V}_1 \vee \cdots \vee \mathcal{V}_n)]\!] = \mathbf{I} \text{ for } 1 \leq j \leq m$$

and, assuming without loss of generality that $j_1 = 1, j_2 = 2, \ldots, j_n = n$,

(20.1) $$[\![\mathcal{U}_1 \supset \mathcal{A}_1 \vee \cdots \vee \mathcal{A}_m]\!] = \mathbf{I}$$

(20.2) $$[\![\mathcal{U}_2 \supset \mathcal{A}_1 \vee \cdots \vee \mathcal{A}_m \vee \mathcal{V}_1]\!] = \mathbf{I}$$

(20.3) $$\llbracket \mathcal{U}_3 \supset \mathcal{A}_1 \vee \cdots \vee \mathcal{A}_m \vee \mathcal{V}_1 \vee \mathcal{V}_2 \rrbracket = \mathbf{I}$$

$$\cdots$$

(20.n) $$\llbracket \mathcal{U}_n \supset \mathcal{A}_1 \vee \cdots \vee \mathcal{A}_m \vee \mathcal{V}_1 \vee \mathcal{V}_2 \vee \cdots \vee \mathcal{V}_{n-1} \rrbracket = \mathbf{I}.$$

We need to show, given (19) and (20),

$$\llbracket ((\mathcal{U}_1 \leq \mathcal{V}_1) \wedge \cdots \wedge (\mathcal{U}_n \leq \mathcal{V}_n)) \supset ((\mathcal{A}_1 \leq \mathcal{D}_1) \vee \cdots \vee (\mathcal{A}_m \leq \mathcal{D}_m)) \rrbracket = \mathbf{I}.$$

Consider any $i \in \mathbf{I}$. It is sufficient to consider only those $i \in \mathbf{I}$ such that

(21) $$i \in \llbracket \mathcal{U}_1 \leq \mathcal{V}_1 \rrbracket, \ldots, i \in \llbracket \mathcal{U}_n \leq \mathcal{V}_n \rrbracket$$

(22) $$i \notin \llbracket \mathcal{A}_2 \leq \mathcal{D}_2 \rrbracket, \ldots, i \notin \llbracket \mathcal{A}_m \leq \mathcal{D}_m \rrbracket$$

and to show that

$$i \in \llbracket \mathcal{A}_1 \leq \mathcal{D}_1 \rrbracket.$$

Since, if $\forall i_1 \in \mathbf{I}\ i\mathbf{R}i_1 \supset i_1 \notin \llbracket \mathcal{D}_1 \rrbracket$, it follows trivially that $i \in \llbracket \mathcal{A}_1 \leq \mathcal{D}_1 \rrbracket$, we may assume that

(23) $$\exists i_1 \in \mathbf{I} \text{ such that } i\mathbf{R}i_1 \text{ and } i_1 \in \llbracket \mathcal{D}_1 \rrbracket.$$

It now is sufficient to show, given (19), (20), (21), (22), and (23), that

(24) $$\exists i_0 \in \mathbf{I} \text{ such that } i\mathbf{R}i_0,\ i_0 \leq_i i_1 \text{ and } i_0 \in \llbracket \mathcal{A}_1 \rrbracket.$$

Proof (Induction step for F $\leq (m, n)$): I will demonstrate the existence of a sequence of worlds

$$\begin{array}{lll} i_1 = i_{1,0}, & i_{1,1}, \ldots & i_{1,n_1}, \\ i_{2,0}, & i_{2,1}, \ldots & i_{2,n_2}, \\ \vdots & \vdots & \vdots \\ i_{k,0}, & i_{k,1}, \ldots & i_{k,n_k}. \end{array}$$

Each new world in the sequence will be $\leq_i$ the previous one, and so by transitivity $\leq_i i_1$. It will turn out that i_{k,n_k} will satisfy the requirements for i_0 in (24).

