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Anshakov-Rychkov Algebras

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Abstract The aim of this paper is to show that the calculi described by Anshakov and Rychkov are algebraizable in the sense of Blok and Pigozzi. As a consequence, a proof of the strong completeness of these calculi is obtained.

1 Introduction

The aim of this paper is to give an algebraic treatment of the finite-valued calculi given by Anshakov and Rychkov [2]. These calculi are very close to the ones considered by Rosser and Turquette [8]. Particular cases are classical logic and Moisil n-valued logics.

Let $n \geq 2$ and $\mathbb{F} = \{J_0, J_1, \ldots, J_{n-1}\} \cup \{\neg, \vee, \wedge, \supset\} \cup \mathbb{G}$ be a finite set of finitary operations such that $\neg$, J_i, $i = 0, \ldots, n-1$ are unary operations and $\vee, \wedge, \supset$ are binary operations. *Anshakov-Rychkov algebras* are algebras of type $\mathbb{F}$ such that they satisfy natural axioms that correspond to the axioms of the propositional calculus. We are going to prove that these algebras are subdirect products of subalgebras of special algebras $\mathbf{M}_n = \langle M_n, \mathbb{F}\rangle$, where $M_n = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$. This fact is used to proof the strong completeness of these calculi with respect to the semantics given by the matrices $\mathbf{M}_n$. More precisely, we are going to prove that the Anshakov-Rychkov calculi are algebraizable in the sense of Blok and Pigozzi [3] with equivalent algebraic semantic $\mathcal{M}$, where $\mathcal{M}$ is a quasi variety which depends naturally on $\mathbf{M}_n$.

2 Anshakov-Rychkov Algebras

In what follows, n will always denote an integer ≥ 2. Let $\mathbb{F} = \{J_0, J_1, \ldots, J_{n-1}\} \cup \{\neg, \vee, \wedge, \supset\} \cup \mathbb{G}$ be a finite set of finitary operations such that $\neg$, J_i, $i = 0, \ldots, n-1$ are unary operations and $\vee, \wedge, \supset$ are binary operations. The set $\mathbb{G}$ may be empty. We denote by $\mathcal{L}_{\mathbb{F}}$ the language of algebras given by $\mathbb{F}$.

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Definition 2.1 An *n-matrix of type* $\mathbb{F}$ is an algebra of language $\mathcal{L}_{\mathbb{F}}$ such that its universe M_n is the set of rational fractions $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ and the operations in $\mathbb{F}$ satisfy the following conditions, for every x, y, z in M_n:

C_{J_i} $\quad J_i\left(\frac{j}{n-1}\right) = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i; \end{cases}$

$C_{\{0,1\}}$ The set $\{0, 1\}$ is closed under the operations $\neg, \vee, \wedge, \supset$, and

$$\langle \{0, 1\}, \neg, \vee, \wedge, \supset, 0, 1\rangle$$

is a Boolean algebra with the usual meaning of the operation $\neg, \vee, \wedge, \supset$;

$C_\wedge$ $\quad x \wedge y = 1$ iff $x = y = 1$;

$C_\vee$ $\quad x \vee y = 1$ iff $x = 1$ or $y = 1$;

$C_{\supset 1}$ $\quad x \supset (y \supset x) = 1$;

$C_{\supset 2}$ $\quad (x \supset (y \supset z)) \supset ((x \supset y) \supset (x \supset z)) = 1$;

$C_{\supset 3}$ $\quad 1 \supset x = 1$ iff $x = 1$.

Observe that we are considering no order relation on the set $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ besides $0 < 1$.

Given an algebra $\mathbf{A} = \langle A, \mathbb{F}\rangle$ of language $\mathcal{L}_{\mathbb{F}}$, the elements of the set

$$E(\mathbf{A}) = \{J_i(a) : a \in A, i = 0, \ldots, n-1\}$$

are called the *exterior elements* of A.

Definition 2.2 Let $\mathbf{M}$ be an n-matrix of type $\mathbb{F}$ (n an integer ≥ 2). $\mathbf{A}$ is an *L_n-algebra for* $\mathbf{M}$ or an *Anshakov-Rychkov algebra of* $\mathbb{F}$ provided that the following conditions hold.

Axiom 2.3 (n-valuedness)

n-Val$_i$ $\quad J_i(a) = \bigwedge_{j \neq i} \neg J_j(a), i = 0, \ldots, n-1.$

Axiom 2.4 (connective J)

Cn-J$_0$ $\quad J_0(J_i(a)) = \neg J_i(a)$,
Cn-J$_{n-1}$ $\quad J_{n-1}(J_i(a)) = J_i(a)$,
Cn-J$_i$ $\quad J_i(J_j(a)) = J_{n-1}(a) \wedge \neg J_{n-1}(a), i \notin \{0, n-1\}$.

Axiom 2.5 (closure)

Cl-J$_*$ $\quad J_i(a) * J_j(b) = J_{n-1}(J_i(a) * J_j(b))$, where $* \in \{\vee, \wedge, \supset\}$,
Cl-J$_\neg$ $\quad \neg J_{n-1}(J_i(a)) = J_{n-1}(\neg J_i(a))$.

Observe that this axiom and Cn-J$_0$ imply that the set $E(\mathbf{A})$ is closed under the operations $\neg, \vee, \wedge$, and $\supset$.

Axiom 2.6 (exterior elements)

$$\text{For any } x \in A, y \in A : J_0(x) \vee \neg J_0(x) = J_0(y) \vee \neg J_0(y).$$

Remark 2.7 We define $1 := J_0(x) \vee \neg J_0(x)$ and $0 := \neg 1$. Observe that from the previous axioms $\langle E(\mathbf{A}), \vee, \wedge, \supset, \neg, 0, 1\rangle$ is a Boolean algebra.

Axiom 2.8 (connective)

Cn-G $\quad J_i(G(a_1, \ldots, a_m)) = \bigvee_{G(\frac{i_1}{n-1}, \ldots, \frac{i_m}{n-1}) = \frac{i}{n-1}} \bigwedge_{j=1}^{m} J_{i_j}(a_j)$, for each $G \in \mathbb{F} - \{J_i, i = 0, \ldots, n-1\}$. We define $\bigvee \varnothing = 0$.

Axiom 2.9 (quasi identity)

$$\text{If } J_i(a) = J_i(b) \text{ for all } i = 0, \ldots, n-1, \text{ then } a = b.$$

Remark 2.10 Observe that there is an axiom Cn for each connective in the set $\{\neg, \vee, \wedge, \supset\} \cup \mathbb{G}$. Hence, these axioms explicitly depend on the definitions of the operations in $\mathbb{G}$ on $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ according to the matrix $\mathbf{M}$. For each set $\mathbb{G}$, let $\mathcal{M} = \mathcal{M}(\mathbf{M})$ be the class of algebras of language $\mathcal{L}_{\mathbb{F}}$ axiomatized by Axiom 2.3 – Axiom 2.9; that is, $\mathbf{A} \in \mathcal{M}$ if and only if $\mathbf{A}$ is an L_n-algebra for $\mathbf{M}$. Since these axioms are identities or quasi identities, it follows that *$\mathcal{M}$ is a quasi variety*.

