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Minimal Doxastic Logic: Probabilistic and Other Completeness Theorems

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Abstract The propositional doxastic logics investigated here are minimal in the sense that they make very limited claims about what holds good of rational or justified belief on the basis of the meaning of those terms. Indeed some of the logics allow for the truth of total scepticism, the view that there are no rational or justified beliefs. The logics are subject to constraints such as that any doxastic logic must be believable in its own terms and that any proposition which must, according to a doxastic logic, be believed (not believed) must itself be a theorem of (be refuted by) the logic. Two techniques are used to establish completeness, one employing possible-worlds models in which there may be several or no accessibility relations, the other using probability distributions and a maximal-probability conception of belief.

0 Introduction The "minimal" of the title signals a distrust of doxastic logics, logics of belief, that generate what might better be thought of as substantial theories of belief. There are two sources of this distrust. First, on an objectivist reading, the logic of justified belief, which ought to contain no more than the uncontroversial beginnings of an analysis of justified belief, should not give rise to theorems that run counter to coherent philosophical theses concerning the justifiability or otherwise of our beliefs. Second, on a subjectivist reading, the logic of rationally held beliefs ought not to ascribe whole classes of at best implicit beliefs to a rational agent, especially beliefs about beliefs that would in effect impute a great deal of self-knowledge, for it is by no means clear that even an ideally rational agent's beliefs are or ought to be transparent to the agent. This is especially so on the model of what it is to hold a belief that I shall adopt, namely Robert Stalnaker's 'pragmatic' conception of beliefs as conditional dispositions to action ([14], pp. 59–77). (It is a common-place that beliefs are related to action.)

The logics—propositional logics—to be investigated below are, therefore, weak. But weakness has its own strengths. One desideratum for weak doxastic

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logics is that, since they do present a minimal analysis of either justified belief or the beliefs of an ideally rational agent, their theorems ought, other things being equal, to be justifiedly/rationally believed on the grounds that they offer little more than an explication or formalization of the concept of justified/rational belief. Certainly, if there is such an animal as a doxastic logic that exists in the common ground between rival epistemologies then there are theses concerning justified/rational belief that, other things being equal, one is justified in believing and/or which a rational individual rationally believes. The qualification 'other things being equal' is necessary because we do not at this stage wish to rule out the possibility of total scepticism, i.e., the thesis that no beliefs are justified/rational.

On the technical plane one of the main novelties introduced is the use of probability theory in supplying soundness and completeness theorems for doxastic logics, notably where higher-order beliefs are involved. Of prime concern is the definition of belief in terms of maximal probability. This is the notion that, in its simplest form, Wolfgang Lenzen calls both strong belief and conviction ([9], pp. 35–41). The idea of employing a probabilistic criterion for the acceptance of the conclusions of inductive inferences or, more straightforwardly, of defining unqualified belief in probabilistic terms, is hardly novel. A number of authors, for diverse reasons, have discussed a high-probability criterion. I have elsewhere given some reasons for preferring the maximal-probability definition to its high-probability analogue (see [12]). Suffice it to say here that whatever the merits, real or imagined, of the high-probability definition/criterion only belief as maximal probability is considered in this paper.

Other completeness theorems are given using what I call weak standard models, possible-worlds models in which there may be several accessibility relations or none at all. The weak standard models and the probabilistic models are to a limited extent complementary, accommodating different weaknesses in the doxastic logics considered. Beyond a certain, still weak, logic, however, both types of model can serve.

For reasons of convenience as much as doctrine, I shall take the objects of belief to be propositions, not sentences of a language. For doctrinal support I would call upon two authors who have researched the dynamics of belief (in contrast to the present topic which might be termed the *statics* of belief). According to Peter Forrest:

> [A] person's having a doxastic attitude is to be analyzed as a relation between the person and the object of belief, which I call a *proposition*. I use the term proposition here [. . .] in its Cambridge sense, that is, in the way that Moore, Russell and Johnson used it. In this Cambridge sense a proposition just is the object of the belief, whatever the object turns out to be. [. . .] [P]ropositions have sentence-*like* structure. ([3], p. 11)

While Peter Gärdenfors affirms:

> The proper objects of logic are not sentences but the contents of sentences, that is, *propositions*. ([4), p. 131)

Throughout what follows, $\underline{B} = \langle B, \&, \vee, \neg \rangle$ is a boolean algebra of propositions formed from atomic propositions by means of the logical connectives

&, ∨, ¬; → *and* ≡ *are understood to be defined in the usual ways, namely,* $a \to b =_{df} \neg a \vee b$ and $a \equiv b =_{df} (a \to b) \,\&\, (b \to a)$. ⊤ is the tautologous proposition, ⊥ the absurd proposition. $At(B)$ designates the set of atomic propositions belonging to B. B contains, for every proposition a, a proposition $\boldsymbol{B}a$, formed using the belief operator $\boldsymbol{B}$. Propositions in $\underline{B} = \langle B, \&, \vee, \neg \rangle$ are identified under equivalence in classical propositional logic; propositions of the form $\boldsymbol{B}a$ are treated as atomic *insofar as the structure of* $\underline{B}$ is concerned (but they are not elements of $At(B)$). Unsurprisingly given this foundation, all the logics considered below are extensions of classical propositional logic, designated **PL**.[1]

Whereas it is commonly assumed that the sentences of a modal propositional language are countably infinite in number (see, e.g., Chellas [2], pp. 25–7; cf. [3], pp. 23, 30), we can, at least some of the time and at the price of added complexity, refrain from any assumption concerning the cardinality of the class of propositions, thought by some authors to be very large—'beth-three, on the lowest reasonable estimate' (Lewis [11], p. 107). The added complexity enters because some probabilistic results are not known to generalize to the uncountable case. The completeness theorems of Sections 1 and 3 place no limit on the number of propositions; in Section 4 we assume countability.

The second innovation comes with the deployment of two adequacy criteria for doxastic logics—in addition to the desideratum stated above—whose combined effect narrows the class of acceptable doxastic logics considerably. The first is that any acceptable doxastic logic ought to be believable in its own terms. That is, it should neither be the case that one cannot justifiedly believe all theorems of an uncontroversial preliminary analysis of justified belief nor that an ideally rational agent cannot rationally believe all theorems of an analysis of the principles governing rational belief. For if we find such an analysis acceptable then we provably cannot justifiedly believe the results of our enquiry and/or we are provably not ideally rational. Formally, this requirement amounts to the adequacy condition that a doxastic logic is acceptable only if the logic remains consistent when augmented by the Gödel rule of necessitation, denoted RN below:

if a is a theorem then so is $\boldsymbol{B}a$.

The second adequacy condition is, perhaps, more controversial. Associated with any doxastic logic **DL** there is what I shall call the *believed doxastic logic*. This is the set of propositions $\{a \in B\colon \boldsymbol{B}a$ is a theorem of $\mathbf{DL}\}$. These are the propositions that one is justified in believing and/or that a rational individual must rationally believe purely on the basis of the analysis of justified/rational belief presented by **DL**. The first clause of the second adequacy condition, the condition of positive accuracy, is that the believed doxastic logic associated with an acceptable doxastic logic must be contained in the logic, i.e.,

$$\{a \in B\colon \vdash_{\mathbf{DL}} \boldsymbol{B}a\} \subseteq \{a \in B\colon \vdash_{\mathbf{DL}} a\}.$$

If this condition is not satisfied then even a supposedly minimal doxastic logic yields justified beliefs that it does not justify and/or commits a rational individual to believing propositions that are, in the logic's own terms, at best accidentally true, i.e., they are not *a priori* truths. Reflection on this fact, if it is possible, would lead to the conclusion that the logic in question is lacking; it would be regarded as incomplete. (And what does it say about our rationality if we knowingly endorse a positively inaccurate doxastic logic as the logic of justified/rational belief?)

Analogous to positive accuracy there is negative accuracy:

$$\{a \in B\colon \vdash_{\mathbf{DL}} \neg \boldsymbol{B}a\} \subseteq \{a \in B\colon \vdash_{\mathbf{DL}} \neg a\}.$$

It is similarly justified. A doxastic logic that is both positively and negatively accurate will be called accurate.

1 Total scepticism and a minimal doxastic logic On the view of beliefs as conditional dispositions to action two elementary principles emerge immediately: the rule RL and the axiom schema M.[2]

(RL) if $a \equiv b$ is a theorem of **PL** then $\boldsymbol{B}a \equiv \boldsymbol{B}b$ is a theorem;
(M) $\boldsymbol{B}(a \,\&\, b) \rightarrow (\boldsymbol{B}a \,\&\, \boldsymbol{B}b)$.

One cannot be disposed to act as if a conjunction is true without being disposed to act as if its conjuncts are true, and no distinction in dispositions to act can be drawn between logically equivalent propositions (cf. [14], pp. 82–3). These considerations give us the minimal doxastic logic that I shall denote by **LM**: the extension of **PL** with rule RL and axiom schema M. Equivalently, **LM** can be formalized by appending to **PL** just this rule:

if $a \rightarrow b$ is a theorem of **PL** then $\boldsymbol{B}a \rightarrow \boldsymbol{B}b$ is a theorem.

Algebraic soundness and completeness proofs for **LM** are readily produced by means of an obvious generalization of Chellas's algebraic models ([2], pp. 212–3), which are in turn a generalization of Hughes and Cresswell's T-algebras ([8], p. 315–8).

Definition 1.1 The sextuple $\langle B', \wedge, \vee, {}^{c}, {}^{*}, F\rangle$ is an augmented boolean algebra when $\underline{B'} = \langle B', \wedge, \vee, {}^{c}\rangle$ is a boolean algebra, F is a filter in $\underline{B'}$, and the function ${}^{*}\colon B' \rightarrow B'$ satisfies the condition:

$$\text{when } a \leq' b \text{ then } ({}^{*}a)^{c} \vee {}^{*}b \in F,$$

$\leq'$ being the partial order on B' defined by:

$$\forall a, b \in B' \ [a \leq' b \text{ iff } a \wedge b = a].$$

$\mathbf{1}_{\underline{B'}}$ and $\mathbf{0}_{\underline{B'}}$ are, respectively, the maximal and minimal elements of $\underline{B'}$, under this partial order. F is a filter in $\underline{B'}$ just in case F is non-empty and:

$$\mathbf{0}_{\underline{B'}} \notin F;$$

$$\forall a, b \in B' \ [a \in F \text{ and } a \leq' b \Rightarrow b \in F];$$

$$\forall a, b \in B' \ [a \in F \text{ and } b \in F \Rightarrow a \wedge b \in F].$$

Definition 1.2 A valuation V on $\underline{B}$, the algebra of propositions, is a homomorphism of boolean algebras from $\underline{B}$ to an augmented boolean algebra $\underline{B'} = \langle B', \wedge, \vee, {}^{c}, {}^{*}, F\rangle$ satisfying the additional constraint that

$$V(\boldsymbol{B}a) = {}^{*}V(a).$$

It can be shown that **LM** is sound and complete relative to augmented boolean algebras in the sense that under every valuation V mapping $\underline{B}$ to an augmented boolean algebra $\underline{B'} = \langle B', \wedge, \vee, ^c, ^*, F\rangle$ the image of every theorem of **LM** belongs to F and there is a characteristic augmented boolean algebra $\underline{B'} = \langle B', \wedge, \vee, ^c, ^*, F\rangle$ and a valuation V such that $V(a) \in F$ just in case a is a theorem of **LM**. This result, however, has no immediate intuitive significance. What does it tell us about the logic **LM** as a logic of belief? Nothing, to all appearances. But perhaps this is no great loss, for **LM** is manifestly too weak. Why? Because it contains only part of the concept of justified/rational belief, and if what it maintains has indeed this lowly status it ought to do no more than lay bare what is implicit in the meaning of the terms justified and/or rational belief. Hence, if, according to **LM**, the propositions a and b are equivalent, then this is so solely in virtue of the concept of justified/rational belief and any two propositions about belief that differ only in that one contains a and the other b ought to be shown to be equivalent in **DL**, our minimal doxastic logic. This would be accomplished by the rule:

if $a \equiv b$ is a theorem of **LM** then $\boldsymbol{B}a \equiv \boldsymbol{B}b$ is a theorem of **DL**.

However, the same can be said of any propositions shown to be equivalent in this logic. Closing off the regress leads us to the rule:

(RE) if $a \equiv b$ is a theorem then $\boldsymbol{B}a \equiv \boldsymbol{B}b$ is a theorem.

Following Chellas ([2]), the logic that extends **PL** by adding the rule RE and the axiom schema M is called **EM**. Analogously to **LM** it can be formalized by adding to **PL** the single rule:

(RM) if $a \rightarrow b$ is a theorem then $\boldsymbol{B}a \rightarrow \boldsymbol{B}b$ is a theorem.

