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On Non-wellfounded Sets as Fixed Points of Substitutions

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Abstract We study the non-wellfounded sets as fixed points of substitution. For example, we show that *ZFA* implies that every function has a fixed point. As a corollary we determine for which functions f there is a function g such that $g = g \star f$. We also present a classification of non-wellfounded sets according to their branching structure.

1 Introduction and Definitions from Barwise and Moss [2]

In Aczel [1] and in Barwise and Moss [2], non-wellfounded sets and the antifoundation axiom *AFA* have been studied. The non-wellfounded sets are modeled by equations. In the equations we use urelements and the class of all urelements is denoted by $\mathcal{U}$. Urelements are not vital for the theory but often they are convenient (see, for example, [2], §11). Recall the definitions of a flat system of equations and a solution to it from [2].

Definition 1.1

1. A flat system of equations is a triple (X, A, f) where X and A are sets of urelements, $X \cap A = \varnothing$, and $f : X \to \mathcal{P}(X \cup A)$ is a function.
2. A solution to a flat system of equations (X, A, f) is a function g such that $\mathrm{dom}(g) = X$, and for all $x \in X$, $g(x) = \{g(y) \mid y \in f(x) \cap X\} \cup (f(x) \cap A)$.

The idea is that X is the set of indeterminates of the equations and A is the set of "constants". The equations are understood as $x = f(x)$, $x \in X$. For example, let $A = \{a\}$, $X = \{x\}$, and $f(x) = \{a, x\}$, then (X, A, f) is a flat system of equations. The solution to this system is a function g such that $g(x) = \{a, g(x)\}$. The antifoundation axiom *AFA* says that every flat system of equations has a unique solution.

Substitution operations $\mathrm{sub}(s, b)$ are also studied in [2]. The operation $\mathrm{sub}(s, b)$ means that in b all x are substituted by $s(x)$. We recall the definition of a substitution

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from [2]. If $A \subseteq \mathcal{U}$, then $V_{afa}[A]$ is the class of all sets x such that support$(x) \subseteq A$, where support(x) is defined to be $TC(x) \cap \mathcal{U}$. So $V_{afa}[\mathcal{U}]$ is the class of all sets.

Definition 1.2 Substitution is a function s such that $\mathrm{dom}(s) \subset \mathcal{U}$. The substitution operation is the operation 'sub' such that the domain of sub consists of a class of pairs $\langle s, b\rangle$ where s is a substitution and $b \in V_{afa}[\mathcal{U}] \cup \mathcal{U}$ such that the following conditions hold:

1. if $x \in \mathrm{dom}(s)$, then $\mathrm{sub}(s, x) = s(x)$;
2. if $x \in \mathcal{U} - \mathrm{dom}(s)$, then $\mathrm{sub}(s, x) = x$;
3. for all sets b, $\mathrm{sub}(s, b) = \{\mathrm{sub}(s, p) \mid p \in b\}$.

In [2] it is shown that there is a unique substitution operation $\mathrm{sub}(s, b)$ defined for all substitutions s and $b \in V_{afa}[\mathcal{U}]\cup\mathcal{U}$. As a corollary to our theory of substitution fixed points, we obtain the same result (see Corollary 3.11). Next we recall the definition of a composition of substitutions from [2].

Definition 1.3 The substitution operation $\mathrm{sub}(s, b)$ is also denoted by $b[s]$, and $[s]$ is the operation mapping each set or urelement b to $b[s]$. A substitution s is proper if for all $x \in \mathrm{dom}(s)$, $s(x) \in V_{afa}[\mathcal{U}]$ whenever $s(x) \neq x$. If s and t are substitutions, then $t \star s$ is the substitution whose domain is $\mathrm{dom}(s)$ and for every $x \in \mathrm{dom}(s)$, $(t \star s)(x) = s(x)[t]$.

It is shown in [2] that we can state the *AFA* axiom in terms of substitution. *AFA* is equivalent to the assertion that for every proper substitution e there is a unique proper substitution s such that $s = s \star e$.

We remark here first that if f is a substitution, not necessarily proper, then a substitution s such that $s = s \star f$ is not necessarily unique for the following reason: Let a and b be distinct urelements. Let $f(a) = b$, $f(b) = a$. Let $u \in \mathcal{U} - \{a, b\}$ and $s(a) = s(b) = u$. Then $s = s \star f$.

Second, if f is such that the domain of f contains sets, then s does not necessarily exist. For example, let a and b be distinct urelements, and let $f(a) = \{a\}$, $f(\{a\}) = b$. If s is such that $s = s \star f$, then

$$s(a) = (s \star f)(a) = f(a)[s] = \mathrm{sub}(s, f(a)) = \mathrm{sub}(s, \{a\}) = s(\{a\})$$

by Definition 1.3. Then by (2) of Definition 1.2,

$$s(\{a\}) = f(\{a\})[s] = \mathrm{sub}(s, f(\{a\})) = \mathrm{sub}(s, b) = b.$$

But by (3) $s(a) = \mathrm{sub}(s, \{a\}) = \{\mathrm{sub}(s, a)\} = \{s(a)\}$, hence $s(a) = \Omega$, where Ω is the unique non-wellfounded set x such that $x = \{x\}$.

As a corollary of our theory of substitution fixed points, we will show that for every substitution f there is g such that $g = g \star f$ (see Lemma 3.9).

2 Fixed Points Approach

Next we study the non-wellfounded sets as fixed points of substitutions. Here we generalize the equation systems to arbitrary functions. The solutions are then defined in terms of the substitution. The fixed points are further generalizations of the solutions. This approach works well also in the situation without urelements. First we introduce some notation.

Definition 2.1

1. With every function f we associate a class function f^* defined as follows. If $x \in \mathrm{dom}(f)$, then $f^*(x) = f(x)$, otherwise $f^*(x) = x$.
2. If f and g are functions, then by $f[g]$ we mean a function such that $\mathrm{dom}(f[g]) = \mathrm{dom}(g)$ and $f[g](x)$ is defined as follows. If $f^*(x)$ is an urelement, then $f[g](x) = f^*(x)$ and otherwise $f[g](x) = \{g^*(y) \mid y \in f^*(x)\}$.
3. For all functions f and g, we say that g is a solution to f ($S(g, f)$), if $\mathrm{dom}(g) = \mathrm{dom}(f)$ and $g = f[g]$.

From the above we see that in [2] a solution to a flat system (X, A, f) of equations is defined so that g is the solution to the system if and only if $S(g, f)$ holds. So, in a sense, the solution to a flat system (X, A, f) of equations is obtained if in $f(x)$ all elements y from $f(x) \cap \mathrm{dom}(f)$ are replaced by $f(y)$. Then all elements z in $f(y) \cap \mathrm{dom}(f)$ are replaced by $f(z)$ and so on. So the solutions are some kind of restricted substitution-fixed points of the function from the system. In fact, in [2] it is shown that we get an equivalent theory if instead of equations we study substitution (cf. above). Because of the urelements, we define $\bigcup X = \bigcup\{x \mid x \in X$ and x is not an urelement$\}$.

Next we introduce the concept of a fixed point and the fixed point axiom.

Definition 2.2

1. We say that g is a fixed point of f, ($FP(g, f)$), if
$$\mathrm{dom}(f) \cup \bigcup f^*[\mathrm{dom}(g)] \subseteq \mathrm{dom}(g) \text{ and } g = f[g]$$
(where $f^*[\mathrm{dom}(g)] = \{f^*(y) \mid y \in \mathrm{dom}(g)\}$).
2. We say that a function f is generating if for all $x \in \mathrm{dom}(f)$ the following holds: if $f(x)$ is an urelement, then $x = f(x)$. We say that a generating f is basic if $\mathrm{dom}(f) \subseteq \mathcal{U}$.
3. The fixed point axiom *FPA* is the following: every function has a fixed point.

Note that if $FP(g, f)$ holds and $f^*(x)$ is not an urelement, then
$$g(x) = \{g(y) \mid y \in f^*(x)\}.$$
The following example shows the difference between solutions and fixed points: Let x be an urelement, $\mathrm{dom}(f) = \{x\}$ and $f(x) = (\varnothing, x)$. Then f itself is the solution to f but if g is a fixed point of f, then $g(x) = (\varnothing, g(x))$. Also following the notation from [2], if f is basic and $FP(g, f)$ holds, then for all $x \in \mathrm{dom}(g)$, if $f(x)$ is not an urelement, then $g(x) = \mathrm{sub}(g, f(x))$. Note that the basic functions are the same as the proper substitutions in [2].

Example 2.3 Assume *ZFC*. Let X be a set and $f : X \to \mathcal{P}(X)$ be such that $f(x) = x \cap X$. Then f has a (unique) fixed point g (such that $\mathrm{dom}(g) = \mathrm{dom}(f)$), namely, the Mostowski collapse of X.