Lemma 2.11 *$\mathbf{M}$ is an L_n-algebra for the n-matrix $\mathbf{M}$ with $E(\mathbf{M}) = \{0, 1\}$, and it is the only possible structure of L_n-algebra for $\mathbf{M}$.*

Proof The axioms are obviously satisfied. For instance, we check Axiom 2.8 (Cn-$*$), where $* \in \mathbb{F} = \{\neg, \vee, \wedge, \supset\} \cup \mathbb{G}$.

$$J_i(*(a_1, a_2, \ldots, a_k)) = \bigvee_{*\left(\frac{i_1}{n-1}, \frac{i_2}{n-1}, \ldots, \frac{i_k}{n-1}\right) = \frac{i}{n-1}} \left(J_{i_1}(a_1) \wedge J_{i_2}(a_2) \wedge \cdots \wedge J_{i_k}(a_k)\right).$$

Since $a_j \in M_n, a_j = \frac{r_j}{n-1}$, $j = 1, \ldots, k$. $J_i(*(a_1, a_2, \ldots, a_k)) = 1$ if and only if $*(a_1, a_2, \ldots, a_k) = \frac{i}{n-1}$. On the other hand, from ($C_\vee$),

$$\bigvee_{*\left(\frac{i_1}{n-1}, \frac{i_2}{n-1}, \ldots, \frac{i_k}{n-1}\right) = \frac{i}{n-1}} \left(J_{i_1}\left(\frac{r_1}{n-1}\right) \wedge J_{i_2}\left(\frac{r_2}{n-1}\right) \wedge \cdots \wedge J_{i_k}\left(\frac{r_k}{n-1}\right)\right) = 1$$

if and only if there are $i_1, i_2, \ldots, i_k$ such that $i_j = r_j, j = 1, \ldots, k$ and $*\left(\frac{i_1}{n-1}, \frac{i_2}{n-1}, \ldots, \frac{i_k}{n-1}\right) = \frac{i}{n-1}$. Since $*\left(\frac{i_1}{n-1}, \frac{i_2}{n-1}, \ldots, \frac{i_k}{n-1}\right) = \frac{i}{n-1}$, this axiom is satisfied.

The connective axioms and Axiom 2.9 ensure that an n-matrix $\mathbf{M}$ admits only one structure of L_n-algebra. □

Remark 2.12 Lemma 2.11 can be generalized as follows. If

$$\mathbf{A}_1 = \langle A, J_0, \ldots, J_{n-1}, \neg, \vee, \wedge, \supset, G_1, \ldots, G_l \rangle$$

and

$$\mathbf{A}_2 = \left\langle A, J_0, \ldots, J_{n-1}, \overline{\neg}, \overline{\vee}, \overline{\wedge}, \overline{\supset}, \overline{G_1}, \ldots, \overline{G_l} \right\rangle$$

are L_n-algebras for $\mathbf{M}$, then $\mathbf{A}_1 = \mathbf{A}_2$.

Lemma 2.13 *Let $\mathbf{A}$ be an L_n-algebra for $\mathbf{M}$. The following properties hold for each $a \in A$.*

1. $\bigvee_{i=0}^{n-1} J_i(a) = 1$.
2. *If $i \neq j$, then $J_i(a) \wedge J_j(a) = 0$.*
3. *$J_i(a) = 1$, then $J_j(a) = 0$ for each $i \neq j$.*
4. *If $J_{n-1}(a) = 1$, then $a = 1$.*
5. *If $J_0(a) = 1$, then $a = 0$.*

Proof

1. $\bigvee_{i=0}^{n-1} J_i(a) = J_0(a) \vee \left(\bigvee_{i=1}^{n-1} J_i(a)\right) = \left(\bigwedge_{j\neq 0} \neg J_j(a)\right) \vee \left(\bigvee_{i=1}^{n-1} J_i(a)\right) = \neg\left(\bigvee_{j\neq 0} J_j(a)\right) \vee \left(\bigvee_{i=1}^{n-1} J_i(a)\right) = 1$.
2. From Axiom 2.3, it follows that $J_i(a) \wedge J_j(a) = \bigwedge_{k\neq i} \neg J_k(a) \wedge J_j(a) = 0$.

3. Suppose $J_i(a) = 1$ and let $j \neq i$. $J_j(a) = \bigwedge_{k \neq j} \neg J_k(a) \leq \neg J_i(a) = \neg 1 = 0$.
4. It follows from item 3 and Axiom 2.9.
5. $$\begin{aligned} J_0(0) = J_0(\neg 1) &= J_0(\neg J_{n-1}(1)) \\ &= \bigvee_{\neg \frac{i}{n-1}=0} J_i(J_{n-1}(1)) \geq J_{n-1}(J_{n-1}(1)) = 1. \end{aligned}$$

From this and Axiom 2.9 we obtain 5. □

Lemma 2.14 *Let* $\mathbf{A}$ *be an* L_n*-algebra for the matrix* $\mathbf{M}$*. The following properties hold for any* a, b *in* A*:*

1. $J_{n-1}(a \vee b) = J_{n-1}(a) \vee J_{n-1}(b)$,
2. $J_{n-1}(a \wedge b) = J_{n-1}(a) \wedge J_{n-1}(b)$.

Proof

1. From Axioms 2.8 ($\mathrm{Cn}_\vee$) and ($\mathrm{C}_\vee$) of Definition 2.1 and $\bigvee_{i=0}^{n-1} J_i(c) = 1$ we obtain that

$$\begin{aligned} J_{n-1}(a \vee b) &= \bigvee_{\frac{i}{n-1} \vee \frac{j}{n-1}=1} (J_i(a) \wedge J_j(b)) \\ &= \left(\bigvee_{j=0}^{n-1} (J_{n-1}(a) \wedge J_j(b))\right) \vee \left(\bigvee_{i=0}^{n-1} (J_i(a) \wedge J_{n-1}(b))\right) \\ &= J_{n-1}(a) \vee J_{n-1}(b). \end{aligned}$$

2. From Axiom ($\mathrm{C}_\wedge$) it follows that

$$J_{n-1}(a \wedge b) = \bigvee_{\frac{i}{n-1} \wedge \frac{j}{n-1}=1} (J_i(a) \wedge J_j(b)) = J_{n-1}(a) \wedge J_{n-1}(b).$$

□

Lemma 2.15 *Let* $\mathbf{A}$ *be an* L_n*-algebra for* $\mathbf{M}$*. For any* a, b, c *in* A *one has*

$\mathrm{C}_\wedge$ $\quad a \wedge b = 1$ *iff* $a = 1$ *and* $b = 1$,
$\mathrm{C}_{\supset 1}$ $\quad a \supset (b \supset a) = 1$,
$\mathrm{C}_{\supset 2}$ $\quad (a \supset (b \supset c)) \supset ((a \supset b) \supset (a \supset c)) = 1$,
$\mathrm{C}_{\supset 3}$ $\quad 1 \supset a = 1$ *iff* $a = 1$.

