Notre Dame Journal of Formal Logic
Volume 35, Number 3, Summer 1994

The Strong Completeness of a System Based on Kleene's Strong Three-Valued Logic

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Abstract The present work, which was inspired by Kripke and McCarthy, is about a non-classical predicate logic system containing a truth predicate symbol. In this system, each sentence A is referred to not by a Gödel number but by its quotation name 'A'.

1 Introduction The aim of this paper is to prove the strong completeness theorem for a system based on Kleene's strong three-valued logic. In the past, three logicians (see Cleave [2], Kearns [3], and Wang [7]) formalized Kleene's strong three-valued logic and presented completeness proofs for their systems, all of which are simple predicate calculus systems without the equality symbol $\doteq$. Our system **K** is different from theirs. It is a Gentzen type of sequent calculus containing a truth predicate symbol as well as the equality symbol. To prove the strong completeness theorem for **K**, we employ the technique used in Kearns [3]. For the completeness theorems for classical systems containing a truth predicate symbol, see Aoyama [1].

2 Syntax of K

2.1 Symbols of the Language $\mathcal{L}$ of K

(i) Logical symbols: $\neg, \vee, \exists$
(ii) Individual variables: $x_0, x_1, x_2, \ldots$
(iii) Individual constants: $c_0, c_1, c_2, \ldots$
(iv) Predicate symbols: Tr, $\doteq$, $P_0^n, P_1^n, P_2^n, \ldots$
(v) Punctuation symbols: (,)
(vi) Quotation marks: ' '

'Tr' in (iv) is the truth predicate symbol. 'n' in (iv) indicates the number of arguments the predicate symbol takes.

Received October 28, 1993; revised August 10, 1994

2.2 Simultaneous Definition of Terms and wffs of $\mathcal{L}$

(i) Individual variables and individual constants are terms.
(ii) If each of $t_1, \ldots, t_n$ is either a variable or a constant, then $P_i^n(t_1 \ldots t_n)$ is a well-formed formula (wff) for each n and i.
(iii) If each of t_1 and t_2 is either a variable or a constant, then $t_1 \doteq t_2$ is a wff.
(iv) If A and B are wffs, then $\neg A$ and $(A \vee B)$ are also wffs.
(v) If A is a wff and x is a variable, then $(\exists x)A$ is a wff.
(vi) If A is a wff with no free variables, then 'A' is a term.
(vii) If A is a wff with no free variables, then Tr('A') is a wff.
(viii) Only those terms and wffs obtained by (i) – (vii) are terms and wffs of $\mathcal{L}$.

Free/bound (occurrences of) variables are defined in the usual way. Sentences are those wffs with no free variables. The logical symbols $\wedge, \rightarrow, \equiv$, and $\forall$ are also defined in the usual way:

$$\begin{aligned}(A \wedge B) &= \neg(\neg A \vee \neg B), (A \rightarrow B) = (\neg A \vee B), (A \equiv B) \\ &= ((A \rightarrow B) \wedge (B \rightarrow A)), (\forall x)A = \neg(\exists x)\neg A.\end{aligned}$$

Variables and constants are called 'vc-terms' and terms of the form 'A' are called 'q-terms.' We often omit parentheses in wffs when no confusion arises.

2.3 Deductions in K **K** is a Gentzen type of sequent calculus whose basic components are sequents of the form $\Gamma \vdash A$, where Γ is a nonempty set of wffs and A is a wff. Γ can be an infinite set. The wffs in Γ are called the 'premises of the sequent $\Gamma \vdash A$' and A is called the 'conclusion of the sequent $\Gamma \vdash A$.'

Each deduction in **K** is a finite tree of inferences, starting with a finite number of initial sequents and ending with a single end sequent. Each inference in a deduction can be written as:

$$\frac{S_1}{S}\,, \qquad \frac{S_1 \quad S_2}{S}\,, \qquad \text{or} \qquad \frac{S_1 \quad S_2 \quad S_3}{S}\,,$$

where $S, S_1, S_2,$ and S_3 are sequents; $S_1, S_2,$ and S_3 are called the 'upper sequents' of the inferences and S the 'lower sequent' of the inferences. We call the end sequent in a deduction a 'theorem of **K**' and say, synonymously, '$\Gamma \vdash A$ is a theorem of **K**' and '$\Gamma \vdash A$ is deducible in **K**.'

Definition 2.1 Initial sequents: Let Γ be a nonempty set of wffs of $\mathcal{L}$.

(a) $\Gamma \vdash A$, where Γ is a nonempty set of wffs and $A \in \Gamma$;
(b) $\Gamma \vdash t \doteq t$, where t is a vc-term and the wff $t \doteq t$ may or may not be in Γ.

Definition 2.2 Rules of inference: Let $\Gamma, \Gamma_1, \Gamma_2,$ and Γ_3 be nonempty, unless otherwise indicated, sets of wffs and let $A, B,$ and C be wffs. (We often write, e.g., 'Γ_1, Γ_2' instead of '$\Gamma_1 \cup \Gamma_2$.')

R1 (a) $$\frac{\Gamma \vdash A}{\Gamma \vdash \neg\neg A}$$ (b) $$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}$$

R2 $$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}$$

R3 $$\frac{\Gamma_1, A \vdash C \quad \Gamma_2, B \vdash C \quad \Gamma_3 \vdash A \vee B}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash C},$$

where Γ_1 and Γ_2 can be empty.

R4 $$\frac{\Gamma_1 \vdash \neg A \qquad \Gamma_2 \vdash \neg B}{\Gamma_1, \Gamma_2 \vdash \neg(A \vee B)}$$

R5 (a) $$\frac{\Gamma \vdash \neg(A \vee B)}{\Gamma \vdash \neg A}$$ (b) $$\frac{\Gamma \vdash \neg(A \vee B)}{\Gamma \vdash \neg B}$$

R6 $$\frac{\Gamma \vdash A \vee B}{\Gamma \vdash B \vee A}$$

R7 (Cut) $$\frac{\Gamma_1 \vdash A \qquad \Gamma_2, A \vdash B}{\Gamma_1, \Gamma_2 \vdash B},$$

where Γ_2 can be empty.

R8 $$\frac{\Gamma_1 \vdash \neg A \qquad \Gamma_2 \vdash A \vee B}{\Gamma_1, \Gamma_2 \vdash B}$$

R9 $$\frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash \neg A}{\Gamma_1, \Gamma_2 \vdash B},$$

where B can be any wff.

R10 (UG) $$\frac{\Gamma \vdash A}{\Gamma \vdash (\forall x)A},$$

where x does not occur free in Γ and x may not occur free in A.

R11 (UI) $$\frac{\Gamma \vdash (\forall x)A}{\Gamma \vdash Ax/t},$$

where t is a vc-term free for x in A, Ax/t is the result of replacing all free occurrences of x in A by t, and x may not occur free in A.

R12 $$\frac{\Gamma \vdash (\forall x)(A \vee B)}{\Gamma \vdash A \vee (\forall x)B},$$

where x does not occur free in A.