Set $i_{1,0} = i_1$. From (19) and (23) we have $i_{1,0} \in \llbracket \mathcal{A}_1 \vee \cdots \vee \mathcal{A}_m \vee \mathcal{V}_1 \vee \cdots \vee \mathcal{V}_n \rrbracket$. Now suppose, in general, that we have

(25.a) $$i_{f,g} \in \llbracket \mathcal{A}_1 \vee \cdots \vee \mathcal{A}_m \vee \mathcal{V}_1 \vee \cdots \vee \mathcal{V}_{h_g} \rrbracket$$

(25.b) $$i_{f,g} \leq_i i_{f',g'} \text{ for all } f' \leq f \text{ and } g' \leq g.$$

If $i_{f,g} \in \llbracket \mathcal{A}_1 \vee \cdots \vee \mathcal{A}_m \rrbracket$, then set $n_f = g$.

Otherwise, $i_{f,g} \in \llbracket \mathcal{V}_1 \vee \cdots \vee \mathcal{V}_{h_g} \rrbracket$. So, for some h with $1 \leq h \leq h_g$, $i_{f,g} \in \llbracket \mathcal{V}_h \rrbracket$. But, from (21), $i \in \llbracket \mathcal{U}_h \leq \mathcal{V}_h \rrbracket$, so there is some $i_{f,g+1}$ such that $i\mathbf{R}i_{f,g+1}$ and $i_{f,g+1} \leq_i i_{f,g}$ and $i_{f,g+1} \in \llbracket \mathcal{U}_h \rrbracket$. Because $i_{f,g+1} \leq_i i_{f,g}$, and from the transitivity of $\leq_i$, we get (25.b). From (20.h), and setting $h_{g+1} = h - 1$, $i_{f,g+1} \in \llbracket \mathcal{A}_1 \vee \cdots \vee \mathcal{A}_m \vee \mathcal{V}_1 \vee \cdots \vee \mathcal{V}_{h-1} \rrbracket$, satisfying (25.a). Since $1 \leq h \leq h_g$, we have $0 \leq h_{g+1} < h_g$. So $h_1, h_2, \ldots,$ is a strictly decreasing sequence of integers bounded below by 0. Thus it must be a finite sequence, and by construction it must end with some $h_{n_f} = 0$.

Thus given (25) we can show that there is a sequence of worlds $i_{f,g}, i_{f,g+1}, \ldots, i_{f,n_f}$ such that $i_{f,n_f} \in [\![\mathfrak{A}_1 \vee \cdots \vee \mathfrak{A}_m]\!]$. The sequence starts because setting $f = 1$, $g = 0$, and $h_g = n$ satisfies the conditions of (25).

Now suppose, in general, that

(26.a) $$i_{f,n_f} \in [\![\mathfrak{A}_1 \vee \cdots \vee \mathfrak{A}_m]\!]$$

(26.b) $$i_{f,n_f} \leq_i i_1 \text{ and } i_{f,n_f} \leq_i i_{f',n_{f'}} \text{ for all } f' \leq f.$$

So $i_{f,n_f} \in [\![\mathfrak{A}_{a_f}]\!]$ for some a_f. If $a_f = 1$, then set $k = f$ and $i_0 = i_{k,n_k}$. Otherwise we have from (23) $i \notin [\![\mathfrak{A}_{a_f} \leq \mathfrak{D}_{a_f}]\!]$. So there is some $i_{f+1,0} \in \mathbf{I}$ such that $i_{f+1,0} \in [\![\mathfrak{D}_{a_f}]\!]$ and $i\mathbf{R}i_{f+1,0}$ and $i_{f+1,0} <_i i_{f,n_f}$ and

(27) $$\forall j \in \mathbf{I}\ (i\mathbf{R}j \text{ and } j \leq_i i_{f+1,0}) \supset j \notin [\![\mathfrak{A}_{a_f}]\!].$$

By the transitivity of $\leq_i$, and the fact that $i_{f+1,0} <_i i_{f,n_f}$, $i_{f+1,0}$ satisfies (25.b). By (19), $i_{f+1,0} \in [\![\mathfrak{A}_1 \vee \cdots \vee \mathfrak{A}_m \vee \mathfrak{V}_1 \vee \cdots \vee \mathfrak{V}_n]\!]$, so $i_{f+1,0}$ satisfies (25.a). Therefore $i_{f+1,n_{f+1}}$ satisfies (26.a) and (26.b).