Proof

($\mathrm{C}_\wedge$) Let a, b be such that $a \wedge b = 1$. It is enough to see that $J_{n-1}(a) = J_{n-1}(b) = 1$.

$$1 = J_{n-1}(a \wedge b) = \bigvee_{\frac{i_1}{n-1} \wedge \frac{i_2}{n-1}=1} J_{i_1}(a) \wedge J_{i_2}(b) = J_{n-1}(a) \wedge J_{n-1}(b).$$

Since $E(\mathbf{M})$ is a Boolean algebra, it follows that $J_{n-1}(a) = J_{n-1}(b) = 1$. The other implication is easier.

($\mathrm{C}_{\supset 1}$) From Axiom 2.8,

$$\begin{aligned} J_{n-1}(a \supset (b \supset a)) &= \bigvee_{\frac{i_1}{n-1} \supset \frac{i_2}{n-1}=1} J_{i_1}(a) \wedge J_{i_2}(b \supset a) \\ &= \bigvee_{\frac{i_1}{n-1} \supset \frac{i_2}{n-1}=1} J_{i_1}(a) \wedge \left(\bigvee_{\frac{i_3}{n-1} \supset \frac{i_4}{n-1}=\frac{i_2}{n-1}} J_{i_3}(b) \wedge J_{i_4}(a)\right) \\ &= \bigvee_{\frac{i_1}{n-1} \supset \frac{i_2}{n-1}=1} \left(\bigvee_{\frac{i_3}{n-1} \supset \frac{i_4}{n-1}=\frac{i_2}{n-1}} J_{i_1}(a) \wedge J_{i_3}(b) \wedge J_{i_4}(a)\right) \\ &= \alpha_1. \end{aligned}$$

From Lemma 2.13, $J_{i_1}(a) \wedge J_{i_4}(a) = 0$ for $i_1 \neq i_4$, and then

$$\begin{aligned}\alpha_1 &= \bigvee_{\frac{i_1}{n-1} \supset \frac{i_2}{n-1} = 1} \left(\bigvee_{\frac{i_3}{n-1} \supset \frac{i_4}{n-1} = \frac{i_2}{n-1}} J_{i_1}(a) \wedge J_{i_3}(b) \right) \\ &= \bigvee_{\frac{i_3}{n-1} \supset \frac{i_4}{n-1} = \frac{i_2}{n-1}} \left(\bigvee_{\frac{i_1}{n-1} \supset \frac{i_2}{n-1} = 1} J_{i_1}(a) \wedge J_{i_3}(b) \right) \\ &= \alpha_2.\end{aligned}$$

Recall ($C_{\supset}1$): $\frac{i_1}{n-1} \supset \left(\frac{i_3}{n-1} \supset \frac{i_1}{n-1} \right) = 1$, then

$$\begin{aligned}\alpha_2 &= \bigvee_{\frac{i_1}{n-1} \supset \left(\frac{i_3}{n-1} \supset \frac{i_1}{n-1} \right) = 1} J_{i_1}(a) \wedge J_{i_3}(b) \\ &= \bigvee_{i_1 j} J_i(a) \wedge J_j(b) = \bigvee_i J_i(a) \wedge \bigvee_j J_j(b) \\ &= 1.\end{aligned}$$

($C_{\supset}2$) The proof is similar to the previous one.

($C_{\supset}3$) In order to prove that $1 \supset a = 1$ implies $a = 1$ observe that it is enough to see $J_{n-1}\,(1 \supset a) = 1$ implies $J_{n-1}\,(a) = 1$. From Axiom 2.8 it follows that

$$\begin{aligned}J_{n-1}(1 \supset a) &= \bigvee_{\frac{i_1}{n-1} \supset \frac{i_2}{n-1} = 1} J_{i_1}(1) \wedge J_{i_2}(a) \\ &= \bigvee_{1 \supset \frac{i_2}{n-1} = 1} J_1(1) \wedge J_{i_2}(a) \\ &= \bigvee_{1 \supset \frac{i_2}{n-1} = 1} J_{i_2}(a).\end{aligned}$$

From the analogous property for **M**, $J_{n-1}(1 \supset a) = J_{n-1}(a)$. Since from hypothesis $J_{n-1}(1 \supset a) = 1$, then $J_{n-1}(a) = 1$. □

Notice that the condition ($C_{\vee}$) does not hold in general. Indeed, it is not true that in a Boolean algebra $x \vee y = 1$ implies $x = 1$ or $y = 1$.

2.1 Examples

Example 2.16 Let $M_3 = \{0, \frac{1}{2}, 1\}$ and let the operations on M_3 be given by

$$x \wedge y = \begin{cases} 1 & \text{if } x = 1 \text{ and } y = 1 \\ 0 & \text{otherwise,} \end{cases} \qquad x \vee y = \begin{cases} 1 & \text{if } x = 1 \text{ or } y = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\neg x = \begin{cases} 1 & \text{if } x = 0 \text{ or } x = \frac{1}{2} \\ 0 & \text{if } x = 1, \end{cases} \qquad x \supset y = \neg x \vee y.$$

It is easy to check that $\mathbf{M} = \langle M_3, \mathbb{F} \rangle$ is an L_n-algebra for **M** for this 3-matrix **M**.

In the following examples, we consider the natural order on M_n, that is, $0 < \frac{1}{n-1} < \cdots < \frac{n-2}{n-1} < 1$ and its natural additive structure.

Example 2.17 Consider now M_3 with the operations

$$x \wedge y = \min\{x, y\}, \qquad x \vee y = \max\{x, y\},$$

$$\neg x = 1 - x, \qquad x \supset y = \min\{1, 1 - x + y\}.$$

Then $\mathbf{M} = \langle M_3, \mathbb{F} \rangle$ is an L_n-algebra for **M** for this 3-matrix **M**.

Example 2.18 Recall (see Cignoli and de Gallego [6], Boicescu et al. [4]) that an n-valued Moisil algebra is a system $\langle A, \vee, \wedge, \neg, \sigma_1, \ldots, \sigma_n, 0, 1\rangle$ such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a distributive lattice with unit 1 and zero 0, and $\neg, \sigma_1, \ldots, \sigma_n$ are unary operators defined on A, fulfilling certain conditions. In any n-valued Moisil algebra, there are operators $J_0, \ldots, J_{n-1}$ such that they can be written in terms of the operators $\sigma_1, \ldots, \sigma_n$:

$$J_i(x) = \sigma_{n-1}(x) \wedge \neg\sigma_{n-i-1}(x).$$

Moreover, the operators $\sigma_1, \ldots, \sigma_n$ can be expressed in terms of $J_0, \ldots, J_{n-1}$:

$$\sigma_i(x) = \bigvee_{j=1}^{i} J_{n-j}(x).$$

Now it is convenient to take an n-valued Moisil algebra to be a system

$$\langle A, J_0, \ldots, J_{n-1}, \neg, \vee, \wedge, \supset, \sigma_1, \ldots, \sigma_{n-1}\rangle\,;$$

the operation $\supset$ is now the *weak implication*:

$$x \supset y = \sigma_{n-1}(\neg x) \vee y.$$

Let $M_n = \{\frac{j}{n-1} : j = 0, \ldots, n-1\}$, the n-valued Moisil algebra with the natural lattice operations $\vee, \wedge, 0, 1, \neg$, and σ_i given by

$$\neg\frac{j}{n-1} = 1 - \frac{j}{n-1}, \qquad \sigma_i\left(\frac{j}{n-1}\right) = \begin{cases} 0 & \text{if } i + j < n \\ 1 & \text{if } i + j \geq n. \end{cases}$$

Let $\mathbf{M}_n = \langle M_n, \mathbb{F}\rangle$ be an algebra of type $\mathbb{F}$. Since conditions (C_{J_i}), $(\mathrm{C}_{\wedge})$, …, $(\mathrm{C}_{\supset 3})$ are satisfied, from Lemma 2.11 it follows that $\mathbf{M}_n$ is an L_n-algebra for the matrix $\mathbf{M}_n$.