R13 (EG) $$\frac{\Gamma \vdash Ax/t}{\Gamma \vdash (\exists x)A},$$

where t, Ax/t, and x are as in R11.

R14 $$\frac{\Gamma_1, A \vdash B \qquad \Gamma_2 \vdash (\exists x)A}{\Gamma_1, \Gamma_2 \vdash B},$$

where Γ_1 can be empty, x does not occur free in Γ_1, B, or Γ_2, and x may not occur free in A.

R15 (a) $$\frac{\Gamma \vdash (\forall x)\neg A}{\Gamma \vdash \neg(\exists x)A}$$ (b) $$\frac{\Gamma \vdash \neg(\exists x)A}{\Gamma \vdash (\forall x)\neg A}$$

R16 (EQ) $$\frac{\Gamma, A \vdash t_1 \doteq t_2}{\Gamma, A \vdash A'},$$

where Γ can be empty, t_1 and t_2 are vc-terms, A' is obtained from A by replacing some (possible zero) free, if t_1 is a variable, occurrences of t_1 in A by t_2, if t_2 is a variable, say x, then those occurrences of t_1 within the scope of $(\forall x)$ and $(\exists x)$ should never be replaced by t_2, and no replacement should be made in q-terms in A.

R17 (a) $$\frac{\Gamma \vdash A}{\Gamma \vdash \mathrm{Tr}(`A\text{'})}$$ (b) $$\frac{\Gamma \vdash \mathrm{Tr}(`A\text{'})}{\Gamma \vdash A}$$

R18 (a) $$\frac{\Gamma \vdash \neg A}{\Gamma \vdash \neg\mathrm{Tr}(`A\text{'})}$$ (b) $$\frac{\Gamma \vdash \neg\mathrm{Tr}(`A\text{'})}{\Gamma \vdash \neg A}$$

Note: It is clear from the formalization of **K** that for every sequent $\Gamma \vdash A$ in a deduction σ, Γ contains at least one wff, i.e., $\Gamma \neq \varphi$ (or, we can easily prove this by an induction on the number of applications of inference rules in σ). Secondly, $A \vdash \mathrm{Tr}(`A\text{'})$ and $\mathrm{Tr}(`A\text{'}) \vdash A$ are theorems of **K** for each sentence A. These theorems can be regarded as the truth definition for **K**. We can also easily show the following three facts:

Fact 2.3 *Let σ be a deduction in **K**. Then each of the initial sequents in σ contains finitely many premises iff the end sequent of σ contains finitely many premises.*

Fact 2.4 *If $\Gamma \vdash A$ is a theorem of K, there is a finite subset Γ' of Γ such that $\Gamma' \vdash A$ is also a theorem of K.*

Fact 2.5 *Let $\Gamma \vdash A$ be deducible in **K** and let Γ' be a set of wffs such that $\Gamma \subseteq \Gamma'$. Then $\Gamma' \vdash A$ is also deducible in **K**.*

2.4 Congruent wffs Let A and B be wffs of $\mathcal{L}$. We say that A is *congruent* with B if A is like B except that they differ only in the choices of bound variables in them. So when A is congruent with B, exactly those variables occurring free in A occur free in B, and vice versa. 'Being congruent' is an equivalence relation on the set of wffs of $\mathcal{L}$. Then we can easily show:

Proposition 2.6 *Let A and B be wffs of $\mathcal{L}$. Then if A is congruent with B, then $A \vdash B$, $B \vdash A$, $\neg A \vdash \neg B$, and $\neg B \vdash \neg A$ are all theorems of* ***K****.*

3 Semantics of K

3.1 Definition of an Interpretation M for $\mathcal{L}$

(i) $|M|$, the universe of M, consists of two nonempty sets D and $S\mathcal{L}$, where $S\mathcal{L}$ is the set of sentences of $\mathcal{L}$. We write $|M| = \langle D, S\mathcal{L}\rangle$.

(ii) For each vc-term t, $M(t)$ is a member of D.

(iii) For each q-term 'A', $M(\text{'}A\text{'}) = A \in S\mathcal{L}$.

(iv) For each n-ary predicate symbol P_i^n, $M(P_i^n) = \langle G(P_i^n), H(P_i^n)\rangle$, where $G(P_i^n) \subseteq D^n$, $H(P_i^n) \subseteq D^n$, and $G(P_i^n) \cap H(P_i^n) = \varphi$.

(v) For the truth predicate symbol Tr, $M(\text{Tr}) = \langle G(\text{Tr}), H(\text{Tr})\rangle$, where $G(\text{Tr}) \subseteq S\mathcal{L}$, $H(\text{Tr}) \subseteq S\mathcal{L}$, and $G(\text{Tr}) \cap H(\text{Tr}) = \varphi$.

(vi) For the equality symbol $\doteq$, $M(\doteq) = \langle G(\doteq), H(\doteq)\rangle$, where $G(\doteq) = \{\langle e, e\rangle : e \in D\}$, $H(\doteq) \subseteq D^2$, and $G(\doteq) \cap H(\doteq) = \varphi$.

(vii) For each wff of the form $P_i^n(t_1 \dots t_2)$,
$M(P_i^n(t_1 \dots t_n)) =$ T (<u>t</u>rue) iff $\langle M(t_1), \dots, M(t_n)\rangle \in G(P_i^n)$,
$M(P_i^n(t_1 \dots t_n)) =$ F (<u>f</u>alse) iff $\langle M(t_1), \dots, M(t_n)\rangle \in H(P_i^n)$,
$M(P_i^n(t_1 \dots t_n)) =$ N (<u>n</u>either true nor false) iff
$\langle M(t_1), \dots, M(t_n)\rangle \notin G(P_i^n) \cup H(P_i^n)$.

(viii) For each wff of the form Tr('A'),
$M(\text{Tr}(\text{'}A\text{'})) =$ T iff $M(\text{'}A\text{'}) \in G(\text{Tr})$ iff $M(A) = \text{T}$,
$M(\text{Tr}(\text{'}A\text{'})) =$ F iff $M(\text{'}A\text{'}) \in H(\text{Tr})$ iff $M(A) = \text{F}$,
$M(\text{Tr}(\text{'}A\text{'})) =$ N iff $M(\text{'}A\text{'}) \notin G(\text{Tr}) \cup H(\text{Tr})$ iff $M(A) = \text{N}$.

(ix) For each wff of the form $t_1 \doteq t_2$,
$M(t_1 \doteq t_2) =$ T iff $\langle M(t_1), M(t_2)\rangle \in G(\doteq)$,
$M(t_1 \doteq t_2) =$ F iff $\langle M(t_1), M(t_2)\rangle \in H(\doteq)$,
$M(t_1 \doteq t_2) =$ N iff $\langle M(t_1), M(t_2)\rangle \notin G(\doteq) \cup H(\doteq)$.

(x) For each wff of the form $\neg A$,
$M(\neg A) =$ T iff $M(A) = \text{F}$,
$M(\neg A) =$ F iff $M(A) = \text{T}$,
$M(\neg A) =$ N iff $M(A) = \text{N}$.