This establishes a sequence $i_{1,n_1}, i_{2,n_2}, \ldots$ and an associated sequence $a_1, a_2, \ldots$ such that $i_{f,n_f} \in [\![\mathfrak{A}_{a_f}]\!]$. Now the latter sequence cannot have any repetitions, for if $g > f$ then, by (25) and (26) $i_{g,n_g} \leq_i i_{f+1,0}$, and so by (27) $i_{g,n_g} \notin [\![\mathfrak{A}_{a_f}]\!]$. So the sequence $a_1, a_2, \ldots$ is at most length m. Yet it can only stop when $a_k = 1$. So for some finite k, $a_k = 1$, and we have $i_{f_k,n_{f_k}} \in [\![\mathfrak{A}_1]\!]$ and $i_{f_k,n_{f_k}} \leq_i i_1$, and so we have satisfied (24).

6 Completeness

Definition 6.1 A *Hintikka element* is a finite set Σ of signed formulas such that:

if $[\mathfrak{B} \wedge \mathfrak{C}]^+ \in \Sigma$ then $[\mathfrak{B}]^+ \in \Sigma$ and $[\mathfrak{C}]^+ \in \Sigma$; and
if $[\mathfrak{B} \wedge \mathfrak{C}]^- \in \Sigma$ then $[\mathfrak{B}]^- \in \Sigma$ or $[\mathfrak{C}]^- \in \Sigma$; and
if $[\mathfrak{B} \vee \mathfrak{C}]^+ \in \Sigma$ then $[\mathfrak{B}]^+ \in \Sigma$ or $[\mathfrak{C}]^+ \in \Sigma$; and
if $[\mathfrak{B} \vee \mathfrak{C}]^- \in \Sigma$ then $[\mathfrak{B}]^- \in \Sigma$ and $[\mathfrak{C}]^- \in \Sigma$; and
if $[\mathfrak{B} \supset \mathfrak{C}]^+ \in \Sigma$ then $[\mathfrak{B}]^- \in \Sigma$ or $[\mathfrak{C}]^+ \in \Sigma$; and
if $[\mathfrak{B} \supset \mathfrak{C}]^- \in \Sigma$ then $[\mathfrak{B}]^+ \in \Sigma$ and $[\mathfrak{C}]^- \in \Sigma$; and
if $[\neg\mathfrak{B}]^+ \in \Sigma$ then $[\mathfrak{B}]^- \in \Sigma$; and
if $[\neg\mathfrak{B}]^- \in \Sigma$ then $[\mathfrak{B}]^+ \in \Sigma$.

A **VC**-*Hintikka element* is a Hintikka element Σ that also satisfies

if $[\mathfrak{B} \leq \mathfrak{C}]^+ \in \Sigma$ then $[\mathfrak{B}]^+ \in \Sigma$ or $[\mathfrak{C}]^- \in \Sigma$; and
if $[\mathfrak{B} \leq \mathfrak{C}]^- \in \Sigma$ then $[\mathfrak{B}]^- \in \Sigma$.

Theorem 6.2 (Completeness Theorem) *For any formula $\mathfrak{A}$ of* **VC**, *if there is no derivation of $\mathfrak{A}$ then $\nvDash_{\mathbf{VC}} \mathfrak{A}$.*

Proof: It is sufficient to prove that:

(28) For any sequent $\Sigma = \{[\mathfrak{B}_1]^+, \ldots, [\mathfrak{B}_n]^+, [\mathfrak{C}_1]^-, \ldots, [\mathfrak{C}_m]^-\}$ which is not derivable, there is a model $\langle \mathbf{I}, \mathbf{R}, \leq, [\![\ ]\!]\rangle$ of **VC** and some $i \in \mathbf{I}$ such that $i \in [\![\mathfrak{B}_1 \wedge \cdots \wedge \mathfrak{B}_n \wedge \neg\mathfrak{C}_1 \wedge \cdots \wedge \neg\mathfrak{C}_m]\!]$. (I will sometimes abuse notation by writing "$i \in [\![\Sigma]\!]$" for this.)