Remark 2.19 Let $\mathbf{M}_n$ be as above. Observe that if $n = 4$ and $\supset$ is the Łukasiewicz implication,

$$x \supset y = \min\{1, 1 - x + y\},$$

and

$$\neg x = 1 - x, \quad x \vee y = \max\{x, y\}, \quad x \wedge y = \min\{x, y\}.$$

In this case, $\mathbf{M}_4$ is not an L_n-algebra for the matrix $\mathbf{M}_4$. It is enough to observe that the Łukasiewicz implication does not verify $(C_{\supset 2})$:

$$\left(\frac{1}{3} \supset \left(\frac{2}{3} \supset 0\right)\right) \supset \left(\left(\frac{1}{3} \supset \frac{2}{3}\right) \supset \left(\frac{1}{3} \supset 0\right)\right) \neq 1.$$

Example 2.20 Let $\mathbf{M}_6 = \langle M_6, \mathbb{F}\rangle$ be the matrix given in Example 2.18 for $n = 6$. Let $A = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ and let $\mathbb{F} = \{\overline{J}_0, \ldots, \overline{J}_5, \neg, \vee, \wedge, \supset\}$ such that $\neg, \vee, \wedge$ are the same as in the same example, $\supset$ is the weak implication, and $\overline{J_0}, \ldots, \overline{J_5}$ are given by

$$\left[\begin{array}{ccccccc} & \overline{J_0} & \overline{J_1} & \overline{J_2} & \overline{J_3} & \overline{J_4} & \overline{J_5} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Let $\mathbf{A} = \langle A, \mathbb{F}\rangle$. Then $\mathbf{A}$ is an L_6-algebra for the matrix $\mathbf{M}_6$. (See [6]). We are going to check $\overline{J_0}(\neg a) = \bigvee_{\neg \frac{i}{n-1}=0} \overline{J_i}(a)$. Observe that

$$\overline{J_0}(\neg a) = \begin{cases} 0 & \text{if } a \in \left\{0, \frac{1}{3}, \frac{2}{3}\right\} \\ 1 & \text{if } a = 1 \end{cases}$$

and also

$$\bigvee_{\neg \frac{i}{5}=0} \overline{J_i}(a) = \overline{J_5}(a) = \overline{J_0}(\neg a).$$

Analogously for $\overline{J_1}(\neg a), \ldots, \overline{J_5}(\neg a)$.

In order to check this axiom for the operation $\supset$, recall that the weak implication on $\left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ is given by

$$x \supset y = \sigma_3(\neg x) \vee y = \begin{cases} 1 & \text{if } x \neq 1 \\ y & \text{if } x = 1, \end{cases}$$

and the weak implication on $\left\{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\right\}$ is given by

$$x \supset y = \sigma_5(\neg x) \vee y.$$

For example, we are going to check that $\overline{J_0}(a_1 \supset a_2) = \bigvee_{\frac{i_1}{5} \supset \frac{i_2}{5}=0} \overline{J_{i_1}}(a_1) \wedge \overline{J_{i_2}}(a_2)$:

$\overline{J_0}(a_1 \supset a_2) = 1$ if and only if $a_1 \supset a_2 = 0$ if and only if $a_1 = 1$ and $a_2 = 0$. On the other hand,

$$\bigvee_{\frac{i_1}{5} \supset \frac{i_2}{5}=0} \overline{J_{i_1}}(a_1) \wedge \overline{J_{i_2}}(a_2) = \overline{J_5}(a_1) \wedge \overline{J_0}(a_2) = 1$$

if and only if $a_1 = 1$ and $a_2 = 0$. In the same way we can prove the other cases.

Remark 2.21 Recall that the intuitionist implication is defined by

$$x \supset y = \min\{z : x \wedge z \leq y\}.$$

On $\{0, \frac{1}{n-1}, \ldots, 1\}$, $\supset$ is given by

$$x \supset y = \begin{cases} 1 & \text{if x} \leq y \\ y & \text{otherwise.} \end{cases}$$

It is known—see [6]—that this implication can be expressed in terms of the operations σ_i. This implication can be used in place of the weak one in the previous example, obtaining another example of n-matrix.

Example 2.22 Let $\mathbb{F} = \{J_0, J_1, \neg, \vee, \wedge, \supset, \}, \mathbf{M}_2 = \langle\{0, 1\}, \mathbb{F}\rangle$ such that $\langle\{0, 1\}, \neg, \vee, \wedge, \supset\rangle$ is a Boolean algebra. Let $\mathbf{A}$ be an L_n-algebra for the matrix $\mathbf{M}_2$. Then $\mathbf{A}$ is also a Boolean algebra. For example, to check the axiom $a \vee \neg a$ it is enough to prove that $J_1(a \vee \neg a) = 1$. First observe that $J_1(\neg a) = \vee_{\neg i=1} J_i(a) = J_0(a)$ and from Lemma 2.14,

$$J_1(a \vee \neg a) = J_1(a) \vee J_1(\neg a) = J_1(a) \vee J_0(a) = 1.$$

In the same way it is easy to prove all the axioms.

Example 2.23 Let $\mathbf{A} \in L_n M$ be the variety of the monadic n-valued Łukasiewicz algebras—see Abad [1]. First recall that the only operator $\exists$ on $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ is the identity. Then, if $\mathbf{A}$ is an L_n-algebra for a matrix $\mathbf{M}$ as in Example 2.18 and $G = \{\exists\}$, from Remark 2.12 it must be that $\exists = \mathrm{Id}_A$. It follows that in general if $\mathbf{A} \in L_n M$, then $\mathbf{A}$ is not an L_n-algebra.

3 Boolean Congruences

As usual, a *congruence* θ on an algebra $\mathbf{A} = \langle A, \mathbb{F}\rangle$ is an equivalence relation that is compatible with the operations belonging in $\mathbb{F}$. In particular, $a\,\theta\,b$ implies $J_i(a)\,\theta\,J_i(b)$, $i = 0, \ldots, n-1$. Let θ_e be the restriction of θ to the Boolean algebra $E(\mathbf{A})$. Notice that if η is a congruence on the Boolean algebra $E(\mathbf{A})$, then from Axioms (Cn-J$_0$), (Cn-J$_{n-1}$), and (Cn-J$_i$), we have that η also preserves the connectives J_i: $J_j(a)\,\eta\,J_k(b)$ implies that $J_i(J_j(a))\,\eta\,J_i(J_k(b))$.