In [2] it is also shown that in the presence of the axiom *AFA*, bisimulation characterizes identity: By $TC(x)$ we mean the transitive closure of x and in the case x is an urelement, $TC(x)$ is defined to be $\varnothing$.

Definition 2.4

1. We write $B(x, y)$ if there is $B \subseteq (\{x\} \cup TC(x)) \times (\{y\} \cup TC(y))$ such that
   (a) $(x, y) \in B$,

(b) if $(a, b) \in B$ and $c \in a$, then there is $d \in b$ such that $(c, d) \in B$,
(c) if $(a, b) \in B$ and $d \in b$, then there is $c \in a$ such that $(c, d) \in B$,
(d) if $(a, b) \in B$, then a is an urelement iff b is an urelement and if they are urelements, then $a = b$.

We call this kind of relation B a bisimulation relation between x and y.

2. We let the strong extensionality axiom *SEA* be the following axiom:

$$\forall x, y(B(x, y) \to x = y).$$

The axiom system ZFC^{-2} consists of pairing, union, power set, infinity, collection, separation, and choice, together with the axiom of urelements:

$$\forall p \forall q(\mathcal{U}(p) \to q \notin p),$$

and the axiom of plenitude of urelements: for every set S there is an injective function $f : S \to \mathcal{U}$ whose image $f[S]$ is disjoint from S. So the list of the axioms of ZFC^{-2} is the same as that in [2], p. 28, of excluding extensionality and replacing strong plenitude by plenitude of urelements. *ZFA* means $ZFC^{-2}+$ extensionality $+AFA$. By ZFC^{+} we mean $ZFC^{-2} + SEA + FPA$. So the difference between *ZFA* and ZFC^{+} is that we have replaced equations by substitution and uniqueness of the solutions by *SEA*.

We feel that our axiom system follows the lines of the axiom systems of [1] in that the axioms for the existence and the uniqueness of the solutions have been separated. Also, this approach is a bit more set theoretical in nature, since the fixed point axiom refers to functions instead of graphs or equation systems. Sometimes this makes the proofs easier.

We start by showing that *ZFA* and ZFC^{+} are equivalent. Especially we show that *ZFA* implies that every function has a fixed point. Then we show that the fixed points of the basic functions are fixed points of themselves, thus the name *fixed point*. For all functions this does not hold. Finally, we study the following question: Do we need to assume the existence of all solutions to the flat systems of equations to get all fixed points? We show that the answer is (essentially) yes.

3 Equivalence of *ZFA* and ZFC^{+}

Item (2) in the following lemma is [2], Exercise 7.3 and item (1) is well known.

Lemma 3.1

1. $ZFC^{-2} \vdash \forall x, y \notin \mathcal{U}(\forall z(z \in x \leftrightarrow z \in y) \to B(x, y))$. *Especially,* $ZFC^{-2} \vdash \forall x, y(x = y \to B(x, y))$.
2. $ZFC \vdash \forall x, y(B(x, y) \to x = y)$.

Proof (1) Let B consist of (x, y) together with $(a, b) \in TC(x) \times TC(y)$ such that $a = b$. To show that B is a bisimulation between x and y, let $z \in x$. Then by the assumption, $z \in y$ also. By the definition of B, $(z, z) \in B$ and hence B is a bisimulation between x and y. Especially, if $x = y$ and $x, y \notin \mathcal{U}$, then $\forall z(z \in x \leftrightarrow z \in y)$, and by the above, we can construct a bisimulation between x and y. If $x = y$ and $x, y \in \mathcal{U}$, then $\{(x, y)\}$ is a bisimulation between x and y.

(2) By $\in$-induction: Assume the claim for all $x' \in x$ and that for some set y, $B(x, y)$ holds. This means that for every $x' \in x$ there is $y' \in y$ such that $B(x', y')$ holds. This is so because if B is the bisimulation between x and y, then $B \restriction ((TC(x') \cup \{x'\}) \times (TC(y') \cup \{y'\}))$ is a bisimulation between x' and y'. But

this means that for every $x' \in x$ there is $y' \in y$ such that $x' = y'$, that is, $x' \in y$. Similarly, if $y' \in y$, then there is $x' \in x$, such that $y' = x'$, that is, $y' \in x$. By extensionality, $x = y$. $\square$

The following lemma is essentially proved in [2].

Lemma 3.2 *Assume $ZFC^{-2} + SEA$.*

1. $\forall x, y \notin \mathcal{U}(\forall z(z \in x \leftrightarrow z \in y) \to x = y)$*, that is, the extensionality axiom holds.*
2. *For all functions f, if $S(g, f)$ and $S(h, f)$ hold, then $g = h$.*
3. *For all functions f, if $FP(g, f)$ and $FP(h, f)$ hold and $A = \mathrm{dom}(g) \cap \mathrm{dom}(h)$, then $g \restriction A = h \restriction A$.*

Notice that since the extensionality axiom holds in $ZFC^{-2} + SEA$, also the functions of the form $f[g]$ are well defined and thus $S(g, f)$ and $FP(g, f)$ are well defined.

Proof (1) Let B consist of (x, y) together with $(x, y) \in TC(x) \times TC(y)$ such that $x = y$. Then B is a bisimulation between x and y. This is so because if $a \in x$, then $a \in y$ also, and therefore $(a, a) \in B$. If $(a, a) \in B$, and $a \in \mathcal{U}$, then the condition 2.4(1)(d) holds. If $a \notin \mathcal{U}$, and $c \in a$, then $(c, c) \in B$ and we are done.

(2) Let B consist of $(g(a), h(a))$ together with $(x, y) \in TC(g(a)) \times TC(h(a))$ such that either

(a) $x = y$ or
(b) there is $z \in \mathrm{dom}(f)$ such that $x = g(z)$ and $y = h(z)$.

To show that B is a bisimulation, let $z \in \mathrm{dom}(f)$ and $(g(z), h(z)) \in B$. If $g(z)$ is an urelement, then $g(z) = f[g](z) = f^*(z) = f[h](z) = h(z)$. Assume that $g(z)$ is not an urelement. Then

$$g(z) = f[g](z) = \{g^*(y) \mid y \in f^*(z)\}.$$

Let $c \in g(z)$. So $c = g^*(y)$ for some $y \in f^*(z)$. But because also

$$h(z) = \{h^*(y) \mid y \in f^*(z)\} \text{ and } g(z) = h(z),$$

we have that $h^*(y) \in h(z)$. Now if $y \notin \mathrm{dom}(f)$, then $y \notin \mathrm{dom}(g) = \mathrm{dom}(h)$. Therefore $g^*(y) = h^*(y) = y$ and $(y, y) \in B$. If $y \in \mathrm{dom}(f)$, then $(g(y), h(y)) \in B$.

(3) Let B consist of $(g(a), h(a))$ together with $(x, y) \in TC(g(a)) \times TC(h(a))$ for which there is $z \in \bigcup f^*[A]$ such that $x = g(z)$ and $y = h(z)$. Assume that $(g(z), h(z)) \in B$ and $f^*(z)$ is not an urelement. So $g(z) = \{g(y) \mid y \in f^*(z)\}$. Now let $y \in \bigcup f^*[A]$, then $y \in A$ and therefore if $g(y) \in g(z)$, then also $h(y) \in h(z)$ and $(g(y), h(y)) \in B$. $\square$

Lemma 3.3 *Assume $ZFC^{-2} + SEA$.*

1. *Assume $FP(g, f)$ holds and let $x \in \mathrm{dom}(g)$. If $TC(f^*(x)) \cap \mathrm{dom}(f) = \varnothing$, then $g(x) = f^*(x)$.*
2. *If $FP(g, f)$ holds and for all $x \in \mathrm{dom}(f)$, $TC(f(x) - \mathrm{dom}(f)) \cap \mathrm{dom}(f) = \varnothing$, then $S(g \restriction \mathrm{dom}(f), f)$ holds.*

Proof (1) We may assume that $f^*(x)$ is not an urelement, since if $f^*(x)$ is an urelement, then $g(x) = f[g](x) = f^*(x)$ by the assumption that $FP(g, f)$ holds. Let $A = TC(f^*(x)) \cap \mathrm{dom}(g)$. Because for all $y \in A$, $y = f^*(y) \subseteq \mathrm{dom}(g)$, A is transitively closed. Since $g(x) = \{g(y) \mid y \in f^*(x)\}$, it is enough to show that for all $y \in f^*(x)$, $g(y) = y$. Since $A \cap \mathrm{dom}(f) = \emptyset$, $FP(g \restriction A, id_A)$ holds. Since A is transitively closed, $S(g \restriction A, id_A)$ holds. So by Lemma 3.2 (2), it is enough to show that $S(id_A, id_A)$ holds, but this is clear.

