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Syntactic Refutations against Finite Models in Modal Logic

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Abstract The purpose of the paper is to study syntactic refutation systems as a way of characterizing normal modal propositional logics. In particular it is shown that there is a decidable modal logic without the finite model property that has a simple finite refutation system.

1 Introduction The concept of syntactic refutation is both simple and well known. It consists in refuting a proposition by deriving from it a proposition that has already been rejected. A syntactic refutation system is a device for refuting propositions by proofs.

For instance we can prove that for any propositional formula α, α is not a theorem of **CL** (Classical Logic) iff there is a substitution instance $e(\alpha)$ of α such that $e(\alpha) \to p \wedge \neg p \in$ **CL**. This theorem in fact describes a refutation procedure for **CL** and it justifies the following refutation system:

Axiom: $\dashv p \wedge \neg p$ ($p \wedge \neg p$ is rejected.)

Rules:

(r_s) $$\frac{\dashv e(\alpha)}{\dashv \alpha}$$

(If a substitution instance of α is refutable then α is refutable.)

(r_{mp}) $$\frac{\vdash \alpha \to \beta \quad \dashv \beta}{\dashv \alpha}$$

(If $\alpha \to \beta$ is provable and β is refutable then α is refutable.)

The above refutation rules were introduced by Łukasiewicz in [3]. Refutation systems for nonclassical logics can be obtained by adding more axioms or more rules to the above system (see Goranko [2], Scott [6], and Skura [8], [9], [10], and [11]).

A syntactic refutation system is similar to a semantic model in that both are refutation devices. Semantic models are very useful for obtaining decidability results. The crucial concept here is the finite model property. If a finitely axiomatizable logic

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has the fmp then it is decidable. However there are logics without the fmp, for instance the remarkable modal logic given by Makinson in [4]. It is natural to ask whether syntactic refutation systems can be useful in such cases. In this paper we study syntactic refutation systems for normal modal logics. In Section 3 we construct such systems for logics with the fmp, and in Section 4, using the results in [4], we prove that there is a normal modal logic without the fmp that has a surprisingly simple refutation system. We also show that this logic is decidable.

2 Preliminaries We shall take the symbol **FOR** to denote the modal language $(FOR, \neg, \wedge, \Box)$, where FOR is the set of all formulas generated from the set $VAR = \{p, q, p_1, p_2, \ldots\}$ of propositional variables by the connectives $\neg, \wedge, \Box$. The connectives $\vee, \rightarrow, \equiv, \Diamond$ are defined in the usual way. If $\alpha_i \in FOR$ $(1 \le i \le n)$ then $\bigwedge\{\alpha_i : 1 \le i \le n\} = \alpha_1 \wedge \ldots \wedge \alpha_n$ and $\bigwedge\varnothing = p \rightarrow p$.

Let $\alpha \in FOR$. For any $k \ge 0$ we define $[\alpha]^k = \bigwedge\{\Box^i\alpha : 0 \le i \le k\}$, where $\Box^n$ is a string of n boxes. Moreover $SUB(\alpha)$ is the set of all subformulas of α, and $VAR(\alpha)$ is the set of all propositional variables occurring in α. For any $\beta \in SUB(\alpha)$, $k(\beta)$ is the modal degree of β in α, i.e., $k(\alpha) = 0$ and $k(\gamma) = k(\delta) = k(\gamma \wedge \delta)$ if $\gamma \wedge \delta \in SUB(\alpha)$, $k(\gamma) = k(\neg\gamma)$ if $\neg\gamma \in SUB(\alpha)$, and $k(\gamma) = k(\Box\gamma) + 1$ if $\Box\gamma \in SUB(\alpha)$. Also $m(\alpha)$ is the greatest natural number in $\{k(\beta) : \beta \in SUB(\alpha)\}$. We write $X \subseteq_f Y$ instead of X is a finite subset of Y.