Lemma 3.1 *For each congruence η on $E(\mathbf{A})$, let $\overline{\eta}$ be the relation on A such that $a\,\overline{\eta}\,b$ if and only if $J_i(a)\,\eta\,J_i(b)$, $i = 0, \ldots, n-1$. Then $\overline{\eta}$ is a congruence on A.*

Proof It is obvious that $\overline{\eta}$ is an equivalence relation on A and it follows from Axiom 2.8 that it also preserves the operations in F. □

Notice that for each congruence θ on A we have that $\theta \subseteq \overline{\theta_e}$ and $\eta \subseteq (\overline{\eta})_e$.

Definition 3.2 Let θ be a congruence on A. We say that θ is a *Boolean congruence* if and only if $\theta = \overline{\theta_e}$.

Let $\mathbf{A} \in \mathcal{M}$. We denote by $\mathrm{Con}(A)$ the complete lattice of congruences of $\mathbf{A}$. Following Pigozzi [7], we say that $\theta \in \mathrm{Con}(A)$ is an $\mathcal{M}$*-congruence* provided that $A/\theta \in \mathcal{M}$. The set of $\mathcal{M}$-congruences of $\mathbf{A}$ is a complete lattice of $\mathrm{Con}(A)$ that we denote $\mathrm{Con}_{\mathcal{M}}(A)$.

Lemma 3.3 *The Boolean congruences are exactly the $\mathcal{M}$-congruences.*

Proof Suppose that $A/\theta \in \mathcal{M}$. We need to show that $\overline{\theta_e} \subseteq \theta$: let $a, b \in A$ such that $a\,\overline{\theta_e}\,b$. Then $J_i(a)\,\theta\,J_i(b)$ for each $i = 0, \ldots, n-1$. Since Axiom 2.9 holds in A/θ we have that $a\,\theta\,b$. For the converse, suppose that $J_i(a)\,\theta\,J_i(b)$ for $i = 0, \ldots, n-1$. Then we have that $J_i(a)\,\theta_e\,J_i(b)$ for $i = 0, \ldots, n-1$ and then $a\,\overline{\theta_e}\,b$. Since $\theta = \overline{\theta_e}$ it follows that $a\,\theta\,b$. □

Lemma 3.4 *If θ is a maximal congruence on $\mathbf{A}$, then θ_e is a maximal congruence on $E(\mathbf{A})$, or it is the universal congruence on $E(\mathbf{A})$.*

Proof If $\theta_e \subseteq \eta$, then $\theta \subseteq \overline{\theta_e} \subseteq \overline{\eta}$. Since θ is maximal, then $\overline{\eta} = \theta$ or $\overline{\eta} = A \times A$. In the first case, $\eta \subseteq (\overline{\eta})_e = \theta_e \subseteq \eta$, therefore $\theta_e = \eta$. In the case $\overline{\eta} = A \times A$, then it is immediate that $\eta = E(\mathbf{A}) \times E(\mathbf{A})$. □

Lemma 3.5 *Let U be an ultrafilter in $E(\mathbf{A})$. Then, for each $a \in A$, there is a unique $i \in \{0, \ldots, n-1\}$ such that $J_i(a) \in U$.*

Proof It follows immediately from Lemma 2.13. □

It is well known that, in a Boolean algebra, filters can be identified with congruences. We denoted U_η, the filter associated with the congruence η, and η_U, the congruence associated with the filter U. Also we put $[x] = \{y : y\,\eta\,x\}$.

Lemma 3.6 *Let U be as above. If $J_i(a) \in U$ then $J_{n-1}(J_i(a)) \in U$, else $J_0(J_i(a)) \in U$.*

Proof In the first case, observe that $J_i(a) = J_{n-1}(J_i(a)) \in U$. If $J_i(a) \notin U$ then $\neg J_i(a) = J_0(J_i(a)) \in U$. □

Theorem 3.7 *Let* $\mathbf{A}$ *be an* L_n*-algebra for the matrix* $\mathbf{M}$*. Let* η *be a maximal congruence on* $E(\mathbf{A})$*. There is a homomorphism* $f_\eta : A/\overline{\eta} \hookrightarrow M_n$*, that is, the quotient* $A/\overline{\eta}$ *is isomorphic to a subalgebra of* M_n*.*

Proof Note first that since $\overline{\eta}$ is a Boolean congruence, $A/\overline{\eta} \in \mathcal{M}$. Let U_η be the prime filter on $E(\mathbf{A})$ as above, and let

$$f_\eta : A/\overline{\eta} \to M_n$$

given by

$$f_\eta([a]) = \frac{i}{n-1}$$

where i is the unique integer such that $J_i(a) \in U_\eta$. The function $f_\eta$is well defined since if $[a] = [b]$, then $J_i(a)\ \eta\ J_i(b)$ for each $i = 0, \ldots, n-1$, and hence $J_i(a) \in U_\eta$ if and only if $J_i(b) \in U_\eta$.

Let us check that f_η is a homomorphism: let

$$* \in \mathbb{F} - \{J_i, i = 0, \ldots, n-1\}, a_1, a_2, \ldots, a_k$$

belong in A, and suppose that

$$f_\eta([a_j]) = \frac{i_j}{n-1},\ j = 0, \ldots, k,$$

that

$$\frac{r}{n-1} = *\left(\frac{i_1}{n-1}, \frac{i_2}{n-1}, \ldots, \frac{i_k}{n-1}\right),$$

and that

$$J_i(*(a_1, a_2, \ldots, a_k)) \in U_\eta.$$

Then $f_\eta(*([a_1], [a_2], \ldots, [a_k])) = f_\eta([*(a_1, a_2, \ldots, a_k)]) = \frac{i}{n-1}$, and in order to prove $f_\eta(*([a_1], [a_2], \ldots, [a_k])) = *(f_\eta([a_1]), f_\eta([a_2]), \ldots, f_\eta([a_k]))$ we must show that $J_r(*(a_1, a_2, \ldots, a_k)) \in U_\eta$. We have

$$\begin{aligned} J_r(*(a_1, a_2, \ldots, a_k)) &= \bigvee_{*(\frac{j_1}{n-1}, \frac{j_2}{n-1}, \ldots, \frac{j_k}{n-1}) = \frac{r}{n-1}} \bigwedge_{l=1}^{k} J_{j_l}(a_l) \\ &\geq \big(J_{i_1}(a_1) \wedge \cdots \wedge J_{i_k}(a_k)\big) \in U_\eta, \end{aligned}$$

because each $J_{i_l}(a_l) \in U_\eta$. Also $f_\eta(J_j([a])) = J_j(f_\eta([a]))$ for each $j = 0, \ldots, n-1$. Indeed, let i be such that $J_i(a) \in U_\eta$. Hence $J_j(f_\eta([a])) = J_j(\frac{i}{n-1}) = 1$ if and only if $i = j$, and the result follows from Lemma 3.6. Finally, f_η is injective: if $(a, b) \notin \overline{\eta}$ then $(J_j(a), J_j(b)) \notin \eta$ for some j. Either $J_j(a) \in U_\eta$ and $J_j(b) \notin U_\eta$, or $J_j(a) \notin U_\eta$ and $J_j(b) \in U_\eta$. In both cases $f_\eta([a]) \neq f_\eta([b])$. □

