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The Expressive Truth Conditions of Two-Valued Logic

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Abstract In a finitary closure space, irreducible sets behave like two-valued models, with membership playing the role of satisfaction. If f is a function on such a space and the membership of $fx_1, \ldots, x_n$ in an irreducible set is determined by the presence or absence of the inputs $x_1, \ldots, x_n$ in that set, then f is a kind of truth function. The existence of some of these truth functions is enough to guarantee that every irreducible set is maximally consistent. The closure space is then said to be *expressive*. This paper identifies the two-valued truth functional conditions that guarantee expressiveness.

1 Introduction

Say that a closure space with domain S is *expressive* if, whenever A is a closed proper subset of B, there is a closed set D such that $A \subseteq D \neq \mathrm{Cl}(B \cup D) = S$. Think of an inconsistent set as one whose closure is the whole domain S. Then the idea is that we can find items that yield a consistent set when added to A, but yield an inconsistent one when added to B. If $\neg$ behaves like two-valued negation, then we just pick an x that belongs to B but not to A. If we add $\neg x$ to A, we obtain a consistent set, but when we add it to B we obtain an inconsistent one (Martin and Pollard [3], pp. 121–22; [4], p. 122). So closure spaces with the right sort of negation function are expressive. This makes it tempting to think of expressive closure spaces as ones in which some analogue of classical negation is definable (Pollard and Martin [5], p. 113); but that is not quite right. A finitary closure space will also be expressive if it has a function or relation that behaves like material implication ([3], pp. 187–88; [4], p. 125). This paper identifies all the two-valued truth functions

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whose closure-theoretic properties guarantee expressiveness. This is of some interest because a finitary closure space is expressive if and only if its maximally consistent sets form a closed basis (that is, if and only if every closed set is the intersection of maximally consistent sets). The existence of such a closed basis is incompatible with an intuitionistic treatment of negations and conditionals. So, by identifying closure theoretic properties that guarantee expressiveness, we identify deductive properties antithetical to intuitionism.

2 Preliminaries

Let $\langle S, \mathrm{Cl}\rangle$ be a *closure space*. That is, (1) Cl is a function that assigns subsets of S to subsets of S; and (2) if A and B are any subsets of S, then $A \subseteq \mathrm{Cl}(B)$ if and only if $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(B)$. $\mathrm{Cl}(A)$ is said to be the *closure* of A. Readers unaccustomed to working with abstract closure functions might find it helpful to think of A as a set of sentences and think of $\mathrm{Cl}(A)$ as the set of sentences that follow from or are derivable from members of A. We assume, for the remainder of this paper, that $\langle S, \mathrm{Cl}\rangle$ is *finitary*. That is, if $A \subseteq S$ and $x \in \mathrm{Cl}(A)$, then x belongs to the closure of some finite subset of A. In the case of implicational or deductive closure, this would mean that a sentence follows from or is derivable from sentences only if it follows from or is derivable from finitely many of those sentences. Say that a subset of S is *closed* just in case it contains (and, hence, is identical to) its own closure. (If it helps, take 'closed' to mean closed under consequence or derivability.) Finally, say that a set B is *irreducible* just in case B is closed but is not the intersection of closed sets all distinct from B. In a finitary closure space, every closed set is the intersection of irreducible sets. (Since we adopt the convention that $\cap\varnothing$ is the universe, S is the intersection of the empty set of irreducible sets.) So, once the universe is fixed, the structure of the whole space is determined by the irreducible sets (Crawley and Dilworth [1], pp. 43–44; [3], p. 161; Wójcicki [10], p. 27). In particular, if we can arrange for the irreducible sets to treat an operator in the right way, we can arrange for that operator to behave well throughout the space.

It is often helpful to think of irreducible sets as closure theoretic surrogates for *models*. In a finitary closure space, $x \in \mathrm{Cl}(A)$ if and only if x belongs to every irreducible set that contains A (just as x is a consequence of A if and only if x is true in every interpretation that satisfies every member of A). So membership in an irreducible set is like truth in a model and it makes some sense to talk about the "semantics" of functions in our closure space.

