Notre Dame Journal of Formal Logic
Volume 34, Number 4, Fall 1993

Multimorphisms Over Enumerated Sets

ANDRZEJ ORLICKI

Abstract The concept of a multimorphism over enumerated sets is a natural generalization of the classical concept of a morphism of enumerated sets. Moreover, there are some connections between multimorphisms and binary relations over enumerated sets (Orlicki [3]). These connections are presented in the first part of the paper. In the second part the main results of the paper are given. The third part shows us that multimorphisms are not "exotic"—they appear in Recursion Theory.

0 Introduction and preliminaries The fundamental idea of the Theory of Enumerations is the following: computability over arbitrary countable sets is realized as a usual computability over natural numbers by using suitable enumerations of the sets considered. An interesting philosophical interpretation of this idea has been given by Eršov in the introduction to his famous monograph [2]—in his opinion, Theory of Enumeration is a kind of modern version of pythagoreanism. Consequently, we get the well-known concept of a morphism of enumerated sets: Let $\underset{\sim}{S}_i = \langle S_i, \nu_i \rangle$ $(i = 1,2)$ be two (non-empty) enumerated sets and let $\mu : S_1 \rightarrow S_2$ be an arbitrary function. We say that μ is a morphism from $\underset{\sim}{S}_1$ to $\underset{\sim}{S}_2$ iff there is a general recursive function f such that $\mu\nu_1 = \nu_2 f$, i.e., there exists an algorithm that "realizes" μ on the level of codes of elements of sets S_1 and S_2. It is clear that f cannot be an arbitrary general recursive function—it must preserve kernels of suitable enumerations. But we do not assume that these kernels are decidable (it is worth noting that in that case Eršov also gave an interesting philosophical motivation). A mathematical practice gives us many important examples of undecidable enumerations (Gödel enumerations of partial recursive functions, for instance). So, if we consider an arbitrary general recursive function f, it may happen that we do not know whether f induces a morphism of suitable enumerated sets. Nevertheless, we have an algorithm on codes of elements of the considered sets. Does there exist "something computable" defined on elements of these sets which corresponds in a natural way to that algorithm? The concept of multimorphism over enumerated sets, introduced and discussed in the paper, is suggested as a possible answer to this question.

Received August 12, 1992

We make free use of notation and definitions given in Eršov [1] and [2]. Moreover, we assume that the reader is familiar with [3]. Here we shall give only such notation as will be necessary later. Let $R \subseteq A \times B$ be an arbitrary binary relation. For every $a \in A$ by $R(a)$ we denote the set $\{b;\langle a,b\rangle \in R\}$. If R is a partial function then by Arg R and Val R we denote the set of arguments and the set of values of R, respectively. The set of all natural numbers will be denoted by N. We fix a general recursive bijection $\tau : N^2 \to N$. By $\mathcal{C}_1$ we shall denote the set of all partial recursive functions of arity 1. We fix a Gödel enumeration κ of the set $\mathcal{C}_1$. For every natural n we put $\phi_n := \kappa(n)$ and $W_n :=$ Arg ϕ_n. By NSET we shall denote the category of enumerated sets. The enumerated set $\langle N, 1_N\rangle$ will be denoted by $\underset{\sim}{N}$. Let ν_1, ν_2 be two enumerations. By $\nu_1 \oplus \nu_2$ we denote the direct sum of these enumerations. If $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are two non-empty disjoint enumerated sets then by $\underset{\sim}{A} \oplus \underset{\sim}{B}$ we denote the canonical NSET-coproduct of $\underset{\sim}{A}$ and $\underset{\sim}{B}$.

1 Multimorphisms and their connections with relations over NSET Let $\underset{\sim}{S}_i = \langle S_i, \nu_i\rangle$ $(i = 1,2)$ be two non-empty enumerated sets.

Definition 1.1

(1) Let f be an arbitrary general recursive function. We define the binary relation $R_f \subseteq S_1 \times S_2$ putting

$$R_f(a) := \nu_2(f(\nu_1^{-1}(a)))$$

for every $a \in S_1$ (evidently it essentially depends on the enumerations ν_1 and ν_2);

(2) Let $R \subseteq S_1 \times S_2$ be an arbitrary relation. We say that R is a *multimorphism over* $\langle \underset{\sim}{S}_1, \underset{\sim}{S}_2\rangle$ iff there exists a general recursive function f such that $R = R_f$. In this case we say that f is a *realization of R over* $\langle \underset{\sim}{S}_1, \underset{\sim}{S}_2\rangle$. If $\underset{\sim}{S}_1 = \underset{\sim}{S}_2 = \underset{\sim}{S}$ then we say that R is a *multimorphism over* $\underset{\sim}{S}$.

The following observations are immediate:

(a) For every $a \in S_1$ we have $0 < |R_f(a)| \leq |\nu_1^{-1}(a)|$. (Using this observation we easily obtain many examples of binary relations which are not multimorphisms.)

(b) R_f is a morphism of enumerated sets iff f is $\langle \nu_1, \nu_2\rangle$-extensional (i.e., $\nu_1(x) = \nu_1(y)$ implies $\nu_2(f(x)) = \nu_2(f(y))$ for every natural x, y).

It should be stressed that the concept of multimorphisms is not a categorical one. Consider the following.