Lemma 3.8 *Let* I *be the set of maximal elements of* $\mathrm{Con}_{\mathcal{M}}(\mathbf{A})$*. Then* $\cap I = \Delta$ *is the minimum element of* $\mathrm{Con}_{\mathcal{M}}(\mathbf{A})$*.*

Proof Suppose $a \neq b$, then there is an i such that $J_i(a) \neq J_i(b)$. Let U be an ultrafilter such that $J_i(a) \in U$ and $J_i(b) \notin U$. Then $(J_i(a), J_i(b)) \notin \eta_U$, and $(a, b) \notin \overline{\eta_U}$. It follows that $(a, b) \notin \bigcap\{\overline{\eta} \in I\}$. □

From the above lemma, Theorem 3.7, and a well-known theorem of Birkhoff (see, for instance, Burris and Sankappanavar [5, Chapter 2, Theorem 8.6]) we obtain another proof of Theorem 5.20 given in [2].

Theorem 3.9 *Let* $\mathbf{A}$ *be an* L_n*-algebra for the matrix* $\mathbf{M}$*. Then* $\mathbf{A}$ *is a subdirect product of a family of subalgebras of the matrix* $\mathbf{M}$*.*

3.1 Examples

Example 3.10 Let $\mathbf{M}_n$ be an n-matrix; since $E(\mathbf{M}_n)=\{0,1\}$, the set $\mathrm{Con}_{\mathcal{M}(M_n)}(\mathbf{M}_n)$ has exactly two elements. (Observe that, if $\mathbf{A} \in \mathcal{M}$ and $\Delta_{\mathbf{A}} = \min \mathrm{Con}(\mathbf{A})$, from Axiom 2.9 it follows that $\overline{\Delta_{E(M_n)}} = \Delta_{M_n}$.) Then $M_n/\overline{\eta} = M_n$ or $M_n/\overline{\eta}$ has only one element. However, $\mathbf{M}_n$ may be not simple. Indeed, let $\mathbf{M}_n$ be as in Example 2.16. Then $\theta = \Delta_{M_n} \cup \{(0,1),(1,0)\}$ verifies $\Delta_{M_n} \subsetneq \theta \subsetneq M_n \times M_n$, and $\theta \notin \mathrm{Con}_{\mathcal{M}(M_n)}(\mathbf{M}_n)$.

Example 3.11 Let $F = \{J_0, J_1, \ldots, J_{n-1}\} \cup \{\neg, \vee, \wedge, \supset\}$, $G = \{\sigma_1, \ldots, \sigma_n\}$, $\mathbb{F} = F \cup G$, $\mathbb{F}' = \{\neg, \vee, \wedge\} \cup G$. Let $\mathbf{M}_n = \langle M_n, \mathbb{F}\rangle$ be the matrix such that $\mathbf{M}'_n = \langle M_n, \mathbb{F}'\rangle$ is an n-Moisil algebra (see Example 2.18). Since in this case the quasi identity given by Axiom 2.9 can be replaced by identities (see [4]) then $\mathcal{M}$ is a variety.

4 An Axiomatization of Anshakov-Rychkov n-valued Propositional Calculi

Let $\mathbf{M}$ be an n-matrix of type $\mathbb{F}$ as in Section 2. Now we are going to define the n-valued propositional calculus of Anshakov-Rychkov for the matrix $\mathbf{M}$. Let $\mathcal{F}$ denote the algebra of well-formed formulas constructed in the usual way from a denumerable set $\{p_m\}_{m\in N}$ of propositional variables by means of the connectives in $\mathbb{F}$.

Let $E(\mathcal{F}) \subseteq \mathcal{F}$ be the set of *external formulas* which are defined inductively as follows:

1. for each $i = 1, \ldots, n-1$, if X is a formula then $J_i X$ is an external formula;
2. if X and Y are external formulas, $\neg X$, $X \vee Y$, $X \wedge Y$, $X \supset Y$ are external formulas.

Notation 4.1 $X \mathrel{\substack{\subset\\ \supset}} Y = (X \supset Y) \wedge (Y \supset X)$, $T = J_{n-1}(p_n) \vee \neg J_{n-1}(p_n)$.

The Axiom schemes are the following:

(Boolean Axioms) If Φ is a classical tautology, let B_Φ be the formula in which we substitute each appearance of each propositional variable by any formula of the form $J_i(X)$. Then B_Φ is an axiom of this calculi. For example, $J_i(X) \vee \neg J_i(X)$ is an axiom, and for any X, Y, $J_i(X) \supset (J_i(Y) \supset J_i(X))$ and $(J_i(X) \supset (J_i(Y) \supset J_i(Z))) \supset ((J_i(X) \supset J_i(Y)) \supset (J_i(X) \supset J_i(Z)))$ are axioms.

n-Val$_i$ $\quad J_i(X) \mathrel{\substack{\subset\\ \supset}} \bigwedge_{j\neq i} \neg J_j(X)$, $i = 0, \ldots, n-1$

CJ$_i$
$$\begin{cases} \mathrm{CJ}_0 & J_0(J_i(X)) \mathrel{\substack{\subset\\ \supset}} \neg J_i(X) \\ \mathrm{CJ}_{n-1} & J_{n-1}(J_i(X)) \mathrel{\substack{\subset\\ \supset}} J_i(X) \\ \mathrm{CJ}_j & J_j(J_i(X)) \mathrel{\substack{\subset\\ \supset}} \neg T,\ j \notin \{0, n-1\} \end{cases}$$

Cl
$$\begin{cases} \text{Cl-J}_* & J_i(X) * J_j(Y) \mathrel{\substack{\subset\\ \supset}} J_{n-1}(J_i(X) * J_j(Y)), \\ & \qquad * \in \{\vee, \wedge, \supset\} \\ \text{Cl-J}_\neg & \neg J_{n-1}(J_i(X)) \mathrel{\substack{\subset\\ \supset}} J_{n-1}(\neg J_i(X)) \end{cases}$$

Cn-G
$$J_i(G(X_1, \ldots, X_m)) \mathrel{\substack{\subset\\ \supset}} \bigvee_{G(\frac{i_1}{n-1}, \ldots, \frac{i_m}{n-1}) = \frac{i}{n-1}} \bigwedge_{j=1}^{m} J_{i_j}(X_j),$$
$$G \in \mathbb{F} \backslash \{J_i, i = 0, \ldots, n-1\}$$

$$\text{(Rules of inference)}\begin{cases}\text{modus ponens:} & \dfrac{J_i(X), \quad J_i(X) \supset J_j(Y)}{J_j(Y)}\\ \text{J-introduction:} & \dfrac{X}{J_{n-1}(X)}\\ \text{J-elimination:} & \dfrac{J_{n-1}(X)}{X}\end{cases}$$

We denote by $\mathcal{S}(\mathbf{M})$, or simply by $\mathcal{S}$, the *logic* given by this language and these axioms and inference rules. $\mathcal{S}$ is the *Anshakov-Rychkov propositional logic for* **M**.