Here is an example. Suppose, for each irreducible set B and each $x \in S$, that B "satisfies" $\neg x$ if and only if B does not "satisfy" x (that is, $\neg x \in B$ if and only if $x \notin B$). Then $\neg$ will behave just like an operator that obeys the truth table for two-valued negation. Note, for instance, that $\mathrm{Cl}(\{\neg\neg x\})$ is the intersection of the irreducible sets that have $\neg\neg x$ as a member. It is easy to see that x belongs to each of those sets (since $\neg x$ does not). So $x \in \mathrm{Cl}(\{\neg\neg x\})$. That is, $\neg$ satisfies the classical double negation elimination principle and it does so because it has the right "semantic" properties: for each x, $\neg x$ belongs to just the right irreducible sets.

Given any two-valued truth function, we can translate its truth conditional properties into abstract closure theoretic properties. Any function that has the latter properties will have "semantic" characteristics that mirror those of the truth function. The next section shows how this is done.

3 Truth Conditions

Given a two-valued truth function, our strategy is to write out its truth table and transform each line into a statement about the interactions between our closure operator and a function $f : S^n \to S$. We are trying to guarantee that the conditions under which $fx_1, \ldots, x_n$ is "satisfied" by an irreducible set will mirror those under which an application of the truth function is satisfied by a two-valued interpretation. We will use the truth table entries to label the closure theoretic conditions generated from them. If the truth table entry says that a pair of **T**s yields an **F**, our name for the corresponding closure theoretic condition will be $\mathbf{TT} \Rightarrow \mathbf{F}$. More generally, $\mathbf{V}_1, \ldots, \mathbf{V}_n \Rightarrow \mathbf{V}_0$ (where each of the **V**s is either truth or falsehood) will be the condition on Cl and f generated from the truth table entry with inputs $\mathbf{V}_1, \ldots, \mathbf{V}_n$ and output $\mathbf{V}_0$. When we use such an entry to characterize an n-ary function $fx_1, \ldots, x_n$, we let T (the set of "true inputs") be $\{x_k : \mathbf{V}_k = \mathbf{T}\}$ and we let F (the set of "false inputs") be $\{x_k : \mathbf{V}_k = \mathbf{F}\}$. Then we say

f satisfies $\mathbf{V}_1, \ldots, \mathbf{V}_n \Rightarrow \mathbf{T}$ just in case

$$\left(\bigcap_{x \in F} \mathrm{Cl}(A \cup \{x\})\right) \cap \mathrm{Cl}(A \cup \{fx_1, \ldots, x_n\}) \subseteq \mathrm{Cl}(A \cup T)$$

whenever $A \subseteq S$ and $x_1, \ldots, x_n \in S$;

f satisfies $\mathbf{V}_1, \ldots, \mathbf{V}_n \Rightarrow \mathbf{F}$ just in case

$$\bigcap_{x \in F} \mathrm{Cl}(A \cup \{x\}) \subseteq \mathrm{Cl}(A \cup T \cup \{fx_1, \ldots, x_n\})$$

whenever $A \subseteq S$ and $x_1, \ldots, x_n \in S$.

Students of multiple-conclusion logics may recognize that this is a closure theoretic version of Shoesmith and Smiley's technique for characterizing two-valued truth functions ([7], p. 312–13; cf. Kneale [2], pp. 246–47; see also Scott [6], p. 415). To see how the technique works in practice, consider the truth table for classical negation.

z	$\neg z$
T	**F**
F	**T**

To associate a closure theoretic principle with each line of the truth table, we say

$\neg$ satisfies $\mathbf{T} \Rightarrow \mathbf{F}$ just in case

$$\bigcap_{x \in \varnothing} \mathrm{Cl}(A \cup \{x\}) \subseteq \mathrm{Cl}(A \cup \{z\} \cup \{\neg z\})$$

whenever $A \subseteq S$ and $z \in S$;

$\neg$ satisfies $\mathbf{F} \Rightarrow \mathbf{T}$ just in case

$$\left(\bigcap_{x \in \{z\}} \mathrm{Cl}(A \cup \{x\})\right) \cap \mathrm{Cl}(A \cup \{\neg z\}) \subseteq \mathrm{Cl}(A \cup \varnothing)$$

whenever $A \subseteq S$ and $z \in S$.

Each principle can be simplified.

$\neg$ satisfies $\mathbf{T} \Rightarrow \mathbf{F}$ just in case $S \subseteq \mathrm{Cl}(\{z, \neg z\})$ for all $z \in S$.

