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# Consequential Implication: A Correction

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**Abstract** It is shown that while the decision procedure devised for system CI*O of consequential implication provides a refutation for the wff expressing the collapse of circumstantial operator $p \supset *p$, this wff is provable in the system. It is argued that the gap between tableaux-validity and model-validity may be corrected either by weakening the system or by forbidding the nesting of circumstantial operators in the language of CI*O.

In Pizzi [2] I prove a constructive completeness theorem for a system named $T^wO$, in which a system CI*O of consequential implication is shown to be embedded (it is proved in fact that a translation function Tr may be defined such that $\vdash_{T^wO} \mathrm{Tr}(A)$ iff $\vdash_{CI^*O} A$). The proof amounts to showing that if the tableau for A closes, a proof for A may be construed from the axioms of $T^wO$.

In the paper it is taken for granted that the notion of tableaux-validity is coincident with the one of model-validity: in other words it is taken for granted that if the tableau for any wff A is open then A has a falsifying $T^wO$-model (the converse statement is unproblematic: if the tableau for A closes we may build a proof of A, and by the soundness of $T^wO$ we conclude that A is model-valid). The identification of these two kinds of validity is normally understood when the rules given for the tableaux construction are specular to the conditions which define the models of the system. A specularity of this kind is for instance apparent in Gödel-Feys' system T, which in the paper is proved to be definitionally equivalent to CI.O. In a system such as S4, however, the decision procedure contains a stop rule (which Hughes and Cresswell call *rule for repeating chains*) whose function is to prevent the construction of infinite tableaux in which the same rectangle, or a subpart of it, is a cyclical output of the procedure. No clause of the same kind, however, occurs in the definition of S4 models. It happens then that whenever the tableau for A is open we have to check that we actually dispose of an S4-countermodel for A. That this is not in general granted has been proved in Tapscott [3] (p. 248), where it is argued that as a result of the appli-

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cation of the rule for repeating chains the accessibility relation may fail to be transitive in the resulting model.

Unfortunately, an analogous difficulty occurs for the tableaux procedure devised in my paper for the system $T^wO$, which is inspired to the one given by Hughes and Cresswell for T and S4. The procedure here contains two stop rules which are elliptically stated in the text, even if they are applied throughout the paper:

1. A first rule which is analogous to the one of repeating chains inasmuch as it asks us to identify rectangles containing the same wffs, and thus to send to the first rectangle of a succession of partial or total duplicates all the arrows which go to the latter. If this rule were not applied, rule R1 would force us to build, for any rectangle containing $p$ and $\mathbf{w}^p$ with value 1, another rectangle containing $p$ and $\mathbf{w}^p$ with value 1, and so on *ad infinitum*. Thanks to the reflexivity of R, it is easy to show that the application of this stop rule leads to models having the properties required for $T^wO$-models.

2. A second rule to the effect that in rectangles beyond the first one no wff has to occur having a circumstantial degree higher than the wff to be tested. If this rule were not applied, we would lack a decision procedure even for the simple $\neg p$. Since the first rectangle, in fact, gives value 1 to $p$, rule R1 obliges us to build a rectangle in which $p$ has value 1 and $\mathbf{w}^p$ has value 1, a third rectangle in which $\mathbf{w}^p$ and $\mathbf{w}^{\mathbf{w}p}$ have value 1, and so on.