If D is a unary predicate, $\mathcal{S}_D$ is the extension of $\mathcal{S}$ to which we add the symbol D. Let $\Gamma \subseteq \mathcal{F}$, $X \in \mathcal{F}$. *Derivability, theorem,* and *theory* are defined in the usual way, and the notation '$\Gamma \vdash X$' means that 'X is derivable from Γ'. Let $\mathcal{M}(\mathbf{M})$ be the quasi variety of algebras defined in Remark 2.10. Let $\mathbf{A} \in \mathcal{M}$.

A *valuation on* **A** is a function $v : \mathcal{F} \to \mathbf{A}$ that preserves the connectives belonging to $\mathbb{F}$. $v(X)$ is an *interpretation* of the formula X.

Notation 4.2 As usual, $\Gamma \vDash_{\mathbf{A},v} X$ if and only if $v(\Gamma) \subseteq \{1\}$ implies $v(X) = 1$, and $\Gamma \vDash_{\mathbf{A}} X$ if and only if for each valuation v on **A**, $\Gamma \vDash_{\mathbf{A},v} X$. If $\Gamma = \varnothing$, X is a *tautology* for **A** and we write $\vDash_{\mathbf{A}} X$. Also, we write $\Gamma \vDash_{\mathcal{M}} X$ (or simply $\Gamma \vDash X$) if and only if $\Gamma \vDash_{\mathbf{A}} X$ for each $\mathbf{A} \in \mathcal{M}$.

X is a *tautology* for $\mathcal{M}$ if and only if X is a *tautology* for **A**, for each $\mathbf{A} \in \mathcal{M}$. In this case, we write $\vDash X$. Finally, if X is a formula with a single variable p, then $X[Y/p]$ is the formula obtained from X by replacing each appearance of p by Y.

Let $\mathcal{S}$ be the *Anshakov-Rychkov propositional logic for* **M**, and $\mathcal{M} = \mathcal{M}(\mathbf{M})$.

Theorem 4.3 *$\mathcal{S}$ is algebraizable with equivalent algebraic semantic $\mathcal{M}$. This means (see [3], Definition 2.8) that*

1. *there exists a finite system of equations—called* defining equations *for $\mathcal{S}$ and $\mathcal{M}$, $\delta_i(p) \approx \epsilon_i(p)$ for $i < m$—with a single variable p such that for all $\Gamma \cup \{X\} \subseteq \mathcal{F}$ and each $j < s$, $\Gamma \vdash X$ if and only if $\{\delta_i(Y/p) \approx \epsilon_i(Y/p) : j < s, Y \in \Gamma\} \vDash \delta_j(X/p) \approx \epsilon_j(X/p)$;*
2. *there exists a finite system $\Delta_k(p,q)$, for $k < t$ of formulas with two variables such that, for every equation $X \approx Y$, $X \approx Y \vDash \delta(X\Delta Y) \approx \epsilon(X\Delta Y)$ and $\delta(X\Delta Y) \approx \epsilon(X\Delta Y) \vDash X \approx Y$.*

In order to prove Theorem 4.3, we will use the following result by Blok and Pigozzi.

Theorem 4.4 ([3], Theorem 5.1) *The following statements are equivalent.*

1. *$\mathcal{S}$ is algebraizable with equivalent semantics $\mathcal{M}$.*
2. *For every algebra **A** the Leibnitz operator $\Omega_{\mathbf{A}}$ is an isomorphism between the lattices of $\mathcal{S}$-filters and $\mathcal{M}$-congruences of **A**.*

We know that the $\mathcal{M}$-congruences of A are exactly the Boolean congruences. In order to prove Theorem 4.4 we first study the class of $\mathcal{S}$-filters. Recall that a subset F of **A** is an $\mathcal{S}$-filter when F contains all the interpretations of the logical axioms of $\mathcal{S}$ and is closed under each inference rule: this means that if the premises are in F, then the conclusion also belongs to F.

Let U be a filter on the Boolean algebra $E(\mathbf{A})$. F is a *Boolean filter* if and only if $F = \{a \in A : J_{n-1}(a) \in U\}$.

Lemma 4.5 *The following statements are equivalent.*

1. *F is a Boolean filter.*
2. *F is an $\mathcal{S}$-filter.*

Proof Let $F = \{a \in A : J_{n-1}(a) \in U\}$. Since $1 \in F$, F contains the interpretations of the logical axioms. Now let $J_i(a) \in F$ and $J_i(a) \supset J_j(b) \in F$. Recalling Axiom 2.4 (Cn-J$_{n-1}$), note that for any $J_i(a) \in E(\mathbf{A})$, $J_i(a) \in F$ if and only if $J_i(a) \in U$. Since $J_i(a) \in U$ and $J_i(a) \supset J_j(b) \in U$ in the Boolean algebra $E(\mathbf{A})$ implies that $J_j(b) \in U$, F is closed under modus ponens. It is easier to see that F is closed by the other rules.

To prove that condition 1 implies condition 2, let F be an $\mathcal{S}$-filter, and let $U = F \cap E(\mathbf{A})$. From the Boolean axiom and the definition of $\mathcal{S}$-filter, U is a filter on $E(\mathbf{A})$ and $F = \{a \in A : J_{n-1}(a) \in U\}$. □

Lemma 4.6 *Let $F = \{a \in A : J_{n-1}(a) \in U\}$ be a Boolean filter. Then $\theta(F) = \{(a, b) \in A^2 : J_i(a) \supset J_i(b) \in U, J_i(b) \supset J_i(a) \in U$ for each $i = 0, \ldots, n-1\}$ is a Boolean congruence and verifies $\theta = \Omega_A(F)$.*

Proof Clearly, if η is the congruence on $E(\mathbf{A})$ defined by U, that is,

$$\eta = \{(x, y) \in E(A)^2 : x \supset y \in U, y \supset x \in U\}$$

and $\overline{\eta}$ is the Boolean congruence given in Section 3, then $\theta(F) = \overline{\eta}$.