Example 1.2 Let $\underset{\sim}{S} := \langle N, \nu\rangle$, where the enumeration ν is given by $\nu(\tau(x,y)) := x$ for every natural x, y. Let $R := N \times N$. Evidently R is a multimorphism over $\langle \underset{\sim}{S}, \underset{\sim}{N}\rangle$. Moreover, $1_N : \underset{\sim}{N} \to \underset{\sim}{S}$ is an isomorphism of enumerated sets. But the composition $R \circ 1_N$ is not a multimorphism over $\underset{\sim}{N}$ (because it is not a function). So, a composition of an NSET-isomorphism and a multimorphism need not be a multimorphism. On the other hand: it is easy to see that if R is a multimorphism over $\langle \underset{\sim}{S}_1, \underset{\sim}{S}_2\rangle$ and $\mu : \underset{\sim}{S}_2 \to \underset{\sim}{S}_3$ is an NSET-morphism then the composition $\mu \circ R$ is a multimorphism over $\langle \underset{\sim}{S}_1, \underset{\sim}{S}_3\rangle$.

In [3] the concept of a binary relation over the category NSET has been considered. Our multimorphisms are special binary relations too. So it is quite natural to compare these two concepts.

Proposition 1.3 *Let $\underset{\sim}{S}_0 = \langle S_0, \nu_0 \rangle$ be an arbitrary non-empty enumerated set. The following conditions are equivalent:*
(1) *ν_0 is a positive enumeration;*
(2) *For every non-empty enumerated set $\underset{\sim}{S}$: if R is a multimorphism over $\langle \underset{\sim}{S}_0, \underset{\sim}{S} \rangle$ then R is a relation over $\langle \underset{\sim}{S}_0, \underset{\sim}{S} \rangle$.*

Proof:

(1) $\Rightarrow$ (2) Let g be a general recursive function such that $W_{g(x)} = \nu_0^{-1}(\nu_0(x))$ for every x (it exists because ν_0 is a positive enumeration). Take a non-empty enumerated set $\underset{\sim}{S}$ and let f be an arbitrary general recursive function. Let h be a general recursive function such that $W_{h(x)} = f(W_{g(x)})$ for every natural x. It is easy to see that R_f is a relation over $\langle \underset{\sim}{S}_0, \underset{\sim}{S} \rangle$ via the function h.

(2) $\Rightarrow$ (1) Let $\underset{\sim}{S} := \underset{\sim}{N}$ and $f := 1_N$. Then $R_f = \nu_0^{-1}$, i.e., $R_f(\nu_0(x)) = \{n; \nu_0(n) = \nu_0(x)\}$ for every natural x. Since R_f is a relation over $\langle \underset{\sim}{S}_0, \underset{\sim}{S} \rangle$, there exists a general recursive function h such that $R_f(\nu_0(x)) = W_{h(x)}$ for every natural x. So, we have $W_{h(x)} = \{n; \nu_0(n) = \nu_0(x)\}$ for every natural x. Therefore ν_0 is a positive enumeration.

Remark 1.4 If $\underset{\sim}{S}_0$ is an enumerated set such that ν_0 is not positive then ν_0^{-1} is an example of a multimorphism over $\langle \underset{\sim}{S}_0, \underset{\sim}{N} \rangle$ which is not a relation over $\langle \underset{\sim}{S}_0, \underset{\sim}{N} \rangle$.

Theorem 1.5 *Let $\underset{\sim}{S}_0 = \langle S_0, \nu_0 \rangle$ be an arbitrary non-empty enumerated set such that the following conditions hold:*
(a) *ν_0 is a positive enumeration;*
(b) *$|\nu_0^{-1}(a)| = \aleph_0$ for every $a \in S_0$.*
Let $\underset{\sim}{S} = \langle S, \nu \rangle$ be an arbitrary non-empty enumerated set and let $R \subseteq S_0 \times S$ be a binary relation such that $R(a)$ is a non-empty set for every $a \in S_0$. The following conditions are equivalent:
(1) *R is a multimorphism over $\langle \underset{\sim}{S}_0, \underset{\sim}{S} \rangle$;*
(2) *R is a relation over $\langle \underset{\sim}{S}_0, \underset{\sim}{S} \rangle$.*

Proof (sketch): The implication (1) $\Rightarrow$ (2) follows from Proposition 1.3. We shall prove the inverse implication. Assume that R is a relation over $\langle \underset{\sim}{S}_0, \underset{\sim}{S} \rangle$. Therefore we can choose a general recursive function t such that $\phi_{t(n)}$ is a general recursive function and $R(\nu_0(n)) = \nu(\text{Val } \phi_{t(n)})$ for every natural n. Since ν_0 is positive, we can choose a general recursive function u such that $\phi_{u(n)}$ is a general recursive function and $\text{Val } \phi_{u(n)} = \{y; \nu_0(y) = \nu_0(n)\}$ for every natural n. We shall define some computable sequence $f_0 \subseteq f_1 \subseteq \ldots$ of finite functions. For every x by A_x we shall denote the set of arguments of the function f_x. The sequence is defined step by step using the following procedure:

Step 0: Put $A_0 := \varnothing$ and $f_0 := \varnothing$.

Step $x + 1$: Let n, m be the natural numbers such that $x = \tau(n, m)$. Let $p := \min\{k; \phi_{u(n)}(k) \notin A_x\}$ (it exists because the set $\text{Val } \phi_{u(n)}$ is infinite while A_x is a finite set). Put $a := \phi_{u(n)}(p)$, $b := \phi_{t(n)}(m)$, $A_{x+1} := A_x \cup \{a\}$ and $f_{x+1} := f_x \cup \{\langle a,b\rangle\}$.