(2) Assume that $f(x)$ is not an urelement. By (1),

$$\begin{aligned} g(x) &= \{g(y) \mid y \in f(x)\} = \\ &\{g(y) \mid y \in f(x) - \mathrm{dom}(f)\} \cup \\ &\{g(y) \mid y \in f(x) \cap \mathrm{dom}(f)\} = \\ &\{y \mid y \in f(x) - \mathrm{dom}(f)\} \cup \\ &\{g(y) \mid y \in f(x) \cap \mathrm{dom}(f)\} = f[g \restriction \mathrm{dom}(f)](x). \end{aligned}$$ □

Corollary 3.4 *Assume* $\mathrm{dom}(f) \subseteq \mathcal{U}$ *and for all* $x \in \mathrm{dom}(f)$, $f(x) \subseteq \mathcal{U}$. *If* $FP(g, f)$ *holds, then* $S(g \restriction \mathrm{dom}(f), f)$ *holds.*

Proof Let f be as in the assumption, then

$$TC(f(x) - \mathrm{dom}(f)) \cap \mathrm{dom}(f) = (f(x) - \mathrm{dom}(f)) \cap \mathrm{dom}(f) = \emptyset.$$

Hence, by Lemma 3.3 (2), $S(g \restriction \mathrm{dom}(f), f)$ holds. □

Lemma 3.5 *Assume* ZFC^{-2}+ *extensionality. Assume that* f *is a generating function,* $A \subseteq \mathrm{dom}(f)$, *for all* $x \in \mathrm{dom}(f) - A$, $f(x)$ *is an urelement, and* $FP(g, f \restriction A)$ *holds. If* h *is a function such that* $\mathrm{dom}(h) = \mathrm{dom}(g) \cup \mathrm{dom}(f)$, $h \restriction \mathrm{dom}(g) = g$, *and for all* $x \in \mathrm{dom}(f) - \mathrm{dom}(g)$, $h(x) = f(x)$, *then* $FP(h, f)$ *holds.*

Proof If $x \in \mathrm{dom}(f) - A$, then $f(x) = x \in \mathcal{U}$ because f is generating. Hence for all x, $f^*(x) = (f \restriction A)^*(x)$. Because

$$\begin{aligned} \bigcup f^*[\mathrm{dom}(h)] &= \bigcup f^*[\mathrm{dom}(g) \cup \mathrm{dom}(f)] = \\ &\bigcup f^*[\mathrm{dom}(g) \cup (\mathrm{dom}(f) - A)] = \\ &\bigcup f^*[\mathrm{dom}(g)] = \\ &\bigcup (f \restriction A)^*[\mathrm{dom}(g)] \subseteq \mathrm{dom}(g) \subseteq \mathrm{dom}(h), \end{aligned}$$

we have that $\mathrm{dom}(f) \cup \bigcup f^*[\mathrm{dom}(h)] \subseteq \mathrm{dom}(h)$.

Assume $x \in \mathrm{dom}(h)$ and $f^*(x) \notin \mathcal{U}$. Then $x \in \mathrm{dom}(g)$ and

$$f[h](x) = \{h(y) \mid y \in f^*(x)\} = \{g(y) \mid y \in (f \restriction A)^*(x)\} = g(x) = h(x),$$

because $f^*(x) = (f \restriction A)^*(x) \subseteq \mathrm{dom}(g)$. Assume that $f^*(x) \in \mathcal{U}$. If $x \notin \mathrm{dom}(g)$, then $h(x) = f(x) = x$. If $x \in \mathrm{dom}(g)$, then $h(x) = g(x) = f^*(x)$. So we have shown that $h = f[h]$. □

The following lemma is proved for basic functions in [2] (cf. Theorem 8.5).

Lemma 3.6 *Assume ZFA. For all generating* f *there is* g *such that* $FP(g, f)$ *holds.*

Proof Let f be a generating function. We show that f has a fixed point. By Lemma 3.5, we may assume that for all $x \in \mathrm{dom}(f)$, $f(x)$ is not an urelement. Choose a transitively closed A so that $\mathrm{dom}(f) \cup \bigcup \mathrm{rng}(f) \subseteq A$. Then $\bigcup f^*[A] \subseteq A$. Choose a one-one function h so that $\mathrm{dom}(h) = B = (A - \mathcal{U}) \cup \mathrm{dom}(f)$, $\mathrm{rng}(h) \subseteq \mathcal{U}$,

$h(y) = y$ if $y \in \mathrm{dom}(f) \cap \mathcal{U}$ and $\mathrm{rng}(h) \cap A = \mathrm{dom}(f) \cap \mathcal{U}$. Define f' so that $\mathrm{dom}(f') = \mathrm{rng}(h)$ and for all $x \in B$, $f'(h(x)) = \{h^*(y) \mid y \in f^*(x)\}$. Then $(\mathrm{rng}(h), (A \cap \mathcal{U}) - \mathrm{rng}(h), f')$ is a flat system of equations. Let g' be such that $S(g', f')$ holds and let g be such that $\mathrm{dom}(g) = A$, $g \restriction B = g' \circ h$ and $g \restriction A - B = id_{A-B}$. We show that g is a fixed point of f. We have already shown that $\mathrm{dom}(f) \cup f^*[\mathrm{dom}(g)] \subseteq \mathrm{dom}(g)$. So it is enough to show that for all $x \in A$, $g(x) = f[g](x)$. If $x \notin B$, then $g(x) = x$ and $x \notin \mathrm{dom}(f)$. So $f^*(x) = x$ is an urelement and we have that $f[g](x) = f^*(x) = x = g(x)$.

Assume that $x \in B$. Then $f'(h(x))$ is not an urelement and so

$$g(x) = g'(h(x)) = \{g'(y) \mid y \in f'(h(x))\} = \\ \{g'(y) \mid y \in f'(h(x)) - \mathrm{rng}(h)\} \cup \{g'(y) \mid y \in f'(h(x)) \cap \mathrm{rng}(h)\}. \quad (1)$$

Now

$$f'(h(x)) - \mathrm{rng}(h) = \{h^*(z) \mid z \in f^*(x)\} - \mathrm{rng}(h) = \\ \{z \mid z \in f^*(x) - B\} = f^*(x) - B.$$

If $y \in f^*(x) - B = f'(h(x)) - \mathrm{rng}(h)$, then $y \notin \mathrm{dom}(f')$, since $\mathrm{rng}(h) = \mathrm{dom}(f')$. Because $f^*(x) \subseteq A$, we have that $f^*(x) - B \subseteq \mathcal{U}$, so $y \in \mathcal{U}$. Hence $g'(y) = f'[g'](y) = (f')^*(y) = y$. On the other hand,

$$f'(h(x)) \cap \mathrm{rng}(h) = \{h^*(y) \mid y \in f^*(x)\} \cap \mathrm{rng}(h) = \{h(y) \mid y \in f^*(x) \cap B\}.$$

Thus we have that Equation (1) is equal to

$$\{y \mid y \in f^*(x) - B\} \cup \{g'(y) \mid y \in \{h(z) \mid z \in f^*(x) \cap B\}\} = \\ \{g(y) \mid y \in f^*(x) - B\} \cup \{g'(h(z)) \mid z \in f^*(x) \cap B\} = \\ \{g(y) \mid y \in f^*(x)\}. \quad \square$$

Lemma 3.7 *Assume $ZFC^{-2} + SEA$. If every generating function has a fixed point, then every function has a fixed point.*

Proof Let f be a function. Let A be a transitively closed set such that $\mathrm{dom}(f) \cup \mathrm{rng}(f) \subseteq A$. Let B be the set of those $x \in A$ such that $f^*(x) \neq x$ is an urelement and let C be the set of those $x \in A$ such that $f^*(x) = x$ is an urelement. Let h be a one-one function such that $\mathrm{dom}(h) = B$ and $\mathrm{rng}(h) \subseteq \mathcal{U} - A$. Define f' so that $\mathrm{dom}(f') = A - B$ and for all $x \in \mathrm{dom}(f')$, if $x \in C$, then $f'(x) = x$ and otherwise $f'(x) = \{h^*(y) \mid y \in f^*(x)\}$. Then f' is generating and so by Lemma 3.6 it has a fixed point g'. Let $D = g'[\mathrm{dom}(f')] - \mathcal{U}$ and define f'' so that $\mathrm{dom}(f'') = D$ and for all $x \in \mathrm{dom}(f'')$, $f''(x) = \{h'(y) \mid y \in x\}$ where $h'(y) = y$, if $y \notin \mathrm{rng}(h)$ and otherwise $h'(y) = f(h^{-1}(y))$. Then f'' is generating and let g'' be a fixed point of f''. We define g so that $\mathrm{dom}(g) = A$, for all $x \in \mathrm{dom}(g)$, if $f^*(x)$ is an urelement, then $g(x) = f^*(x)$ and otherwise, $g(x) = g''(g'(x))$. We show that g is a fixed point of f.