Now a few definitions and facts about modal logics and modal algebras (for a systematic exposition see Bull and Segerberg [1] or Makinson [5]). A normal modal logic is a set $L \subseteq FOR$ such that $K \subseteq L$ (i.e., L contains the minimal normal modal logic K) and L is closed under modus ponens, substitution and necessitation. $S4$ is a normal modal logic that is especially important for it has the modal reduction law $\Box\alpha \equiv \Box\Box\alpha$. In K we have the following replacement law: $\Box^{k(p)}(p \equiv q) \rightarrow (\alpha(p) \equiv \alpha(q))$, where $k(p)$ is the modal degree of p in α, while in $S4$ we have its simplified version $\Box(p \equiv q) \rightarrow (\alpha(p) \equiv \alpha(q))$.

A modal algebra is an algebra $\mathbf{A} = (A, -, \cap, l)$, where $(A, -, \cap)$ is a Boolean algebra and l satisfies the conditions $l1_A = 1_A$, $l(a \cap b) = la \cap lb$. An interior algebra is a modal algebra with l such that $la \le a$, $la \le lla$. An algebra $\mathbf{A}$ is trivial iff $|A| = 1$.

For any $\alpha \in FOR$ we say that α is valid in a modal algebra $\mathbf{A}$ (in symbols $\alpha \in E(A)$) iff for every valuation $v : FOR \rightarrow A$ we have $v(\alpha) = 1_A$. It is known that $K = \bigcap\{E(A) : A$ is a finite modal algebra$\}$ and $S4 = \bigcap\{E(A) : A$ is a finite interior algebra$\}$.

With every finite modal algebra A we associate a one-one function $g_A : A \rightarrow VAR$ and we define the description of A as follows:

$$\begin{aligned} \Delta_A \;=\; & \bigwedge\{p_x \wedge p_y \equiv p_{x\cap y} : x, y \in A\} \wedge \bigwedge\{\neg p_x \equiv p_{-x} : x \in A\} \\ & \wedge \bigwedge\{\Box p_x \equiv p_{lx} : x \in A\} \text{ where } p_x = g_A(x)(x \in A). \end{aligned}$$

3 Logics with the finite model property First we construct refutation systems for the logics determined by finite modal algebras.

Lemma 3.1 *If A is a finite nontrivial modal algebra then for any $\alpha \in FOR$ we have $\alpha \notin E(A)$ iff there is a substitution $e : FOR \rightarrow FOR$ such that $e(\alpha) \rightarrow ([\Delta_A]^k \rightarrow p_x) \in K$ for some $k \ge 0$ and some $x \in A - \{1_A\}$.*

Proof: ($\leftarrow$) We let v be a valuation on A such that $v(p_a) = a$ ($a \in A$). Then $v([\Delta_A]^k \to p_x) = x \neq 1_A$. Hence $[\Delta_A]^k \to p_x \notin E(A)$. Moreover $K \subseteq E(A)$ and $E(A)$ is closed under modus ponens and substitution. Therefore $\alpha \notin E(A)$.

($\to$) Assume that $\alpha \notin E(A)$. Then $v(\alpha) \neq 1_A$ for some valuation v on A. Let e be a substitution such that $e(u) = p_{vu}$ ($u \in VAR$). We show that for any $\beta \in SUB(\alpha)$

$$[\Delta_A]^k \to \Box^{k(\beta)}(e\beta \equiv p_{v\beta}) \in K$$

where $k = m(\alpha)$. Indeed if $\beta \in VAR$ then $e\beta = p_{v\beta}$, so $e\beta \equiv p_{v\beta} \in K$, $\Box^{k(\beta)}(e\beta \equiv p_{v\beta}) \in K$, and $[\Delta_A]^k \to \Box^{k(\beta)}(e\beta \equiv p_{v\beta}) \in K$. Proceed now by induction.