Suppose that a sequent Σ_1 is one of the sequents derived from some sequent Σ_0 by one of the rules T$\wedge$, F$\wedge$, T$\vee$, F$\vee$, T$\supset$, F$\supset$, T$\neg$, F$\neg$, T$\leq$, or F$\leq$. Then it is easy to show that if $i \in [\![\Sigma_1]\!]$ then $i \in [\![\Sigma_0]\!]$ and so $i \in [\![\Sigma_0 \cup \Sigma_1]\!]$. If Σ_0 is not derivable, then any sequence of applications of these rules must end (by a simple complexity argument) and leave at least one underivable sequent. Since the series of applications of rules has ended, none of the rules except F $\leq (m, n)$ can be applicable to the underivable sequent. Now consider the union of all the ancestor sequents of this underivable sequent in the attempted derivation. Since the definition of a **VC**-Hintikka element matches the definition of the rules, this union must be a **VC**-Hintikka element. So if Σ_0 is not derivable, there is some **VC**-Hintikka element Σ' which is not derivable, and such that if $i \in [\![\Sigma']\!]$ then $i \in [\![\Sigma_0]\!]$.

The previous paragraph shows that it is sufficient to prove (28) only for sequents Σ' which are **VC**-Hintikka elements. This will be done by induction on the maximum nesting of the symbol "$\leq$" in Σ'.

Induction Base: Since Σ' is not derivable, Σ' does not contain $[\top]^-$, $[\bot]^+$, or $[\mathcal{B}]^-$ and $[\mathcal{B}]^+$ for any $\mathcal{B}$. Define a model M for **VC** by

- $\mathbf{I} = \{i_0\}$ which makes $\mathbf{R}$ and $\leq$ trivial.
- For atomic propositions $\mathcal{P}$, $i_0 \in [\![\mathcal{P}]\!]$ if and only if $[\mathcal{P}]^+ \in \Sigma'$.
- For nonatomic formulas $\mathcal{A}$ of **VC**, $[\![\mathcal{A}]\!]$ is defined by the above and the semantics of **VC**. ($[\![\ ]\!]$ is well-defined because for each compound formula $\mathcal{A}$, $[\![\mathcal{A}]\!]$ is defined in terms of strictly simpler formulas.)

It is now easy to show (28) by induction on the total size of the formulas in the sequent Σ'.

Induction Step: Suppose Σ is not derivable. Then there is a sequence of rules, not including F$\leq (m, n)$, which can be applied to yield a **VC**-Hintikka element Σ' such that each possible application of F$\leq (m, n)$ to Σ' yields at least one underivable sequent.

Suppose the set of all signed formulas in Σ' with $\leq$ dominating is $\Sigma_1 = \{[\mathcal{A}_1 \leq \mathcal{D}_1]^-, \ldots, [\mathcal{A}_m \leq \mathcal{D}_m]^-, [\mathcal{U}_1 \leq \mathcal{V}_1]^+, \ldots, [\mathcal{U}_n \leq \mathcal{V}_n]^+\}$. I will show the existence of two finite sequences of sequents $\Sigma_1, \ldots \Sigma_p, \ldots$ and $\Sigma_{1,1}, \ldots \Sigma_{p,q}, \ldots$ which can be used to construct a model for Σ'. If $m = 0$, the first sequence will be simply Σ_1 and the second will be empty.