$\neg$ satisfies $\mathbf{F} \Rightarrow \mathbf{T}$ just in case $\mathrm{Cl}(A \cup \{z\}) \cap \mathrm{Cl}(A \cup \{\neg z\}) \subseteq \mathrm{Cl}(A)$ whenever $A \subseteq S$ and $z \in S$.

In a deductive setting, $\mathbf{T} \Rightarrow \mathbf{F}$ would say that every sentence is derivable from a contradiction (ex falso quodlibet), whereas $\mathbf{F} \Rightarrow \mathbf{T}$ would say that something is derivable whenever it is derivable from each term of a contradiction. If we let A be the empty set, we obtain the two axioms for classical negation from Tarski [8] ([9], ch. 3; see also [3], p. 68).

Tarski's Axiom 9* $\mathrm{Cl}(\{z, \neg z\}) = S$.

Tarski's Axiom 10* $\mathrm{Cl}(\{z\}) \cap \mathrm{Cl}(\{\neg z\}) = \mathrm{Cl}(\varnothing)$.

As so often occurs in mathematics, the order of exposition here is the reverse of the order of discovery. The two Tarski axioms were the starting point and the abstract schemes emerged as generalizations of them.

We refer to each of our principles $\mathbf{V}_1, \ldots, \mathbf{V}_n \Rightarrow \mathbf{V}_0$ as a *truth condition*. We now show that each of our truth conditions supplies f with the right "semantic" properties (that is, each truth condition guarantees that the membership or nonmembership of $fx_1, \ldots, x_n$ in an irreducible set will depend in the intended way on the membership or nonmembership of the arguments $x_1, \ldots, x_n$ in that set).

Theorem 3.1 *Suppose the n-ary function f satisfies the truth condition* $\mathbf{V}_1, \ldots, \mathbf{V}_n \Rightarrow \mathbf{T}$. *Pick any irreducible set B. Then $fx_1, \ldots, x_n \in B$ if $T \subseteq B$ and $(F \cap B) = \varnothing$.*

Proof Under the hypotheses of the theorem,

$$\left(\bigcap_{x \in F} \mathrm{Cl}(B \cup \{x\})\right) \cap \mathrm{Cl}(B \cup \{fx_1, \ldots, x_n\}) \subseteq B.$$

If $F = \varnothing$, then it is trivial that $fx_1, \ldots, x_n \in B$. Suppose $F \neq \varnothing$. Then it would contradict B's irreducibility if $fx_1, \ldots, x_n \notin B$. □

For example, if the function $\neg$ satisfies $\mathbf{F} \Rightarrow \mathbf{T}$, then $\neg z$ will belong to an irreducible set whenever z does not.

Theorem 3.2 *Suppose the n-ary function f satisfies the truth condition* $\mathbf{V}_1, \ldots, \mathbf{V}_n \Rightarrow \mathbf{F}$. *Pick any irreducible set B. Then $fx_1, \ldots, x_n \notin B$ if $T \subseteq B$ and $(F \cap B) = \varnothing$.*

Proof Under the hypotheses of the theorem,

$$\bigcap_{x \in F} \mathrm{Cl}(B \cup \{x\}) \subseteq \mathrm{Cl}(B \cup \{fx_1, \ldots, x_n\}).$$

If $F = \varnothing$ and $fx_1, \ldots, x_n \in B$, then $B = \cap\varnothing = S$, which is impossible. If $F = \{x_k\}$ and $fx_1, \ldots, x_n \in B$, then $x_k \in B$, which contradicts our assumption that no member of F belongs to B. Suppose F has more than one member. Then it would contradict B's irreducibility if $fx_1, \ldots, x_n \in B$. □

If the function $\neg$ satisfies $\mathbf{T} \Rightarrow \mathbf{F}$, then $\neg z$ will not belong to an irreducible set if z does. The converse of Theorem 3.2 holds, as does that of Theorem 3.1.

Theorem 3.3 *If, for each irreducible set B, $fx_1, \ldots, x_n \in B$ whenever $T \subseteq B$ and $(F \cap B) = \varnothing$, then f satisfies the truth condition* $\mathbf{V}_1, \ldots, \mathbf{V}_n \Rightarrow \mathbf{T}$.