Case 1: $\beta = \Box\gamma$. Then $e\beta = \Box e\gamma$, $v\beta = Iv\gamma$. By the definition of Δ_A we have:

$$[\Delta_A]^k \to \Box^{k(\beta)}(\Box p_{v\gamma} \equiv p_{v\beta}) \in K.$$

Moreover

$$[\Delta_A]^k \to \Box^{k(\gamma)}(e\gamma \equiv p_{v\gamma}) \in K$$

by the inductive hypothesis. Also $k(\gamma) = k(\beta) + 1$. Hence

$$[\Delta_A]^k \to \Box^{k(\beta)}(\Box e\gamma \equiv p_{v\beta}) \in K.$$

Cases 2 and 3: The cases where $\beta = \gamma \wedge \delta$ and where $\beta = \neg\gamma$ are similar. Therefore

$$[\Delta_A]^k \to (e\alpha \equiv p_{v\alpha}) \in K$$

whence $e\alpha \to ([\Delta_A]^k \to p_{v\alpha}) \in K$.

In a similar way we can prove that:

Lemma 3.2 *If A is a finite nontrivial interior algebra then for any $\alpha \in FOR$ we have $\alpha \notin E(A)$ iff there is a substitution e such that $e\alpha \to (\Box\Delta_A \to p_x) \in S4$ for some $x \in A - \{1\}$.*

We say that a normal modal logic L has the finite model property iff, for some set M of finite modal algebras, $L = \bigcap\{E(A) : A \in M\}$.

For any finite nontrivial modal algebra A we define the following set of formulas:

$$R(A) = \{[\Delta_A]^k \to p_x : k \geq 0, x \in A - \{1\}\}.$$

And for any finite nontrivial interior algebra A we define

$$r(A) = \{\Box\Delta_A \to p_x : x \in A - \{1\}\}.$$

Now we can prove the following theorem describing refutation procedures for normal modal logics with the fmp.

Theorem 3.3 *If $L = \bigcap\{E(A) : A \in M\}$ and M is a set of finite nontrivial modal algebras then for any $\alpha \in FOR$ we have $\alpha \notin L$ iff there is $A \in M$ such that $e\alpha \to \chi \in K$ for some substitution e and some $\chi \in R(A)$.*

Proof: By Lemma 3.1.

For normal extensions of $S4$ with the fmp we have the following:

Theorem 3.4 *If $L = \bigcap\{E(A) : A \in M\}$ and M is a set of finite nontrivial interior algebras then for any $\alpha \in FOR$ we have $\alpha \notin L$ iff there is $A \in M$ such that $e\alpha \to \chi \in S4$ for some substitution e and some $\chi \in r(A)$.*

Proof: By Lemma 3.2.

In other words Theorem 3.3 justifies the following refutation system for a logic $L = \bigcap\{E(A) : A \in M\}$, where M is a set of finite modal algebras.

Axioms: $\dashv \chi$ $(\chi \in R(A), A \in M)$

Rules: r_s, r_{mp}.

(And for normal extensions of $S4$ we would have $r(A)$ instead of $R(A)$.)

Comparing $R(A)$ with $r(A)$ we can see that the formulas in $R(A)$ have uncomfortable strings of boxes and, what is more, $R(A)$ is infinite while $r(A)$ is finite. We can avoid these features and obtain uniform refutation systems for normal modal logics by using sequential refutation systems. In order to introduce such systems we need some definitions. A sequent is a pair X/α, where $X \cup \{\alpha\} \subseteq_f FOR$. Recall that every set P of sequents determines a structural consequence relation $\vdash_P$ defined as follows: for any $X \cup \{\alpha\} \subseteq_f FOR$, $X \vdash_P \alpha$ iff there is a finite sequence $\varphi_1, \ldots, \varphi_n$ of formulas such that $\varphi_n = \alpha$ and for each $1 \leq i \leq n$ either $\varphi_i \in X$ or for some substitution e and some $Y/\beta \in P$ we have $eY \subseteq \{\varphi_1, \ldots, \varphi_{i-1}\}$, $e\beta = \varphi_i$.

By a sequential refutation system we mean a pair (P, N), where P, N are sets of sequents. (Intuitively speaking P is a set of accepted inferences and N is a set of rejected inferences.) A sequential refutation system (P, N) is finite iff both P and N are finite.