Since $\mathrm{rng}(f) \subseteq A$ and A is transitively closed, $\bigcup f^*[A] \subseteq A$. Also $\mathrm{dom}(f) \subseteq A$. So it is enough to prove that for all $x \in A$, $g(x) = f[g](x)$. If $x \in B \cup C$, the claim is clear. So assume $x \in A - (B \cup C)$. Then

$$\begin{aligned} g(x) &= g''(g'(x)) = \{g''(y) \mid y \in f''(g'(x))\} = \\ &\{g''(h'(y)) \mid y \in g'(x)\} = \\ &\{g''(h'(g'(y))) \mid y \in f'(x)\} = \\ &\{g''(h'(g'(h^*(y)))) \mid y \in f^*(x)\}. \end{aligned}$$

We have several cases.

Case 1 $y \in C$: Then $h^*(y) = y$, $g'(y) = (f')^*(y) = y$, $h'(y) = y$ and $g''(y) = (f'')^*(y) = y$. Also $g(y) = y$ and so $g''(h'(g'(h^*(y)))) = g(y)$.

Case 2 $y \in B$: Then $(f')^*(h^*(y)) = h^*(y)$ and so $g'(h^*(y)) = h^*(y)$. Furthermore, $h'(h^*(y)) = f(y)$ and since $f(y) \notin \mathrm{dom}(f'')$, $g''(f(y)) = f(y)$. So $g''(h'(g'(h^*(y)))) = f(y) = g(y)$.

Case 3 $y \in A - (B \cup C)$: Clearly $h^*(y) = y$ and $g'(y)$ is a set. So $h'(g'(y)) = g'(y)$. Then $g''(h'(g'(h^*(y)))) = g''(g'(y)) = g(y)$.

By Cases 1–3, $g(x) = \{g''(h'(g'(h^*(y)))) \mid y \in f^*(x)\} = \{g(y) \mid y \in f^*(x)\} = f[g](x)$. □

Corollary 3.8 *ZFC^+ is equivalent to ZFA.*

Proof Assume ZFC^+ and that (X, A, f) is a flat system of equations. Then $f(x) \subseteq \mathcal{U}$ for every $x \in \mathrm{dom}(f)$. Let g be such that $FP(g, f)$ holds. Then by Corollary 3.4, $S(g \restriction \mathrm{dom}(f), f)$ holds. Thus *AFA* holds. The extensionality axiom follows from Lemma 3.2 (1) and hence *ZFA* holds.

Assume *ZFA*. Then by Lemmas 3.6 and 3.7, every function has a fixed point. So *FPA* holds. By Theorem 7.3 of [2], the strong extensionality axiom holds in *ZFA*. Hence ZFC^+ holds. □

Lemma 3.9 *If f is a substitution, then there is a function g such that $g = g \star f$.*

Proof Let $u \in \mathcal{U}$ be such that $u \notin \mathrm{dom}(f)$. We define f' as follows. Let $\mathrm{dom}(f') = \mathrm{dom}(f)$. If $f(x)$ is not an urelement or $f(x) = x$ or $f(x) \in \mathcal{U} - \mathrm{dom}(f)$, then let $f'(x) = f(x)$. If there is $n < \omega$ such that $\forall m < n : f^m(x) \in \mathcal{U}$ and $f^m(x) \neq f^{m-1}(x)$ but $f^n(x) \notin \mathcal{U}$, or $f^n(x) = f^{n-1}(x)$, or $f^n(x) \in \mathcal{U} - \mathrm{dom}(f)$, then let $f'(x) = f^n(x)$. Otherwise let $f'(x) = u$.

Let g' be such that $FP(g', f')$ holds and let $g = g' \restriction \mathrm{dom}(f)$. We show that $g = g \star f$. If $f(x) \in \mathrm{dom}(f)$ is an urelement, then by the definition of f', we have that $f'(f(x)) = f'(x)$. Now if $f'(x) \in \mathcal{U}$,

$$g(x) = f'(x) = f'(f(x)) = g(f(x)) = \mathrm{sub}(g, f(x)).$$

If $f'(x) \notin \mathcal{U}$,

$$g(x) = \{g'(y) \mid y \in f'(x)\} = \{g'(y) \mid y \in f'(f(x))\} = g(f(x)) = \mathrm{sub}(g, f(x)).$$

If $f(x) \in \mathcal{U} - \mathrm{dom}(f)$, then $g^*(x) = f'(x) = f(x) = \mathrm{sub}(g, f(x))$. So we have shown that if $f(x)$ is an urelement, then $g(x) = g(f(x))$, hence we need to show that for all $x \in \mathrm{dom}(f)$, if $f(x) \notin \mathcal{U}$, then $g'(x) = \mathrm{sub}(g, f(x))$. For this, we define a bisimulation B so that $(a, b) \in B$ if and only if there is a $y \in \mathrm{dom}(g')$ such that $a = g'(y)$ and $b = \mathrm{sub}(g, f^*(y))$ or $a = b \in \mathcal{U} \cap \mathrm{dom}(g')$ or $a = g'(y) = b$.

To show that B is a bisimulation, let $(g'(y), \mathrm{sub}(g, f^*(y))) \in B$ for some $y \in \mathrm{dom}(g')$. We have several cases.

Case 1 $y \in \mathcal{U} - \mathrm{dom}(f)$: Then $(f')^*(y) = f^*(y) = y$ so $g'(y) = y = \mathrm{sub}(g, f^*(y))$.

Case 2 $y \in \mathrm{dom}(f)$ and $f(y) \in \mathcal{U}$: As above we have that $g'(y) = g(f(y)) = \mathrm{sub}(g, f^*(y))$.

Case 3 $y \in \mathrm{dom}(f)$, $f(y) \notin \mathcal{U}$: Because $f(y) \notin \mathcal{U}$, we have that $f(y) = f'(y)$, so

$$\begin{aligned} g'(y) &= \{g'(z) \mid z \in f'(y)\}, \\ \mathrm{sub}(g, f^*(y)) &= \{\mathrm{sub}(g, z) \mid z \in f(y)\}. \end{aligned}$$

Assume $z \in f(y)$, so $z \in \mathrm{dom}(g')$. If $z \notin \mathrm{dom}(f)$, then $(g'(z), \mathrm{sub}(g, f^*(z))) \in B$. If $z \in \mathrm{dom}(f)$, then $\mathrm{sub}(g, z) = g(z) = g'(z)$ and $(g'(z), g'(z)) \in B$. □

For a class function F, $FP(G, F)$ is defined as for the set functions. We show that under ZFC^+ also the class functions have fixed points.

Lemma 3.10 *Assume ZFC^+. Let $F : V_{afa}[\mathcal{U}] \cup \mathcal{U} \to V_{afa}[\mathcal{U}] \cup \mathcal{U}$ be a definable class function. Then there exists a unique definable class function $G : V_{afa}[\mathcal{U}] \cup \mathcal{U} \to V_{afa}[\mathcal{U}] \cup \mathcal{U}$ such that $FP(G, F)$ holds.*

Proof Let $x \in V_{afa}[\mathcal{U}] \cup \mathcal{U}$. If $F(x) \in \mathcal{U}$, then let $G(x) = F(x)$. Otherwise we define $G(x)$ as follows. Let $A_0 = TC(\{x\})$, $A_{n+1} = A_n \cup TC(F[A_n])$, and $A(x) = \bigcup_{n<\omega} A_n$. Then $A(x)$ is transitively closed and $F[A(x)] \subseteq A(x)$. Now let g be a function such that $FP(g, F \upharpoonright A(x))$ holds and define $G(x) = g(x)$.

We show that $g(y)$ does not depend on the choice of $A(x)$ as long as $y \in A(x)$ and $F[A(x)] \subseteq A(x)$. Let A and A' be transitively closed sets such that $y \in A$, $F[A] \subseteq A$, and $F[A'] \subseteq A'$. Let g and g' be such that $FP(g, F \upharpoonright A)$ and $FP(g', F \upharpoonright A')$ hold. Let $C = A \cap A'$, then C is transitively closed, $y \in C$, and $F[C] \subseteq C$. We show that $g \upharpoonright C = g' \upharpoonright C$ from which the claim follows. So let $z \in C$ and let B consist of $(g(z), g'(z))$ together with $(a, b) \in TC(g(z)) \times TC(g'(z))$ such that either $a = b$ or there is $c \in C$ such that $a = g(c)$ and $b = g'(c)$. To show that B is a bisimulation between $g(z)$ and $g'(z)$, let $c \in C$ and $(g(c), g'(c)) \in B$. Assume that $F(c)$ is not an urelement. Then $g(c) = \{g(d) \mid d \in (F \upharpoonright C)(c)\}$. Now if $d \in (F \upharpoonright C)(c)$, then $d \in C$ and so $(g(d), g'(d)) \in B$. So B is a bisimulation and hence for all $z \in C$, $g(z) = g'(z)$.