Suppose, in general, that $\Sigma_p \subseteq \Sigma_1$ and that $\Sigma_p = \{[\mathcal{A}_1 \leq \mathcal{D}_1]^-, \ldots, [\mathcal{A}_{m_p} \leq \mathcal{D}_{m_p}]^-, [\mathcal{U}_1 \leq \mathcal{V}_1]^+, \ldots, [\mathcal{U}_{n_p} \leq \mathcal{V}_{n_p}]^+\}$ with $m_p > 0$. We can apply F$\leq (m_p, n_p)$ to Σ_1 and we know by the definition of Σ' that the application will not close. This means that either one of the sequents $\{[\mathcal{A}_1]^-, \ldots, [\mathcal{A}_{m_p}]^-, [\mathcal{D}_j]^+, [\mathcal{V}_1]^-, \ldots, [\mathcal{V}_{n_p}]^-\}$ is not derivable for some j, or the special condition (*) fails. If the former, then set $\Sigma_{p,0} = \{[\mathcal{A}_1]^-, \ldots, [\mathcal{A}_{m_p}]^-, [\mathcal{D}_j]^+, [\mathcal{V}_1]^-, \ldots, [\mathcal{V}_{n_p}]^-\}$ and set $\Sigma_{p+1} = \Sigma_p - \{[\mathcal{A}_j \leq \mathcal{D}_j]^-\}$. If the special rule fails, then for some $k < n_p$ there must be a sequence $j_1, j_2, \ldots, j_k$ (and I will assume, without loss of generality, that it is the sequence $1, 2, \ldots, k$) with the property that: each sequent $\{[\mathcal{A}_1]^-, \ldots, [\mathcal{A}_{m_p}]^-, [\mathcal{U}_j]^+, [\mathcal{V}_1]^-, \ldots, [\mathcal{V}_{j-1}]^-\}$ is derivable for $1 \leq j \leq k$, but no sequent $\{[\mathcal{A}_1]^-, \ldots, [\mathcal{A}_{m_p}]^-, [\mathcal{U}_j]^+, [\mathcal{V}_1]^-, \ldots, [\mathcal{V}_k]^-\}$ is derivable for $k < j \leq n_p$. If there was no such sequence for $k < n_p$ then the special rule would not fail. In this case, set $\Sigma_{p,q} = \{[\mathcal{A}_1]^-, \ldots, [\mathcal{A}_{m_p}]^-, [\mathcal{U}_{k+q}]^+, [\mathcal{V}_1]^-,$

$\ldots, [\mathcal{V}_k]^-\}$ for $1 \leq q \leq n_p - k$ and set $\Sigma_{p+1} = \Sigma_p - \{[\mathcal{U}_{k+1} \leq \mathcal{V}_{k+1}]^+, \ldots, [\mathcal{U}_{n_p} \leq \mathcal{V}_{n_p}]^+\}$.

We can repeat this process until $m_p = 0$, when we stop. Note that if $m_p = 0$ then the first sequence ends with Σ_p and the second with some $\Sigma_{p-1,q}$. Each $\Sigma_{p,q}$ is underivable, and so from each one we can build an underivable **VC**-Hintikka element $\Sigma'_{p,q}$, as in the opening of this completeness proof. The symbol "$\leq$" is nested strictly less deeply in $\Sigma_{p,q}$ than in Σ', and none of the rules used to build $\Sigma'_{p,q}$ increases the nesting of "$\leq$". So "$\leq$" is nested strictly less deeply in each $\Sigma'_{p,q}$ than in Σ'. So by the induction hypothesis, and since $\Sigma_{p,q} \subset \Sigma'_{p,q}$, we may assume that for each $\Sigma_{p,q}$ there is a **VC** model $M_{p,q} = \langle \mathbf{I}_{p,q}, \mathbf{R}_{p,q}, \leq_{p,q}, [\![\ ]\!]_{p,q}\rangle$ and some $i_{p,q} \in \mathbf{I}_{p,q}$ such that $i_{p,q} \in [\![\Sigma_{p,q}]\!]_{p,q}$.

Now define a **VC** model $\langle \mathbf{I}, \mathbf{R}, \leq, [\![\ ]\!]\rangle$ containing a world i_0 as follows.

$$\mathbf{I} = \{i_0\} \cup \{i : i \in \mathbf{I}_{p,q} \text{ for some } p, q\}$$

$$i\mathbf{R}j \Leftrightarrow \begin{cases} i, j \in \mathbf{I}_{p,q} \text{ and } i\mathbf{R}_{p,q}j \\ \text{or } i = i_0 \text{ and } j = i_{p,q} \\ \text{or } i = i_0 \text{ and } j = i_0 \end{cases}$$

$i \leq_k j$ defined by the $\leq_{p,q}$ relations and by:

$i_0 \leq_{i_0} i_0$
$i_0 <_{i_0} i_{p,q}$
$i_{p,q} <_{i_0} i_{p',q'}$ if $p < p'$
$i_{p,q} \leq_{i_0} i_{p,q'}$ and $i_{p,q'} \leq_{i_0} i_{p,q}$

$[\![\ ]\!]$ defined by the $[\![\ ]\!]_{p,q}$ relations and by:

For atomic propositions $\mathcal{P}, i_0 \in [\![\mathcal{P}]\!] \Leftrightarrow [\mathcal{P}]^+ \in \Sigma'$.