We obtain a natural possible-worlds semantics for **EM** when we marry two sets of ideas. Jaakko Hintikka proposed possible-worlds semantics for belief in which, given a single accessibility relation, world w_1 is possible relative to/accessible from world w_2 if and only if the propositions true in w_1 are jointly compatible with all beliefs held in w_2, i.e., with the totality of propositions a such that $\boldsymbol{B}a$ is true in w_2. Effectively, then, w_1 is a world that an agent who held the beliefs of w_2 would consider doxastically possible, i.e., not contrary to her beliefs. Viewed this way, what is remarkable about the construction is that the agent deploys *all* of her beliefs. It is precisely this feature that Stalnaker's analysis of belief challenges; for, calling what Hintikka models a belief state, Stalnaker maintains that in order to describe the totality of an agent's beliefs we may require not one but several belief states. In other words, the Hintikka-model imposes an unargued unity on beliefs. As previously noted, beliefs are, Stalnaker supposes, conditional dispositions to action; so viewed they depend on contexts for their manifestation; according to Stalnaker, an individual agent may simultaneously be in incompatible belief states, the different belief states being used to explain her behavior in different contexts. (For more details see [14], p. 83.) The first step in providing models for **EM**—a natural generalization of the Hintikka-model to encompass Stalnaker's view of belief—is to allow a plurality, $\mathfrak{R}$, of accessibility relations.

Transparently, **EM** does not entail any theorems of the form $\boldsymbol{B}a$; hence our semantics for it must make provision for worlds in which no beliefs are held. This is the function of the 'queer' worlds (cf. [2], p. 75) in models for **EM**, worlds in which

$$\forall s \in T \, \forall R \in \mathfrak{R} \, [\text{not } tRs].$$

Definition 1.3 The ordered triple $\langle T, \mathfrak{R}, u\rangle$ is an **EM**-model if and only if T is a non-empty set, $\mathfrak{R}$ is a non-empty set of binary relations defined on T (i.e., $\forall R \in \mathfrak{R} \, [R \subseteq T^2]$) and u is a function from elements of T—worlds—and atomic propositions to truth-values (i.e., $u: T \times At(B) \to \{0,1\}$). We define the set $Q = \{t \in T: \forall s \in T \, \forall R \in \mathfrak{R} \, [\text{not } tRs]\}$, where tRs abbreviates $\langle t,s\rangle \in R$.

Given an **EM**-model $\langle T, \mathfrak{R}, u\rangle$, the function $w: T \times B \to \{0,1\}$ is defined by these conditions:

(i) $\forall a \in At(B) \, [w_t(a) = u_t(a)]$;
(ii) $w_t(\neg a) = 1 - w_t(a)$;
(iii) $w_t(a \,\&\, b) = \min\{w_t(a), w_t(b)\}$;
(iv) $w_t(a \vee b) = \max\{w_t(a), w_t(b)\}$;
(v) $w_t(\boldsymbol{B}a) = 1$ iff $t \notin Q$ and $\exists R \in \mathfrak{R} \, \forall s \in T \, [tRs \Rightarrow w_s(a) = 1]$.

Lemma 1.4 (Soundness of **EM**) *If a is a theorem of* **EM** *then $w_t(a) = 1$ for all $t \in T$ in the* **EM**-*model* $\langle T, \mathfrak{R}, u\rangle$.

Proof: Proof proceeds by induction on formal derivations in **EM**. Obviously, if a is a tautology then $w_t(a) = 1$, and if $w_t(a_1) = w_t(a_2) = \ldots = w_t(a_n) = 1$ and b is a consequence of $a_1, a_2, \ldots, a_n$ in **PL** then $w_t\,(b) = 1$.

If $\forall t \in T \, w_t(a \to b) = 1$ then $\forall t \in T \, [w_t(a) \leq w_t(b)]$. Hence, when $\exists R \in \mathfrak{R} \, \forall s \in T \, [tRs \Rightarrow w_s(a) = 1]$ it follows that $\exists R \in \mathfrak{R} \, \forall s \in T \, [tRs \Rightarrow w_s(b) = 1]$, and so, for all $t \in T - Q$, $w_t(\boldsymbol{B}a) = 1$ only if $w_t(\boldsymbol{B}b) = 1$, *i.e.*, $w_t(\boldsymbol{B}a \to \boldsymbol{B}b) = 1$. On the other hand, $\forall t \in Q$, $w_t(\boldsymbol{B}a) = 0$ and so $w_t(\boldsymbol{B}a \to \boldsymbol{B}b) = 1$. Hence, $\forall t \in T \, w_t(\boldsymbol{B}a \to \boldsymbol{B}b) = 1$.

Definition 1.5 Let $F = \{a \in B: a \text{ is a theorem of } \mathbf{EM}\}$. Let $T = \{U \subseteq B: U \text{ is an ultrafilter and } F \subseteq U\}$, $Q = \{U \in T: \boldsymbol{B}\top \notin U\}$. For each $a \in B$, define the binary relation R_a by:

$$U_1 R_a U_2 \text{ iff } U_1 \notin Q \text{ and } [\boldsymbol{B}a \notin U_1 \text{ or } a \in U_2].$$

$\mathfrak{R} = \{R_a : a \in B\}$. We define $u: T \times At(B) \to \{0,1\}$ by:

$$u_U(a) = 1 \text{ iff } a \in U.$$

The triple $\langle T, \mathfrak{R}, u\rangle$ thus defined is called the canonical **EM**-model.[3]

In any boolean algebra $\underline{B'} = \langle B', \wedge, \vee, ^{c}\rangle$, if F is a filter in $\underline{B'}$ and $a \notin F$ then there is an ultrafilter U in $\underline{B'}$ such that $F \subseteq U$ and $a^c \in U$. (See Bell and Machover [1], Ch. 4, Problem 3.14.)

Lemma 1.6 (Canonical **EM**-Model Lemma) *The canonical* **EM**-*model is an* **EM**-*model. Furthermore,* $\forall a \in B \, [w_U(a) = 1$ *iff* $a \in U]$.

Proof: From the definition of R_a it follows immediately that if $U \in Q$ then $\forall U' \in T \, \forall a \in B$ [not UR_aU']. Conversely, if $U \notin Q$ then $\boldsymbol{B}\top \in U$ and so $UR_\top U'$, for all $U' \in T$ since $\top \in F$.

By the definition of the canonical **EM**-model we have that when a is an atomic proposition $w_U(a) = 1$ iff $a \in U$. Proof for all propositions proceeds by induction on their complexity. The only interesting case is when a is of the form $\boldsymbol{B}b$. Then we have:

(i) $U \in Q$: As $U \in Q$, $w_U(\boldsymbol{B}b) = 0$. As **EM** extends **PL**, $b \to \top \in F$; by RM, $\boldsymbol{B}b \in U$ only if $\boldsymbol{B}\top \in U$; as $U \in Q$, $\boldsymbol{B}\top \notin U$. So $\boldsymbol{B}b \notin U$.

(ii) $U \notin Q$: If $\boldsymbol{B}b \in U$ then $\forall U' \in T\ [UR_b U'$ iff $b \in U']$, i.e., by the induction hypothesis, $\forall U' \in T\ [UR_b U'$ iff $w_{U'}(b) = 1]$. But then $\exists R \in \Re\ \forall U' \in T\ [URU' \Rightarrow w_{U'}(b) = 1]$, i.e., $w_U(\boldsymbol{B}b) = 1$. If $\boldsymbol{B}b \notin U$ and $\boldsymbol{B}c \in U$ then $c \to b \notin F$ and so, for some $U' \in T$, $c \mathbin{\&} \neg b \in U'$. By the induction hypothesis, we have $UR_c U'$ and $w_{U'}(b) = 0$. As $\boldsymbol{B}b \notin U$, $\top \to b \notin F$ and so, for some $U' \in T$, $\neg b \in U'$. By the induction hypothesis we have $w_{U'}(b) = 0$; also $UR_c U'$ and $w_{U'}(b) = 0$ when $\boldsymbol{B}c \notin U$. Consequently, $\forall R \in \Re\ \exists U' \in T\ [URU'$ and $w_{U'}(b) = 0]$, i.e., $w_U(\boldsymbol{B}b) = 0$.

Theorem 1.7 (Completeness of **EM**) *a is a theorem of* **EM** *if and only if $w_t(a) = 1$ for all $t \in T$ in every* **EM**-*model $\langle T, \Re, u\rangle$.*

Proof: This is immediate from Lemmas 1.4 and 1.6.

Consider again the queer worlds in an **EM**-model, the worlds in which no beliefs are justifiedly/rationally held. They are, then, worlds in which total scepticism is true. As far as **EM**-models are concerned, queer worlds are accessible from worlds in which beliefs are held. It is, therefore, quite feasible in an **EM**-model that an agent should justifiedly/rationally believe that there are no justified/rational beliefs, the state of affairs represented by the proposition $\boldsymbol{B}\neg\boldsymbol{B}\top$. But what we should seek to model is not total scepticism as a thesis perhaps believed by an agent and in any case no more than a doxastic possibility among others, but rather justified/rational belief understood in such a way that no decision is made between two "global" views—namely, total scepticism (which declares that there are no justified/rational beliefs) and its denial.

What I shall call weak standard models achieve this synthesis. The queer worlds of **EM**-worlds in which total scepticism reigns become instead the special case, added to **EM**-models, that allows $\Re$, the class of accessibility relations, to be empty.

Definition 1.8 The ordered triple $\langle T, \Re, u\rangle$ is a *weak standard model*, henceforth a w-model, if and only if T is a non-empty set, $\Re$ is a set, possibly empty, of binary relations defined on T, and u is a function from worlds and atomic propositions to truth-values, i.e., $u : T \times At(B) \to \{0,1\}$.

Given a w-model $\langle T, \Re, u\rangle$ the function $w : T \times B \to \{0,1\}$ is defined by these conditions:

(i) $\forall a \in At(B)\ [w_t(a) = u_t(a)]$;
(ii) $w_t(\neg a) = 1 - w_t(a)$;
(iii) $w_t(a \mathbin{\&} b) = \min\{w_t(a), w_t(b)\}$;
(iv) $w_t(a \vee b) = \max\{w_t(a), w_t(b)\}$;
(v) $w_t(\boldsymbol{B}a) = 1$ iff $\exists R \in \Re\ \forall s \in T\ [tRs \Rightarrow w_s(a) = 1]$.

The doxastic logic that is sound and complete with respect to weak standard models is the logic **EMV**, the extension of **EM** obtained by adding the axiom schema:

(V) $\boldsymbol{B}a \to \boldsymbol{B}(a \mathbin{\&} \boldsymbol{B}\top)$.

This axiom schema has little intuitive motivation. Its significance will become clearer in the light of the soundness and completeness theorems for **EMV** and for some of the logics in Section 4.[4]

Definition 1.9 The ordered triple $\langle T, N, u\rangle$ is a *minimal model* if and only if T is a non-empty set, N is a function that associates elements of T with sets of subsets of T, i.e., $N: T \to \mathcal{P}(\mathcal{P}(T))$, subject to the condition that if $X \in N_t$ and $X \subseteq Y$ then $Y \in N_t$, and u is as in the definition of w-models.

Given a minimal model $\langle T, N, u\rangle$ the map $m: T \times B \to \{0,1\}$ is defined by these conditions:

(i) $\forall a \in At(B)\ [m_t(a) = u_t(a)]$;
(ii) $m_t(\neg a) = 1 - m_t(a)$;
(iii) $m_t(a \,\&\, b) = \min\{m_t(a), m_t(b)\}$;
(iv) $m_t(a \vee b) = \max\{m_t(a), m_t(b)\}$;
(v) $m_t(\boldsymbol{B}a) = 1$ iff $\{s \in T: m_s(a) = 1\} \in N_t$.