Now, let

$$f := \bigcup\{f_x; x \in N\}.$$

It is a matter of easy technical considerations to show that f is a general recursive function and $R = R_f$. So, R is a multimorphism over $\langle \underset{\sim}{S}_0, \underset{\sim}{S}\rangle$.

Relations over NSET have a nice "lifting property"—see [3], Proposition 2.6. Our multimorphisms have the same property, too.

Proposition 1.6 *Let S be a set such that $0 < |S| \leq \aleph_0$ and let $R \subseteq S \times S$ be a relation such that $R(a) \neq \emptyset$ for every $a \in S$. Then for every enumeration ν of the set S there exists an enumeration ν^* such that $\nu \leq \nu^*$ and R is a multimorphism over $\langle S, \nu^*\rangle$.*

Proof: The proof is a modification of the proof of Proposition 2.6 from [3]. Fix a family $\{\omega_n : S \to S; n \in N\}$ of functions such that $R(a) = \{\omega_n(a); n \in N\}$ for every $a \in S$. Let ν be an arbitrary enumeration of the set S. For every natural n let $\nu_n : N \to S$ be the function given by $\nu_n(\tau(k,m)) := \omega_n^k(\nu(m))$ for every natural k,m (of course $\omega_n^0 = 1_N$). We define ν^* putting $\nu^*(\tau(n,x)) := \nu_n(x)$ for every natural n,x. Evidently $\nu \leq \nu^*$. Let f be the general recursive function defined by

$$f(\tau(n,\tau(k,m)) := \tau(n,\tau(k+1,m))$$

for every natural n,k,m. Then R is a multimorphism over $\langle S,\nu^*\rangle$ via the function f.

Similarity between relations over NSET and multimorphisms gives rise to the following notions:

Definition 1.7 Let $\underset{\sim}{S}_i = \langle S_i, \nu_i\rangle$ $(i = 1,2)$ be two non-empty enumerated sets and let $R \subseteq S_1 \times S_2$ be an arbitrary relation. We say that R is a *strong multimorphism over* $\langle \underset{\sim}{S}_1, \underset{\sim}{S}_2\rangle$ iff there exists a general recursive function f such that $\nu_2^{-1}(R(a)) = f(\nu_1^{-1}(a))$ for every $a \in S_1$. In this case we say that f is a *strong realization of R over* $\langle \underset{\sim}{S}_1, \underset{\sim}{S}_2\rangle$. If $\underset{\sim}{S}_1 = \underset{\sim}{S}_2 = \underset{\sim}{S}$ then we say that R is a *strong multimorphism over* $\underset{\sim}{S}$ (compare [3], Definition 2.8). Moreover, if R is a function then we say that R is a *strong morphism from $\underset{\sim}{S}_1$ to $\underset{\sim}{S}_2$*.

Observe that if R is a strong multimorphism over $\langle \underset{\sim}{S}_1, \underset{\sim}{S}_2\rangle$ then $|\nu_2^{-1}(R(a))| \leq |\nu_1^{-1}(a)|$ for every $a \in S_1$ (using this observation we easily obtain many examples of multimorphisms which are not strong).

It is straightforward to see that we have the following proposition.

Proposition 1.8

(1) *The class of all strong multimorphisms is closed under the composition of relations. Moreover, if f is a strong realization of R and g is a strong realization of T then gf is a strong realization of $T \circ R$;*

(2) *Strong morphisms constitute a subcategory of the category* NSET.

Contrary to the concept of strong relations, the definition of strong multimorphisms is not "symmetrical". It follows from the following simple example.

Example 1.9 Let $\underset{\sim}{S}_1 := \langle\{a,b\},\nu_1\rangle$, where $a \neq b$, $\nu_1(0) := a$ and $\nu_1(x+1) := b$ for every natural x. Moreover, let $\underset{\sim}{S}_2 := \langle\{*\},\nu_2\rangle$ be the one-point enumerated set and let $\mu : \underset{\sim}{S}_1 \to \underset{\sim}{S}_2$ be the unique NSET-morphism from $\underset{\sim}{S}_1$ to $\underset{\sim}{S}_2$. Evidently the relation μ^{-1} is a strong multimorphism over $\langle\underset{\sim}{S}_2,\underset{\sim}{S}_1\rangle$. But $(\mu^{-1})^{-1} = \mu$ is not a strong morphism from $\underset{\sim}{S}_1$ to $\underset{\sim}{S}_2$ (because $|\nu_1^{-1}(a)| < |\nu_2^{-1}(\mu(a))|$).

Observe that Proposition 1.3 and Theorem 1.5 remain true if the words "multimorphism" and "relation" are replaced by "strong multimorphism" and "strong relation", respectively. We leave to the reader the easy modification of the appropriate proofs.

As could be expected, for strong multimorphisms the analog of Proposition 1.6 is not true. To prove it, we start from the following proposition.

Proposition 1.10 *If R is a strong multimorphism over $\langle\underset{\sim}{S}_1,\underset{\sim}{S}_2\rangle$ then R^{-1} is a relation over $\langle\underset{\sim}{S}_2,\underset{\sim}{S}_1\rangle$.*

Proof: Let f be a strong realization of R over $\langle\underset{\sim}{S}_1,\underset{\sim}{S}_2\rangle$. Choose a general recursive function g such that $W_{g(x)} = f^{-1}(x)$ for every natural x. It can readily be seen that R^{-1} is a relation over $\langle\underset{\sim}{S}_2,\underset{\sim}{S}_1\rangle$ via the function g.