(xi) For each wff of the form $(A \vee B)$,
$M(A \vee B) =$ T iff $M(A) =$ T or $M(B) = \text{T}$,
$M(A \vee B) =$ F iff $M(A) =$ F and $M(B) = \text{F}$,
$M(A \vee B) =$ N iff either (1) $M(A) = M(B) =$ N, (2)$M(A) = \text{F}$ and $M(B) = \text{N}$, or (3) $M(A) =$ N and $M(B) = \text{F}$.

(xii) For each wff of the form $(\exists x)A$,
$M((\exists x)A) = \text{T}$ iff for some x-variant Mx of M, $Mx(A) = \text{T}$,
$M((\exists x)A) = \text{F}$ iff for every x-variant Mx of M, $Mx(A) = \text{F}$,
$M((\exists x)A) = \text{N}$ iff both, for every x-variant Mx of M, $Mx(A) \neq \text{T}$ and, for some x-variant Mx of M, $Mx(A) = \text{N}$, where an x-variant Mx of M is, by definition, an interpretation of $\mathcal{L}$ just like M except that Mx may possibly differ from M on the value assignment to the variable x.

The truth values of $\neg A$, $A \vee B$, $A \wedge B$, $A \rightarrow B$, and $A \equiv B$ can be determined as in Table 1:

A	B	$\neg A$	$A \vee B$	$A \wedge B$	$A \rightarrow B$	$A \equiv B$
T	T	F	T	T	T	T
T	F	F	T	F	F	F
T	N	F	T	N	N	N
F	T	T	T	F	T	F
F	F	T	F	F	T	T
F	N	T	N	F	T	N
N	T	N	T	N	T	N
N	F	N	N	F	N	N
N	N	N	N	N	N	N

Table 1

For convenience sake, we write the truth condition of wffs of the form $(\forall x)A$ below:

(xiii) $M((\forall x)A) = \mathrm{T}$ iff for every x-variant Mx of M, $Mx(A) = \mathrm{T}$,
$M((\forall x)A) = \mathrm{F}$ iff for some x-variant Mx of M, $Mx(A) = \mathrm{F}$,
$M((\forall x)A) = \mathrm{N}$ iff both, for every x-variant Mx of M, $Mx(A) \neq \mathrm{F}$ and, for some x-variant Mx of M, $Mx(A) = \mathrm{N}$.

We use the capital letter D both as a metavariable for wffs and as a constituent of $|M|$, but there will be no confusion. For an x-variant Mx of M and a predicate symbol P_i^n, we write $Mx(P_i^n) = \langle Gx(P_i^n), Hx(P_i^n)\rangle$, and similarly for $Mx(\mathrm{Tr})$ and $Mx(\doteq)$.

3.2 Some Definitions and Simple Facts

(i) Given a wff A, if $M(A) = \mathrm{T}$, then M is said to be a *model of* A. Given a set Γ of wffs, if $M(\Gamma) = \mathrm{T}$, i.e., if $M(A) = \mathrm{T}$ for each $A \in \Gamma$, then M is said to be a *model of* Γ.

(ii) Given a nonempty set Γ of wffs and a wff A, $\Gamma \models A$, which we read 'Γ logically implies A,' means that every model of Γ is a model of A.

(iii) For every x-variant Mx of M, we have:

- (a) for each vc-term t, $Mx(t) = M(t)$, if $t \neq x$,
- (b) for each q-term 'A', $Mx(\text{'}A\text{'}) = M(\text{'}A\text{'}) = A \in S\mathcal{L}$,
- (c) for each n-ary predicate symbol P_i^n, $Mx(P_i^n) = M(P_i^n)$,
- (d) for the equality symbol $\doteq$, $Mx(\doteq) = M(\doteq)$.

(iv) Let Γ_1 and Γ_2 be nonempty sets of wffs such that $\Gamma_1 \subseteq \Gamma_2$. Then if $\Gamma_1 \models A$, then $\Gamma_2 \models A$.

3.3 The Soundness of K First, we list a few easy propositions.

Proposition 3.1 *Let A be a wff in which variable x does not occur free, and let Mx be an x-variant of an interpretation M for $\mathcal{L}$. Then $Mx(A) = M(A)$.*

Proposition 3.2 *Let Mx be an x-variant of an interpretation M for language $\mathcal{L}$. Then $Mx(\mathrm{Tr}) = M(\mathrm{Tr})$.*

Proposition 3.3 *Let A be a wff and t a vc-term which is free for variable x in A. And let Mx be an x-variant of an interpretation M for $\mathcal{L}$ such that $Mx(x) = M(t)$. Then $Mx(A) = M(Ax/t)$.*

Given these propositions, it is straightforward to prove the soundess theorem for **K**.

Theorem 3.4 (The Soundness Theorem for **K**) *Let Γ be a nonempty set of wffs and A a wff. Then if $\Gamma \vdash A$ is deducible in **K**, then $\Gamma \models A$.*

Corollary 3.5 *Let A and B be wffs such that A is congruent with B. Then $M(A) = M(B)$ for any interpretation M for $\mathcal{L}$.*

Proof: Immediate from Proposition 2.6 and Theorem 3.4. □

3.4 Some Definitions and Facts In what follows, Γ and Δ are nonempty sets of wffs of $\mathcal{L}$.

(i) Γ is *inconsistent* iff for some wff A of $\mathcal{L}$, both $\Gamma \vdash A$ and $\Gamma \vdash \neg A$ are deducible in **K** (or, equivalently, for every wff A of $\mathcal{L}$, $\Gamma \vdash A$ is deducible in **K**).

(ii) Γ is *disjunctive* iff for every wff of $\mathcal{L}$ of the form $(A \vee B)$, if $(A \vee B) \in \Gamma$ then either $A \in \Gamma$ or $B \in \Gamma$.

(iii) Γ is *henkin* iff for every wff of $\mathcal{L}$ of the form $(\exists x)A$, if $(\exists x)A \in \Gamma$ then $Ax/y \in \Gamma$ for some variable y of $\mathcal{L}$ not occurring in $(\exists x)A$.

(iv) Γ is *closed under deducibility in **K*** iff for each wff A of $\mathcal{L}$, ($\Gamma \vdash A$ is deducible in **K** iff $A \in \Gamma$).

(v) Γ is the *closure of a set Δ under deducibility in **K*** iff $\Gamma = \{A : A$ is a wff of $\mathcal{L}$ and $\Delta \vdash A$ is deducible in **K**$\}$.

Concerning the closure of a set of wffs, we can easily get

Fact 3.6 *Let Γ be the closure of a set Δ of wffs under deducibility in **K**. Then Γ is closed under deducibility in* **K**.

(vi) We will later in the strong completeness proof extend the language $\mathcal{L}$ to new languages by adding new variables. Then the definitions we made above will be extended to those new languages. It is clear that the previous results (propositions, facts, etc.) also hold for those new languages.