We show that G is a fixed point of F. Let $x \in V_{afa}[\mathcal{U}] \cup \mathcal{U}$. If $F(x) \in \mathcal{U}$, then $G(x) = F[G](x) = F(x)$. If $F(x) \notin \mathcal{U}$, let $A(x)$ be the transitively closed set such that $x \in A(x)$ and $F[A(x)] \subseteq A(x)$. Let g be such that $FP(g, F \upharpoonright A(x))$ holds. Then $G(x) = g(x) = \{g(y) \mid y \in (F \upharpoonright A(x))^*(x)\} = \{G(y) \mid y \in F(x)\}$ because $A(x)$ is transitively closed and $F[A(x)] \subseteq A(x)$. □

As a corollary we have Theorem 8.1 of [2].

Corollary 3.11 *There is a unique operation* $\mathrm{sub}(s, b)$ *as in Definition 1.2 defined for all substitutions s and sets b.*

Proof Assume s is a substitution. Define a class function F by $F(x) = x$ if $x \notin \mathrm{dom}(s)$ and $F(x) = s(x)$ otherwise. By the above lemma, let G be such that $FP(G, F)$ holds. We claim that $G(x) = \mathrm{sub}(s, x)$ for all x.

If x is an urelement, then $G(x) = F(x) = \mathrm{sub}(s, x)$. Let x be a set. Define the relation B by $(a, b) \in B$ if and only if $a = G(y)$ and $b = \mathrm{sub}(s, y)$ for some $y \in \{x\} \cup TC(x)$. We show that B is a bisimulation between $G(x)$ and $\mathrm{sub}(s, x)$. The case for urelements is as above, so let $(G(y), \mathrm{sub}(s, y)) \in B$ where $G(y)$ is a set. Then $G(y) = \{G(z) \mid z \in F(y)\}$ and $\mathrm{sub}(s, y) = \{\mathrm{sub}(s, z) \mid z \in y\}$. Because $y \notin \mathrm{dom}(s)$, $F(y) = y$ and we see that the bisimulation can be continued. □

We finish this section by showing that a fixed point of a fixed point of a basic f is a fixed point of f. Thus the name fixed point.

Lemma 3.12 *Assume ZFC$^+$. For every function f there is a function g such that $FP(g, f)$ holds and* $\mathrm{rng}(g) \cup \bigcup \mathrm{rng}(g) \subseteq \mathrm{dom}(g)$.

Proof Let f be a function and g' such that $FP(g', f)$. We define inductively functions f_n and g_n for $n < \omega$ as follows. Let $f_0 = f$ and $g_0 = g'$.

Let $A_n = \mathrm{dom}(g_n) \cup \mathrm{rng}(g_n) \cup \bigcup \mathrm{rng}(g_n)$ and $\mathrm{dom}(f_{n+1}) = \mathrm{dom}(f_n) \cup A_n$. Define $f_{n+1}(x) = f_n(x)$, if $x \in \mathrm{dom}(f_n)$, and otherwise $f_{n+1}(x) = x$. Let g_{n+1} be such that $FP(g_{n+1}, f_{n+1})$ holds.

Because for every n, $\mathrm{dom}(f_n) \subseteq \mathrm{dom}(g_{n+1})$, we have that

$$\mathrm{rng}(g_n) \cup \bigcup \mathrm{rng}(g_n) \subseteq \mathrm{dom}(g_{n+1}). \tag{2}$$

From the definition of f_n if follows that for all n, $f_n \subseteq f_{n+1}$ and also $f \subseteq f_n$. Clearly if $x \notin \mathrm{dom}(f)$, then $f_n(x) = x$, hence for every x and n, $f_n^*(x) = f^*(x)$.

It is clear that $\mathrm{dom}(f) \subseteq \mathrm{dom}(g_n)$. Also

$$f^*[\mathrm{dom}(g_n)] = f[\mathrm{dom}(g_n) \cap \mathrm{dom}(f)] \cup (\mathrm{dom}(g_n) - \mathrm{dom}(f)) = f_n^*[\mathrm{dom}(g_n)].$$

Hence $\bigcup f^*[\mathrm{dom}(g_n)] \subseteq \bigcup f_n^*[\mathrm{dom}(g_n)] \subseteq \mathrm{dom}(g_n)$. If $x \in \mathrm{dom}(g_n)$ and $f^*(x)$ is an urelement, then $g_n(x) = f_n^*(x) = f^*(x)$. If $f^*(x)$ is not an urelement, then $g_n(x) = \{g_n(y) \mid y \in f_n^*(x)\} = \{g_n(y) \mid y \in f^*(x)\}$. Thus $FP(g_n, f)$.

Now because for every $n < \omega$, $FP(g_n, f)$, $FP(g_{n+1}, f)$, and $\mathrm{dom}(g_n) \subseteq \mathrm{dom}(g_{n+1})$, we have by Lemma 3.2 (3) that $g_n \subseteq g_{n+1}$. So we can define $g = \bigcup_{n<\omega} g_n$. By Equation (2) we have that $\mathrm{rng}(g) \cup \bigcup \mathrm{rng}(g) \subseteq \mathrm{dom}(g)$.

Finally, we show that $FP(g, f)$ holds. Because for all n, $FP(g_n, f)$, we have that $\mathrm{dom}(f) \cup \bigcup f^*[\mathrm{dom}(g)] \subseteq \mathrm{dom}(g)$. Let $x \in \mathrm{dom}(g)$. Then for some n, $x \in \mathrm{dom}(g_n)$. If $f^*(x) \in \mathcal{U}$, then $g(x) = g_n(x) = f^*(x)$. Otherwise $g(x) = g_n(x) = \{g_n(y) \mid y \in f_n^*(x)\} = \{g(y) \mid y \in f^*(x)\} = f[g](x)$. □

Lemma 3.13 *Assume ZFC^{-2} + SEA, f is basic, $FP(g, f)$ holds and* $\mathrm{rng}(g) \cup \bigcup \mathrm{rng}(g) \subseteq \mathrm{dom}(g)$. *Then for all $x \in \mathrm{dom}(g)$, $g(g(x)) = g(x)$ and if $g(x) \notin \mathcal{U}$, then $g(x) = \{g(y) \mid y \in g(x)\}$. Especially, $FP(g, g)$ and $S(g, g)$ hold.*

Proof Let $a \in \mathrm{dom}(g)$. Let B consist of $(g(a), g(g(a)))$ together with $(x, y) \in TC(g(a)) \times TC(g(g(a)))$ such that either $x = y$ or there is $z \in \mathrm{dom}(g)$ such that $x = g(z)$ and $y = g(g(z))$.

We show that B is a bisimulation between $g(a)$ and $g(g(a))$. Assume that $z \in \mathrm{dom}(g)$, $f^*(z) \notin \mathcal{U}$, and $(g(z), g(g(z))) \in B$.

Let $y \in g(z) = \{g(w) \mid w \in f^*(z)\}$. So $y = g(w)$ for some $w \in f^*(z)$. Because $g(w) \in \mathrm{rng}(g) \subseteq \mathrm{dom}(g)$, $g(g(w))$ is defined. So $(g(w), g(g(w))) \in B$.

Let $y \in g(g(z)) = \{g(w) \mid w \in f^*(g(z))\}$. So $y = g(w)$ for some $w \in f^*(g(z)) \subseteq \mathrm{dom}(g)$. Thus $(g(w), g(g(w))) \in B$.

For the second claim, assume that $g(x) \notin \mathcal{U}$. So $g(x) \notin \mathrm{dom}(f)$ and thus $g(x) = g(g(x)) = \{g(y) \mid y \in g(x)\} = g[g](x)$. Thus $S(g, g)$ holds. Because $\mathrm{dom}(g) \cup \bigcup g^*[\mathrm{dom}(g)] \subseteq \mathrm{dom}(g) \cup \mathrm{rng}(g) \cup \bigcup \mathrm{rng}(g) \subseteq \mathrm{dom}(g)$, we have that $FP(g, g)$ holds. □

The assumption that f is basic is needed in Lemma 3.13.

Example 3.14 Assume ZFC^+ (the first example works also in ZFC).

1. We define sets e^n, $n < \omega$, so that $e^0 = \varnothing$ and $e^{n+1} = \{e^n\}$. Let f be such that $f(e^3) = e^2$, $f(e^2) = \{e^0, e^1\}$ and for $n < 2$, $f(e^n) = e^n$. Then $FP(f, f)$ holds, but $f(f(e^3)) = \{e^0, e^1\} \neq f(e^3)$.
2. As in [2], let Ω be such that $\Omega = \{\Omega\}$. Define f so that $f(\varnothing) = \{\varnothing\}$ and $f(\Omega) = \varnothing$. Let g be such that $FP(g, f)$ holds. Then since $g(x) = \{g(y) \mid y \in f^*(x)\}$, $g(\varnothing) = \Omega$ and $g(\Omega) = \varnothing$. But $\{g(y) \mid y \in g(\varnothing)\} = \{\varnothing\} \neq g(\varnothing)$, so it is not the case that $FP(g, g)$.