Remark 1.11 Let $\underset{\sim}{S}_1,\underset{\sim}{S}_2$ and μ be the same as in Example 1.9. Then μ^{-1} is a strong relation over $\langle\underset{\sim}{S}_2,\underset{\sim}{S}_1\rangle$ while μ is not a strong morphism from $\underset{\sim}{S}_1$ to $\underset{\sim}{S}_2$.

Now, we can prove the following theorem.

Theorem 1.12 *There exist a set S with $|S| = \aleph_0$ and a function $\mu : S \to S$ such that for every enumeration ν of the set S the function μ is not a strong morphism from $\langle S,\nu\rangle$ to $\langle S,\nu\rangle$.*

Proof: Let $S := N \cup \{a\} \cup \{b_n; n \in N\}$, where $a, b_0, b_1, \ldots$ are different elements and they are not natural numbers. Let $Z = \{z_0 < z_1 < \ldots\}$ be a fixed not arithmetical set of natural numbers. We define the relation $R \subseteq S \times S$ by $R := \{\langle a,0\rangle,\langle a,a\rangle\} \cup \{\langle n,n+1\rangle; n \in N\} \cup \{\langle z_n,b_n\rangle; n \in N\}$. Put $\mu := R^{-1}$. Evidently μ is a function. Let ν be an arbitrary enumeration of the set S. Repeating the proof of Theorem 2.5 in [3], we obtain that R is not a relation over $\langle S,\nu\rangle$. So, from Proposition 1.10 it follows that μ is not a strong endomorphism of the enumerated set $\langle S,\nu\rangle$.

We end this section with some fundamental information about strong morphisms.

Proposition 1.13 *Let $\mu : \underset{\sim}{S}_1 \to \underset{\sim}{S}_2$ be a strong morphism between two nonempty enumerated sets.*
(1) *If μ is an injection then $\langle\underset{\sim}{S}_1,\mu\rangle$ is an e-subobject of $\underset{\sim}{S}_2$;*
(2) *If μ is a surjection then μ is a factorization.*

Proof: Let $\underset{\sim}{S}_i = \langle S_i,\nu_i\rangle$, $i = 1,2$. Let f be a strong realization of μ.

(1) Assume that μ is an injection. Let $i : \mathrm{Val}\ \mu \to S_2$ be the identity embedding and let $\nu := \mu\nu_1$. It is known that $\langle\underset{\sim}{S}_1,\mu\rangle$ and $\langle\langle\mathrm{Val}\ \mu,\nu\rangle, i\rangle$ are equivalent (as two subobjects of $\underset{\sim}{S}_2$). But from the property of the function f we obtain that $\nu_2^{-1}(\mathrm{Val}\ \mu) = \mathrm{Val} f$ and $\nu = \nu_2 f$. Therefore $\langle\langle\mathrm{Val}\ \mu,\nu\rangle, i\rangle$ is an e-subobject of $\underset{\sim}{S}_2$. So, $\langle\underset{\sim}{S}_1,\mu\rangle$ is an e-subobject of $\underset{\sim}{S}_2$, too.

(2) Assume that μ is a surjection. Denote by Ker μ the kernel of the function μ, i.e., Ker $\mu := \{\langle a,b\rangle; \mu(a) = \mu(b)\}$. Let $\eta : \underset{\sim}{S}_1 \to \underset{\sim}{S}_1/\text{Ker } \mu$ be the canonical factorization induced by the equivalence relation Ker μ. Moreover, let $\alpha : \underset{\sim}{S}_1/\text{Ker } \mu \to \underset{\sim}{S}_2$ be the unique NSET-morphism such that $\mu = \alpha\eta$. From the property of the function f we obtain that $f(N) = f(\nu_1^{-1}(S_1)) = \nu_2^{-1}(\text{Val } \mu) = \nu_2^{-1}(S_2) = N$. Using this observation we easily obtain that α is an isomorphism in the category NSET. Therefore μ is a factorization.

Example 1.14 Let $\underset{\sim}{S} = \langle N,\nu\rangle$ be the same as in Example 1.2. Evidently $1_N : \underset{\sim}{N} \to \underset{\sim}{S}$ is an NSET-isomorphism. Therefore it is a factorization and $\langle \underset{\sim}{N}, 1_N\rangle$ is an e-subobject of $\underset{\sim}{S}$. But 1_N is not a strong morphism from $\underset{\sim}{N}$ to $\underset{\sim}{S}$ (because $|1_N^{-1}(a)| < |\nu^{-1}(1_N(a))|$ for every $a \in N$).

Example 1.15 Let $\underset{\sim}{S} = \langle S,\nu\rangle$ be a non-empty enumerated set, let A be a non-empty denumerable set, and let $\mu : S \to A$ be an arbitrary injection. Take an arbitrary enumeration α_0 of the set $A \setminus \text{Val } \mu$. Put $\alpha := (\mu\nu) \oplus \alpha_0$ and $\underset{\sim}{A} := \langle A,\alpha\rangle$. It is clear that $\mu : \underset{\sim}{S} \to \underset{\sim}{A}$ is a strong morphism.

2 Minimal multimorphisms and selectors Let $\underset{\sim}{S}_i = \langle S_i,\nu_i\rangle$ $(i = 1,2)$ be two non-empty enumerated sets and let R be an arbitrary multimorphism over $\langle \underset{\sim}{S}_1,\underset{\sim}{S}_2\rangle$. By $M(\underset{\sim}{S}_1,\underset{\sim}{S}_2,R)$ we shall denote the set of all multimorphisms T over $\langle \underset{\sim}{S}_1,\underset{\sim}{S}_2\rangle$ such that $T \subseteq R$. Frequently we shall use the same notation to denote the partial order $\langle M(\underset{\sim}{S}_1,\underset{\sim}{S}_2,R),\subseteq\rangle$. If $\underset{\sim}{S}_1 = \underset{\sim}{S}_2 = \underset{\sim}{S}$ then we simply write $M(\underset{\sim}{S},R)$.