3.5 The Strong Completeness of K We are now ready to prove the stong completeness theorem for **K**.

Theorem 3.7 (The Strong Completeness Theorem for **K**) *Let Γ be a nonempty set of wffs of $\mathcal{L}$ and $\overline{A}$ a wff of $\mathcal{L}$. Then if $\Gamma \models \overline{A}$, then $\Gamma \vdash \overline{A}$ is deducible in **K**.*

Proof: Let Γ and $\overline{A}$ be as in the theorem. We show that if $\Gamma \vdash \overline{A}$ is not deducible in **K**, then there is a model of Γ in which $\overline{A}$ is not true. □

Outline of the proof: We first extend Γ to a new set Π which is consistent, disjunctive, and henkin and which does not contain $\overline{A}$. Using Π, we then define an interpretation M such that $B \in \Pi$ iff $M(B) =$T.

Assume that $\Gamma \vdash \overline{A}$ is not deducible in **K**. Then $\overline{A} \notin \Gamma$ and Γ is consistent. Before we extend Γ to Π containing Γ, we first define extensions of $\mathcal{L}$ and **K**. We introduce the following new variables:

$$\begin{array}{llll} {}^0w_{1,1},\ {}^0w_{1,2},\ {}^0w_{1,3},\ \cdots; & {}^0w_{2,1},\ {}^0w_{2,2},\ {}^0w_{2,3},\ \cdots; & {}^0w_{3,1},\ {}^0w_{3,2},\ {}^0w_{3,3},\ \cdots; & \cdots \\ {}^1w_{0,1},\ {}^1w_{0,2},\ {}^1w_{0,3},\ \cdots; & {}^1w_{1,1},\ {}^1w_{1,2},\ {}^1w_{1,3},\ \cdots; & {}^1w_{2,1},\ {}^1w_{2,2},\ {}^1w_{2,3},\ \cdots; & \cdots \\ {}^2w_{0,1},\ {}^2w_{0,2},\ {}^2w_{0,3},\ \cdots; & {}^2w_{1,1},\ {}^2w_{1,2},\ {}^2w_{1,3},\ \cdots; & {}^2w_{2,1},\ {}^2w_{2,2},\ {}^2w_{2,3},\ \cdots; & \cdots \\ {}^3w_{0,1},\ {}^3w_{0,2},\ {}^3w_{0,3},\ \cdots; & {}^3w_{1,1},\ {}^3w_{1,2},\ {}^3w_{1,3},\ \cdots; & {}^3w_{2,1},\ {}^3w_{2,2},\ {}^3w_{2,3},\ \cdots; & \cdots \\ \cdots\ \cdots\ \cdots\ \cdots; & \cdots\ \cdots\ \cdots\ \cdots; & \cdots\ \cdots\ \cdots\ \cdots; & \cdots \\ \cdots\ \cdots\ \cdots\ \cdots; & \cdots\ \cdots\ \cdots\ \cdots; & \cdots\ \cdots\ \cdots\ \cdots; & \cdots \end{array}$$

Let $\mathcal{L}_{0,0} = \mathcal{L}$ and $\mathcal{L}_{0,n+1}$ be obtained from $\mathcal{L}_{0,n}$ by adding the variables ${}^0w_{n+1,1}$, ${}^0w_{n+1,2}, {}^0w_{n+1,3}, \ldots$. Let $\mathbf{K}_{0,0} = \mathbf{K}$ and $\mathbf{K}_{0,n}$ be the system in which wffs of $\mathcal{L}_{0,n}$ can be used in deductions. Then let $\mathcal{L}_0 = \cup_{n\in\omega}\mathcal{L}_{0,n}$ and then $\mathcal{L}_0$ induces an extension $\mathbf{K}_0$ of $\mathbf{K}_{0,0}$. Similarly, let $\mathcal{L}_{1,0} = \mathcal{L}_0 \cup \{{}^1w_{0,1}, {}^1w_{0,2}, {}^1w_{0,3}, \ldots\}$ and $\mathcal{L}_{1,n+1} = \mathcal{L}_{1,n} \cup \{{}^1w_{n+1,1}, {}^1w_{n+1,2}, {}^1w_{n+1,3}, \ldots\}$. Each $\mathcal{L}_{1,n}$ induces $\mathbf{K}_{1,n}$. And $\mathcal{L}_1 = \cup_{n\in\omega}\mathcal{L}_{1,n}$. $\mathcal{L}_1$ induces $\mathbf{K}_1$. For $m \geqq 2$, $n \geqq 0$, $\mathcal{L}_{m,n}, \mathbf{K}_{m,n}, \mathcal{L}_m$, and $\mathbf{K}_m$ are defined similarly. Finally, let $\mathcal{L}_\omega = \cup_{n\in\omega}\mathcal{L}_m$ and it induces $\mathbf{K}_\omega$.

We assume that there is some fixed enumeration of the wffs of $\mathcal{L}_\omega$, which also yields an enumeration of the wffs of $\mathcal{L}_m$ for each $m \in \omega$, by deleting those wffs not belonging to $\mathcal{L}_m$; the same operation yields an enumeration for the wffs of $\mathcal{L}_{m,n}$ for each $n \geqq 0$. We now extend Γ to a larger set Π of wffs of $\mathcal{L}_\omega$ as follows:

Step 1: Let $\alpha_0 = \{\overline{A}\}$ and ${}^0\Gamma_0$ be the closure of Γ under deducibility in $K_{0,0}$. And let ${}^0\Gamma_{n+1}$ be obtained from ${}^0\Gamma_n$, $n \in \omega$, as follows, where A_{n+1} is the $n+1$st wff in the enumeration of the wffs of $L_{0,1}$:

(i) If $A_{n+1} \notin {}^0\Gamma_n$, then let ${}^0\Gamma_{n+1} = {}^0\Gamma_n$.

(ii) If $A_{n+1} \in {}^0\Gamma_n$, $A_{n+1} \neq (\exists x)B$, and $A_{n+1} \neq (C \vee D)$, then let ${}^0\Gamma_{n+1} = {}^0\Gamma_n$.

(iii) If $A_{n+1} \in {}^0\Gamma_n$, and $A_{n+1} = (\exists x)B$, then let ${}^0\Gamma_{n+1} = {}^0\Gamma_n \cup \{Bx/y\}$, where y is the first variable of $\mathcal{L}_{0,1}$ not occurring in any wff in ${}^0\Gamma_n \cup \alpha_0$.

(iv) If $A_{n+1} \in {}^0\Gamma_n$, $A_{n+1} = (C \vee D)$, and either ${}^0\Gamma_n \vdash C$ or ${}^0\Gamma_n \vdash D$ is deducible in $\mathbf{K}_{0,1}$, then let ${}^0\Gamma_{n+1} = {}^0\Gamma_n$.