Both these stop rules are obvious and allow us to prove easily that $\neg(p \wedge \neg(\mathbf{w}^p \wedge p))$ is not $T^wO$-tableaux-valid, so that its Tr-counterimage $p \supset *p$ is not a CI*O theorem (see [2], p. 631). The undesired collapse formula, however, turns out to be model-valid. Let us suppose by Reductio that $V(\neg(p \wedge \neg\mathbf{w}^p), m_i) = O$, where $m_i$ is a world of an arbitrary $T^wO$-model. Then, by clause VR1, there is at least one world $m_j$ such that $m_iRm_j$ and such that $V(\mathbf{w}^{p\wedge\neg\mathbf{w}^p}, m_j) = 1$ and $V(p \wedge \neg\mathbf{w}^p, m_j) = 1$ (hence also $V(p, m_i) = 1$ and $V(\neg\mathbf{w}^p, m_i) = 1$)). Since we have $V(\mathbf{w}^p, m_i) = O$, by VR2 this implies that there is a world $m_k$ such that $m_jR^wm_k$ and $V((p \wedge \neg\mathbf{w}^p) \supset p, m_k) = O$, which is impossible. Hence $\neg(p \wedge \neg\mathbf{w}^p)$ has no $T^wO$-countermodel and is $T^wO$-valid. On the other hand, we may build a syntactical proof of $p \supset *p$ in this way (I am grateful to Timothy Williamson for suggesting to me the steps of the proof):

| | | |
|---|---|---|
| (1) | $*(p \wedge \neg*p) \supset *p$ | $\vdash(p \wedge \neg*p) \supset p$ × R*O |
| (2) | $*(p \wedge \neg*p) \supset \neg*p$ | $\vdash*(p \wedge q) \supset q$ $(\neg*p/q)$ |
| (3) | $\neg*(p \wedge \neg*p)$ | (1), (2) × PC |
| (4) | $\neg\Diamond*(p \wedge \neg*p)$ | (3), × Nec |
| (5) | $\neg\Diamond(p \wedge \neg*p)$ | (4), $\vdash\Diamond p \supset \Diamond*p$ |
| (6) | $p \supset *p$ | (6), T. |

There are two proposals to eliminate the discrepancy between tableaux-validity and model-validity:

(a) The first proposal is to weaken rule $\vdash A \supset B \Rightarrow \vdash*A \supset *B$ (R*O) into $\vdash A \equiv B \Rightarrow \vdash*A \equiv *B$ (R*OEq) and the parallel rule $\vdash A \supset B \Rightarrow \vdash\mathbf{w}^A \supset \mathbf{w}^B$ ($R^wO$) into $\vdash A \equiv B \Rightarrow \vdash\mathbf{w}^A \equiv \mathbf{w}^B$ ($R^wOEq$). Such weakened systems will be called CI.O*Eq and $T^{Ow}Eq$ respectively. (This restriction actually amounts to simply postulating the unrestricted validity of replacement of proved material equivalents, and hence to asking that the conditional fragment of CI.O*Eq should be, according to an accepted terminology, a *classic* conditional logic.)

(b) The second proposal is to forbid the nesting of circumstantial operators in CI*O and the parallel nesting of quasi-variables in $T^{w}O$.

A concise justification for the two proposed restrictions may be given as follows:

(a) $T^{Ow}$Eq-models may be defined as quadruples $\langle M,R,R^{w},V\rangle$, which are like $T^{w}O$-models save for the following condition in place of VR2: If $(\mathbf{w}^{A},m_i) \neq V(\mathbf{w}^{B},m_i)$ then some $m_j$ exists such that $m_iRm_j$ and $V(A \equiv B,m_j) = O$ (VR2Eq). A rule for the tableaux construction specular to this condition will be of course introduced. In order to understand that if a wff A is $T^{Ow}$Eq-model-valid then it is $T^{Ow}$Eq-tableaux-valid, we may (sketchily) reason as follows. By omitting rule VR2 we obtain a system $T^{Ow}$ which is easily shown to be complete in respect of the class of models $\langle M,R,V\rangle$ which are defined as T-models satisfying the further condition VR1. The decision procedure for $T^{Ow}$ is the same as the one given for $T^{w}O$ with the omission of rule R2. When we reason by neglecting the stop rules in order to ascertain model-validity, whenever we find a possible world containing $\mathbf{w}^{A}$ with value 1 and $\mathbf{w}^{B}$ with value O, we cannot apply rule VR2Eq or VR2 in order to derive a contradiction, so tableaux-validity and model-validity are coincident. Let us now define a relation between any two wffs C and B which is the equivalence relation expressed by $\vdash_{T^{Ow}} C \equiv B$. Any subformula of any wff A is then assigned to an equivalence class of subformulas modulo this relation. In considering each one of the classes, we choose in each of them one of the shortest wffs (say, the first in lexicographic order) and put it in place of the longer ones of the same class wherever they occur in A. By reiterating this operation, at the end of the replacements we will reach a wff A′ which is $T^{Ow}$-Eq-equivalent to A, but does not contain any $T^{Ow}$-Eq-equivalent subformulas. Now, if A is $T^{Ow}$Eq-model-valid, so is A′. Since A′ does not contain equivalent subformulas, however, it is $T^{Ow}$-valid, and it turns out that the $T^{Ow}$-tableau for it is the same as the $T^{Ow}$Eq-tableau for it. Compared to A′, A contains subformulas which are $T^{Ow}$Eq-equivalent to other subformulas, but these equivalences are detected just by applying rule VR2Eq in the $T^{Ow}$Eq-tableau for it. Thus if A is $T^{Ow}$Eq-model-valid it is $T^{Ow}$Eq-tableaux-valid.