Definition 2.1 Let $\mu : \underset{\sim}{S}_1 \to \underset{\sim}{S}_2$ be a morphism in the category NSET. We say that μ is a *selector for R* (abbreviated μ is an *R-selector*) iff $\mu \subseteq R$.

Evidently every R-selector is a minimal element in the partial order $M(\underset{\sim}{S}_1,\underset{\sim}{S}_2,R)$.

We can say that, in a sense, multimorphisms are non-deterministic morphisms. So, a complexity of the structure of the poset $M(\underset{\sim}{S}_1,\underset{\sim}{S}_2,R)$ can be interpreted as a measure of non-determinism of R. From this point of view the problem of existence of minimal elements in $M(\underset{\sim}{S}_1,\underset{\sim}{S}_2,R)$ and, in particular, the problem of existence of selectors are quite natural.

Let $\underset{\sim}{S} = \langle S,\nu\rangle$ be an arbitrary non-empty enumerated set. Evidently the relation ν^{-1} is a strong multimorphism over $\langle \underset{\sim}{S},\underset{\sim}{N}\rangle$ (this observation was already used in the proof of Proposition 1.3). The following two Propositions will be fundamental for our further considerations.

Proposition 2.2 *If T is a minimal element in $M(\underset{\sim}{S},\underset{\sim}{N},\nu^{-1})$ then T is a ν^{-1}-selector.*

Proof: Let T be minimal and suppose that T is not a selector, i.e., T is not a function. Let f be a realization of T over $\langle \underset{\sim}{S},\underset{\sim}{N}\rangle$. Choose $a \in S$ such that $|T(a)| \geq 2$ and fix two different elements m,n from the set $T(a)$. We define the general recursive function g putting for $x \in N$:

$$g(x) := \text{if } f(x) = m \text{ then } n \text{ else } f(x).$$

It is clear that $R_g = T \setminus \{\langle a,m\rangle\}$. Thus T is not minimal, a contradiction.

Proposition 2.3 *The following conditions are equivalent*:
(1) *The enumeration ν is decidable*;
(2) *The morphism $\nu : \underline{N} \to \underline{S}$ is a split epi in* NSET;
(3) *The multimorphism ν^{-1} has a selector*;
(4) *The partial order $M(\underline{S}, \underline{N}, \nu^{-1})$ has a minimal element.*

Proof: The equivalence (1) $\Leftrightarrow$ (2) is obvious. Let $\mu : \underline{S} \to \underline{N}$ be an NSET-morphism. Then μ is a ν^{-1}-selector iff $\nu\mu = 1_S$. So, we have the equivalence (2) $\Leftrightarrow$ (3). From Proposition 2.2 we obtain the equivalence (3) $\Leftrightarrow$ (4).

Lemma 2.4 *Assume that the following objects are given*:
(a) *two non-empty enumerated sets $\underline{A} = \langle A, \alpha \rangle$ and $\underline{B} = \langle B, \beta \rangle$ such that $A \cap B = \varnothing$*;
(b) *a multimorphism R over $\langle \underline{A}, \underline{B} \rangle$*;
(c) *an* NSET*-morphism $\mu : \underline{B} \to \underline{B}$.*
Consider the relation $R \cup \mu \subseteq (A \cup B) \times (A \cup B)$. We have:
(1) *$R \cup \mu$ is a multimorphism over $\underline{A} \oplus \underline{B}$*;
(2) *If R and μ are strong then $R \cup \mu$ is also strong*;
(3) *The partial orders $M(\underline{A}, \underline{B}, R)$ and $M(\underline{A} \oplus \underline{B}, R \cup \mu)$ are isomorphic. Moreover, the suitable isomorphism preserves and reflects selectors.*

Proof: Let f, g be realizations of R and μ, respectively. We define the general recursive function h by $h(2x) := 2f(x) + 1$ and $h(2x + 1) := 2g(x) + 1$ for every natural x. It is easy to see that h is a realization of $R \cup \mu$ over $A \oplus \underline{B}$. Moreover, if f and g are strong realizations then h is a strong realization, too. Now, we define the function

$$\xi : M(\underline{A}, \underline{B}, R) \to M(\underline{A} \oplus \underline{B}, R \cup \mu)$$

putting $\xi(T) := T \cup \mu$ for every $T \in M(\underline{A}, \underline{B}, R)$. It is not difficult to show that ξ is well-defined and that the following statements are true:

(i) ξ is a bijection (we have $\xi^{-1}(P) := P \backslash \mu$ for every suitable P);
(ii) ξ is an isomorphism of suitable partially ordered sets;
(iii) ξ preserves and reflects selectors.

We omit technical details of the proof. However, one remark must be given here: the conditions "$A \cap B = \varnothing$" and "μ is a function" are important for the proof.