(v) If $A_{n+1} \in {}^0\Gamma_n$, $A_{n+1} = (C \vee D)$, and neither ${}^0\Gamma_n \vdash C$ nor ${}^0\Gamma_n \vdash D$ is deducible in $\mathbf{K}_{0,1}$, then let

$$ {}^0\Gamma_{n+1} = \begin{cases} {}^0\Gamma_n \cup \{C\}, & \text{if } {}^0\Gamma_n \cup \{C\} \vdash \overline{A} \text{ is not deducible in } \mathbf{K}_{0,1}, \\ {}^0\Gamma_n \cup \{D\}, & \text{otherwise.} \end{cases} $$

Let ${}^0\Gamma_\omega = \cup_{n\in\omega}{}^0\Gamma_n$ and ${}^1\Gamma_0$ be the closure of ${}^0\Gamma_\omega$ under deducibility in $K_{0,1}$. We then repeat the above procedure (i)–(v) to obtain ${}^1\Gamma_0, {}^1\Gamma_1, {}^1\Gamma_2 \ldots$, in $K_{0,2}$. Let ${}^1\Gamma_\omega = \cup_{n\in\omega}{}^1\Gamma_n$ and ${}^2\Gamma_0$ be the closure of ${}^1\Gamma_\omega$ under deducibility in $K_{0,2}$. We repeat this process so that we obtain an infinite series of sets ${}^0\Gamma_0, {}^1\Gamma_0, {}^2\Gamma_0 \ldots$. Then let $\Pi_0 = \cup_{n\in\omega}{}^n\Gamma_0$, which is a set of wffs of $\mathcal{L}_0$.

Step 2: Let each wff of $\mathcal{L}_0$ of the form $(\forall x)B$, where x occurs free in B, be associated with a unique variable ${}^1w_{0,i}$ of $\mathcal{L}_{1,0}$ so that if $(\forall x)B$ is the nth wff in the enumeration of the wffs of $\mathcal{L}_0$, then it is associated with ${}^1w_{0,n}$ and $Bx/{}^1w_{0,n}$ is called the ‘distinguished instance in $\mathcal{L}_{1,0}$ of $(\forall x)B$.' We define

$$\begin{aligned} \beta_1 &= \{B : B \text{ is a wff of } \mathcal{L}_0 \text{ and } B \notin \Pi_0\}. \\ \gamma_1 &= \{Bx/{}^1w_{0,n} : (\forall x)B \in \beta_1, \text{ where } x \text{ occurs free in } B \text{ and} \\ &\quad Bx/{}^1w_{0,n} \text{ is the distinguished instance in } \mathcal{L}_{1,0} \text{ of } (\forall x)B\}, \\ &\quad \text{and} \\ \alpha_1 &= \text{the closure of } \beta_1 \cup \gamma_1 \text{ under disjunction,} \\ &\quad \text{i.e., if } B \in \alpha_1 \text{ and } C \in \alpha_1, \text{ then } (B \vee C) \in \alpha_1. \end{aligned}$$

Now let ${}^0\Delta_0 = \Pi_0$ and ${}^0\Delta_{n+1}$ be obtained from ${}^0\Delta_n$ as follows, where A_{n+1} is the $n+1$st wff in the enumeration of the wffs of $\mathcal{L}_{1,0}$:

(i) If $A_{n+1} \notin {}^0\Delta_n$, then let ${}^0\Delta_{n+1} = {}^0\Delta_n$.

(ii) If $A_{n+1} \in {}^0\Delta_n$, $A_{n+1} \neq (\exists x)B$, and $A_{n+1} \neq (C \vee D)$, then let ${}^0\Delta_{n+1} = {}^0\Delta_n$.

(iii) If $A_{n+1} \in {}^0\Delta_n$ and $A_{n+1} = (\exists x)B$, then let ${}^0\Delta_{n+1} = {}^0\Delta_n \cup \{Bx/y\}$, where y is the first variable of $\mathcal{L}_{1,0}$ not occurring in any wff in ${}^0\Delta_n \cup \alpha_1$.

(iv) If $A_{n+1} \in {}^0\Delta_n$, $A_{n+1} = (C \vee D)$, and either ${}^0\Delta_n \vdash C$ or ${}^0\Delta_n \vdash D$ is deducible in $K_{1,0}$, then let ${}^0\Delta_{n+1} = {}^0\Delta_n$.

(v) If $A_{n+1} \in {}^0\Delta_n$, $A_{n+1} = (C \vee D)$, and neither ${}^0\Delta_n \vdash C$ nor ${}^0\Delta_n \vdash D$ is deducible in $K_{1,0}$, then let

$$ {}^0\Delta_{n+1} = \begin{cases} {}^0\Delta_n \cup \{C\}, & \text{if for no wff } A \in \alpha_1, {}^0\Delta_n \cup \{C\} \vdash A \\ & \text{is deducible in } K_{1,0}, \\ {}^0\Delta_n \cup \{D\}, & \text{otherwise.} \end{cases} $$

Let ${}^0\Delta_\omega = \cup_{n\in\omega}{}^0\Delta_n$ and ${}^1\Delta_0$ be the closure of ${}^0\Delta_\omega$ under deducibility in $K_{1,0}$. Then we repeat the above procedure (i)–(v) to obtain ${}^1\Delta_0, {}^1\Delta_1, {}^1\Delta_2 \ldots$, in $K_{1,1}$. Let ${}^1\Delta_\omega = \cup_{n\in\omega}{}^1\Delta_n$ and ${}^2\Delta_0$ be the closure of ${}^1\Delta_\omega$ under deducibility in $K_{1,1}$. We repeat this process so that we obtain an infinte series of sets ${}^0\Delta_0, {}^1\Delta_0, {}^2\Delta_0 \ldots$. Then let $\Pi_1 = \cup_{n\in\omega}{}^n\Delta_0$, which is a set of wffs of $\mathcal{L}_1$.

Step 2 which was used to get Π_1 is now repeated infinitely to get $\Pi_2, \Pi_3, \Pi_4, \ldots, \Pi_n, \Pi_{n+1}, \ldots$, where we define β_{n+1}, γ_{n+1}, and α_{n+1} as follows:

$$\begin{aligned} \beta_{n+1} &= \{B : B \text{ is a wff of } \mathcal{L}_n \text{ and } B \notin \Pi_n\}, \\ \gamma_{n+1} &= \{Bx/{}^{n+1}w_{0,m} : (\forall x)B \in \beta_{n+1}, \text{ where } x \text{ occurs free in } B \text{ and} \\ &\quad Bx/{}^{n+1}w_{0,m} \text{ is the distinguished instance in } \mathcal{L}_{n+1,0} \text{ of } (\forall x)B\}, \\ &\quad \text{and} \\ \alpha_{n+1} &= \text{the closure of } \beta_{n+1} \cup \gamma_{n+1} \text{ under disjunction.} \end{aligned}$$

Having $\Pi_0, \Pi_1, \Pi_2, \ldots,$ set $\Pi = \cup_{n\in\omega}\Pi_n$. Note that $\Gamma \subseteq \Pi_n \subseteq \Pi_{n+1} \subseteq \Pi$ for each $n \in \omega$. We now prove a series of claims. The first two claims are immediate and the third is straightforward.

Claim 3.8 *Each $\Pi_n (n \in \omega)$ and Π are all disjunctive and henkin.*

Claim 3.9 *Each $\Pi_n (n \in \omega)$ is closed under deducibility in K_n and Π is closed under deducibility in K_ω.*

Claim 3.10 $\overline{A} \notin \Pi_0$.