Proof Suppose $z \notin \mathrm{Cl}(A \cup T)$. Then we can pick an irreducible set B that contains $(A \cup T)$ but does not have z as a member. Suppose $z \in \mathrm{Cl}(A \cup \{fx_1, \ldots, x_n\})$. Then z belongs to every irreducible set that contains $(A \cup \{fx_1, \ldots, x_n\})$. So $fx_1, \ldots, x_n \notin B$ and, hence, $(F \cap B) \neq \varnothing$. Let x belong to both F and B. Then not every irreducible set that contains $(A \cup \{x\})$ has z as a member. So $z \notin \mathrm{Cl}(A \cup \{x\})$. □

If $\neg z$ belongs to an irreducible set whenever z does not, then the function $\neg$ satisfies $\mathbf{F} \Rightarrow \mathbf{T}$.

Theorem 3.4 *If, for each irreducible set B, $f x_1, \ldots, x_n \notin B$ whenever $T \subseteq B$ and $(F \cap B) = \varnothing$, then f satisfies the truth condition $\mathbf{V}_1, \ldots, \mathbf{V}_n \Rightarrow \mathbf{F}$.*

Proof Suppose $z \notin \mathrm{Cl}(A \cup T \cup \{f x_1, \ldots, x_n\})$. Then we can pick an irreducible set B that contains $(A \cup T \cup \{f x_1, \ldots, x_n\})$ but does not have z as a member. Note that $(F \cap B) \neq \varnothing$ and proceed as in the previous proof. □

If z and $\neg z$ never belong to the same irreducible set, then the function $\neg$ satisfies $\mathbf{T} \Rightarrow \mathbf{F}$.

4 Expressiveness

We now say that a set of n-ary truth conditions is *consistent* if and only if none of the associated truth table entries assign both $\mathbf{T}$ and $\mathbf{F}$ to the same sequence of inputs. (For example, you do not have both $\mathbf{TF} \Rightarrow \mathbf{T}$ and $\mathbf{TF} \Rightarrow \mathbf{F}$.) We extend the notion of expressiveness from closure spaces to sets of truth conditions by stipulating that a consistent set of n-ary truth conditions is *expressive* if and only if every finitary closure space with a function satisfying those conditions is expressive. A closed subset of our domain is *maximally consistent* if and only if its only closed proper superset is the whole domain S. A finitary closure space is expressive if and only if each of its irreducible sets is maximally consistent ([3], Theorems 6.4, 6.33, 6.35, and 7.32). We use this fact in the proofs of some of the following results. By $\mathbf{F}, \ldots, \mathbf{F}$ we mean a sequence of inputs all of which are $\mathbf{F}$. $\mathbf{T}, \ldots, \mathbf{T}$ will be a sequence of inputs all of which are $\mathbf{T}$.

Lemma 4.1 *A consistent set of n-ary truth conditions will not be expressive unless $\mathbf{F}, \ldots, \mathbf{F} \Rightarrow \mathbf{T}$ is one of its members.*

Proof Suppose $(\mathbf{F}, \ldots, \mathbf{F} \Rightarrow \mathbf{T}) \notin \Theta$. We need just one finitary, nonexpressive closure space with a function satisfying each member of Θ. Consider the following lattice of closed sets.

$$\begin{array}{c} \{a, b\} \\ | \\ \{b\} \\ | \\ \varnothing \end{array}$$

Notice that $\varnothing$ is irreducible but not maximally consistent. So this closure space is not expressive. Let

$$\tau(x) = \begin{cases} \mathbf{T} & \text{if } x = b \\ \mathbf{F} & \text{if } x = a \end{cases}$$

and let

$$g x_1, \ldots, x_n = \begin{cases} b & \text{if } (\tau(x_1), \ldots, \tau(x_n) \Rightarrow \mathbf{T}) \in \Theta \\ a & \text{otherwise.} \end{cases}$$