Theorem 2.5 *Let S be a set such that $|S| = \aleph_0$. There exist $2^{\aleph_0}$ enumerations ν of the set S with the following property*: *There exist $\aleph_0$ strong multimorphisms R over $\langle S, \nu \rangle$ for which the poset $M(\langle S, \nu \rangle, R)$ has no minimal elements.*

Proof: Without loss of generality we can assume that $S = S_0 \cup N$ where $|S_0| \geq 2$ and $S_0 \cap N = \varnothing$. Let ν_0 be an arbitrary undecidable enumeration of the set S_0. Put $\nu := \nu_0 \oplus 1_N$. Take an arbitrary general recursive function f. Put $R := \nu_0^{-1} \cup f$. From Lemma 2.4 we obtain that R is a strong multimorphism over $\langle S, \nu \rangle$ and that the posets $M(\langle S_0, \nu_0 \rangle, \underline{N}, \nu_0^{-1})$ and $M(\langle S, \nu \rangle, R)$ are isomorphic. But $M(\langle S_0, \nu_0 \rangle, \underline{N}, \nu_0^{-1})$ has no minimal elements (see Proposition 2.3). Therefore $M(\langle S, \nu \rangle, R)$ has no minimal elements either.

Lemma 2.6 *Let S_0 be a set such that $|S_0| = \aleph_0$. There exist $2^{\aleph_0}$ enumerations ν_0 of the set S_0 with the following property: There exists a strong multimorphism P over $\langle\langle S_0, \nu_0\rangle, \underline{N}\rangle$ for which the following conditions hold:*
(1) P has a selector;
(2) The poset $M(\langle\langle S_0, \nu_0\rangle, \underline{N}, P)$ has a minimal element which is not a selector.

Proof: Without loss of generality we can assume that $S_0 = \{a\} \cup N$, where $a \notin N$. Take an arbitrary non-recursive set A of natural numbers. We define the enumeration ν_0 putting

$$\nu_0(\tau(n,x)) := \text{if } x \in A \text{ then } n \text{ else } a$$

for every natural n,x. Let $T := (\{a\} \times N) \cup 1_N$. It is easy to see that T is a strong multimorphism over $\langle\langle S_0, \nu_0\rangle, \underline{N}\rangle$ via the general recursive function f given by $f(\tau(n,x)) := n$ for every natural n. We shall prove that

$$(*) \qquad M(\langle\langle S_0 \nu_0\rangle, \underline{N}, T) = \{T\}.$$

Suppose that T_0 is an element of $M(\langle\langle S_0, \nu_0\rangle, \underline{N}, T)$ and $T_0 \neq T$. Then there exists a set N_0 of natural numbers such that $\varnothing \neq N_0 \neq N$ and $T_0 = \{\langle a, n\rangle;\ n \in N_0\} \cup 1_N$. Fix an arbitrary natural number m_0 from the set $N \backslash N_0$. Let g be a realization of T_0 over $\langle\langle S_0, \nu_0\rangle, \underline{N}\rangle$. Using the definition of the enumeration ν_0, the property of the number m_0 and the property of the function g we obtain that for every natural x the following equivalence holds: $x \in A$ iff $g(\tau(m_0, x)) = m_0$. Thus A is a recursive set, a contradiction. So, the observation (*) is proved. Now, let $\mu : \langle S_0, \nu_0\rangle \to \underline{N}$ be the morphism of enumerated sets given by $\mu(x) := 0$ for every $x \in S_0$. Put $P := T \cup \mu$. Fix a natural number t from the set A and let h be the general recursive function given by

$$h(\tau(n,x)) := \text{if } x = t \text{ then } 0 \text{ else } n$$

for every natural n,x. It is easy to see that h is a strong realization of P over $\langle\langle S_0, \nu_0\rangle, \underline{N}\rangle$. Evidently μ is a P-selector. From the observation (*) we obtain that T is a minimal element in $M(\langle\langle S_0, \nu_0\rangle, \underline{N}, P)$. Evidently T is not a selector (it is not a function).

Theorem 2.7 *Let S be a set such that $|S| = \aleph_0$. There exist $2^{\aleph_0}$ enumerations ν of the set S with the following property: There exist $\aleph_0$ strong multimorphisms R over $\langle S, \nu\rangle$ for which the following conditions hold:*
(1) R has a selector;
(2) The poset $M(\langle\langle S, \nu\rangle, R)$ has a minimal element which is not a selector.

Proof: Without loss of generality we can assume that $S = S_0 \cup N$ where $|S_0| = \aleph_0$ and $S_0 \cap N = \varnothing$. Let ν_0, P be an enumeration of the set S_0 and a strong multimorphism over $\langle\langle S_0, \nu_0\rangle, \underline{N}\rangle$ obtained by applying Lemma 2.6 to our set S_0. Let $\nu := \nu_0 \oplus 1_N$. Take an arbitrary general recursive function f and put $R := P \cup f$. Using Lemma 2.4 it is not difficult to show that R has the required properties.

Proposition 2.8 *Let $\underline{S} = \langle S, \nu\rangle$ be a non-empty enumerated set such that ν is a decidable enumeration. Then every multimorphism over $\underline{S}$ has a selector.*

Proof: Let f be the general recursive function defined by $f(x) := \min\{n; \nu(n) = \nu(x)\}$ for every natural x. Then for every general recursive function g the function $h := gf$ is ν-extensional and $\mu := R_h$ is an R_g-selector.

The methods from the proofs of Theorems 2.5 and 2.7 have "constructive counterparts" in the case $S := \mathfrak{C}_1$. For example, we shall give a sketch of the proof of the constructive analog of Theorem 2.7.