Claim 3.11 *If $B \in \alpha_{n+1}$, then $B \notin \Pi_{n+1}$, for each $n \in \omega$.*

Proof: We only prove the case where $n = 0$; that is, we only show that if $B \in \alpha_1$, then $B \notin \Pi_1$. The other cases where $n > 0$ can be proven similarly. Let B be an arbitrary wff in α_1. Without loss of generality, we may assume that B is of the form $(B_1 \vee B_2 \vee \ldots \vee B_m)$ for $m \geqq 1$ and each $B_i (1 \leqq i \leqq m)$ is distinct from the others and is either in β_1 or in γ_1. We first show the following: □

Claim 3.12 *For each $n \in \omega$, ${}^0\Delta_n \vdash B$ is not deducible in $K_{1,0}$.*

Proof: Induction on n. Note that B is any wff in α_1. Basis step ($n = 0$): We consider two cases: (1) B contains none of the variables ${}^1w_{0,1}, {}^1w_{0,2}, {}^1w_{0,3}, \ldots$, of $\mathcal{L}_{1,0}$, and (2) B contains some of them. □

Case 1: Then each of $B_1, \ldots, B_m$ is a member of β_1. That is, each of them is a wff of $\mathcal{L}_0$ (and therefore B itself is a wff of $\mathcal{L}_0$), and by the definition of β_1 we have $B_1 \notin \Pi_0, B_2 \notin \Pi_0, \ldots, B_m \notin \Pi_0$.

We first show that $\Pi_0 \vdash B$ is not deducible in K_0 and $B \notin \Pi_0$. Suppose for a contradiction that $B \in \Pi_0$. By Claim 3.8, Π_0 is disjunctive. So, one of $B_1, \ldots, B_m$ must be in Π_0, which is a contradiction. Thus $B \notin \Pi_0$. Hence $\Pi_0 \vdash B$ is not deducible in K_0, i.e., ${}^0\Delta_0 \vdash B$ is not deducible in $K_{1,0}$.

Case 2: B contains some of the variables ${}^1w_{0,1}, {}^1w_{0,2}, \ldots$, of $\mathcal{L}_{1,0}$. Suppose for a contradiction that $\Pi_0 \vdash B$ is deducible in $K_{1,0}$. Then for some finite $\Pi' \subseteq \Pi_0$, $\Pi' \vdash B$ is deducible in $K_{1,0}$. Let σ be one such deduction. σ contains only finitely many wffs of $\mathcal{L}_{1,0}$. Let ${}^1w_{0,k1}, {}^1w_{0,k2}, \ldots {}^1w_{0,ki}$ be the distinct variables of $\mathcal{L}_{1,0}$ occurring in B. Let $z_1, z_2, \ldots, z_i$ be distinct variables of $\mathcal{L}_0$ not occurring in σ. Then, using UG, we see that

$$\Pi' \vdash (\forall z_1)(\forall z_2)\ldots(\forall z_i)(B^1w_{0,k1}{}^1w_{0,k2}\ldots{}^1w_{0,ki}/z_1z_2\ldots z_i)$$

is deducible in $K_{1,0}$, where $B^1w_{0,k1}{}^1w_{0,k2}\ldots{}^1w_{0,ki}/z_1z_2\ldots z_i$ is

$$(\ldots(B^1w_{0,k1}/z_1)^1w_{0,k2}/z_2)\ldots)^1w_{0,ki}/z_i$$

and can be written as $(B'_1 \vee B'_2 \vee \ldots \vee B'_m)$ in which for each $j (1 \leqq j \leqq m)$,

$$B'_j = \begin{cases} B_j, & \text{if } B_j \in \beta_1, \\ B_j{}^1w_{0,kh}/z_h \text{ for some } h \leqq i, & \text{if } B_j \in \gamma_1. \end{cases}$$

Then by using **R12** and some other rules of inference, we can see that $\Pi' \vdash (B''_1 \vee B''_2 \vee \ldots \vee B''_m)$ can be deducible in $K_{1,0}$, where for each $j (1 \leqq j \leqq m)$,

$$B''_j = \begin{cases} B_j, & \text{if } B'_j = B_j, \\ (\forall z_h)B'_j, \text{ if } B'_j = B_j{}^1w_{0,kh}/z_h & \text{for some } h \leqq i. \end{cases}$$

Since $(B''_1 \vee B''_2 \vee \ldots \vee B''_m)$ is a wff of $\mathcal{L}_0$, we can deduce $\Pi' \vdash (B''_1 \vee B''_2 \vee \ldots \vee B''_m)$ in K_0 and also $\Pi_0 \vdash (B''_1 \vee B''_2 \vee \ldots \vee B''_m)$ in K_0. Since Π_0 is disjunctive, one of $B''_1, B''_2, \ldots, B''_m$ must be in Π_0. Assume that $B''_j (1 \leqq j \leqq m)$ is in Π_0. Suppose $B''_j = B_j$. Then $B''_j = B'_j = B_j \in \beta_1$ and by the definition of β_1, $B''_j \notin \Pi_0$, which is a contradiction. Suppose $B''_j = (\forall z_h)B'_j$, for some $h \leqq i$. Then $(\forall z_h)B'_j$ is congruent with some $(\forall x)C \in \beta_1$, where $B'_j = B_j{}^1w_{0,kh}/z_h$ and B_j is the distinguished instance of $(\forall x)C$ in $\mathcal{L}_{1,0}$. Since $(\forall z_h)B'_j \in \Pi_0$, $\Pi_0 \vdash (\forall z_h)B'_j$ is deducible in K_0. Then using Cut, we can deduce $\Pi_0 \vdash (\forall x)C$ in K_0, which yields $(\forall x)C \in \Pi_0$. This contradicts $(\forall x)C \in \beta_1$. We get a contradiction in either way. Hence $\Pi_0 \vdash B$ is not deducible in $K_{1,0}$.

Induction step ($n > 0$): Assume as the induction hypothesis that for all $B \in \alpha_1$, ${}^0\Delta_n \vdash B$ is not deducible in $K_{1,0}$. We want to show that for all $B \in \alpha_1$, ${}^0\Delta_{n+1} \vdash B$ is not deducible in $K_{1,0}$. Now suppose for a contradiction that for some $B \in \alpha_1$, ${}^0\Delta_{n+1} \vdash B$ is deducible in $K_{1,0}$. Then ${}^0\Delta_{n+1} \neq {}^0\Delta_n$. So, ${}^0\Delta_{n+1}$ is obtained either by the clause (iii) or by (v) of Step 2.

Case 3: If ${}^0\Delta_{n+1}$ is obtained by the clause (iii) of Step 2, we can easily get a contradiction.