We want to show that g satisfies every truth condition in Θ. Suppose, first, that $(\mathbf{V}_1, \dots, \mathbf{V}_n \Rightarrow \mathbf{T}) \in \Theta$. By Theorem 3.3, we need only verify that $gx_1, \dots, x_n$ belongs to our two irreducible sets ($\varnothing$ and $\{b\}$) under the right conditions. As before, we let $T = \{x_k : \mathbf{V}_k = \mathbf{T}\}$ and $F = \{x_k : \mathbf{V}_k = \mathbf{F}\}$. Since $(\mathbf{F}, \dots, \mathbf{F} \Rightarrow \mathbf{T}) \notin \Theta$, T is nonempty. So it is vacuously true that $gx_1, \dots, x_n \in \varnothing$ whenever $T \subseteq \varnothing$ and $(F \cap \varnothing) = \varnothing$. Suppose $T \subseteq \{b\}$ and $(F \cap \{b\}) = \varnothing$. Then $\mathbf{V}_k = \tau(x_k)$ whenever $1 \leq k \leq n$. So $gx_1, \dots, x_n = b$ and, hence, $gx_1, \dots, x_n \in \{b\}$. Now suppose $(\mathbf{V}_1, \dots, \mathbf{V}_n \Rightarrow \mathbf{F}) \in \Theta$. By Theorem 3.4, we need only verify that $gx_1, \dots, x_n$ is a nonmember of our irreducible sets under the right conditions. Clearly $gx_1, \dots, x_n$ will not belong to $\varnothing$ under any conditions. Suppose $T \subseteq \{b\}$ and $(F \cap \{b\}) = \varnothing$. Then, again, $\mathbf{V}_k = \tau(x_k)$ whenever $1 \leq k \leq n$. Since Θ is consistent, $(\tau(x_1), \dots, \tau(x_n) \Rightarrow \mathbf{T}) \notin \Theta$. So $gx_1, \dots, x_n = a$ and, hence, $gx_1, \dots, x_n \notin \{b\}$. □

It was essential to this proof that $\varnothing$ be irreducible—as the following theorem shows.

Theorem 4.2 *A finitary closure space in which $\varnothing$ is not irreducible will be expressive if there is a function on that space satisfying $\mathbf{T}, \dots, \mathbf{T} \Rightarrow \mathbf{F}$ and at least one truth condition with output $\mathbf{T}$.*

Proof Suppose B is irreducible and $z \notin B$. We want to show that $\mathrm{Cl}(B \cup \{z\}) = S$. Suppose f satisfies $\mathbf{T}, \dots, \mathbf{T} \Rightarrow \mathbf{F}$ and $\mathbf{V}_1, \dots, \mathbf{V}_n \Rightarrow \mathbf{T}$. Pick $y \in B$ and let

$$x_i = \begin{cases} y & \text{if } \mathbf{V}_i = \mathbf{T} \\ z & \text{if } \mathbf{V}_i = \mathbf{F}. \end{cases}$$

Then, by Theorem 3.1, $fx_1, \dots, x_n \in B$. But $\mathrm{Cl}(B \cup \{y, z, fx_1, \dots, x_n\}) = S$. So $\mathrm{Cl}(B \cup \{z\}) = S$. □

The following lemmas will help us with the proof of Theorem 4.5.

Lemma 4.3 *If f satisfies $\mathbf{V}_1, \dots, \mathbf{V}_n \Rightarrow \mathbf{F}$ and*

$$x_i = \begin{cases} y & \text{if } \mathbf{V}_i = \mathbf{F} \\ z & \text{if } \mathbf{V}_i = \mathbf{T} \end{cases}$$

then $y \in \mathrm{Cl}(\{z, fx_1, \dots, x_n\})$.

Proof If $F = \varnothing$, then $S \subseteq \mathrm{Cl}(\{z, fx_1, \dots, x_n\})$. If F is nonempty, then $F = \{y\}$ and, hence, $\mathrm{Cl}(\{y\}) \subseteq \mathrm{Cl}(\{z, fx_1, \dots, x_n\})$. □

Lemma 4.4 *A consistent set of n-ary truth conditions will not be expressive unless at least one of its members has output $\mathbf{F}$.*

Proof Suppose every member of Θ has output $\mathbf{T}$. Consider the following lattice of closed sets.

$$\begin{array}{c} \{a, b, c\} \\ | \\ \{b, c\} \\ | \\ \{c\} \end{array}$$

We need to arrange that $gx_1, \ldots, x_n$ belongs to each of our irreducible sets ($\{b, c\}$ and $\{c\}$) no matter what $x_1, \ldots, x_n$ might be. No problem: just stipulate that $gx_1, \ldots, x_n = c$. By Theorem 3.3, g satisfies every member of Θ. □

Theorem 4.5 *A consistent set of n-ary truth conditions will be expressive if and only if its members include* $\mathbf{F}, \ldots, \mathbf{F} \Rightarrow \mathbf{T}$ *and at least one truth condition with output* $\mathbf{F}$.