For nonatomic propositions $\mathcal{A}$, whether or not $i_0 \in [\![\mathcal{A}]\!]$ is given by the above definitions and the semantics of **VC**.

Note that for each world in one of the models $M_{p,q}$, the above definitions leave the semantics of that world unchanged, since each such world bears exactly the same **R** and $\leq$ relations as it did in $M_{p,q}$. Also the semantics of i_0 for atomic propositions are well defined since Σ' does not contain $[\mathcal{P}]^+$ and $[\mathcal{P}]^-$ for any proposition P. The definition of semantics of **VC** defines each nonatomic proposition in terms of strictly simpler propositions, and so the semantics of all propositions in i_0 are well defined. Thus the definition of $[\![\ ]\!]$ is well defined.

It remains to show that for any formula $\mathcal{A}$ of **VC**: that if $[\mathcal{A}]^+ \in \Sigma'$ then $i_0 \in [\![\mathcal{A}]\!]$; and that if $[\mathcal{A}]^- \in \Sigma'$ then $i_0 \notin [\![\mathcal{A}]\!]$. This will be done by another, *inner*, induction on the complexity of the structure of $\mathcal{A}$. The induction base is the case of atomic formulas. In this case, these requirements are met by the definition of $[\![\ ]\!]$ above. The induction step is straightforward for signed formulas dominated by a propositional symbol. So it only remains to prove the induction step for signed formulas dominated by $\leq$.

(Inner) Induction step for formulas $[\mathcal{U}_k \leq \mathcal{V}_k]^+$: If $[\mathcal{U}_k \leq \mathcal{V}_k]^+ \in \Sigma'$ then $i_0 \in [\![\mathcal{U}_k \leq \mathcal{V}_k]\!]$.

Proof: Since Σ' is a **VC**-Hintikka element, either $[\mathcal{U}_k]^+ \in \Sigma'$ or $[\mathcal{V}_k]^- \in \Sigma'$. Each of these signed formulas is strictly simpler than $[\mathcal{U}_k \leq \mathcal{V}_k]^+$, so by the inner induction hypothesis either $i_0 \in [\![\mathfrak{U}_k]\!]$ or $i_0 \notin [\![\mathfrak{V}_k]\!]$. If $i_0 \in [\![\mathfrak{U}_k]\!]$, then by the semantics of **VC**, we have $i_0 \in [\![\mathfrak{U}_k \leq \mathfrak{V}_k]\!]$, and we have finished. So we may assume that $[\mathfrak{V}_k]^- \in \Sigma'$ and so $i_0 \notin [\![\mathfrak{V}_k]\!]$.

Suppose that for some p', there is no $i_{p,q}$ such that $p < p'$ and $i_{p,q} \in [\![\mathfrak{U}_k]\!]$. Then, from the choice of model $M_{p,q}$, for each $p < p'$, $[\mathfrak{U}_k]^+ \notin \Sigma_{p,q}$ and so, by construction, $[\mathfrak{U}_k \leq \mathfrak{V}_k]^+ \in \Sigma_p$. Then, again by construction, $[\mathfrak{V}_k]^- \in \Sigma_{p,q}$ and so, again by the choice of $M_{p,q}$, $i_{p,q} \notin [\![\mathfrak{V}_k]\!]$ for $p < p'$. Either no $i_{p,q} \in [\![\mathfrak{U}_k]\!]$ or some $i_{p,q} \in [\![\mathfrak{U}_k]\!]$. If the former, then by the above argument we have that for each $p, q, i_{p,q} \notin [\![\mathfrak{V}_k]\!]$, and remembering that we assumed that $i_0 \notin [\![\mathfrak{V}_k]\!]$, we have that there is no world j such that $i_0\mathbf{R}j$ and $j \in [\![\mathfrak{V}_k]\!]$. Thus $i_0 \in [\![\mathfrak{U}_k \leq \mathfrak{V}_k]\!]$. If the latter, then there is some smallest p' such that for some q' $i_{p',q'} \in [\![\mathfrak{U}_k]\!]$. Then by the above argument, and the assumption that $i_0 \notin [\![\mathfrak{V}_k]\!]$, we have that there is no world j such that $i_0\mathbf{R}j$ and $j \in [\![\mathfrak{V}_k]\!]$ and $j <_{i_0} i_{p',q'}$. Thus $i_0 \in [\![\mathfrak{U}_k \leq \mathfrak{V}_k]\!]$.