Case 4: Suppose ${}^0\Delta_{n+1}$ is obtained by the clause (v). Then $A_{n+1} \in {}^0\Delta_n$, $A_{n+1} = (C \vee D)$, neither ${}^0\Delta_n \vdash C$ nor ${}^0\Delta_n \vdash D$ is deducible in $K_{1,0}$, and

$$
{}^0\Delta_{n+1} = \begin{cases} {}^0\Delta_n \cup \{C\}, & \text{if for no wff } A \in \alpha_1, {}^0\Delta_n \cup \{C\} \vdash A \text{ is deducible in } K_{1,0}, \\ {}^0\Delta_n \cup \{D\}, & \text{otherwise.} \end{cases}
$$

If ${}^0\Delta_{n+1} = {}^0\Delta_n \cup \{C\}$, then for no wff $A \in \alpha_1$, ${}^0\Delta_n \cup \{C\} \vdash A$ is deducible in $K_{1,0}$, which contradicts our assumption that ${}^0\Delta_{n+1} \vdash B$ is deducible in $K_{1,0}$. So ${}^0\Delta_{n+1} = {}^0\Delta_n \cup \{D\}$ and there is some wff $A \in \alpha_1$ such that ${}^0\Delta_n \cup \{C\} \vdash A$ is deducible in $K_{1,0}$. Let A' be one such wff in α_1. Since α_1 is closed under disjunction, $(B \vee A') \in \alpha_1$. Also, using the rules of inference **R2** and **R6**, we see that both ${}^0\Delta_n \cup \{D\} \vdash B \vee A'$ and ${}^0\Delta_n \cup \{C\} \vdash B \vee A'$ are deducible in $K_{1,0}$. Then we can obtain the following deduction in $K_{1,0}$ by **R3**:

$$
\frac{{}^0\Delta_n, C \vdash B \vee A' \qquad {}^0\Delta_n, D \vdash B \vee A' \qquad {}^0\Delta_n \vdash C \vee D}{{}^0\Delta_n, {}^0\Delta_n, {}^0\Delta_n \vdash B \vee A'}.
$$

That is, for some wff A in α_1, ${}^0\Delta_n \vdash A$ is deducible in $K_{1,0}$, which contradicts the induction hypothesis.

Thus in either case we get a contradiction. So, for no wff $B \in \alpha_1$, ${}^0\Delta_{n+1} \vdash B$ is deducible in $K_{1,0}$. This completes the proof of the Claim. Now it is easy to see that for each $B \in \alpha_1$, ${}^0\Delta_\omega \vdash B$ is not deducible in $K_{1,0}$, from which we can also see that for each $B \in \alpha_1$, ${}^1\Delta_0 \vdash B$ is not deducible in $K_{1,0}$. Similarly, we can show that for each $n \geqq 2$ and each $B \in \alpha_1$, ${}^n\Delta_0 \vdash B$ is not deducible in $K_{1,n-1}$. Finally, it is then a routine to show that for each $B \in \alpha_1$, $\Pi_1 \vdash B$ is not deducible in K_1 and $B \notin \Pi_1$. This completes the proof of Claim 3.11.

The next two claims are straightforward.

Claim 3.13 *$\overline{A} \notin \Pi$ and Π is a consistent set of wffs of $\mathcal{L}_\omega$.*

Claim 3.14 *Let $(\forall x)A$ be a wff of $\mathcal{L}_\omega$ such that x occurs free in A and $(\forall x)A \notin \Pi$. Then there is some variable y of $\mathcal{L}_\omega$ such that y is free for x in A and $Ax/y \notin \Pi$.*

Claim 3.15 *There is an interpretation M for $\mathcal{L}_\omega$ such that for every wff $A \in \mathcal{L}_\omega$,*

(a) $A \in \Pi$ iff $M(A) = \mathrm{T}$,
(b) $\neg A \in \Pi$ iff $M(A) = \mathrm{F}$, and
(c) $(A \notin \Pi$ and $\neg A \notin \Pi)$ iff $M(A) = \mathrm{N}$.

Proof: We first define an interpretation M for $\mathcal{L}_\omega$ as follows: □

(i) $|M| = \langle D, S\mathcal{L}_\omega \rangle$, where $D = \{\bar{t} : t$ is a vc-term of $\mathcal{L}_\omega\}$ and $\bar{t} = \{u : t \doteq u \in \Pi\}$ for each vc-term t of $\mathcal{L}_\omega$, and $S\mathcal{L}_\omega$ is the set of sentences of $\mathcal{L}_\omega$.
(ii) For each vc-term t of $\mathcal{L}_\omega$, $M(t) = \bar{t} \in D$.
(iii) For each q-term 'A' of $\mathcal{L}_\omega$, $M(\text{'}A\text{'}) = A \in S\mathcal{L}_\omega$.
(iv) For each n-ary predicate symbol P_i^n of $\mathcal{L}_\omega$, i.e., of $\mathcal{L}$, $M(P_i^n) = \langle G(P_i^n), H(P_i^n) \rangle$, where

$$G(P_i^n) = \{\langle \bar{t}_1, \bar{t}_2, \ldots, \bar{t}_n \rangle : P_i^n(t_1\, t_2 \ldots t_n) \in \Pi\}, \text{ and}$$
$$H(P_i^n) = \{\langle \bar{t}_1, \bar{t}_2, \ldots, \bar{t}_n \rangle : \neg P_i^n(t_1\, t_2 \ldots t_n) \in \Pi\}.$$

(v) For the truth predicate Tr, $M(\mathrm{Tr}) = \langle G(\mathrm{Tr}), H(\mathrm{Tr}) \rangle$, where

$$G(\mathrm{Tr}) = \{A : A \in S\mathcal{L}_\omega \text{ and } \mathrm{Tr}(\text{'}A\text{'}) \in \Pi\} \text{ and}$$
$$H(\mathrm{Tr}) = \{A : A \in S\mathcal{L}_\omega \text{ and } \neg\mathrm{Tr}(\text{'}A\text{'}) \in \Pi\}.$$

(vi) For the equality symbol $\doteq$, $M(\doteq) = \langle G(\doteq), H(\doteq) \rangle$, where

$$G(\doteq) = \{\langle \bar{t}_1, \bar{t}_2 \rangle : t_1 \doteq t_2 \in \Pi\} \text{ and}$$
$$H(\doteq) = \{\langle \bar{t}_1, \bar{t}_2 \rangle : \neg t_1 \doteq t_2 \in \Pi\}.$$

(vii) M also satisfies the clauses (vii) – (xiii) in the definition of an interpretation for $\mathcal{L}$ in 3.1.