Proof ($\rightarrow$) Apply Lemmas 4.1 and 4.4.

($\leftarrow$) Suppose f satisfies $\mathbf{F}, \ldots, \mathbf{F} \Rightarrow \mathbf{T}$ and $\mathbf{V}_1, \ldots, \mathbf{V}_n \Rightarrow \mathbf{F}$. Suppose B is irreducible and $z \notin B$. We want to show that $\mathrm{Cl}(B \cup \{z\}) = S$. Suppose $y \notin B$ and $x_1, \ldots, x_n$ are as in Lemma 4.3. Then, by Theorem 3.1, $fx_1, \ldots, x_n \in B$. So, by Lemma 4.3, $y \in \mathrm{Cl}(B \cup \{z\})$. □

Exactly $2^{(2^n-1)} - 1$ two-valued n-ary truth functions have expressive truth conditions. The seven binary ones are

	$\leftarrow$	$\rightarrow$	$\leftrightarrow$	$\mid$	$\neg_2$	$\neg_1$	$\downarrow$
TT	**T**	**T**	**T**	**F**	**F**	**F**	**F**
TF	**T**	**F**	**F**	**T**	**T**	**F**	**F**
FT	**F**	**T**	**F**	**T**	**F**	**T**	**F**
FF	**T**	**T**	**T**	**T**	**T**	**T**	**T**

The only one of the seven that might occasion any surprise at all is material equivalence ($\leftrightarrow$). But consider three other binary truth functions.

	$\overline{\leftrightarrow}$	$\overline{\rightarrow}$	$\overline{\leftarrow}$
TT	**F**	**F**	**F**
TF	**T**	**T**	**F**
FT	**T**	**F**	**T**
FF	**F**	**F**	**F**

Each of these is definable in a finitary closure space that is not expressive. So these functions do not have expressive truth conditions. However, as Theorem 4.2 teaches us, all the counterexamples have a somewhat odd property: $\varnothing$ is irreducible. There are two circumstances under which $\varnothing$ is *not* irreducible. First, $\varnothing$ is not closed. This is equivalent to saying that $\mathrm{Cl}(\varnothing)$ has members. In a deductive setting, this would mean that at least one sentence is derivable from the empty set of premises—a perfectly common circumstance. Second, $\varnothing$ is closed but is not the intersection of other closed sets. In a sentential logic with infinitely many sentence letters, one would expect $\mathrm{Cl}(\varnothing)$ to be the intersection of $\{\mathrm{Cl}(\{P\}) : P \text{ is a sentence letter}\}$. The frequency of contrary cases will depend, of course, on what one is willing to count as a "sentential logic."

If we consider only the finitary closure spaces in which $\varnothing$ is not irreducible, the only two-valued binary truth functions whose presence does not guarantee expressiveness will be **T**-constant, **F**-constant, conjunction, disjunction, and the two binary projection functions (exactly the binary truth functions of Post's logic A_1).

5 Examples

A connective with the deductive properties of intuitionistic negation will take on the deductive properties of classical negation when combined with any operator satisfying the conditions of Theorem 4.5. Here is why. If $\neg$ obeys the principle

$$\neg x \in \mathrm{Cl}(A) \ \text{ iff } \ \mathrm{Cl}(A \cup \{x\}) = S$$

(as does intuitionistic negation) then $\neg x$ will belong to a maximally consistent set if and only if x does not ([3], p. 203). So if every irreducible set is maximally consistent, then $\neg$ will have the "truth conditions" of classical negation.