Lemma 1.10 (Correspondence Lemma) *To every w-model $\langle T, \mathfrak{R}, u\rangle$ there corresponds a minimal model $\langle T, N, u\rangle$, point-wise equivalent throughout T, in which either all the N_t's are empty or none are. And to every minimal model $\langle T, N, u\rangle$ in which either all the N_t's are empty or none are there corresponds a w-model $\langle T, \mathfrak{R}, u\rangle$ point-wise equivalent throughout T.*

Proof: Given $\langle T, \mathfrak{R}, u\rangle$ define N by:

$$N_t = \{X \subseteq T: \exists R \in \mathfrak{R}\ [\{s \in T: tRs\} \subseteq X]\}.$$

Note that if $\mathfrak{R} = \varnothing$ then $\forall t \in T\ [N_t = \varnothing]$ and if $\mathfrak{R} \neq \varnothing$ then $\forall t \in T\ [N_t \neq \varnothing]$, for then $T \in N_t$. We show by induction on the complexity of propositions that $\langle T, \mathfrak{R}, u\rangle$ and $\langle T, N, u\rangle$ are point-wise equivalent. The only interesting case is when a is of the form $\boldsymbol{B}b$:

$$\begin{aligned} w_t(\boldsymbol{B}b) = 1 &\text{ iff } \exists R \in \mathfrak{R}\ \forall s \in T\ [tRs \Rightarrow w_s(b) = 1] \\ &\text{ iff } \exists R \in \mathfrak{R}\ [\{s \in T: tRs\} \subseteq \{s \in T: w_s(b) = 1\}] \\ &\text{ iff } \{s \in T: w_s(b) = 1\} \in N_t, \text{ by definition of } N_t, \\ &\text{ iff } \{s \in T: m_s(b) = 1\} \in N_t, \text{ by induction hypothesis,} \\ &\text{ iff } m_t(\boldsymbol{B}b) = 1. \end{aligned}$$

Given $\langle T, N, u\rangle$ in which all N_t's are empty let $\mathfrak{R} = \varnothing$ and we are done. Otherwise, when no N_t is empty, define R_X, for each $X \subseteq T$, by:

$$tR_Xs \text{ iff } [X \notin N_t \text{ or } s \in X].$$

$\mathfrak{R} = \{R_X: X \subseteq T\}$. We show by induction on the complexity of propositions that $\langle T, N, u\rangle$ and $\langle T, \mathfrak{R}, u\rangle$ are point-wise equivalent. Again, the only interesting case is when a is of the form $\boldsymbol{B}b$. Assume, first, that $m_t(\boldsymbol{B}b) = 1$ and let $X = \{s \in T: m_s(b) = 1\}$, so $X \in N_t$. As $X \in N_t$, we have, for any $s \in T$, that tR_Xs iff $s \in X$ iff $m_s(b) = 1$ iff, by the induction hypothesis, $w_s(b) = 1$. Hence $\exists R \in \mathfrak{R}\ \forall s \in T[tRs \Rightarrow w_s(b) = 1]$, i.e., $w_t(\boldsymbol{B}b) = 1$. Assume, next, that $m_t(\boldsymbol{B}b) = 0$, so that $\forall X \in N_t\ \exists s \in T\ [s \in X \text{ and } m_s(b) = 0]$. As $N_t \neq \varnothing$, $\exists X \in N_t$ and $\exists s \in T\ [m_s(b) = 0]$. Also $\forall X \in N_t\ \exists s \in T\ [tR_Xs \text{ and } m_s(b) = 0]$. Now, suppose that

$Y \notin N_t$. As $\exists s \in T\ [m_s(b) = 0]$ we have that $\exists s \in T\ [tR_Y s$ and $m_s(b) = 0]$. Consequently, for every subset X of T, $\exists s \in T\ [tR_X s$ and $m_s(b) = 0]$. By the definition of $\mathfrak{R}$ and the induction hypothesis this yields $w_t(\boldsymbol{B}b) = 0$, i.e., $\forall R \in \mathfrak{R}\ \exists s \in T\ [tRs$ and $w_s(b) = 0]$.

Definition 1.11 A minimal model $\langle T, N, u\rangle$ is a v-model if and only if $\forall X \subseteq T\ \forall t \in T\ [X \in N_t \Rightarrow X \cap \{s \in T: N_s \neq \varnothing\} \in N_t]$.

The minimal models corresponding to weak standard models are all v-models.

Lemma 1.12 (Splitting Lemma) *Given the v-model $\langle T, N, u\rangle$ let $E = \{s \in T: N_t = \varnothing\}$, $u^1 = u \restriction (T - E)$, $u^2 = u \restriction E$, $N^2 = N \restriction E$ and let N^1 be defined by: for all $t \in T - E$, $X \in N^1_t$ iff $X \cup E \in N_t$. Then, provided $T \neq E$, $\langle T - E, N^1, u^1\rangle$ is a minimal model in which all N^1_t's are non-empty, point-wise equivalent to $\langle T, N, u\rangle$ in $T - E$; and, provided $E \neq \varnothing$, $\langle E, N^2, u^2\rangle$ is a minimal model, in which all N^2_t's are empty, point-wise equivalent to $\langle T, N, u\rangle$ in E.*

Proof: Again the proof is by induction on the complexity of propositions; again the only interesting case is when a is of the form $\boldsymbol{B}b$. Obviously, for $t \in E$ we have that $\forall b \in B\ [m^2_t(\boldsymbol{B}b) = m_t(\boldsymbol{B}b) = 0]$. For $t \in T - E$ we have:

$$
\begin{aligned}
m^1_t(\boldsymbol{B}b) = 1 \text{ iff } & \{s \in T - E: m^1_s(b) = 1\} \in N^1_t \\
\text{iff } & \{s \in T - E: m^1_s(b) = 1\} \cup E \in N_t, \text{ by definition of } N^1, \\
\text{iff } & \{s \in T - E: m_s(b) = 1\} \cup E \in N_t, \text{ by induction hypothesis,} \\
\text{iff } & \{s \in T: m_s(b) = 1\} \cup E \in N_t \\
\text{iff } & \{s \in T: m_s(b) = 1\} - E \in N_t, \text{ by definition of } v\text{-model,} \\
\text{iff } & \{s \in T: m_s(b) = 1\} \in N_t, \text{ by definition of } v\text{-model} \\
\text{iff } & m_t(\boldsymbol{B}b) = 1.
\end{aligned}
$$

Lemma 1.13 (Soundness of **EMV**) *If a is a theorem of* **EMV** *then $m_t(a) = 1$ for all $t \in T$ in the v-model $\langle T, N, u\rangle$.*

Proof: Proof proceeds by induction on formal derivations in **EMV**. Obviously, if a is a tautology then $m_t(a) = 1$, and if $m_t(a_1) = m_t(a_2) = \ldots = m_t(a_n) = 1$ and b is a consequence of $a_1, a_2, \ldots, a_n$ in PL then $m_t(b) = 1$.

Next we show $m_t(\boldsymbol{B}\top) = 1$ iff $N_t \neq \varnothing$. $\{s \in T: m_s(\top) = 1\} = T$; if $\exists X \subseteq T\ [X \in N_t]$ then $T \in N_t$, by the definition of minimal models, so $m_t(\boldsymbol{B}\top) = 1$; but if $N_t = \varnothing$ then $T \notin N_t$, and $m_t(\boldsymbol{B}\top) = 0$. Suppose that $m_t(\boldsymbol{B}a) = 1$; then $\{s \in T: m_s(a) = 1\} \in N_t$. As $\langle T, N, u\rangle$ is a v-model, $\{s \in T: m_s(a) = 1\} \cap \{s \in T: N_s \neq \varnothing\} \in N_t$.

$$
\begin{aligned}
&\{s \in T: m_s(a) = 1\} \cap \{s \in T: N_s \neq \varnothing\} \\
&\quad = \{s \in T: m_s(a) = 1\} \cap \{s \in T: m_s(\boldsymbol{B}\top) = 1\} \\
&\quad = \{s \in T: m_s(a) = 1 \text{ and } m_s(\boldsymbol{B}\top) = 1\} \\
&\quad = \{s \in T: m_s(a \mathbin{\&} \boldsymbol{B}\top) = 1\}.
\end{aligned}
$$

Hence $m_t(\boldsymbol{B}(a \mathbin{\&} \boldsymbol{B}\top)) = 1$. Thus, $\forall t \in T\ m_t(\boldsymbol{B}a \rightarrow \boldsymbol{B}(a \mathbin{\&} \boldsymbol{B}\top)) = 1$.

Lastly, suppose that $\forall t \in T\, m_t(a \to b) = 1$. Then $\forall t \in T\, [m_t(a) \leq m_t(b)]$. Hence $\{s \in T: m_s(a) = 1\} \subseteq \{s \in T: m_s(b) = 1\}$, and so, for all $t \in T$, $m_t(\boldsymbol{B}a) = 1$ only if $m_t(\boldsymbol{B}b) = 1$. I.e., $m_t(\boldsymbol{B}a \to \boldsymbol{B}b) = 1$ for all $t \in T$.

Definition 1.14 Let $F = \{a \in B: a$ is a theorem of **EMV**$\}$. Let $T = \{U \subseteq B$: U is an ultrafilter and $F \subseteq U\}$. For each $a \in B$, let $[a] = \{U \in T: a \in U\}$. We define the function $N: T \to \mathcal{P}(\mathcal{P}(T))$ by:

$$\forall X \subseteq T\, [X \in N_U \text{ iff } \exists a \in B\, ([a] \subseteq X \text{ and } \boldsymbol{B}a \in U)].$$

We define $u: T \times At(B) \to \{0,1\}$ by: $u_U(a) = 1$ iff $a \in U$. The triple $\langle T, N, u\rangle$ thus defined is *the canonical* **EMV**-*model.*

Lemma 1.15 (Canonical **EMV**-Model Lemma) *The canonical* **EMV**-*model is a v-model. Furthermore, in the canonical* **EMV**-*model,* $\forall a \in B\, [m_U(a) = 1$ *iff* $a \in U]$.

Proof: Given the definition of N_U it follows immediately that if $X \in N_U$ and $X \subseteq Y$ then $Y \in N_U$, so the canonical **EMV**-model is a minimal model. It remains to show that it is a v-model. First, for all $U \in T$, $\boldsymbol{B}\top \in U$ iff $\exists a \in B$ $[\boldsymbol{B}a \in U]$ iff $N_U \neq \varnothing$. Hence $[\boldsymbol{B}\top] = \{U \in T: N_U \neq \varnothing\}$. Next, suppose that $X \in N_U$, for some $X \subset T$. Then, for some $a \in B$, $[a] \subseteq X$ and $\boldsymbol{B}a \in U$. As $F \subseteq U$, $\boldsymbol{B}a \to \boldsymbol{B}(a\ \&\ \boldsymbol{B}\top) \in U$, whence $[a\ \&\ \boldsymbol{B}\top] \in N_U$. $[a\ \&\ \boldsymbol{B}\top] = [a] \cap [\boldsymbol{B}\top] = [a] \cap \{U \in T: N_U \neq \varnothing\}$, and so $X \cap \{U \in T: N_U \neq \varnothing\} \in N_U$.

By the definition of the canonical **EMV**-model we have that when a is an atomic proposition $m_U(a) = 1$ iff $a \in U$. Proof for all propositions proceeds by induction on their complexity. As usual, the only interesting case is when a is of the form $\boldsymbol{B}b$. Then we have:

$$\begin{aligned} m_U(\boldsymbol{B}b) = 1 &\text{ iff } \{U' \in T: m_{U'}(b) = 1\} \in N_U \\ &\text{ iff } \{U' \in T: b \in U'\} \in N_U, \text{ by induction hypothesis,} \\ &\text{ iff } \exists c \in B\, ([c] \subseteq \{U' \in T: b \in U'\} \text{ and } \boldsymbol{B}c \in U) \\ &\text{ iff } \exists c \in B\, ([c] \subseteq [b] \text{ and } \boldsymbol{B}c \in U) \\ &\text{ iff } \exists c \in B\, ([c \to b] = T \text{ and } \boldsymbol{B}c \in U) \\ &\text{ iff } \exists c \in B\, [c \to b \in F \text{ and } \boldsymbol{B}c \in U] \\ &\text{ iff } \boldsymbol{B}b \in U, \text{ as RM holds in } \textbf{EMV}. \end{aligned}$$

Theorem 1.16 (Soundness and completeness of **EMV**) *a is a theorem of* **EMV** *iff a is true in every world in every weak standard model.*

Proof: If a is false in some world in a weak standard model then, by the Correspondence Lemma (Lemma 1.10) and the Soundness Lemma (Lemma 1.13), a is not a theorem of **EMV**. On the other hand, if a is not a theorem of **EMV** then, by the Ultrafilter Theorem and the fact that the theorems of **EMV** form a filter in B, there is an ultrafilter U in B such that U contains all theorems of **EMV** and $a \notin U$. U belongs to the canonical **EMV**-model and, by Lemma 1.15, $m_U(a) = 0$. As, again by Lemma 1.15, the canonical **EMV**-model is a v-model, we have, by the Soundness Lemma (Lemma 1.13), the Splitting Lemma (Lemma

1.12), and the Correspondence Lemma (Lemma 1.10), in that order, that there is a world in a weak standard model at which a is false.

As the proof of the Correspondence Lemma (Lemma 1.10) shows, v-models in which $N_t = \varnothing$, for all $t \in T$, correspond to w-models in which $\mathfrak{R} = \varnothing$; v-models in which $N_t \neq \varnothing$, for all $t \in T$, correspond to w-models in which $\mathfrak{R} \neq \varnothing$. As we know from the proof of the soundness of **EMV** (Lemma 1.13), in any v-model $\langle T, N, u \rangle$, $w_t(\boldsymbol{B}\top) = 1$ iff $N_t \neq \varnothing$. Hence the logic **EMN**, obtained by adding the axiom

(N) $\boldsymbol{B}\top$

to **EM**, is sound when $\mathfrak{R} \neq \varnothing$. And the logic $\mathbf{EMN_c}$, obtained by adding the axiom

(N_c) $\neg\boldsymbol{B}\top$

to **EM**, is sound when $\mathfrak{R} = \varnothing$.[5] In fact, these logics are not only sound but complete in the relevant circumstances. If this is not obvious in the case of **EMN** see Theorems 9.8, 9.10, and 9.13 of [2]. It is obvious in the case of $\mathbf{EMN_c}$. $\mathbf{EMN_c}$ is the logic of total scepticism.[6] **EMN** is the doxastic logic that affirms the existence of justified/rational beliefs. The theorems of **EMV** are the propositions that are theorems of both **EMN** and $\mathbf{EMN_c}$. **EMV** allows that there may be no justified/rational beliefs but unlike **EM** it does not treat the issue of whether there are or not as doxastically contingent. An argument in favor of this stance is that, in the precise sense alluded to previously, $\mathbf{EMN_c}$ is not believable, i.e., the logic obtained by adding RN to $\mathbf{EMN_c}$ is inconsistent. Consequently, justified/rational belief in total scepticism is self-defeating, for justified/rational belief in total scepticism contradicts the thesis believed. Any rational agent whose powers of reasoning encompass elementary logic and the meaning of justified/rational belief should recognize the absurdity of justified/rational belief in total scepticism. (Recall that in a bid for consistency Pyrrhonian sceptics asserted that they knew nothing, not even that they knew nothing.)