Proposition 2.9 *There exist $\aleph_0$ computable enumerations ν of the set $\mathfrak{C}_1$ with the following property: There exist $\aleph_0$ strong multimorphisms R over $\langle \mathfrak{C}_1, \nu \rangle$ for which the following conditions hold:*
(1) *R has a selector;*
(2) *The poset $M(\langle\langle \mathfrak{C}_1, \nu\rangle, R)$ has a minimal element which is not a selector.*

Proof (sketch): Fix two non-empty enumerated sets $\langle F_0, \gamma_0 \rangle$ and $\langle F, \gamma \rangle$ such that the following conditions hold:
(i) $F_0 \cup F = \mathfrak{C}_1 \setminus \{\varnothing\}$ and $F_0 \cap F = \varnothing$;
(ii) γ_0 and γ are computable bijections.

(a) Take an arbitrary recursively enumerable non-recursive set A of natural numbers. We define the computable enumeration ν_0 of the set $F_0 \cup \{\varnothing\}$ putting

$$\nu_0(\tau(n,x)) := \text{if } x \in A \text{ then } \gamma_0(n) \text{ else } \varnothing$$

for every natural n,x. Let $T := (\{\varnothing\} \times F) \cup (\gamma\gamma_0^{-1})$ and $P := T \cup (F_0 \times \{\gamma(0)\})$. Then T and P are strong multimorphisms over $\langle\langle F_0 \cup \{\varnothing\}, \nu_0\rangle, \langle F, \gamma\rangle\rangle$.

(b) Let ν be the computable enumeration of the set $\mathfrak{C}_1$ defined by $\nu := \nu_0 \oplus \gamma$. Take an arbitrary general recursive function f and put $R := P \cup (\gamma f \gamma^{-1})$. Then R has the required properties.

Observe that part (a) of our proof is a counterpart of the proof of Lemma 2.6 while part (b) is a counterpart of the proof of Theorem 2.7.

For the important enumerated set $\langle \mathfrak{C}_1, \kappa \rangle$ we can prove the following weak analog of Theorem 2.5:

Proposition 2.10 *There exists a multimorphism R over $\langle \mathfrak{C}_1, \kappa \rangle$ such that the poset $M(\langle\langle \mathfrak{C}_1, \kappa\rangle, R)$ has no minimal elements.*

Proof: Let u be a general recursive function such that $\phi_{u(x)}(n) = x$ for every natural x, n. We define the relation $R \subseteq \mathfrak{C}_1 \times \mathfrak{C}_1$ by $R := \{\langle \phi_x, \phi_{u(x)} \rangle; x \in N\}$. Evidently R is a multimorphism over $\langle \mathfrak{C}_1, \kappa \rangle$ and u is its realization. Let $T \in M(\langle\langle \mathfrak{C}_1, \kappa\rangle, \underline{N}, \kappa^{-1})$. We define the relation $T^* \subseteq \mathfrak{C}_1 \times \mathfrak{C}_1$ putting $T^*(\psi) := \{\phi_{u(x)}; x \in T(\psi)\}$ for every partial recursive function ψ. It is easy to see that T^* is an element of $M(\langle\langle \mathfrak{C}_1, \kappa\rangle, R)$ (if f is a realization of T then the composition uf is a realization of T^*).

Now, let $P \in M(\langle\langle \mathfrak{C}_1, \kappa\rangle, R)$. We define the relation $P^0 \subseteq \mathfrak{C}_1 \times N$ putting $P^0(\psi) := \{x; \phi_{u(x)} \in P(\psi)\}$ for every partial recursive function ψ. Let g be a realization of P over $\langle \mathfrak{C}_1, \kappa \rangle$ and let h be the partial recursive function given by $h(x) := \phi_{g(x)}(0)$ for every natural x. Since $P \subseteq R$, the function h is total. Moreover, h is a realization of P^0 over $\langle\langle \mathcal{C}_1, \kappa\rangle, \underline{N}\rangle$. Using the above considerations we can define the function

$$\xi : M(\langle \mathcal{C}_1, \kappa\rangle, \underline{N}, \kappa^{-1}) \to M(\langle \mathcal{C}_1, \kappa\rangle, R)$$

putting $\xi(T) := T^*$ for every suitable T. It is clear that ξ is a bijection (we have $\xi^{-1}(P) := P^0$ for every suitable P) and that it is an isomorphism of the suitable partial orders. So, from Proposition 2.3 we obtain that $M(\langle \mathcal{C}_1, \kappa\rangle, R)$ has no minimal elements.

3 ***Some important examples*** In this section we shall give three examples of multimorphisms which are related to some fundamental and important constructions from Recursion Theory. These multimorphisms have no selectors. So, our examples are non-trivial. In our opinion, these examples give us a good motivation to consider the concept of multimorphisms over enumerated sets.

Throughout this section f_0 will denote a fixed general recursive function such that

$$\mathrm{Val}\, f_0 = \{x;\ \phi_x \text{ is a non-empty function}\}.$$

By κ_0 we shall denote the computable enumeration of the set $\mathcal{C}_1 \setminus \{\emptyset\}$ defined by $\kappa_0 := \kappa f_0$. Moreover, let K be a fixed recursively enumerable not recursive set of natural numbers.

Let p be a partial recursive function such that for every natural number x the following assertion holds: if W_x is non-empty then $x \in \mathrm{Arg}\, p$ and $p(x) \in W_x$. Put $t := pf_0$. Evidently t is a general recursive function. Denote by E_1 the multimorphism over $\langle\langle \mathcal{C}_1 \setminus \{\emptyset\}, \kappa_0\rangle, \underline{N}\rangle$ induced by t, i.e., $E_1 := R_t$.