(b) The restriction on nesting may be obtained by modifying the formation rules in this way: in CI*O clause (4) becomes: if A is a CI.O-wff, *A is a CI*O wff, while in $T^{w}O$ clause (4) becomes: if A is a T-wff, $\mathbf{w}^{A}$ is a $T^{w}O$-wff. It is trivial to show that the argument which leads to prove the model-validity of $\neg(p \wedge \neg(\mathbf{w}^{p} \wedge p))$ is blocked by this restriction. More generally, it may be shown that the second stop rule for the tableaux construction may be dropped since the restriction on nesting amounts to the same effect.

Of course, any restriction on formation rules or on substitution rules needs some non-*ad hoc* motivation, but this is not difficult to find in the literature on conditionals. $*p$ may be read "$p$ all other things being equal" as in classical conditional logics, but also "$p$ in normal circumstances" or "$p$ is true in the least exceptional worlds". Given the latter interpretation A > B takes the sense of a conditional for default properties. In the latter interpretation it is questionable that wffs such as $**p$, $*(p \wedge \neg *p)$ make an independent sense, and also that iterated synthetic conditional make an independent sense. It is not by chance that Del Grande's logic, for instance, rules out iterated conditionals from the language (see Del Grande [1]).

It may be of some interest to stress that in system CI*O we obtain as a theorem $\Diamond A \supset (((A \vee R) > B) \supset (A > B))$, namely the simplification of disjunctive antecedents in weakened form, which is obtained in so-called "semiclassical" conditional logics by restricting replacement. In CI.O*Eq simplification of disjunctive antecedents holds only in the variant $\Diamond A \supset (((*A \vee *B) > C) \supset (A > C))$, which seems to be more plausible, even if not expressible in standard conditional logics. (It is indeed a merit of logics with circumstantial operators to allow for these distinctions.) We have simply to add that the language of every logic in which rule R*O ($\vdash A \supset B \Rightarrow \vdash *A \supset *B$) is provable must be restricted in the mentioned way. In particular, system CI*2 must be restricted in the same way since it is easy to show that its characteristic axiom $((*p \wedge q) \supset\cdot (*q \wedge p))$ allows us to obtain R*O as a derived rule. Of course, system $T^w2$ must be restricted in a parallel manner, and given the new clause $Tr(*A) = \mathbf{w} \wedge Tr(A)$ in definition of the translation function, we have simply to stipulate that if A is a T-wff then $\mathbf{w} \wedge A$ is a $T^w2$-wff. Since in CI*2 we obtain transitivity for the corner operators, we conclude that transitivity may be accepted in these logics provided we restrict nesting of circumstantial operators. This fact may have some bearing on the philosophical controversy on the transitivity of conditionals which has taken place in the last years.

## REFERENCES

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