While it is one thing for $\mathbf{EMN_c}$ to be the logic of justified/rational belief and another for that thesis to be justifiedly/rationally believed, **EM** does nothing to bring out the oddity of the latter belief. On the other hand, **EMV** does, for it yields a demonstration that anyone who justifiedly/rationally believes total scepticism justifiedly/rationally believes anything:

1.	$\boldsymbol{B}\neg\boldsymbol{B}\top \rightarrow \boldsymbol{B}(\neg\boldsymbol{B}\top \,\&\, \boldsymbol{B}\top)$	V
2.	$(\neg\boldsymbol{B}\top \,\&\, \boldsymbol{B}\top) \rightarrow a$	**PL**
3.	$\boldsymbol{B}(\neg\boldsymbol{B}\top \,\&\, \boldsymbol{B}\top) \rightarrow \boldsymbol{B}a$	2 RM
4.	$\boldsymbol{B}\neg\boldsymbol{B}\top \rightarrow \boldsymbol{B}a$	1,3 **PL**

This is about as strong a condemnation of justified/rational belief in total scepticism as we can hope for, for as yet we have done nothing to outlaw absurd beliefs. (In **LM**, and hence in any extension, the absurd proposition is justifiedly/rationally believed if and only if every proposition is.)

Stalnaker's pragmatic picture of belief does give us good grounds to reject the possibility of belief in absurdities (cf. [14], p. 83). Lenzen credits R. L. Putrill with the observation:

> if 'belief' is [. . .] taken to imply that the person in question is "disposed to act as if a were true" then [. . .] [a] behaviour that could count as establishing someone's disposition to act as if a were both true and false hardly is imaginable. ([9], p. 51).

This observation would license the addition of

(P) $\neg \boldsymbol{B}\top$

as an axiom of minimal doxastic logic. It is not difficult to show—proofs are omitted—that the logic **EMVP** is complete with respect to all w-models that satisfy the further condition

$$\text{for all } R \in \mathfrak{R},\ R \text{ is serial, i.e., } \forall t \in T\ \exists s \in T\ [tRs].$$

(The corresponding condition on v-models is that, for all $t \in T$, $\varnothing \notin N_t$.) P is a theorem of $\mathbf{EMN_c}$. The theorems of **EMVP** are those propositions that are theorems of **EMNP** and $\mathbf{EMN_c}$.

Weak standard models in which $\mathfrak{R}$ contains at most one element validate the axiom schema:

(C) $(\boldsymbol{B}a\ \&\ \boldsymbol{B}b) \rightarrow \boldsymbol{B}(a\ \&\ b)$.

When $\mathfrak{R}$ is a singleton we get the logic **EMNC**, better known as **K**, and the models are standard. As standardly presented, **K** extends **PL** by adding the axiom schema

(K) $\boldsymbol{B}(a \rightarrow b) \rightarrow (\boldsymbol{B}a \rightarrow \boldsymbol{B}b)$

and the Gödel necessitation rule RN. **EMNCP** is better known as **KD**, a logic to which we shall return. K and C are interderivable in **LM**, and hence in any of its extensions; P and D are interderivable in any extension of **LM** that contains C. D is the schema

(D) $\boldsymbol{B}a \rightarrow \neg \boldsymbol{B}\neg a$.

One point that is of methodological significance for Stalnaker's conception of belief is that although $\mathfrak{R}$'s being a singleton is sufficient for C to hold good of an agent's justified/rational beliefs it is by no means necessary: there are weak standard models in which C holds and in which $\mathfrak{R}$ is not a singleton. This is true of the canonical weak standard model for **EMNC**.

Definition 1.17 Let $F = \{a \in B\colon a \text{ is a theorem of } \mathbf{EMNC}\}$. Let $T = \{U \subseteq B\colon U \text{ is an ultrafilter and } F \subseteq U\}$. For each $a \in B$, let R_a be the binary relation on T^2 defined by:

$$U_1 R_a U_2 \text{ iff } [\boldsymbol{B}a \notin U_1 \text{ or } a \in U_2].$$

Let $\mathfrak{R} = \{R_a\colon a \in B\}$ and define $u\colon T \times At(B) \rightarrow \{0,1\}$ by: $u_U(a) = 1$ iff $a \in U$. The triple $\langle T, \mathfrak{R}, u\rangle$ thus defined is *the canonical* **EMNC**-*model.*

Theorem 1.18 (Soundness and completeness of **EMNC**) *a is a theorem of* **EMNC** *if and only if* $\forall U \in T\ [a \in U]$ *in the canonical* **EMNC**-*model* $\langle T, \mathfrak{R}, u\rangle$.

Proof: The canonical **EMNC**-model is clearly a w-model and, as $\mathfrak{R} \neq \varnothing$, every theorem of **EMN** is true at every world in it. Proof that $w_U(a) = 1$ iff $a \in U$, for all $a \in B$ and all $U \in T$ follows the proof of the canonical **EM**-model lemma

(Lemma 1.6). It follows immediately that C holds at every world in the canonical **EMNC**-model, since every "world" U extends F, the set of theorems of **EMNC**. This establishes soundness ('only if'). For completeness ('if'), when a is not a theorem of **EMNC** there is, as in the proof of Theorem 1.16, an ultrafilter U in B such that $F \subseteq U$ and $a \notin U$.

In a weak standard model the elements of $\mathfrak{R}$ correspond to distinct belief states in Stalnaker's sense. It is readily seen that in the canonical **EMNC**-model $\langle T, \mathfrak{R}, u\rangle$, $\mathfrak{R}$ is not a singleton. (The same holds good of the similarly defined canonical **EMNPC**-model, so nothing hangs on the use of **EMNC** here.) From this we may infer that there is no proposition nor set of propositions that expresses the condition that the totality of an agent's justified/rationally held beliefs constitute a single belief state. On the other hand, when C is satisfied they can be *represented* as forming a single belief state. The upshot is that while the number of belief states required for the representation of an agent's justified/rationally held beliefs may be bounded below, the latter impose no maximum. It is, therefore, appropriate to speak of their representation and not, strictly speaking, their description, for nothing intrinsic to the body of beliefs fixes the number of belief states the agent is in.

2 Probabilistic and algebraic preliminaries

Definition 2.1 A probability distribution is a function $P: B \rightarrow [0,1]$, where B is the domain of a boolean algebra $\underline{B} = \langle B, \wedge, \vee, ^{c}\rangle$, that satisfies the conditions:

(i) $\forall a \in B\ [0 \leq P(a) \leq 1]$;
(ii) $P(\mathbf{1}_{\underline{B}}) = 1$;
(iii) $\forall a, b \in B$ [if $a \wedge b = \mathbf{0}_{\underline{B}}$ then $P(a \vee b) = P(a) + P(b)$].

Definition 2.2 A distribution-pair, assigning upper and lower probabilities, is understood to be a pair of functions $\langle P^*, P_*\rangle$ such that $P^*: B \rightarrow [0,1]$ and $P_*: B \rightarrow [0,1]$, where B is the domain of a boolean algebra $\underline{B} = \langle B, \wedge, \vee, ^{c}\rangle$, and P^* and P_* satisfy these conditions:

(iv) $\forall a \in B\ [0 \leq P_*(a) \leq P^*(a) \leq 1]$;
(v) $P_*(\mathbf{1}_{\underline{B}}) = 1$;
(vi) $\forall a, b \in B$ [if $a \wedge b = \mathbf{0}_{\underline{B}}$ then $P_*(a) + P_*(b) \leq P_*(a \vee b) \leq P^*(a) + P_*(b) \leq P^*(a \vee b) \leq P^*(a) + P^*(b)$].

Corresponding to any (point-valued) probability distribution P there is the distribution-pair $\langle P, P\rangle$.

Definition 2.3 Where $D \subseteq E \subseteq B$, B the domain of a boolean algebra $\underline{B} = \langle B, \wedge, \vee, ^{c}\rangle$, and $D \neq \varnothing$, D is a disjoint subset of E iff $\forall a, b \in D\ [a \neq b \Rightarrow a \wedge b \notin E]$.

Definition 2.4 A non-empty subset S of a boolean algebra $\underline{B} = \langle B, \wedge, \vee, ^{c}\rangle$ is an n-block when

(i) $\mathbf{0}_{\underline{B}} \notin S$;
(ii) if $a \in S$ and $a \leq b$, then $b \in S$;
(iii) no disjoint subset of S contains more than n elements.

A filter is a 1-block.

Lemma 2.5 *If S is an n-block in $\underline{B}$ and $D = \{a_1, \ldots, a_n\}$ is a disjoint subset of S then*:
(1) $\forall b \in B\ [b \in S \Rightarrow \exists a \in D\ (a \wedge b \in S)]$;
(2) $\forall b, c \in B\ \forall a \in D\ [a \wedge b,\ a \wedge c \in S \Rightarrow a \wedge (b \wedge c) \in S]$.

Proof:

(1) Proof is immediate from the fact that no disjoint subset of S can contain more than n elements.

(2) Suppose that $a_1 \wedge b \in S$ and $a_1 \wedge c \in S$ but $a_1 \wedge (b \wedge c) \notin S$. Then $(a_1 \wedge b) \wedge (a_1 \wedge c) \notin S$. When $i > 1$ then, by (ii) in the definition of an n-block, $(a_1 \wedge b) \wedge a_i \notin S$, as $a_1 \wedge a_i \notin S$. Similarly, $(a_1 \wedge c) \wedge a_i \notin S$ for $i > 1$. Consequently, $\{a_1 \wedge b,\ a_1 \wedge c,\ a_2, \ldots, a_n\}$ is a disjoint subset of S containing $n + 1$ elements, contradicting the definition of an n-block.

Lemma 2.6 (n-Block Representation Lemma) *Every n-block in a boolean algebra can be represented as the union of n or fewer filters in that algebra.*

Proof: Let S be an n-block in $\underline{B} = \langle B, \wedge, \vee, {}^{c}\rangle$. By condition (iii) in the definition of an n-block, no disjoint subset of S contains more than n elements. Let the disjoint subset $D = \{a_1, \ldots, a_m\}$ be maximal in that no disjoint subset of S contains more members. Then S is an m-block. For each i, $1 \leq i \leq m$, define G_i by: $G_i = \{b \in B : a_i \wedge b \in S\}$. G_i is a filter, for (i) $\mathbf{1}_{\underline{B}} \in G_i$ as $a_i \in S$; (ii) as $a_i \wedge \mathbf{0}_{\underline{B}} = \mathbf{0}_{\underline{B}} \notin S$, $\mathbf{0}_{\underline{B}} \notin G_i$; (iii) if $b \in G_i$ and $b \leq c$ then $a_i \wedge b \in S$ and $a_i \wedge b \leq a_i \wedge c$, so $a_i \wedge c \in S$ and hence $c \in G_i$, (iv) if $b, c \in G_i$ then $a_i \wedge b \in S$ and $a_i \wedge c \in S$, and so, by (2) of Lemma 2.5, $a_i \wedge (b \wedge c) \in S$, i.e., $b \wedge c \in G_i$. If $b \in G_i$ then $a_i \wedge b \in S$ so $b \in S$, since $a_i \wedge b \leq b$. If $b \in S$ then, by (1) of Lemma 2.5, $a_i \wedge b \in S$ for some i, $1 \leq i \leq m$. Hence $S = G_1 \cup G_2 \cup G_m$ and so S is indeed the union of no more than n filters.

Definition 2.7 The distribution-pair $\langle P^*, P_*\rangle$ assigning upper and lower probabilities to the elements of the boolean algebra $\underline{B} = \langle B, \wedge, \vee, {}^{c}\rangle$ represents the filter F in $\underline{B}$ just in case:

$$P_*(a) = 1 \text{ iff } a \in F.$$

The probability distribution $P : B \to [0,1]$ represents the filter F just in case:

$$P(a) = 1 \text{ iff } a \in F.$$

Lemma 2.8[7] (First Probability Representation Lemma) *Let F be a filter in the boolean algebra $\underline{B} = \langle B, \wedge, \vee, {}^{c}\rangle$. Then there is a distribution-pair that represents F. If B is countable there is a probability distribution that represents F.*

Proof. Define the functions P^* and P_* by:

for all $a \in B$, $P^*(a) = 1$ unless $a^{c} \in F$, in which case $P^*(a) = 0$;

for all $a \in B$, $P_*(a) = 0$ unless $a \in F$, in which case $P_*(a) = 1$.