A connective with the deductive properties of intuitionistic implication will take on the deductive properties of material implication when combined with any operator satisfying the conditions of Theorems 4.2 or 4.5. If $\to$ obeys the principle

$$(x \to y) \in \mathrm{Cl}(A) \ \text{ iff } \ y \in \mathrm{Cl}(A \cup \{x\})$$

(as does intuitionistic implication) then $x \to y$ will belong to a maximally consistent set if and only if x does not or y does ([3], p. 73). So if every irreducible set is maximally consistent, then $\to$ will have the "truth conditions" of material implication. Note, furthermore, that $(x \to x) \in \mathrm{Cl}(\varnothing)$. So $\varnothing$ is not closed and, hence, Theorem 4.2 applies.

More concretely, if you want to have a connective that satisfies modus ponens and the deduction theorem, but not Peirce's law, then (for example) you cannot also have a connective f that satisfies the following principles:

TT ⇒ F $\quad \{\varphi, \psi, f(\varphi, \psi)\}$ proves θ

TF ⇒ T $$\left.\begin{array}{r} \{\xi_1, \ldots, \xi_n, \psi\} \text{ proves } \theta \\ \{\xi_1, \ldots, \xi_n, f(\varphi, \psi)\} \text{ proves } \theta \end{array}\right\} \Rightarrow \{\xi_1, \ldots, \xi_n, \varphi\} \text{ proves } \theta$$

where "proves" is a derivability relation that induces a finitary closure space (that is, Cl will be a finitary closure operator if we stipulate that $\varphi \in \mathrm{Cl}(A)$ if and only if A *proves* φ). Your hope of avoiding Peirce's law would also be dashed if f satisfied modus ponens (**TF ⇒ F**) and

FF ⇒ T $$\left.\begin{array}{r} \{\xi_1, \ldots, \xi_n, \varphi\} \text{ proves } \theta \\ \{\xi_1, \ldots, \xi_n, \psi\} \text{ proves } \theta \\ \{\xi_1, \ldots, \xi_n, f(\varphi, \psi)\} \text{ proves } \theta \end{array}\right\} \Rightarrow \{\xi_1, \ldots, \xi_n\} \text{ proves } \theta.$$

It seems likely that someone has noticed these facts before. Now, however, we can locate them within the general theory of finitary closure spaces.

Here is another application. Though it involves no result that is the least bit profound, it may give a sense of how expansive the province of logic appears from the closure theoretic perspective. Consider the following two conditions on the function H. Given any integers j and k, any prime p, and any positive integer n,

$H1$ If p^n divides j but not k, then p^n divides Hjk.
$H2$ If p^n divides both j and k, then p^n does not divide Hjk.

Is there a function H that assigns integers to pairs of integers and satisfies both $H1$ and $H2$? There are infinitely many functions that satisfy $H1$. Three of the most obvious are 0-constant ($Hjk = 0$), the first projection function ($Hjk = j$), and multiplication ($Hjk = j \cdot k$). There are also infinitely many functions that satisfy $H2$.

Two of the most obvious are 1-constant ($Hjk = 1$) and the function that assigns to the pair $\langle j, k\rangle$ the first prime that does not divide either j or k. However, no function satisfies both $H1$ and $H2$. Here is one reason why. The subgroups of the additive group of integers form a finitary, nonexpressive closure space in which the empty set is not irreducible ([3], pp. 135–38). The irreducible sets are the cyclic groups generated by a power of a prime. So $H1$ is a version of **TF** $\Rightarrow$ **T**, while $H2$ is a version of **TT** $\Rightarrow$ **F**. Theorem 4.2 guarantees that H will not satisfy both.

6 Summary

In a finitary, expressive closure space every closed set is the intersection of maximally consistent sets. So, once the universe of such a closure space is given, all its remaining structure is determined by the maximally consistent sets. (The universe is always the intersection of a set of maximally consistent sets, namely, the empty one. But you cannot figure out what $\bigcap\varnothing$ is supposed to be unless you already know what the universe is. If the universe is not the union of the maximally consistent sets, then the maximally consistent sets alone do not tell you what the universe is.) It is natural to inquire into the circumstances under which the system UNIVERSE + MAXIMALLY CONSISTENT SETS determines the structure of the whole closure space. This paper has contributed to that inquiry by identifying all the two-valued truth functions whose truth conditions guarantee that every irreducible set is maximally consistent. If any of these functions are definable on a finitary closure space, every closed set will be the intersection of maximally consistent sets and the system UNIVERSE + MAXIMALLY CONSISTENT SETS will tell you everything there is to know about the closure space.

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