4 A Model in Which Not All Equations Have Solutions

We now turn to the question: Do we need to assume that all flat systems of equations have solutions in order to get all fixed points. First we show how to construct a model of set theory from a given transitive class of non-wellfounded sets.

Definition 4.1 Let $C \subset V_{afa}[\mathcal{U}] \cup \mathcal{U}$ be a transitive class.

$$\mathbf{C}(C) = \{x \in V_{afa}[\mathcal{U}] \cup \mathcal{U} \mid \text{ there is no sequence } x_i, i < \omega \text{ such that } x_0 \in TC(x) \text{ and } \forall i, x_{i+1} \in x_i, x_i \notin C\}.$$

Intuitively this class is the same as the following class V'': Let $V'_0 = C \cup \mathcal{U}$, $V'_{\alpha+1} = \{x \mid x \subseteq V'_\alpha\}$, $V'_\beta = \bigcup_{\alpha<\beta} V'_\alpha$, when β is a limit ordinal, and let $V'' = \bigcup\{V'_\alpha \mid \alpha \text{ is an ordinal}\}$.

Lemma 4.2 *Assume that $C \subset V_{afa}[\mathcal{U}] \cup \mathcal{U}$ is a transitive class and $V' = \mathbf{C}(C)$, then $V' \models ZFC^{-2} + SEA$.*

Proof Now V' is a transitive class, so the axioms of extensionality and strong extensionality hold in V'. If x is a subset of V', then clearly $x \in V'$. Hence the power set axiom holds in V'.

The axiom of urelements, $\forall p \forall q(\mathcal{U}(p) \to \neg(q \in p))$ holds in V'. The pairing and union axioms also hold in V'. Because $\omega \in V'$, $\varnothing$ and the successor operation are absolute for V', V' satisfies the axiom of infinity.

For the collection it is enough to show that for each $\varphi(x, y, A, w_1, \ldots, w_n)$ and each $A, w_1, \ldots, w_n \in V'$, if

$$\forall x \in A \exists! y \in V' \varphi^{V'}(x, y, A, w_1, \ldots, w_n),$$

then

$$\exists Y \in V'(\{y \mid \exists x \in A, \varphi^{V'}(x, y, A, w_1, \ldots, w_n)\} \subseteq Y).$$

So let $Y = \{y \in V' \mid \exists x \in A, \varphi^{V'}(x, y, A, w_1, \ldots, w_n)\}$. Then $Y \subset V'$ and hence $Y \in V'$.

Since for every $z \in V'$, $P(z) \subseteq V'$, we have that V' satisfies the separation axiom. For the axiom of choice, we can show that if $x \in V'$ and x can be well ordered, then $(x \text{ can be well ordered})^{V'}$: If $R \subseteq x \times x$ well orders x, then since $x \times x \in V'$ we have that $R \in V'$. The formula 'R totally orders x' is absolute for V'. For well ordering we have to check that $(\forall y \varphi(y, x, R))^{V'}$, where $\varphi(y, x, R)$ is

$$y \subseteq x \wedge y \neq \varnothing \rightarrow \exists z \in y \forall w \in y((w, z) \notin R).$$

Now φ is absolute for V' so it is enough to show that $\forall y \in V' \varphi(y, x, R)$ which follows since R well orders x. Thus the axiom of choice holds in V'.

For the axiom of plenitude which is

$$\forall S \notin \mathcal{U}(\exists f : S \rightarrow \mathcal{U} \text{ such that } f \text{ is injective and } f[S] \cap S = \varnothing),$$

let S be a set in V'. Let $g : S \rightarrow \mathcal{U}$ be an injection in $V_{afa}[\mathcal{U}]$. Then also $g \in V'$ because V' is closed under the power set operation. We have shown that $V' \models ZFC^{-2} + SEA$. □

Lemma 4.3 *Assume that $C \subset V_{afa}[\mathcal{U}]$ is a transitive class and there exist x_i, $i < \omega$ such that $x_{i+1} \in TC(x_i)$ and $x_i \notin C$. If $V' = \mathbf{C}(C)$, then $V' \models ZFC^{-2} + SEA$ and $V' \not\models AFA$.*

Proof We define the canonical flat system of equations for x_0 as follows. Let h be an injection such that $\mathrm{dom}(h) = TC(x_0)$, $\mathrm{rng}(h) \subseteq \mathcal{U}$ and if $a \in TC(x_0) \cap \mathcal{U}$, then $h(a) = a$. Let $A = TC(x_0) \cap \mathcal{U}$, $a_0 \in \mathcal{U} - \mathrm{rng}(h)$, $X = (\mathrm{rng}(h) \cup \{a_0\}) - A$. Define f in X such that $f(a_0) = \{h(y) \mid y \in x_0\}$ and $f(h(z)) = \{h(y) \mid y \in z\}$ for $z \in \mathrm{dom}(h)$. So f is a system of equations which belongs to V' and it was constructed so that for the solution g to f, we have that $g(a_0) = x_0$.

Now x_0 cannot be in V' because of the definition of V'. Because being a solution to a flat system of equations is an absolute property for V', we have that f has not a solution in V'. □

Next we introduce the notion of a flat $\mathcal{P}$-coalgebra from [2] which corresponds to a flat system of equations with no atoms.

Definition 4.4 A flat $\mathcal{P}$-coalgebra is a pair (X, f) such that $X \subseteq \mathcal{U}$ is a set of urelements and $f : X \rightarrow \mathcal{P}(X)$. A function g is a substitution solution to the flat $\mathcal{P}$-coalgebra if $FP(g, f)$ holds.

It is shown in [2] that if g is a substitution solution to a flat $\mathcal{P}$-coalgebra, then $\mathrm{rng}(g) \subseteq \mathcal{P}^*$, where $\mathcal{P}^*$ is the greatest fixed point of the operator $\mathcal{P}$ and it is equal to the class of all pure sets $V_{afa}[\varnothing]$. (In case there are no urelements, then $\mathcal{P}^*$ is the whole universe and the example below does not hold anymore—but see the next section.)

Example 4.5 Assume $ZFC^{-2} + SEA$. The following does not imply AFA: Every flat $\mathcal{P}$-coalgebra has a substitution solution.

Proof Let $C = V_{afa}[\varnothing]$. If (X, f) is a flat $\mathcal{P}$-coalgebra and g its solution, then $\mathrm{rng}(g) \subseteq C$. If we let $V' = \mathbf{C}(C)$, then $\mathrm{dom}(g) \in V'$ and hence $g \in V'$ because

V' is closed under the power set operation. Because being a flat $\mathcal{P}$-coalgebra and a solution to it are absolute properties for V', we have that in V' every flat $\mathcal{P}$-coalgebra has a solution.

Let x be an urelement and $f(x) = \{a, x\}$ where $a \in \mathcal{U}$ and $a \neq x$. Then f is an equation system and it has a solution g in $V_{afa}[\mathcal{U}]$. Then $g(x) = \{a, g(x)\} \notin C$, and we get the x_is as required in Lemma 4.3, by setting $x_i = g(x)$. Hence by Lemma 4.3 *AFA* does not hold in V'. □

Γ-coalgebras can be seen as systems of equations (see [2], §16) and if we restrict our interest to flat Γ-coalgebras, then the class of solutions can be seen as the final Γ-coalgebra. One may wonder if the same can be done (e.g., by fixed points) for a class of Γ-coalgebras larger than the flat ones. This does not seem to be the case or at least a much deeper understanding of non-wellfounded sets is needed. The crucial property of the flat Γ-coalgebras (X, e) is that X is new for Γ (i.e., for all substitutions t and sets a, $\Gamma(a[t]) = \Gamma(a)[t]$); this forces X to be flat in the usual sense of the word. And without something like this the theory does not work. For example, the crucial Lemma 16.1 in [2] fails.

Let $\Gamma(X) = \mathcal{P}(\mathcal{P}(X) - \{\varnothing\})$. This is a monotone and proper operator, that is, if $X \subseteq Y$, then $\Gamma(X) \subseteq \Gamma(Y)$, and for all sets a, $\Gamma(a) \subseteq V_{afa}[\mathcal{U}]$. Let $X = \{\varnothing, \{\varnothing\}\}$, and let $e(\varnothing) = \{\{\varnothing\}\}$ and $e(\{\varnothing\}) = \varnothing$. Then (X, e) is a Γ-coalgebra that is not flat. If s is a solution to e, then $s(\varnothing) = \{\varnothing\}$ and $s(\{\varnothing\}) = \varnothing$ but $\{\varnothing\} \notin \Gamma^*$. Hence s is not a Γ-morphism of (X, e) into (Γ^*, id).