Proposition 3.1 *The multimorphism E_1 has no selectors.*

Proof: Let $\beta : N^3 \to N$ be a general recursive function such that for every natural numbers x, m, n the following conditions hold:

(i) if $x \in K$ then $\phi_{\beta(x,m,n)} = \{\langle m,1\rangle, \langle n,1\rangle\}$;
(ii) if $x \notin K$ then $\phi_{\beta(x,m,n)} = \{\langle m,1\rangle\}$

(we consider functions of arity 1 as sets of ordered pairs). Thus $\phi_{\beta(x,m,n)}$ is a finite non-empty function for every natural x, m, n. Therefore we can define the general recursive function $h : N^3 \to N$ putting $h(x,m,n) := \min\{z; f_0(z) = \beta(x,m,n)\}$ for every x, m, n.

Now, suppose that $\mu : \langle \mathcal{C}_1 \setminus \{\emptyset\}, \kappa_0\rangle \to \underline{N}$ is an E_1-selector. Then $\mu\kappa_0$ is a general recursive function and $\mu(\psi) \in \mathrm{Arg}\, \psi$ for every non-empty partial recursive function ψ. Consider the following two cases:

Case 0: $\mu(\{\langle 0,1\rangle, \langle 1,1\rangle\}) = 0$.

Then for every natural number x the following equivalence holds:

$$x \in K \text{ iff } \mu\kappa_0 h(x,1,0) = 0;$$

Case 1: $\mu(\{\langle 0,1\rangle, \langle 1,1\rangle\}) = 1$.

Then for every natural number x the following equivalence holds:

$$x \in K \text{ iff } \mu\kappa_0 h(x,0,1) = 1.$$

Thus in the both cases we obtain that K is a recursive set, a contradiction. So, E_1 has no selectors.

Now, let g be a general recursive function such that for every natural number x. Val $\phi_{g(x)} = W_x$ and if W_x is non-empty then $\phi_{g(x)}$ is a general recursive function. Denote by E_2 the multimorphism over $\langle \mathcal{C}_1, \kappa \rangle$ induced by g, i.e., $E_2 := R_g$.

Proposition 3.2 *The multimorphism E_2 has no selectors.*

Proof: Suppose that $\mu : \langle \mathcal{C}_1, \kappa \rangle \to \langle \mathcal{C}_1, \kappa \rangle$ is an E_2-selector and let h be a general recursive function such that $\mu\kappa = \kappa h$. Observe that for every natural x the function $\mu(\kappa_0(x))$ is total, $\mu(\kappa_0(x)) = \phi_{hf_0(x)}$ and Val $\mu(\kappa_0(x)) = $ Arg $\kappa_0(x)$. Therefore, if we put

$$\mu^*(\kappa_0(x)) := (\mu(\kappa_0(x)))(0)$$

then we obtain the NSET-morphism $\mu^* : \langle \mathcal{C}_1 \setminus \{\emptyset\}, \kappa_0 \rangle \to \underline{N}$ which is an E_1-selector, a contradiction. So, E_2 has no selectors.

Our third example is strictly related to the *s-m-n*-theorem. Denote by $\mathcal{C}_2$ the set of all partial recursive functions of arity 2. By λ we shall denote a fixed Gödel enumeration of the set $\mathcal{C}_2$. Let s be a general recursive function such that for every natural number x the following conditions hold:

(1) $\phi_{s(x)}$ is a general recursive function;
(2) $\lambda(x)(a,-) = \phi_{\phi_{s(x)}(a)}$ for every natural a.

Denote by E_3 the multimorphism over $\langle \langle \mathcal{C}_2, \lambda \rangle, \langle \mathcal{C}_1, \kappa \rangle \rangle$ induced by s, i.e., $E_3 := R_s$.

Proposition 3.3 *The multimorphism E_3 has no selectors.*

Proof: Suppose that $\mu : \langle \mathcal{C}_2, \lambda \rangle \to \langle \mathcal{C}_1, \kappa \rangle$ is an E_3-selector and let h be a general recursive function such that $\mu\lambda = \kappa h$. So, for every $\gamma \in \mathcal{C}_2$ and every $a \in N$ we have $\gamma(a,-) = \phi_{\mu(\gamma)(a)}$. Let $c : N^2 \to N$ be a fixed general recursive function. Choose a general recursive function f such that

$$\lambda(f(x)) = \text{if } x \in K \text{ then } c \text{ else } \emptyset$$

for every natural x. Observe that:

(a) if $x \in K$ then $\phi_{hf(x)} = \mu(c)$;
(b) if $x \notin K$ then $\phi_{hf(x)} = \mu(\emptyset)$.

Take an arbitrary natural number a. We have:

$$\phi_{\mu(c)(a)} = c(a,-) \neq \emptyset(a,-) = \phi_{\mu(\emptyset)(a)}.$$

So, $\mu(c)(a) \neq \mu(\emptyset)(a)$ for every natural a. Using this observation and the above assertions (a) and (b) we obtain that for every natural number x the following equivalence holds:

$$x \in K \text{ iff } \phi_{hf(x)}(0) = \mu(c)(0).$$

Therefore K is a recursive set, a contradiction. So, E_3 has no selectors.

REFERENCES

[1] Eršov, J., "Theorie der Numerierungen I," *Zietschrift für Mathematische Logik und Grundlagen der Mathematik*, vol. 19 (1973), pp. 289–388.

[2] Eršov, J., *Teorija Numeracij* (in Russian), Nauka, Moscow, 1977.

[3] Orlicki, A., "Binary relations over the category of enumerated sets," *Zietschrift für Mathematische Logik und Grundlagen der Mathematik*, vol. 34 (1988), pp. 265–276.

Instytut Matematyki UMK
ul. Chopina 12/18
87-100 Toruń
Poland