By Theorem 1.6 of [3], to prove that $\theta = \Omega_A(F)$, it is enough to see that θ is *elementarily definable* in $\mathcal{S}_D$. This means that there exists a first-order formula X with parameters and without equality in $\mathcal{S}_D$ such that $a\,\theta\,b$ if and only if $\vDash_{\mathbf{A},v(p/a,q/b)} X$. Recalling that the interpretation of D is F, let X be the formula

$$D\left(\bigwedge_{i=0}^{n-1} (J_i(p) \supset J_i(q)) \wedge (J_i(q) \supset J_i(p))\right).$$

Then $\vDash_{\mathbf{A},v(p/a,q/b)}\ D\left(\bigwedge_{i=0}^{n-1} (J_i(p) \supset J_i(q)) \wedge (J_i(q) \supset J_i(p))\right)$ if and only if $\bigwedge_{i=0}^{n-1} (J_i(a) \supset J_i(b)) \wedge (J_i(b) \supset J_i(a))$ belong to F. Note that if

$$c = \bigwedge_{i=0}^{n-1} (J_i(a) \supset J_i(b)) \wedge (J_i(b) \supset J_i(a)) \in E(A),$$

then $c \in F$ if and only if $c \in U$, and therefore $a\,\theta\,b$ if and only if $\vDash_{\mathbf{A},v(p/a,q/b)} X$. □

Lemma 4.7 *Let $\mathbf{A}$ be an algebra, $\mathbf{A} \in \mathcal{M}$, and let $\mathcal{D}$ be the family of $\mathcal{S}$-filters on $\mathbf{A}$. Let Θ be given by*

$$\Theta : \mathcal{D} \longrightarrow Con_{\mathcal{M}}(\mathbf{A})$$
$$F \longmapsto \theta(F)$$

with $\theta(F)$ as in Lemma 4.6. Then the map Θ is an order isomorphism.

Proof Let $\mathbb{D}$ be the map: $\mathbb{D} : \mathrm{Con}_{\mathcal{M}}(\mathbf{A}) \longrightarrow \mathcal{D}, \overline{\eta} \longmapsto F$, where F is the Boolean filter given by the filter $U \subseteq E(\mathbf{A})$ associated with η. Clearly, $\mathbb{D} = \Theta^{-1}$. Also, it is easy to see that Θ preserves the order: if $F \subseteq F'$ then $\theta(F) \subseteq \theta(F')$. □

From Lemmas 4.6 and 4.7 we can conclude the following.

Theorem 4.8 *For any algebra $\mathbf{A} \in \mathcal{M}$, the Leibnitz operator $\Omega_{\mathbf{A}}$ is an isomorphism between the lattices of $\mathcal{S}$-filters and $\mathcal{M}$-congruences of $\mathbf{A}$.*

Theorem 4.3 follows from Lemmas 4.6 and 4.7 and from Theorem 4.4.

Lemma 4.9 *Let Δ be the system of formulas in two variables and let E be the single equation given by $\Delta_i(p,q) = J_i(p) \overset{\subset}{\supset} J_i(q), i = 0, \ldots, n-1$, and $J_{n-1}(p) \approx T$, respectively. Then the following conditions hold for all $X_1, X_2, X_3 \in \mathcal{F}$:*

1. $\vdash X_1 \Delta X_1$,
2. $X_1 \Delta X_2 \vdash X_2 \Delta X_1$,
3. $X_1 \Delta X_2, X_2 \Delta X_3 \vdash X_1 \Delta X_3$. *For every primitive connective $*$ and all $X_1, \ldots, X_k, Y_1, \ldots, Y_k \in \mathcal{F}$, where k is the rank of $*$, it holds that*
4. $X_1 \Delta Y_1, \ldots, X_k \Delta Y_k \vdash *(X_1, \ldots, X_k) \, \Delta * (Y_1, \ldots, Y_k)$, *and for all $X \in \mathcal{F}$ it holds that*
5. $X \vdash J_{n-1}(X) \Delta T$ *and* $J_{n-1}(X) \Delta T \vdash X$.

Proof The conditions (1)–(3) are immediate.

(4) The fact that Δ is a congruence relation on $\mathcal{F}$ is a direct consequence of the Boolean axioms, Axiom (Cn-G), and Axiom (CJ_i).

(5) First we observe that $J_{n-1}(X) \Delta T$ are the formulas

$$J_0(J_{n-1}(X)) \overset{\subset}{\supset} J_0(T), \ldots, J_i(J_{n-1}(X)) \overset{\subset}{\supset} J_i(T), \ldots, J_{n-1}(J_{n-1}(X)) \overset{\subset}{\supset} J_{n-1}(T).$$

From the Boolean and (CJ_i) axioms, it follows that to prove (5) it is sufficient to see that $X \vdash J_{n-1}(X) \overset{\subset}{\supset} T$ and $J_{n-1}(X) \overset{\subset}{\supset} T \vdash X$, which in turn is a consequence of the deduction rules and the Boolean axioms. □

From Lemma 4.9 and from [3] (see Theorem 4.7) the following corollary is immediate.

Corollary 4.10 *Δ and E are systems of equivalence formulas and defining equations for $\mathcal{S}$ and $\mathcal{M}$.*

The following result was already proved in [2].

Corollary 4.11 (Completeness Theorem) *For each $\Gamma \subset \mathcal{F}$ and each $X \in \mathcal{F}$, if $\Gamma \models X$, then $\Gamma \vdash X$.*

Proof First we observe that if v is a valuation on $\mathbf{A} \in \mathcal{M}$, $v(X) = 1$ if and only if $v(J_{n-1}(X)) = 1$. Also, we observe that from the corollary, it is enough to see that if $\Gamma \models X$, then $\{J_{n-1}(Y) \approx T : Y \in \Gamma\} \models J_{n-1}(X) \approx T$. Let v be a valuation on $\mathbf{A} \in \mathcal{M}$. If $v(J_{n-1}(Y)) = 1$ for each $Y \in \Gamma$, then $v(Y) = 1$ for each $Y \in \Gamma$, and from hypothesis $v(X) = 1$. Therefore, $v(J_{n-1}(X)) = 1$. □

5 Axiomiomatization with Truth Values Designated

We note some facts in order to generalize the results in the case of more designated values. Now, let $D \subset M_n$ be such that $0 \notin D$, $1 \in D$ and D verifies the conditions, ($\mathrm{C}_{D\supset}$): there exists $d \in D$, $d \supset x \in D$, then $x \in D$, and ($\mathrm{C}_{D\vee}$): $x \vee y \in D$ if and only if $x \in D$ or $y \in D$. D is the set of *designated truth values*.

Consider the set of formulas and axioms as above. The rules of inference are modus ponens, J_D-introduction ($\frac{X}{\bigvee_{\alpha \in D} J_\alpha(X)}$), and J_D-elimination ($\frac{\bigvee_{\alpha \in D} J_\alpha(X)}{X}$). Let $\mathcal{S}_D$ be this propositional logic.

A formula X is a *tautology* for $\mathcal{M}$ if $v(\bigvee_{\alpha \in D} J_\alpha(X)) = 1$ for each valuation v on $\mathbf{A}$, for any $\mathbf{A} \in \mathcal{M}$. Observe that from condition ($\mathrm{C}_{D\vee}$), X is a tautology if and only if there exists $\alpha \in D$ such that $J_\alpha(X) = 1$. It follows from ($C_{D\vee}$) that every derivable formula is a tautology. Now F is a *Boolean filter* if and only if $F = \{a \in A : \bigvee_{\alpha\in D} J_\alpha(a) \in U\}$, where U is a filter on $E(A)$. $\mathcal{S}_D$ is algebraizable with equivalent algebraic semantic $\mathcal{M}$. Now the defining equation $\delta(p) \approx \epsilon(p)$ is the single equation $\bigvee_{\alpha\in D} J_\alpha(p) \approx T$.

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