5 Classifying Non-wellfounded Sets

Here we have a fixed class of urelements $\mathcal{U}$ and all sets can contain urelements as their members. So from now on we denote by V_{afa} the class $V_{afa}[\mathcal{U}] \cup \mathcal{U}$ of [BM]. But as in the above, urelements are not vital in here. We regard arbitrary functions as equation systems and when we speak of the indeterminates of the equation systems, we mean the elements of their domain.

We define a series of classes of equation systems E_α, $\alpha \in \mathbf{ON}$ in increasing complexity. From these equation systems we obtain a series of classes of non-wellfounded sets,

$$V^0_{afa} \subset V^1_{afa} \subset \cdots \subset V^\alpha_{afa} \subset \cdots,$$

so that $V^{\alpha+1}_{afa} \not\subset V^\alpha_{afa}$. We also define the *rank* of a non-wellfounded set x as the least α such that $x \in V^\alpha_{afa}$.

The non-wellfounded sets become more complicated in the series $V^0_{afa} \subset V^1_{afa} \subset \cdots$ according to the branching structure of the non-wellfounded sets. V^0_{afa} is the class of wellfounded sets. In V^1_{afa} there are sets which can be described either as Ω and sets that can be obtained from it by standard set theoretical operations or as sets which have a non-wellfounded $\in$-sequence of length ω such that going down this sequence one has ω chances to branch out of that sequence. But in V^1_{afa}, after branching, the sets are wellfounded. In V^2_{afa} there are sets in which there are ω chances to branch to sets in which there are again ω chances to branch into sets in which there are only finite number of possibilities to branch. So the rank tells how many times it is possible to branch arbitrarily deep.

A non-wellfounded set of rank ω has elements of arbitrarily high rank below ω. In a non-wellfounded set of rank $\omega + 1$, one can find a non-wellfounded $\in$-sequence in which there are ω chances to branch into sets of rank ω. And so on in the higher degrees.

There is also the possibility that this branching process goes on arbitrarily long. In this case we say that the rank is ∞. First we need to characterize the different classes of equation systems. This is done in game theoretic terms.

Definition 5.1 Let f be a system of equations. A sequence $\vec{x} = \langle x_i \mid i < \omega \rangle$ where $x_i \in \mathrm{dom}(f), i < \omega$, of indeterminates of f is called descending if for all $i < \omega, x_{i+1} \in f(x_i)$.

We describe a game $G_\alpha(\mathcal{E})$, where $\alpha \in \mathbf{ON}$, that is played on a given system of equations f as follows. There are two players, black and white. First the black player chooses a descending sequence $\vec{x}$ of indeterminates. Then white chooses an ordinal $\alpha_0 < \alpha$ and a natural number $n < \omega$. Black must respond with a descending sequence of indeterminates $\vec{y}$ such that for some $m \geq n$, $y_0 \in f(x_m)$ and $y_0 \neq x_{m+1}$. So $\vec{y}$ branches out of $\vec{x}$. Then again white chooses an ordinal $\alpha_2 < \alpha_1$ and a natural number and so on.

The length of this game is the number of pairs of moves by black and white. This length is finite since there are no infinite descending sequences of ordinals.

We say that black has a winning strategy in the game $G_\alpha(f)$ if she is able to respond to white's moves until white has no more moves. White wins otherwise, that is, if black is not able to respond with a descending sequence of indeterminates to white's move.

There is also a game of infinite length. In $G_\infty(f)$ the white player does not choose ordinals, only indices. Hence the length of this game is ω.

More formally, we say that a move of the white player is a pair (α, n) where α is an ordinal and $n < \omega$. We use the projection function $\pi_2(\alpha, n) = n$ to get the second coordinate of the pair (α, n). In $G_\alpha(f)$ we say that a sequence $\vec{w}$ is a *legal* sequence of white's moves of length k if

$$\vec{w} = \langle (\alpha_i, n_i) \mid i < k \rangle,\ \forall i \in \omega (n_i < \omega),\ \text{and } \alpha > \alpha_0 > \alpha_1 > \cdots > \alpha_{k-1}.$$

We define the winning strategy σ for black as follows.

Definition 5.2 Let f be a system of equations and α an ordinal. A winning strategy for the black player in the game $G_\alpha(f)$ is a function σ of two arguments, a natural number k and a legal sequence $\vec{w}$ of white's moves of length k, that satisfies the following conditions:

1. $\sigma(0, \varnothing) = \vec{x}$, where $\vec{x}$ is a descending sequence of the indeterminates of f.
2. $\sigma(k+1, \vec{w}) = \vec{y}$, where $\vec{y}$ is a descending sequence of indeterminates such that the following holds. Denote by $\vec{x}$ the previous move of black, that is, $\sigma(k, \vec{w} \restriction k)$ and denote by n white's last move, that is, $\pi_2(w_k)$. We require from $\vec{y}$ that $\exists m \geq n (y_0 \in f(x_m)$ and $y_0 \neq x_{m+1})$.

We say that the black player *wins* a game if she has a winning strategy. The white player wins if the black player does not win.

We may also define a similar game played on non-wellfounded sets $G_\alpha(x)$. We say that a sequence $\langle x_i \mid i < \omega \rangle$ is a non-wellfounded sequence, if for all $i < \omega$, $x_{i+1} \in x_i$. If we replace, in the above definitions, the system of equations f by a set

x and the descending sequences of indeterminates by non-wellfounded sequences, then we have the corresponding definition for sets. For sets we also require that $\sigma(0, \varnothing)$ is a non-wellfounded sequence starting from x.

If white wins the game $G_0(x)$, then x is well founded. If white wins $G_1(x)$, and black wins $G_0(x)$, then in $TC(x)$ there are non-wellfounded sets but no sets in which we can branch two times as described above. Also, if black wins G_α, then black wins G_β for all $\alpha \leq \beta$ and if white wins G_α, then white wins G_β for all $\beta \geq \alpha$.

Definition 5.3

1. $E_\alpha = \{f \mid \text{white wins } G_\alpha(f)\}$,
2. $E_\infty = \{f \mid \text{black wins } G_\infty(f)\}$,
3. $V^\alpha_{afa} = \{x \mid \text{white wins } G_\alpha(x)\}$,
4. $V^\infty_{afa} = \{x \mid \text{black wins } G_\infty(x)\}$,
5. AFA_α is the statement that all the systems of equations in E_α have solutions.

From the preceding definition it follows that if x is a set and f is its canonical system of equations, then $x \in V^\alpha_{afa}$ if and only if $f \in E_\alpha$. Also black player's winning strategy in the game on sets can be straightforwardly converted into a winning strategy in the game on equation systems. Thus all the solutions to the equation systems from E_α are in V^α_{afa}.

The solution set x to an equation system f does not always have the same rank as f. For example, define an equation system f such that

$$\mathrm{dom}(f) = \{u_\eta \in \mathcal{U} \mid \eta \in 2^{<\omega}\},$$

where u_ηs are distinct, by $f(u_\eta) = \{u_{\eta^\frown\{0\}}, u_{\eta^\frown\{1\}}\}$. Then $f \notin E_\alpha$ for all α but the solution set of f is Ω, by Exercise 7.1 of [2] and $\Omega \in V^1_{afa}$.

Definition 5.4 The non-wellfoundedness rank of a set x, denoted by nwfrank(x) is the least α such that $x \in V^\alpha_{afa}$, if there is such and ∞ otherwise.

Note that for x such that $\mathrm{nwfrank}(x) \in \mathbf{ON}$ we have that $\mathrm{nwfrank}(x) = \min\{\alpha \mid \text{white wins } G_\alpha(x)\} = \sup\{\alpha + 1 \mid \text{black wins } G_\alpha(x)\}$. We also have that if $x \in y$, then $\mathrm{nwfrank}(x) \leq \mathrm{nwfrank}(y)$ but not necessarily $\mathrm{nwfrank}(x) < \mathrm{nwfrank}(y)$. In fact, Marshall and Schwarze [3] have shown that it is not possible to define in set theory a rank function r such that if $x \in y$, then $r(x) < r(y)$, without the foundation axiom. Another notion of rank for non-wellfounded sets, defined using modal logic, appears in [2], §11.

Lemma 5.5 *Black wins $G_\alpha(x)$ if and only if there is a non-wellfounded sequence $\vec{x}$ starting from x such that for all $\beta < \alpha$ the set*

$$A_\beta = \{i < \omega \mid \exists y \in x_i (y \neq x_{i+1} \text{ and black wins } G_\beta(y))\}$$

is an unbounded subset of ω.