That P^* and P_* satisfy the axioms (iv)–(vi) for upper and lower probabilities is readily shown. The only interesting item is the super- and sub-additivity property and it follows easily when it is noted that at most one of a and b can belong to any filter when $a \wedge b = \mathbf{0}_{\underline{B}}$ and that if a^{c} and b^{c} both belong to a filter then $a \vee b$ does not.

When U is an ultrafilter, so that $a \in U$ just in case $a^c \notin U$, and the distribution pair $\langle P^*, P_* \rangle$ represents U then $P^* = P_*$, i.e., there is a two-valued probability distribution that represents U.

If $\underline{B}$ is countable then so too is the quotient algebra $\underline{B}/F$ determined by the equivalence relation

$$a \approx_F b \text{ iff } (a^c \vee b) \wedge (a \vee b^c) \in F.$$

$\underline{B}/F$ is countable, hence separable (i.e., contains a countable dense subset) and so, by Theorem 2.5 of Horn and Tarski [7], there is a strictly positive probability distribution $P: B/F \to [0,1]$ such that $P([a]) = 1$ iff $[a] = \mathbf{1}_{\underline{B}/F}$, where $[a]$ is the equivalence class containing a. Defining $P': B \to [0,1]$ by $P'(a) = P([a])$ yields the required probability distribution: $P'(a) = 1$ iff $P([a]) = 1$ iff $[a] = \mathbf{1}_{\underline{B}/F}$ iff $a \approx_F \mathbf{1}_{\underline{B}}$ iff $a \in F$.

Countability is sufficient but not necessary for separability which, in turn, is sufficient but not necessary for the existence of a strictly positive distribution ([7], p. 484). A condition that is necessary for the existence of a strictly positive measure is that $B \backslash \{\mathbf{0}_{\underline{B}}\}$ should contain no uncountable disjoint subset. (See Bell and Machover [1], p. 141.) Horn and Tarski, writing in 1947, observe that 'no workable criteria (conditions that are both necessary and sufficient) for the existence of a strictly positive measure in an arbitrary Boolean algebra are known' ([7], p. 480).

Lemma 2.9 (Second Probability Representation Lemma) *Let S be an n-block in $\underline{B} = \langle B, \wedge, \vee, ^c \rangle$, a countable boolean algebra. Then there is a family* $\mathbf{P}$ *of n or fewer (point-valued) probability distributions such that*

$$a \in S \textit{ iff } \exists P \in \mathbf{P}\ [P(a) = 1].$$

Proof: By Lemma 2.6, as S is an n-block it can be represented as the union $G_1 \cup G_2 \cup \ldots \cup G_m$ of m filters in B, for some m, $1 \leq m \leq n$. Since $\underline{B}$ is countable there is, by Lemma 2.8, a probability distribution P_i representing G_i, $1 \leq i \leq m$. Let $\mathbf{P}$ be the set $\{P_i : 1 \leq i \leq m\}$. For all $a \in B$, $a \in S$ iff $a \in G_i$, for some i, $1 \leq i \leq m$, iff $P_i(a) = 1$, for some i, $1 \leq i \leq m$.

Definition 2.10 For each $n > 1$, the modal axiom schema C_n is a weak analogue of C. It is:

$$(\mathrm{C_n}) \quad [(\boldsymbol{B}a_1 \,\&\ldots\&\, \boldsymbol{B}a_n \,\&\, \boldsymbol{B}b) \,\&\, \boldsymbol{\&}_{1 \leq i < j \leq n} \neg \boldsymbol{B}(a_i \,\&\, a_j)] \\ \to (\boldsymbol{B}(a_1 \,\&\, b) \vee \ldots \vee \boldsymbol{B}(a_n \,\&\, b)).$$

Obviously, C can be thought of as $\mathrm{C_1}$.

Lemma 2.11 (Generic Completeness Lemma) *Let F be the set of theorems of* **DL**, *where* **DL** *is any consistent doxastic logic that contains* $\mathbf{LMNPC_n}$, *including* $\mathbf{LMNPC_n}$ *itself. F is a filter since* **DL** *extends classical propositional logic and $\bot \notin F$. Let U be any ultrafilter in B that contains F and let S be the set $\{a \in B : \boldsymbol{B}a \in U\}$. Then S is an n-block. If* **DL** *contains C then S is a filter.*

Proof: S is an n-block, for (i) $\boldsymbol{B}\top \in U$ as $\mathbf{LMNPC_n}$ contains N, so $S \neq \varnothing$; (ii) as $\neg\boldsymbol{B}\bot$ is a theorem of $\mathbf{LMNPC_n}$, $\boldsymbol{B}\bot \notin U$ and $\bot \notin$ S; (iii) if $a \in S$ and $a \leq b$, i.e., $a \to b$ is a theorem of **PL**, then $b \in S$ as $\mathbf{LMNPC_n}$ contains M and RL;

(iv) C_n ensures that no disjoint subset of S contains more than n members. When $n = 1$, i.e., when $C_n = C$, S is a 1-block, i.e., a filter.

3 Probabilistic completeness theorems The probabilistic result at the heart of this section concerns the logic **LMNPC**. This is the logic with rule RL, axioms N and P, and axiom schemas M and C. **LMNPC** is, of course, weaker than **EMNPC** which we met briefly in Section 1.

Definition 3.1 A probability model, henceforth a p-model, comprises a pair $\langle u,P\rangle$, where u is a function from atomic propositions to truth-values, i.e., $u: At(B) \rightarrow \{0,1\}$, and $P: B \rightarrow [0,1]$ is a probability distribution.

Given a p-model $\langle u,P\rangle$ the function $\tau: B \rightarrow \{0,1\}$ is defined by these conditions:

(i) $\forall a \in At(B)\, [\tau(a) = u(a)]$;
(ii) $\tau(\neg a) = 1 - \tau(a)$;
(iii) $\tau(a \,\&\, b) = \min\{\tau(a),\tau(b)\}$;
(iv) $\tau(a \vee b) = \max\{\tau(a),\tau(b)\}$;
(v) $\tau\,(\boldsymbol{B}a) = 1$ iff $P(a) = 1$.

Definition 3.2 A weak probability model, henceforth a p^*-model, comprises a triple $\langle u,P^*,P_*\rangle$, where u is a function from atomic propositions to truth-values, i.e., $u: At(B) \rightarrow \{0,1\}$, and the pair $\langle P^*,P_*\rangle$, containing the functions $P^*: B \rightarrow [0,1]$ and $P^*: B \rightarrow [0,1]$, is a distribution-pair.

Every probability model is a weak probability model.

Given a p^*-model $\langle u,P^*,P_*\rangle$ the function $\tau: B \rightarrow \{0,1\}$ is defined by these conditions:

(i) $\forall a \in At(B)\, [\tau(a) = u(a)]$;
(ii) $\tau(\neg a) = 1 - \tau(a)$;
(iii) $\tau(a \,\&\, b) = \min\{\tau(a),\tau(b)\}$;
(iv) $\tau(a \vee b) = \max\{\tau(a),\tau(b)\}$;
(v) $\tau(\boldsymbol{B}a) = 1$ iff $P_*(a) = 1$.

Lemma 3.3 (Soundness of **LMNPC**) *If a is a theorem of* **LMNPC** *then $\tau(a) = 1$ in every p^*-model $\langle u,P^*,P_*\rangle$.*

Proof: Obviously, if a is a theorem of **PL** then $\tau(a) = 1$ and if $\tau(a_1) = \ldots = \tau(a_n) = 1$ and b is entailed by $a_1,\ldots,a_n$ in **PL** then $\tau(b) = 1$.

Let $\langle P^*,P_*\rangle$ be any distribution-pair assigning upper and lower probabilities to the elements of a boolean algebra $\underline{B} = \langle B,\wedge,\vee,{}^{c}\rangle$. Then:

$P_*(\mathbf{1}_{\underline{B}}) = 1$; $P^*(\mathbf{0}_{\underline{B}}) = 0$;

when $P_*(a) = P_*(b) = 1$,

$$0 \leq P^*(a^c \vee b^c) \leq P^*(a^c) + P^*(a \wedge b^c)$$
$$\leq P^*(a^c) + P^*(a \wedge b^c) + P_*(a^c \wedge b) \leq P^*(a^c) + P^*(b^c) = 0,$$

hence $P_*(a \wedge b) = 1 - P^*(a^c \vee b^c) = 1$;

if $a \leq b$ then $P_*(a^c \vee b) = P_*(\mathbf{1}_{\underline{B}}) = 1$, so $1 \leq P^*(a^c) + P_*(a \wedge b)$,

and hence $P_*(a) \leq P_*(a \wedge b) \leq P_*\,(a \wedge b) + P_*\,(a^c \wedge b) \leq P_*(b)$.

Consequently, when we turn our attention to the algebra of propositions $\underline{B} = \langle B, \&, \vee, \neg \rangle$, we have that for any u, $\tau(\boldsymbol{B}\top) = \tau(\neg \boldsymbol{B}\bot) = 1$; if $\tau(\boldsymbol{B}a) = \tau(\boldsymbol{B}b) = 1$ then $\tau(\boldsymbol{B}(a \,\&\, b)) = 1$, so $\tau((\boldsymbol{B}a \,\&\, \boldsymbol{B}b) \rightarrow \boldsymbol{B}(a \,\&\, b)) = 1$, for all $a, b \in B$; if $a \rightarrow b$ is a theorem of **PL** then $\tau(\boldsymbol{B}a) \leq \tau(\boldsymbol{B}b)$, so $\tau(\boldsymbol{B}a \rightarrow \boldsymbol{B}b) = 1$.

Theorem 3.4[8] (Completeness of **LMNPC**) *If a is not a theorem of* **LMNPC** *then there is a p^*-model $\langle u, P^*, P_* \rangle$ in which $\tau(a) = 0$. Furthermore, if the algebra of propositions is countable we may assume that upper and lower probabilities coincide for all propositions, i.e., that the p^*-model is a p-model.*

Proof: Suppose that a is not a theorem of **LMNPC** and let F be the filter $\{b \in B\text{: } b \text{ is a theorem of } \textbf{LMNPC}\}$. As $a \notin F$, there is an ultrafilter U such that $F \subseteq U$ and $a \notin U$. Defining the set $S = \{b \in B : \boldsymbol{B}b \in U\}$ we know, by Lemma 2.11, that S is a filter. By Lemma 2.8, there is a distribution-pair $\langle P^*, P_* \rangle$ that represents S. Also by Lemma 2.8, when B is countable we may assume that $P^* = P_*$.

Let $u : AT(B) \rightarrow \{0,1\}$ be the function defined by the condition

$$u(b) = 1 \text{ iff } b \in U.$$

$\langle u, P^*, P_* \rangle$ is a p^*-model. We need to show that, for all $b \in B$, $\tau(b) = 1$ iff $b \in U$. Proof is by induction on the complexity of propositions. As usual the only interesting case arises when b is of the form $\boldsymbol{B}c$:

$$\tau(\boldsymbol{B}c) = 1 \text{ iff } P_*(c) = 1 \text{ iff } c \in S \text{ iff } \boldsymbol{B}c \in U.$$

Consequently, $\tau(a) = 0$ in the p^*-model $\langle u, P^*, P_* \rangle$.

The probabilistic argument in the Soundness Lemma (Lemma 3.3) is carried out for an arbitrary distribution-pair, not merely for distributions of upper and lower probabilities to propositions. This makes clear the sense in which **LMNPC** is not (just) the logic of belief-defined-as-maximal-probability but rather the logic of maximal probability itself. The bracketing of 'just' foreshadows the suggestion that **LMNPC** is in fact too weak to be appropriate even as a minimal doxastic logic. That this is so will be clearer in the light of the following corollary to Theorem 3.4.

Corollary 3.5 (Trivialization Result for **LMNPC**) (1) *If* ***Ba*** *is a theorem of* **LMNPC** *then a is tautologous.* (2) *If* $\neg$***Ba*** *is a theorem of* **LMNPC** *then a is absurd.*

Proof:

(2) If $a \neq \bot$ then there is an ultrafilter U in $\underline{B} = \langle B, \&, \vee, \neg \rangle$ and a representing probability distribution P such that $a \in U$ and $P(a) = 1$. But then, for arbitrary $u : At(B) \rightarrow \{0,1\}$, $\tau(\boldsymbol{B}a) = 1$ in the p-model $\langle u, P \rangle$. By Lemma 3.3, $\neg \boldsymbol{B}a$ is not a theorem of **LMNPC**. Hence, if $\neg \boldsymbol{B}a$ is a theorem of **LMNPC** then $\neg a$ is a theorem of **PL**.

(1) The schema D$-\boldsymbol{B}a \rightarrow \neg \boldsymbol{B} \neg a-$is derivable in **LMNPC**, hence if ***Ba*** is a theorem of **LMNPC** then $\neg \boldsymbol{B} \neg a$ is also a theorem. By (2), $\neg\neg a$ is then a theorem of **PL** and so a is a theorem of **PL**.