(Inner) Induction step for formulas: $[\mathfrak{U}_k \leq \mathfrak{V}_k]^-$*:* If $[\mathfrak{C}_k \leq \mathfrak{D}_k]^- \in \Sigma'$ then $i_0 \notin [\![\mathfrak{C}_k \leq \mathfrak{D}_k]\!]$.

Proof: By construction, there is some Σ_p which does not contain any formula $[\mathfrak{C}_i \leq \mathfrak{D}_i]^-$ and so there is some largest p' such that $[\mathfrak{C}_k \leq \mathfrak{D}_k]^- \in \Sigma_{p'}$. Then, by construction, $[\mathfrak{D}_k]^+ \in \Sigma_{p',0}$, and so by the choice of $M_{p,q}$, $i_{p',0} \in [\![\mathfrak{D}_k]\!]$. Now consider $j \in \mathbf{I}$ such that $j \leq_{i_0} i_{p',0}$. Either $j = i_0$ or $j = i_{p,q}$ for some $p \leq p'$. If $j = i_0$, then since Σ' is a **VC**-Hintikka element, $[\mathfrak{C}_k]^- \in \Sigma'$, and so by the (inner) induction hypothesis, $i_0 \notin [\![\mathfrak{C}_k]\!]$. If $j = i_{p,q}$ for some $p \leq p'$, then $[\mathfrak{C}_k \leq \mathfrak{D}_k]^- \in \Sigma_p$. Then, by construction, $[\mathfrak{C}_k]^- \in \Sigma_{p,q}$, and so by the choice of $M_{p,q}$, $i_{p,q} \notin [\![\mathfrak{C}_k]\!]$. So we have established that there is some p' such that $i_{p',0} \in [\![\mathfrak{D}_k]\!]$, and that if $j \leq_{i_0} i_{p',0}$ then $j \notin [\![\mathfrak{C}_k]\!]$. This establishes that $i_0 \notin [\![\mathfrak{C}_k \leq \mathfrak{D}_k]\!]$.

Acknowledgments Primarily, I am grateful to Dr. de Swart for doing the original work on sequent systems for **VCS** and **VC**, on which my work is very closely based, and for his comments on this work. My supervisor, Dr. Tony Cohn, Mike Slade, and a referee of this journal all helped me to put this work into this form, and Rajeev Goré helped me to correct a mistake in an earlier version of this paper. I am, of course, very grateful to the Science and Engineering Research Council for their funding of my research.

NOTES

1. I use the term "sequent" as shorthand for "set of signed formulas". The ambiguity with the term "Gentzen sequent" is unimportant since the proof system presented here would appear very Gentzen-like if each formula $[\mathfrak{A}]^+$ were written to the left of a sequent arrow and each formula $[\mathfrak{A}]^-$ to the right of a sequent arrow.
2. The reader may be interested in how this counterexample was discovered. Curiously, I wrote down the system of Section 4 before I realized that de Swart's system was wrong. In studying the relationship between the two systems, I was able to construct the above counterexample to their equivalence. It was only then that I noticed that de Swart's system gives the wrong answer in this case.

3. This condition could be expressed as a complicated condition on various sets of sequents, mirroring the presentation of $F \leq (1,2)$ and $F \leq (1,3)$ in de Swart's system. However, this method of presentation makes the general rule much clearer.

REFERENCES

[1] de Swart, H. C. M., "A Gentzen- or Beth-type system, a practical decision procedure and a constructive completeness proof for the counterfactual logics VC and VCS," *The Journal of Symbolic Logic*, vol. 48 (1983), pp. 1–20.

[2] Lewis, D. K., *Counterfactuals*, Basil Blackwell, Oxford, 1973.

Department of Artificial Intelligence
University of Edinburgh
Edinburgh EH1 1HN
United Kingdom