Note that since Π is consistent, the sets $G(P_i^n) \cap H(P_i^n)$, $G(\mathrm{Tr}) \cap H(\mathrm{Tr})$, and $G(\doteq) \cap H(\doteq)$ are all empty. We can now easily establish the following four claims:

Claim 3.16 *Let t_1, t_2, and t_3 be arbitrary vc-terms of $\mathcal{L}_\omega$. Then the following hold:*

(1) $t_1 \doteq t_1 \in \Pi$,
(2) If $t_1 \doteq t_2 \in \Pi$, *then* $t_2 \doteq t_1 \in \Pi$,
(3) If $t_1 \doteq t_2 \in \Pi$ *and* $t_2 \doteq t_3 \in \Pi$, *then* $t_1 \doteq t_3 \in \Pi$, *and*
(4) If $t_1 \doteq t_2 \in \Pi$ *and* $A \in \Pi$, *then* $A' \in \Pi$, *where* t_1, t_2, A, *and* A' *satisfy the conditions in the inference rule EQ.*

Claim 3.17 *Let t and u be arbitrary vc-terms of $\mathcal{L}_\omega$. Then $\bar{t} = \bar{u}$ iff $t \doteq u \in \Pi$.*

Claim 3.18 $\{\langle \bar{t}_1, \bar{t}_2 \rangle : t_1 \doteq t_2 \in \Pi\} = \{\langle \bar{t}, \bar{t} \rangle : \bar{t} \in D\}$, *where t, t_1, and t_2 are arbitrary vc-terms of $\mathcal{L}_\omega$.*

Claim 3.19 *$M(P_i^n)$ and $M(\doteq)$ are well-defined. That is, e.g., if $\bar{t}_i = \bar{u}_i$ for all $i (1 \leqq i \leqq n)$, then*

(1) $\langle M(t_1), \ldots, M(t_n) \rangle \in G(P_i^n)$ *iff* $\langle M(u_1), \ldots, M(u_n) \rangle \in G(P_i^n)$,
(2) $\langle M(t_1), \ldots, M(t_n) \rangle \in H(P_i^n)$ *iff* $\langle M(u_1), \ldots, M(u_n) \rangle \in H(P_i^n)$,

(3) $\langle M(t_1), \ldots, M(t_n)\rangle \notin G(P_i^n) \cup H(P_i^n)$
iff $\langle M(u_1), \ldots, M(u_n)\rangle \notin G(P_i^n) \cup H(P_i^n)$.

We are now ready to prove Claim 3.15 by induction on the complexity of wff A of $\mathcal{L}_\omega$. The cases $A = P_i^n(t_1 \ldots t_n)$, $A = t_1 \doteq t_2$, $A = \mathrm{Tr}(\text{‘}B\text{’})$, $A = \neg B$, and $A = (B \vee C)$ are easy. So we only consider the case $A = (\exists x)B$. Then it is also easy to show both (a) $(\exists x)B \in \Pi$ iff $M((\exists x)B) =$ T and (c) $((\exists x)B) \notin \Pi$ and $\neg(\exists x)B \notin \Pi)$ iff $M((\exists x)B) =$ N. So we only show (b) $\neg(\exists x)B \in \Pi$ iff $M((\exists x)B) =$ F.

Suppose $\neg(\exists x)B \in \Pi$. Then by the inference rule **R15**-(b), $(\forall x)\neg B \in \Pi$. So by **UI**, $\neg Bx/t \in \Pi$ for every vc-term t of $\mathcal{L}_\omega$ free for x in B. Then by the induction hypothesis,

$$(*)\ M(Bx/t) = \text{ F for every vc-term } t \text{ of } \mathcal{L}_\omega \text{ free for } x \text{ in } B.$$

We now would like to show that $M((\exists x)B) =$ F, i.e., for every x-variant Mx of M, $Mx(B) =$ F. Suppose for a contradiction that for some Mx, $Mx(B) \neq$ F, i.e., for some Mx, either $Mx(B) =$ T or $Mx(B) =$ N. Let $Mx(x) = \bar{t}$ for some vc-term t of $\mathcal{L}_\omega$.

Case 5: $Mx(B) =$ T: We consider two subcases: *Subcase (1) t* is free for x in B, and *Subcase (2) t* is not free for x in B.

Subcase 1: Since $Mx(x) = M(t) = \bar{t}$, $Mx(B) = M(Bx/t) =$ T by Proposition 3.3, which contradicts $(*)$ above.

Subcase 2: Then we can find a wff D of $\mathcal{L}_\omega$ which is congruent with B and in which t is free for x. Then we easily get $(\exists x)D \in \Pi$, which in turn yields $(\exists x)B \in \Pi$. This is a contradiction.

Case 6: $Mx(B) = N$: When t is free for x in B, we can get a contradiction as in Subcase (1) of the above case. So suppose that t is not free for x in B. Then there is a wff D of $\mathcal{L}_\omega$ such that D is congruent with B and t is free for x in D. By Corollary 3.5, $Mx(B) = Mx(D) =$ N. Since $Mx(x) = M(t) = \bar{t}$, $Mx(D) = M(Dx/t) =$ N. Then by the induction hypothesis, $Dx/t \notin \Pi$ and $\neg Dx/t \notin \Pi$. But $(\forall x)\neg B \in \Pi$ and $(\forall x)\neg B$ is congruent with $(\forall x)\neg D$, from which we see by Proposition 2.6 that $(\forall x)\neg B \vdash (\forall x)\neg D$ is deducible in K_ω. Hence, using Cut, $(\forall x)\neg D \in \Pi$. Then, using **UI**, $\neg Dx/u \in \Pi$ for every vc-term u of $\mathcal{L}_\omega$ free for x in D, which contradicts $\neg Dx/t \notin \Pi$.

Conversely, suppose that $M((\exists x)B) =$ F. Then for every x-variant Mx of M, $Mx(B) =$ F. If x does not occur free in B, then since M itself is an x-variant of M, $M(B) =$ F. Thus by the induction hypothesis, $\neg B \in \Pi$, from which we can easily get $\neg(\exists x)B \in \Pi$ by some inference rules. Suppose now that x does occur free in B. Let y be an arbitrary variable of $\mathcal{L}_\omega$ free for x in B and let Mx be an x-variant of M such that $Mx(x) = M(y) = \bar{y}$. Then by Proposition 3.3, $Mx(B) = M(Bx/y) =$ F. By the induction hypothesis, $\neg Bx/y \in \Pi$, i.e., $(\neg B)x/y \in \Pi$ for every variable y of $\mathcal{L}_\omega$ free for x in $\neg B$. So $(\forall x)\neg B \in \Pi$ by Claim 3.14. Using the inference rule **R15**-(a), we see that $\Pi \vdash \neg(\exists x)B$ is deducible in K_ω. So $\neg(\exists x)B \in \Pi$.

This completes the proof of Claim 3.15.

Claim 3.20 $\Gamma \not\models \overline{A}$.

Proof: The interpretation M given in Claim 3.15 is a model of Γ, since $\Gamma \subseteq \Pi$ and $(A \in \Pi$ iff $M(A) = $ T, for each wff A of $\mathcal{L}_\omega)$. But, since $\overline{A} \notin \Pi$ by Claim 3.13, $M(\overline{A}) \neq$ T. Thus $\Gamma \not\models \overline{A}$. □

Claim 3.20 completes the proof of Theorem 3.7.

Corollary 3.21 *$\Gamma \vdash A$ iff $\Gamma \models A$, where Γ is a nonempty set of wffs of $\mathcal{L}$ and A is a wff of $\mathcal{L}$.*

Proof: From Theorem 3.4 and Theorem 3.7. □

Corollary 3.22 (The Compactness Theorem for **K**) *Let Γ and A be as in Corollary 3.21. Then if $\Gamma \models A$, then $\Gamma' \models A$ for some finite subset Γ' of Γ.*

Proof: From Theorem 3.7, Fact 2.4, and Theorem 3.4. □

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