Since **LMNPC** contains N, M and RL we have, as a consequence of Corollary 3.5, that ***Ba*** is a theorem of **LMNPC** just in case a is a theorem of **PL** and that $\neg \boldsymbol{B}a$ is a theorem of **LMNPC** just in case $\neg a$ is a theorem of **PL**. Because **LMNPC** extends **PL** it is therefore both positively and negatively accurate.

According to **LMNPC** an ideally rational individual rationally believes all tautologous propositions and these are the only propositions that she must believe. Similarly, one is justified in believing them and they are the only propositions one is justified in believing *a priori*. There are then no propositions, as one might say, properly concerning belief that must be believed by rational individuals belief in which is justified *a priori*. Put another way, the believed doxastic logic associated with **LMNPC** is just **PL** itself. Like **LM**, **LMNPC** appears to be too weak. And notice that here we do not have the excuse that we should not rule out the possibility of total scepticism: **LMNPC** is incompatible with total scepticism.

The next step is to follow up the thought that perhaps not all probability distributions on the algebra of propositions can represent belief states, not when we allow for the explicit expression of belief. With the exception of tautologous and absurd propositions, **LMNPC** does nothing to fix the degree of belief in a, even when a is a truth of doxastic logic, i.e., a conceptual truth about justified/rational belief. Notice that only two degrees of belief/upper and lower probabilities are expressible "in the object language", i.e., in terms of the propositions belonging to $\underline{B}$, $P(_*)(a) = 1$ and $P(^*)(a) = 0$ – $\boldsymbol{B}a$ and $\boldsymbol{B}\neg a$, respectively. Perhaps, then, we should confine attention to a subclass of the distributions/distribution-pairs that assign (upper and lower) probabilities to the elements of the algebra of propositions, a subclass $\underline{P}$ closed under the condition:

> if a is an expressible constraint on degrees of belief determined by the members of $\underline{P}$ then $\forall P(_*) \in \underline{P}\ [P(_*)(a) = 1]$.

This corresponds to closure under the rule RN.

When we add the rule RN to **LMNPC** we obtain the logic **EMNPC**, i.e., **KD**. (RE and RN are interderivable in **LMNC**.) Let us therefore take the class $\underline{P}$ of distribution-pairs:

$$\underline{P} = \{\langle P^*, P_* \rangle : \text{if } a \text{ is a theorem of } \mathbf{KD} \text{ then } P_*(a) = 1\}.$$

As **LMNPC** is sound with respect to all probability distributions/distribution-pairs, $\underline{P}$ is the largest class satisfying the closure condition above.

Definition 3.6 The p^*-model $\langle u, P^*, P_* \rangle$ is an augmented p^*-model when $\langle P^*, P_* \rangle \in \underline{P}$.

Lemma 3.7 (Soundness of **KD**) *If a is a theorem of* **KD** *then* $\tau(a) = 1$ *in every augmented* p^**-model* $\langle u, P^*, P_* \rangle$.

Proof: Proof proceeds by induction on formal derivations in **KD**. In view of the fact that **KD** is **LMNPC** augmented by RN and of the Soundness Lemma for **LMNPC** (Lemma 3.3), we need only show that RN is sound in augmented p^*-models. Suppose, then, that a is of the form $\boldsymbol{B}b$, where b is a theorem of **KD**. Then $P_*(b) = 1$ and so $\tau(a) = \tau(\boldsymbol{B}b) = 1$.

Theorem 3.8 (Completeness of **KD**) *If a is not a theorem of* **KD** *then there is an augmented* p^**-model* $\langle u, P^*, P_* \rangle$ *in which* $\tau(a) = 0$. *Furthermore, if the* *algebra of propositions is countable we may assume that the p^*-model is a p-model.*

Proof: Proof is analogous to that of Theorem 3.4 but takes F to be the set $\{b \in B : b$ is a theorem of **KD**$\}$. Where $\langle P^*, P_* \rangle$ is the representing distribution-pair we must show that $P_*(b) = 1$ when $b \in F$. Now, with U, S, and P_* as in the proofs of Lemma 2.11 and Theorem 3.4, if $b \in F$ then $b \in S$: for **KD** contains RN and so $\boldsymbol{B}b \in F$ when $b \in F$; as $F \subseteq U$, $\boldsymbol{B}b \in U$ and, consequently, $b \in S$. Thus $P_*(b) = 1$ when b is a theorem of **KD** and $\langle P^*, P_* \rangle \in \underline{P}$. By Lemma 2.8, when B is countable we may assume that $P^* = P_*$.

Corollary 3.9 (Accuracy of **KD**) (1) *If* $\boldsymbol{B}a$ *is a theorem of* **KD** *then* a *is also a theorem.* (2) *If* $\neg\boldsymbol{B}a$ *is a theorem of* **KD** *then* $\neg a$ *is a theorem.*

Proof: Suppose that $\neg a$ is not a theorem of **KD**. Then, by Theorem 3.8, there is a function $\tau : B \to \{0,1\}$ such that (i) $\tau(a) = 1$; (ii) if b is a theorem of **KD** then $\tau(b) = 1$; and (iii) τ is a probability distribution on B. The third clause holds good since every assignment of truth-values is a two-valued probability distribution. By (ii), $\langle \tau, \tau \rangle \in \underline{P}$. Let u be any function from atomic propositions to truth-values. Then $\langle u, \tau \rangle$ is an augmented p-model in which $\tau'(\boldsymbol{B}a) = 1$, where τ' is the truth-value assignment associated with $\langle u, \tau \rangle$. By the soundness of **KD** (Lemma 3.7), $\neg\boldsymbol{B}a$ is not a theorem of **KD**. As in the proof of Corollary 3.5, in any logic that contains D (2), which we have just proved, is sufficient for (1).

The logic **KD** has been thought suitable for deontic logic, although, as its positive accuracy shows:

> [R]oughly speaking, there are no logically true statements of obligation with non-trivial content. ([2], p. 195).

While this may render its appropriateness to deontic concerns doubtful, it is exactly this "triviality" that makes it appealing, from the minimalist perspective, as a doxastic logic. And not only is it positively accurate, it is both negatively accurate and believed (not just believable)—the believed doxastic logic generated by **KD** is **KD** itself. **KD** assumes, in the Hintikka style, that an agent's justified/rational beliefs are representable, in Stalnaker's terms, by just one belief state. It is not the only accurate, believed, doxastic logic that does this, although it is perhaps the strongest one that does so "sensibly". Probabilistic soundness and completeness proofs are readily forthcoming for the strengthening of **KD** that replaces the axiom schema D with:

(D!) $\boldsymbol{B}a \equiv \neg\boldsymbol{B}\neg a$.

This schema is most implausible as a constraint on any sort of belief, let alone justified or rational belief.

Definition 3.10 A p-model $\langle u, P \rangle$ is two-valued if the probability distribution P is two-valued.

As is readily shown, a is a theorem of **KD!** just in case $\tau(a) = 1$ in every two-valued augmented p-model $\langle u, P \rangle$. Moreover, **KD!** is accurate and, because ultrafilters are associated with two-valued probability distributions no matter the cardinality of the algebra in which they exist, the completeness theorem for **KD!**

requires no cardinality qualification on the algebra of propositions. The latter fact may be regarded as a virtue of **KD!**, but it is a virtue gained at a high price, two-valued probability distributions representing an extreme of opinionatedness that would normally be thought incompatible with justified/rational belief.

We shall next look at another extension of **KD** that Lenzen has suggested is the logic of strong belief/conviction ([9], p. 83). This is the logic obtained by adding to **LMNPC** the two axiom schemas:

(4) $\boldsymbol{B}a \rightarrow \boldsymbol{BB}a$

(5) $\neg \boldsymbol{B}a \rightarrow \boldsymbol{B}\neg \boldsymbol{B}a$.

These correspond to the constraints on probability distributions/distribution-pairs:

(i) if $P(_*)(a) = 1$ then $P(_*)(\boldsymbol{B}a) = 1$

(ii) if $P(_*)(a) \neq 1$ then $P(_*)(\neg \boldsymbol{B}a) = 1$,

which, in the case of probability distributions, in turn correspond to the expressible instances of the more general requirements:

(iii) if $P(a) = r$ then $P(P(a) = r) = 1$

(iv) if $P(a) \neq r$ then $P(\neg [P(a) = r]) = 1$.

Dutch book arguments can be given in support of these last two.[9]

Before proceeding to soundness and completeness theorems we must first show that the logics **LMNPC45** and **KD45** are identical. The only non-trivial item arises in showing that the rule RN is derivable when the schemas 4 and 5 are added to **LMNPC**.

Definition 3.11 For any proposition a, let $+\boldsymbol{B}a$ denote $\boldsymbol{B}a$ and $-\boldsymbol{B}a$ denote $\neg \boldsymbol{B}a$, so that $\pm \boldsymbol{B}a$ ambiguously denotes one or other of $\boldsymbol{B}a$ and $\neg \boldsymbol{B}a$. $\mp \boldsymbol{B}a$ stands for $\neg \boldsymbol{B}a$ when $\pm \boldsymbol{B}a$ is $\boldsymbol{B}a$ and for $\boldsymbol{B}a$ when $\pm \boldsymbol{B}$ is $\neg \boldsymbol{B}a$: briefly, $\mp \boldsymbol{B}a$ is the contrary of $\pm \boldsymbol{B}a$.

Definition 3.12 For any propositions $a_1, a_2, \ldots, a_n$, $\&_i a_i = a_1 \& a_2 \& \ldots \& a_n$.

Lemma 3.13 *In any logic that includes the axiom schemas* 4, 5, *and* M, *and the rule* RL,

$$(\&_i \pm \boldsymbol{B}a_i \rightarrow \pm \boldsymbol{B}b) \rightarrow \boldsymbol{B}(\&_i \pm \boldsymbol{B}a_i \rightarrow \pm \boldsymbol{B}b)$$

is a theorem.

Proof: $\pm \boldsymbol{B}b \rightarrow (\&_i \pm \boldsymbol{B}a_i \rightarrow \pm \boldsymbol{B}b)$ is tautologous, so, by RL and M, $\boldsymbol{B} \pm \boldsymbol{B}b \rightarrow \boldsymbol{B}(\&_i \pm \boldsymbol{B}a_i \rightarrow \pm \boldsymbol{B}b)$. By either 4 or 5, whichever is appropriate to the sign of $\pm \boldsymbol{B}b$, $\pm \boldsymbol{B}b \rightarrow \boldsymbol{B} \pm \boldsymbol{B}b$. And so, by **PL**, $\pm \boldsymbol{B}b \rightarrow \boldsymbol{B}(\&_i \pm \boldsymbol{B}a_i \rightarrow \pm \boldsymbol{B}b)$. Likewise $\mp \boldsymbol{B}a_i \rightarrow (\&_i \pm \boldsymbol{B}a_i \rightarrow \pm \boldsymbol{B}b)$ is tautologous, for each i, and so $\boldsymbol{B} \mp \boldsymbol{B}a_i \rightarrow \boldsymbol{B}(\&_i \pm \boldsymbol{B}a_i \rightarrow \pm \boldsymbol{B}b)$, by RL and M, for each i. By either 4 or 5, whichever is appropriate to the sign of $\mp \boldsymbol{B}a_i$, $\mp \boldsymbol{B}a_i \rightarrow \boldsymbol{B} \mp \boldsymbol{B}a_i$. Whence, by **PL**, $\mp \boldsymbol{B}a_i \rightarrow \boldsymbol{B}(\&_i \pm \boldsymbol{B}a_i \rightarrow \pm \boldsymbol{B}b)$. Combining the a_i theorems, using **PL** alone, gives $\neg (\&_i \pm \boldsymbol{B}a_i) \rightarrow \boldsymbol{B}(\&_i \pm \boldsymbol{B}a_i \rightarrow \pm \boldsymbol{B}b)$. By **PL**, this in conjunction with $\pm \boldsymbol{B}b \rightarrow \boldsymbol{B}(\&_i \pm \boldsymbol{B}a_i \rightarrow \pm \boldsymbol{B}b)$ yields the sought for theorem $(\&_i \pm \boldsymbol{B}a_i \rightarrow \pm \boldsymbol{B}b) \rightarrow \boldsymbol{B}(\&_i \pm \boldsymbol{B}a_i \rightarrow \pm \boldsymbol{B}b)$.

Theorem 3.14 *If a is a theorem of* **LMNPC45** *then so is* $\boldsymbol{B}a$.

Proof: Proof is by induction on formal proofs in **LMNPC45**, a formal proof being a finite sequence of propositions $a_1, a_2, \ldots, a_n$, such that a_1 is either $\boldsymbol{B}\top$,

$\neg \boldsymbol{B}\bot$, an instance of M, C, 4, or 5, or tautologous, and for each $i, 1 \leq i \leq n, a_i$ is either (a) $\boldsymbol{B}\top, \neg \boldsymbol{B}\bot$, or an instance of M, C, 4, or 5, or (b) tautologous, a theorem of **PL**, or (c) obtained by an application of RL to some proposition a_j that occurs earlier in the proof, or (d) entailed in **PL** by one or more propositions that occur earlier in the proof. We show that when $a_1, a_2, \ldots, a_n$ is a formal proof in **LMNPC45** the propositions $\boldsymbol{B}a_1, \boldsymbol{B}a_2, \ldots, \boldsymbol{B}a_n$ are all theorems of **LMNPC45**.