Proof Assume that black has a winning strategy σ in the game $G_\alpha(x)$. Let $\vec{x} = \sigma(0, \varnothing)$. Let $n < \omega$ and $\beta < \alpha$. If we let (β, n) be the first move of the white player, then $\sigma(1, (\beta, n))$ is a non-wellfounded sequence $\vec{y}$ such that $\exists i \geq n(y_0 \in x_i$ and $y_0 \neq x_{i+1})$ by the definition of a winning strategy. The winning strategy σ' for black in the game $G_\beta(y_0)$ is defined by the following equations:

$$\begin{aligned}\sigma'(0,\emptyset) &= \sigma(1,(\beta,n));\\ \sigma'(m+1,\vec{w}) &= \sigma(m+2,(\beta,n)^\frown\vec{w}).\end{aligned}$$

Hence A_β is unbounded in ω.

Assume on the other hand that there is a non-wellfounded sequence $\vec{x}$ starting from x and satisfying the condition. We prove that black has a winning strategy σ in the game $G_\alpha(x)$. Let $\sigma(0,\emptyset)=\vec{x}$. Let $(\beta,n)\in\alpha\times\omega$ be white's first move. Since A_β is unbounded in ω there is $i\geq n$ such that $\exists y\in x_i(y\neq x_{i+1}$ and black wins $G_\beta(y)$ with winning strategy $\sigma')$. Then let

$$\begin{aligned}\sigma(1,(\beta,n)) &= \sigma'(0,\emptyset);\\ \sigma(m+2,(\beta,n)^\frown\vec{w}) &= \sigma'(m+1,\vec{w}).\end{aligned} \qquad \square$$

Corollary 5.6 nwfrank$(x)\geq\alpha$ *if and only if for all* $\beta<\alpha$ *there is a non-wellfounded sequence* $\vec{x}$ *starting from* x *such that*

$$\{i<\omega\mid \exists y\in x_i(y\neq x_{i+1}\ \textit{and}\ \mathrm{nwfrank}(y)\geq\beta)\}$$

is an unbounded subset of ω.

By Corollary 5.6 we could have also defined the classes V^α_{afa} via the concept of nwfrank by letting wellfounded sets have rank 0. So we have that

$$V^\alpha_{afa}=\{x\mid \mathrm{nwfrank}(x)\leq\alpha\}.$$

Theorem 5.7 $V^\alpha_{afa}\models ZFC^{-2}+SEA+AFA_\alpha$.

Proof Let $V'=\mathbf{C}(V^\alpha_{afa})$. We claim that $V^\alpha_{afa}=V'$ from which the conclusion follows. Since all the solutions to the equation systems from E_α are in V^α_{afa}, $V^\alpha_{afa}\models AFA_\alpha$.

By the definition of V', we have that $V^\alpha_{afa}\subseteq V'$. We show that $V'\subseteq V^\alpha_{afa}$ by showing that if nwfrank$(x)>\alpha$, then $x\notin V'$.

Assume toward a contradiction that there is $x\in V'$ for which nwfrank$(x)>\alpha$. So the white player does not have a winning strategy in $G_\alpha(x)$ and this means that the black player has. From this it follows by Lemma 5.5 that there is a non-wellfounded sequence $\vec{x}$ starting from x such that for all $\beta<\alpha$ the set

$$A_\beta=\{i<\omega\mid \exists y\in x_i(y\neq x_{i+1}\text{ and black wins }G_\beta(y))\}$$

is an unbounded subset of ω.

Let $i<\omega$. Black wins $G_\beta(x_i)$ for all $\beta<\alpha$ because $\vec{x}\restriction[i,\omega]$ is now a non-wellfounded sequence where black wins. So by Lemma 5.5, black wins $G_\alpha(x_i)$. Hence nwfrank$(x_i)>\alpha$, and so $x_i\notin V^\alpha_{afa}$ which violates the definition of $\mathbf{C}(V^\alpha_{afa})$. $\square$

From the preceding proof we can extract the following corollaries.

Corollary 5.8 *If* $\alpha<\gamma$, *then* $V^\alpha_{afa}\subsetneq V^\gamma_{afa}$.

Proof If $\text{nwfrank}(x) = \gamma > \alpha$, then by the proof of the theorem, we have that $x \notin V^{\alpha}_{afa}$. We construct an example of a set x for which $\text{nwfrank}(x) = \gamma$ as follows.

Let $X = \{u_\alpha \mid \alpha \leq \gamma\}$ be a set of distinct urelements. Let f be such that $\text{dom}(f) = X$ and for $\alpha \leq \gamma$, let

$$f(u_\alpha) = \begin{cases} \{u_\beta \mid \beta < \alpha\} & \text{if } \alpha \text{ is a limit,} \\ \{u_\alpha, u_{\alpha-1}\} & \text{if } \alpha \text{ is a successor,} \\ \varnothing & \text{if } \alpha = 0. \end{cases}$$

Let g be the solution to f and let $x = g(u_\gamma)$. We show by induction that $\text{nwfrank}(g(u_\alpha)) = \alpha$ for $\alpha \leq \gamma$. It is clear that $\text{nwfrank}(g(u_0)) = 0$.

Assume the claim for α. By the construction, there is a non-wellfounded sequence $\vec{x} = \langle g(u_{\alpha+1}), g(u_{\alpha+1}), \ldots\rangle$ starting from $g(u_{\alpha+1})$. Let $i < \omega$. Then $g(u_\alpha) \in x_i$ and $\text{nwfrank}(g(u_\alpha)) = \alpha$, hence by Corollary 5.6, $\text{nwfrank}(g(u_{\alpha+1})) \geq \alpha + 1$. We show that white wins $G_{\alpha+1}(g(u_{\alpha+1}))$, whence $\text{nwfrank}(g(u_{\alpha+1})) = \alpha + 1$. For the first move black has to choose $\vec{x}$ (other choices would be worse). But black cannot win $G_{\alpha+1}(g(u_{\alpha+1}))$ in $\vec{x}$ by Lemma 5.5 because for all $i < \omega$, there is no $y \in x_i$ such that black wins $G_\alpha(y)$.

Assume the claim for $\beta < \alpha$. By the construction, $g(u_\alpha) = \{g(u_\beta) \mid \beta < \alpha\}$. For every $\beta < \alpha$, $\text{nwfrank}(g(u_\beta)) = \beta$ so $\text{nwfrank}(g(u_\alpha)) \geq \alpha$. We show that white wins $G_\alpha(g(u_\alpha))$. Black has to choose some non-wellfounded sequence starting from $g(u_\alpha)$, say $\vec{x} = \langle g(u_\beta), g(u_\beta), \ldots\rangle$. Then white chooses some ordinal γ such that $\beta < \gamma < \alpha$. Now black cannot win $G_\gamma(g(u_\beta))$ because $\text{nwfrank}(g(u_\beta)) = \beta < \gamma$. □

Corollary 5.9 *If $\alpha < \gamma$, then $V^{\alpha}_{afa} \not\models AFA_\gamma$.*

Proof If we let f be the canonical equation system for a set x such that $\text{nwfrank}(x) = \gamma$, then $f \in E_\gamma$. But f does not have a solution in V^{α}_{afa}. □

Next we show that all the AFA_α axioms together with AFA_∞ imply AFA. But note that $\forall\alpha AFA_\alpha \not\vdash AFA$.

Lemma 5.10 $\vdash AFA \leftrightarrow (AFA_\infty \wedge \forall\alpha\, AFA_\alpha)$.

Proof Let f be an arbitrary system of equations, and assume that the white player does not win $G_\alpha(f)$ for any α. We show that then black wins $G_\infty(f)$. For a descending sequence of indeterminates $\vec{u}$ of f, let

$$r(\vec{u}) = \sup\{\alpha \mid \text{ black wins } G_\alpha(f) \text{ where the first move of black is } \vec{u}\,\}.$$

There is an ordinal α such that if $\vec{u}$ is a descending sequence of indeterminates of f and $r(\vec{u}) \geq \alpha$, then $r(\vec{u}) = \infty$. This is so because otherwise the set $\{\vec{u} \mid \vec{u}$ is a descending sequence of indeterminates of $f\}$ and hence f would be a proper class.

We describe a winning strategy for black in the game $G_\infty(f)$ as follows. There is a descending sequence of indeterminates $\vec{u}_0$ of f such that $r(\vec{u}_0) \geq \alpha$ since otherwise we could take

$$\gamma = \sup\{r(\vec{u}) \mid \vec{u} \text{ from } f \text{ such that } r(\vec{u}) \neq \infty\}$$

and white would win $G_{\gamma+1}(f)$. Let $\vec{u}_0$ be the first move of black. Let n be the first move of white. Because $r(\vec{u}_0) \geq \alpha$, then by the above, $r(\vec{u}_0) = \infty$. So there is a descending sequence $\vec{u}_1$ such that it branches out of $\vec{u}_0$ below n and $r(\vec{u}_1) \geq \alpha$. So this way we can continue the game arbitrarily long. □

By the previous lemma, we also see that $V_{afa}^{\infty} \cup \{x \mid \exists\alpha(x \in V_{afa}^{\alpha})\} = V_{afa}$.

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