By N and 4, $\boldsymbol{BB}\top$ is a theorem; by P and 5, $\boldsymbol{B}\neg\boldsymbol{B}\bot$ is a theorem. By Lemma 3.13, if a_i is an instance of M, C, 4, or 5, then $\boldsymbol{B}a_i$ is a theorem of **LMNPC45**. If a_i is tautologous then $\top \equiv a_i$ is a theorem of **PL**, and hence, by RL, $\boldsymbol{B}\top \equiv \boldsymbol{B}a_i$, is a theorem of **LMNPC45**, as, consequently, is $\boldsymbol{B}a_i$. This takes care of cases (a) and (b) and the possibilities for a_1. Case (c): if a_i is obtained by an application of RL to some proposition a_j, $j < i$, then a_j must be a **PL** theorem of the form $a \equiv b$; hence $\boldsymbol{B}(a \to b)$ is a theorem of **LMNPC45**; as K is a theorem of **LMNPC45**, $\boldsymbol{B}a \to \boldsymbol{B}b$ is a theorem; by Lemma 3.13, $\boldsymbol{B}(\boldsymbol{B}a \to \boldsymbol{B}b)$ is a theorem; similarly, $\boldsymbol{B}(\boldsymbol{B}b \to \boldsymbol{B}a)$ is a theorem; by C, $\boldsymbol{B}(\boldsymbol{B}a \equiv \boldsymbol{B}b)$, i.e., $\boldsymbol{B}a_i$, is a theorem. Case (d): suppose that $\boldsymbol{B}b_1, \boldsymbol{B}b_2, \ldots, \boldsymbol{B}b_m$ are theorems of **LMNPC45** and that $b_1, b_2, \ldots, b_m$ jointly entail c in **PL**; by repeated applications of C, $\boldsymbol{B}(b_1 \,\&\, b_2 \,\&\ldots\&\, b_m)$ is a theorem; also, $(b_1 \,\&\, b_2 \,\&\ldots\&\, b_m) \to c$ is a theorem of **PL**, hence, by RL and M, $\boldsymbol{B}(b_1 \,\&\, b_2 \,\&\ldots\&\, b_m) \to \boldsymbol{B}c$ is a theorem of **LMNPC45**; finally, then, so is $\boldsymbol{B}c$.

Definition 3.15 Let $\underline{Q}$ be the class of distribution-pairs assigning upper and lower probabilities to the propositions in B constrained by the conditions

(i) if $P_*(a) = 1$ then $P_*(\boldsymbol{B}a) = 1$
(ii) if $P_*(a) \neq 1$ then $P_*(\neg\boldsymbol{B}a) = 1$.

Lemma 3.15 (Soundness of KD45) *If a is a theorem of* **KD45** *then* $\tau(a) = 1$ *in every* p^**-model* $\langle u, P^*, P_*\rangle$ *in which* $\langle P^*, P_*\rangle \in \underline{Q}$.

Proof: In view of the fact that **KD45** in **LMNPC45** (Theorem 3.14) and of the Soundness Lemma for **LMNPC** (Lemma 3.3), we need only show that 4 and 5 hold in the p^*-models under consideration. If $\tau(\boldsymbol{B}a) = 1$ then $P_*(a) = 1$, hence $P_*(\boldsymbol{B}a) = 1$ and $\tau(\boldsymbol{BB}a) = 1$; if $\tau(\neg\boldsymbol{B}a) = 1$ then $P_*(a) \neq 1$, hence $P_*(\neg\boldsymbol{B}a) = 1$ and $\tau(\boldsymbol{B}\neg\boldsymbol{B}a) = 1$.

Theorem 3.16 (Completeness of **KD45**) *If a is not a theorem of* **KD** *then there is a* p^**-model* $\langle u, P^*, P_*\rangle$ *in which* $\tau(a) = 0$ *and in which* $\langle P^*, P_*\rangle \in \underline{Q}$. *Furthermore, if the algebra of propositions is countable we may assume that the* p^**-model is a p-model.*

Proof: Proof is analogous to that of Theorem 3.4 but takes F to be the set $\{b \in B\colon b \text{ is a theorem of } \textbf{KD45}\}$. Where $\langle P^*, P_*\rangle$ is the representing distribution-pair we must show that if $P_*(a) = 1$ then $P_*(\boldsymbol{B}a) = 1$ and if $P_*(a) \neq 1$ then $P_*(\neg\boldsymbol{B}a) = 1$. $P_*(b) = 1$ when $b \in F$. Now, with U, S, and P_* as in the proofs of Lemma 2.11 and Theorem 3.4, this is readily established:

$P_*(a) = 1$ iff $a \in S$ iff $\boldsymbol{B}a \in U$ only if $\boldsymbol{BB}a \in U$, as $F \subseteq U$,
iff $\boldsymbol{B}a \in S$ iff $P_*(\boldsymbol{B}a) = 1$;

$P_*(a) \neq 1$ iff $a \notin S$ iff $\boldsymbol{B}a \notin U$ iff $\neg\boldsymbol{B}a \in U$ only if $\boldsymbol{B}\neg\boldsymbol{B}a \in U$, as $F \subseteq U$, iff $\neg\boldsymbol{B}a \in S$ iff $P_*(\neg\boldsymbol{B}a) = 1$.

By Lemma 2.8, when B is countable we may assume that $P^* = P_*$.

Theorem 3.17 **KD45** *is neither positively nor negatively accurate.*

Proof: $\boldsymbol{B}(\boldsymbol{B}a \to a)$ is derivable in any extension of **LM** that contains the axiom schema 5 but the schema

(T) $\boldsymbol{B}a \to a$

is not derivable in **KD45**.

$\neg\boldsymbol{B}(a \,\&\, \neg\boldsymbol{B}a)$ is derivable in any extension of **LM** that contains the axiom schemas D and 4 but the schema

(T_c) $a \to \boldsymbol{B}a$

is not derivable in **KD45**.

In view of Theorem 3.17, **KD45**—deontic **S5**, as Chellas calls it ([2], p. 193)—is seen to be much too strong. It would commit the rational individual to accepting and rejecting propositions that are, in its own terms, at most contingently true and false, respectively.

The weakest positively accurate extension of **KD45** is **S5 (KT5)**, which entails that justified/rational beliefs are infallible. The weakest accurate extension of **KD45** is **S5** + T_c **(KT!)**, which entails omniscience: all and only true propositions are justified/rational beliefs. The p^*-models of **KT!** bring out this feature. **KT!** is consistent, a fact most easily seen by striking out all occurrences of $\boldsymbol{B}$ in the propositions forming a proof in **KT!**, an operation which converts every line of a proof into a theorem of **PL** (cf., [8], pp. 41, 267). Any **KT!**-consistent set of propositions is **KD45** consistent, and therefore, by Theorem 3.16, has a p^*-model $\langle u, P^*, P_* \rangle$. As $\boldsymbol{B}a \vee \boldsymbol{B}\neg a$ is a theorem of **KT!**, $P^* = P_*$; as $\boldsymbol{B}a \equiv a$ is a theorem, $\tau(a) = \tau(\boldsymbol{B}a) = P_*(a)$, i.e., the p^*-model is a p-model and P $(= P_*)$ and τ coincide. Conversely, **KT!** is clearly sound with respect to all p-models $\langle u, P \rangle$ in which P and τ coincide. **KT!** is hence sound and complete with respect to all "omniscient" p^*-models.

4 More probabilistic completeness theorems **KD** is the logic that results from weak standard models in which $\mathfrak{R}$ is a singleton. It is also the logic that results from the class $\underline{P}$ of probability distributions and distribution-pairs defined in the previous section. In view of Dutch book arguments and other considerations epistemologists most commonly represent the belief state of a rational individual by a single probability distribution. Also, those who treat objective probabilities as measures of confirmation and/or justified belief most often appeal to a single probability distribution. On either approach the maximal-probability definition of belief yields:

$$\boldsymbol{B}a \text{ is true iff } P(a) = 1,$$

where P is the probability distribution in question.

For a variety of reasons a number of authors have suggested that an agent's belief state is best represented by a family of probability distributions. (See, for example, Levi [10] and Gärdenfors and Sahlin [5].) In view of Stalnaker's model of belief this is an entirely natural generalization in the present context (and is implicit in much of [4], although not exploited).

For the rest of this paper we shall concentrate on cases in which the beliefs that an agent justifiedly or rationally holds are represented by a *family* of probability distributions *over an at most countably infinite algebra of propositions* $\underline{B} = \langle B, \&, \vee, \neg \rangle$. In such a setting it makes sense to define belief relative to the family $\mathbf{P}$ of distributions as follows:

$$\boldsymbol{Ba} \text{ is true iff } P(a) = 1 \text{ for some } P \in \mathbf{P}.$$

When $\mathbf{P}$ is finite, but not necessarily otherwise, it also makes sense to define a distribution-pair by the equations:

$$P^*(a) = \max\{P(a) : P \in \mathbf{P}\};$$

$$P_*(a) = \min\{P(a) : P \in \mathbf{P}\}.$$

It then follows that a proposition is justifiedly/rationally believed just in case it receives maximal *upper* probability.[10]

Definition 4.1 A multiple probability model, henceforth an *mp*-model, comprises a pair $\langle u, \mathbf{P} \rangle$, where u is a function from atomic propositions to truth-values, i.e., $u : At(B) \to \{0,1\}$, and $\mathbf{P}$ is a (non-empty) family of probability distributions assigning probabilities to propositions.

Given an *mp*-model $\langle u, \mathbf{P} \rangle$ the function $\tau : B \to \{0,1\}$ is defined by these conditions:

(i) $\forall a \in At(B)\,[\tau(a) = u(a)]$;
(ii) $\tau(\neg a) = 1 - \tau(a)$;
(iii) $\tau(a \,\&\, b) = \min\{\tau(a), \tau(b)\}$;
(iv) $\tau(a \vee b) = \max\{\tau(a), \tau(b)\}$;
(v) $\tau(\boldsymbol{Ba}) = 1$ iff $\exists P \in \mathbf{P}\ [P(a) = 1]$.

Lemma 4.2 (Soundness of **LMNP**) *If a is a theorem of* **LMNP** *then $\tau(a) = 1$ in every mp-model $\langle u, \mathbf{P} \rangle$.*

Proof: This follows immediately from the proof of soundness of **LMNPC** (Lemma 3.3), for there only the soundness of C relied on $\mathbf{P}$ being a singleton.

Theorem 4.3 (Completeness of **LMNP**) *If a* is not a theorem of **LMNP** then there is an *mp*-model $\langle u, \mathbf{P} \rangle$ in which $\tau(a) = 0$.

Proof: Suppose that a is not a theorem of **LMNP** and let F be the filter $\{b \in B$: b is a theorem of **LMNP**$\}$. As $a \notin F$, there is, by a Lindenbaum's Lemma construction, an ultrafilter U such that $F \subseteq U$ and $a \notin U$. Let S be the set $\{b \in B$: $\boldsymbol{Bb} \in U\}$. For each $b \in S$, the set $F_b = \{c \in B : b \to c$ is a theorem of **PL**$\}$ is a filter and a subset of S. $S = \bigcup_{b \in S} F_b$. As B is countable, there is, by Lemma 2.8, a probability distribution P_b that represents F_b. Let $\mathbf{P}$ be the set $\{P_b : b \in S\}$.

Let $u : At(B) \to \{0,1\}$ be the function defined by the condition

$$u(b) = 1 \text{ iff } b \in U.$$

$\langle u,\mathbf{P}\rangle$ is an *mp*-model. We need to show that, for all $b \in B$, $\tau(b) = 1$ iff $b \in U$. Proof is by induction on the complexity of propositions. The only interesting case arises when b is of the form $\boldsymbol{B}c$:

$$\tau(\boldsymbol{B}c) = 1 \text{ iff } \exists P \in \mathbf{P}\ [P(c) = 1] \text{ iff } \exists d \in S\ [c \in F_d]$$

$$\text{iff } \exists d \in B\ [\boldsymbol{B}d \in U \text{ and } d \to c \text{ is a theorem of } \mathbf{PL}]$$

$$\text{iff } \boldsymbol{B}c \in U, \text{ as } F \subseteq U.$$

Consequently, $\tau(a) = 0$ in the *mp*-model $\langle u,\mathbf{P}\rangle$.

Clearly, it makes no difference to the logic if we insist that the family $\mathbf{P}$ be convex (as Isaac Levi does when he uses sets of probability distributions to represent credal states ([10])).

In the definition of *mp*-models it is stipulated that $\mathbf{P}$ is non-empty. If we relax that requirement we obtain models in which there are no justified/rational beliefs, i.e., total scepticism is again a possibility. It is obvious that the logic that is sound and complete with respect to such models is **LMP**, the logic whose theorems are all propositions that are theorems of both **LMNP** and $\mathbf{EMN_c}$.

In the representation of an agent's justified/rational beliefs in an explanation of the agent's behavior it would be natural although by no means necessary to suppose that only some finite number n of probability distributions are needed for the task.

Lemma 4.4 (Soundness of $\mathbf{LMNPC_n}$) *If a is a theorem of* $\mathbf{LMNPC_n}$ *then* $\tau(a) = 1$ *in every mp-model* $\langle u,\mathbf{P}\rangle$ *in which* $\mathbf{P}$ *contains at most n probability distributions.*

Proof: In view of the soundness proof for **LMNP** (Lemma 4.2) we need only show that $\mathrm{C_n}$ obtains. Suppose that $\tau(\boldsymbol{B}a_1) = \tau(\boldsymbol{B}a_2) = \ldots = \tau(\boldsymbol{B}a_n) = \tau(\boldsymbol{B}b) = 1$ and $\tau(\boldsymbol{B}(a_i \,\&\, a_j)) = 0$, $1 \le i < j \le n$. As $\tau(\boldsymbol{B}c) = 1$ if and only if for some $P \in \mathbf{P}$, $P(c) = 1$ and $\mathbf{P}$ contains at most n distributions, it must be that $\mathbf{P}$ contains exactly n distributions and that if $P(a_i) = 1$ then $P(a_j) \neq 1$, $1 \le i < j \le n$. But then it must be that, for some i, $1 \le i \le n$, and some P in $\mathbf{P}$, $P(a_i) = P(b) = 1$, and so $P(a_i \,\&\, b) = 1$, i.e., $\tau(\boldsymbol{B}(a_i \,\&\, b)) = 1$.

Theorem 4.5 (Completeness of $\mathbf{LMNPC_n}$) *If a is not a theorem of* $\mathbf{LMNPC_n}$ *then there is an mp-model* $\langle u,\mathbf{P}\rangle$ *in which* $\mathbf{P}$ *contains at most n probability distributions and* $\tau(a) = 0$.

Proof: Suppose that a is not a theorem of $\mathbf{LMNPC_n}$ and let F be the filter $\{b \in B\text{: } b \text{ is a theorem of } \mathbf{LMNPC_n}\}$. As $a \notin F$, there is, by a Lindenbaum's Lemma construction, an ultrafilter U such that $F \subseteq U$ and $a \notin U$. The set $S = \{b \in B : \boldsymbol{B}b \in U\}$, we know by Lemma 2.11, is an n-block. By Lemma 2.9, there is a family $\mathbf{P}$ of n or fewer probability distributions such that $b \in S$ iff $\exists P \in \mathbf{P}\ [P(b) = 1]$. Let $u : At(B) \to \{0,1\}$ be the function defined by the condition

$$u(b) = 1 \text{ iff } b \in U.$$

$\langle u, \mathbf{P} \rangle$ is an *mp*-model. We need to show that, for all $b \in B$, $\tau(b) = 1$ iff $b \in U$. Proof is by induction on the complexity of propositions. Again, the only interesting case arises when b is of the form $\boldsymbol{B}c$:

$$\tau(\boldsymbol{B}c) = 1 \text{ iff } \exists P \in \mathbf{P}\ [P(c) = 1] \text{ iff } c \in S \text{ iff } \boldsymbol{B}c \in U.$$

Consequently, $\tau(a) = 0$ in the *mp*-model $\langle u, \mathbf{P} \rangle$.

Because **LMNP** and $\mathbf{LMNPC_n}$ are contained in **LMNPC** and because they contain N, M, and RL, the triviality result for **LMNPC** (Corollary 3.5) tells us that **LMNP** and $\mathbf{LMNPC_n}$ are likewise accurate but trivial. $\boldsymbol{B}a$ is a theorem of $\mathbf{LMNP/LMNPC_n}$ just in case a is a theorem of **PL**; $\neg \boldsymbol{B}a$ is a theorem of $\mathbf{LMNP/LMNPC_n}$ just in case $\neg a$ is a theorem of **PL**.

Just as we were led to strengthen **LMNPC** by considering a subclass of probability distributions/distribution-pairs closed under a condition equivalent to RN holding in the related doxastic logic, so too we now consider restricting the probability distributions from which the members of **P** in *mp*-models may be taken.

This time we look for a class $\underline{P'}$ closed under the condition:

if a is an expressible constraint on degrees of belief determined by all finite nonempty subsets containing n or fewer members of $\underline{P'}$ then $\forall P \in \underline{P'}\ [P(a) = 1]$.

Again this represents closure under RN but just as RN generates different theorems in different logics so the class we seek is broader than before. Moreover, in adding RN to $\mathbf{LMNPC_n}$ we do not get RE as a bonus (unless $n = 1$). On the other hand, if we add RE we do get RN. And we can argue for the incorporation of RE as in Section 1, when we argued for its addition to **LM**.

When we add the rule RE to $\mathbf{LMNPC_n}$ we obtain the logic $\mathbf{EMNPC_n}$ in which RN is a derived rule. We shall take as the class of probability distribution:

$$\underline{P'} = \{P\text{: if } a \text{ is a theorem of } \mathbf{EMNPC_n} \text{ then } P(a) = 1\}.$$

Definition 4.6 The *mp*-model $\langle u, \mathbf{P} \rangle$ is an augmented *mp*-model iff $\forall P \in \mathbf{P}\ [P \in \underline{P'}]$.

Lemma 4.7 (Soundness of $\mathbf{EMNPC_n}$) *If a is a theorem of* $\mathbf{EMNPC_n}$ *then $\tau(a) = 1$ in every augmented mp-model $\langle u, \mathbf{P} \rangle$ in which* **P** *contains at most n probability distributions.*

Proof: In view of the soundness proof for $\mathbf{LMNPC_n}$ (Lemma 4.4) we need only show that RE is sound in all augmented *mp*-models. Suppose, then, that a is a theorem of $\mathbf{EMNPC_n}$ of the form $b \equiv c$. Then, for all $P \in \mathbf{P}$, $P(b \equiv c) = 1$. Consequently, for any $P \in \mathbf{P}$, $P(b) = 1$ iff $P(c) = 1$, which establishes that $\tau(\boldsymbol{B}b) = 1$ iff $\tau(\boldsymbol{B}c) = 1$, i.e., $\tau(\boldsymbol{B}b \equiv \boldsymbol{B}c) = 1$.

Theorem 4.8 (Completeness of $\mathbf{EMNPC_n}$) *If a is not a theorem of* $\mathbf{EMNPC_n}$ *then there is an augmented mp-model $\langle u, \mathbf{P} \rangle$ in which* **P** *contains at most n probability distributions and $\tau(a) = 0$.*

Proof: The proof follows that of Theorem 4.5 but with F taken to be the set $\{a \in B\text{: } a \text{ is a theorem of } \mathbf{EMNPC_n}\}$. It remains to show only that $\forall P \in \mathbf{P}$ $[P \in \underline{P'}]$: as $\mathbf{EMNPC_n}$ contains RN, $\boldsymbol{B}a \in F$ when $a \in F$; as $F \subseteq U$, $\boldsymbol{B}a \in U$ and, consequently, $a \in S$.

Corollary 4.9 (Accuracy of $\mathbf{EMNPC_n}$) (1) *If* $\boldsymbol{Ba}$ *is a theorem of* $\mathbf{EMNPC_n}$ *then a is also a theorem.* (2) *If* $\neg\boldsymbol{Ba}$ *is a theorem of* $\mathbf{EMNPC_n}$ *then* $\neg a$ *is a theorem.*

Proof:

(1) Suppose that a is not a theorem of $\mathbf{EMNPC_n}$. Then, by Theorem 4.8 and in analogy with the proof of the accuracy of **KD** (Corollary 3.9), there is a function $\tau: B \to \{0,1\}$ such that (i) $\tau(a) = 0$; (ii) if b is a theorem of $\mathbf{EMNPC_n}$ then $\tau(b) = 1$; and (iii) τ is a probability distribution over B. But then, for arbitrary $u: At(B) \to \{0,1\}$, $\langle u, \{\tau\}\rangle$ is an augmented *mp*-model in which $\tau'(\boldsymbol{Ba}) = 0$, τ' being the assignment of truth-values derived from $\langle u, \{\tau\}\rangle$.

(2) is proved analogously.

By restricting the probability distributions in *mp*-models we can produce soundness and completeness theorems for a variety of logics. The general method has been sufficiently demonstrated here. I shall mention some special cases of interest. First, soundness and completeness of **EMNP** relative to probability distributions belonging to the class

$$\underline{P}'' = \{P\text{: if } a \text{ is a theorem of } \mathbf{EMNP} \text{ then } P(a) = 1\}.$$

Second, by substituting $\underline{P}$ of Section 3 for $\underline{P}'$ above soundness and completeness of a logic whose theorems are those of $\mathbf{EMNPC_n}$ together with the doxastic necessitations of theorems of **KD**. This gives rise to a positively inaccurate logic whose associated believed doxastic logic is **KD**. This logic might be motivated by Stalnaker's model because, even if the agent is in a multiplicity of belief states, what she believes about belief will be determined in any context relative to the belief state associated with that context, plausibly leading the agent to believe that **KD** is the logic of belief.

Third, by relaxing the condition that **P** be non-empty, soundness and completeness theorems are forthcoming for **EMVP** and $\mathbf{EMVPC_n}$. In the first case **P** must be a subset of $\underline{P}''$, in the second a subset of $\underline{P}'$. These results emphasize the connection between the family $\mathfrak{R}$ of weak standard models and the family **P** of multiple probability models. Weak standard models allow a logic as weak as **EMV**. Multiple probability models allow a logic as weak as **LMP**. **EMVP** is the meeting point, suggesting itself as a prime contender for the title Minimal Doxastic Logic. If the possibility of total scepticism is to be disallowed we move to **EMNP**. The logics $\mathbf{EMNPC_n}$ are descriptive as much as normative, since there is nothing in the concept of justified/rational belief that fixes a specific value of n. The next natural stopping point after **EMNP** is, therefore, **KD**, a logic which may be thought to disqualify itself from any pretension to minimality because it rules out the possibility that justified or rationally held beliefs may be contradictory.

As a final observation, I have, following Stalnaker's model of belief, taken these weak doxastic logics to furnish norms governing the justified/rationally held beliefs of a single agent. They might equally well be regarded as governing the beliefs of a collective, for formally the plurality of belief states could belong to a plurality of individuals, each individual being in one belief state as in Hintikka's models. Rather than human agents, the individuals in question might be databases storing possibly jointly inconsistent bodies of information.

NOTES

1. Gärdenfors uses a definition of proposition derived from the dynamics of belief states as a means to proving the completeness of intuitionist logic ([4], pp. 132–41). As the representation considers only expansions of belief states it is no surprise that the most natural logic is intuitionist. However, when he considers probabilistic models of belief states the propositional logic is effectively classical since $P(a \vee \neg a) = P(a) + P(\neg a) = 1$ (see, e.g., [4], pp. 37–40).

2. Where possible I follow the nomenclature of [2] but RL is already a point of departure.

3. In this setting an ultrafilter is just a maximal consistent set of propositions, consistency being determined in **EM**. When the algebra of propositions is countable we can use the standard Lindenbaum's Lemma method to construct maximal consistent sets rather than appeal to the Ultrafilter Theorem (see, e.g., [2], pp. 55–7). The standard proof of the theorem invokes Zorn's Lemma (see [1], pp. 136–7).

4. The proof of Theorem 1.16 owes some to [2], Chapters 7–9. My minimal models are Chellas's supplemented minimal models.

5. The use of N_c fits Chellas's pattern (see, e.g., [2], p. 71) since $\boldsymbol{B}\top$ is logically equivalent in **PL** to $\neg\boldsymbol{B}\top \rightarrow \boldsymbol{B}\top$, $\neg\boldsymbol{B}\top$ to $\boldsymbol{B}\top \rightarrow \neg\boldsymbol{B}\top$, and $\neg\boldsymbol{B}\top \rightarrow \boldsymbol{B}\top$ and $\boldsymbol{B}\top \rightarrow \neg\boldsymbol{B}\top$ are converses.

6. We have the means at our disposal for a completely general description of total scepticism: a doxastic logic is compatible with total scepticism if and only if the believed doxastic logic associated with it is empty. This characterization holds good even for doxastic logics that do not contain the rule RL. Without RL and M the axiom $\mathrm{N_c}$ is not sufficient for total scepticism.

7. Cf. Walley [15], §2.9.8.

8. Cf. [4], p. 39, Lemma 2.3.

9. See, e.g., Milne [13], pp. 307–8. The exact standing of these arguments is called into question below. In [13], pp. 310, 313, it is hinted that they might best be regarded as artefacts of the betting situation rather than constraints on rational belief.

10. In the context of neither of the two mentioned sources for the representation of beliefs by families of probability distributions—those of Levi ([9]) and of Gärdenfors and Sahlin ([5])—is this stipulation remotely acceptable.

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