We say that a formula α is refutable in a refutation system (P, N) iff for some substitution e and some $Y/\beta \in N$ we have

$$e(\alpha), Y \vdash_P \beta.$$

Remark 3.5 In a general definition we would say that X/α is refutable in (P, N) iff for some $Y/\beta \in N$, $Y \vdash_{P'} \beta$, where $P' = P \cup \{X/\alpha\}$, and our sequential refutation systems are different from Goranko's Gentzen-style refutation systems (see [2]).

Now we define the following set of sequents:

$$P_K = \{\varnothing/\varphi : \varphi \text{ is an axiom of } K\} \cup \{p, p \to q/q\} \cup \{p/\Box p\}.$$

Remark 3.6 The consequence relation $\vdash_{P_K}$ is neither of the two which are usually considered (one of them being a derivability in a logic given by a set of axioms with rules modus ponens, substitution and necessitation, and the other the logical consequence generated by modus ponens only).

Further for any finite nontrivial modal algebra A we define

$$N_A = \{\Delta_A/p_x : x \in A - \{1\}\}.$$

Note that for any finite modal algebra A the refutation system (P_K, N_A) is finite.

Moreover we say that a sequent X/α is valid in a modal algebra A (in symbols $X \vdash_A \alpha$) iff for any valuation v on A if $v(X) \subseteq \{1\}$ then $v(\alpha) = 1$.

Lemma 3.7 *If A is a finite nontrivial modal algebra then for any $\alpha \in FOR$ we have $\alpha \notin E(A)$ iff α is refutable in (P_K, N_A).*

Proof: ($\leftarrow$) Assume that α is refutable in (P_K, N_A). Then for some substitution e and some $x \in A - \{1\}$

$$e\alpha, \Delta_A \vdash_{P_K} p_x.$$

All sequents in $\{eX/e\alpha : X/\alpha \in P_K,\ e \text{ is a substitution}\}$ are valid in A, so $\vdash_{P_K} \subseteq \vdash_A$. Hence $e\alpha, \Delta_A \vdash_A p_x$. Now suppose $\alpha \in E(A)$. Then $\Delta_A \vdash_A p_x$. On the other hand $v(\Delta_A) = 1$, $v(p_x) = x$ if v is a valuation on A such that $v(p_a) = a$, so $\Delta_A \nvdash_A p_x$. This is a contradiction. Therefore $\alpha \notin E(A)$.

($\rightarrow$) By Lemma 3.1.

Finally we have the following theorem describing uniform refutation systems for normal modal logics with the fmp.

Theorem 3.8 *If $L = \bigcap\{E(A) : A \in M\}$ and M is a set of finite nontrivial modal algebras then for any $\alpha \in FOR$ we have $\alpha \notin L$ iff there is $A \in M$ such that α is refutable in (P_K, N_A).*

Proof: By Lemma 3.7.

Thus we have presented a general way of constructing refutation systems for normal modal logics with the fmp. This construction involves certain formulas "describing" modal algebras characterizing a given logic so that such refutation systems are not defined in a purely syntactic way. But when we deal with a specific logic L we can try to find a genuine syntactic refutation system for it. One way of doing so is to look for refutation rules of the form: If $\dashv \alpha_1, \ldots, \dashv \alpha_n$ then $\dashv \beta$. Such rules should on the one hand be valid in L and on the other hand suffice to refute all nonvalid formulas of L when added to the refutation system for CL. An interesting feature of such rules is that they express a syntactic property uniquely characterizing L, being valid in no proper extension of L. However to find *elegant* rules of this kind is usually a difficult formal problem. Goranko has recently given refutation rules for some of the most important modal logics, in particular very elegant rules for K and KW (see his [2]). Rules for $S4$ which are a bit simpler are presented in [9] and in [11].

4 A Logic without the fmp but with a Finite Refutation System In [4] Makinson defined the Kripke frame (N, R), where N is the set of natural numbers and xRy iff $x \leq y + 1$.

Let M be the modal algebra corresponding to (N, R), i.e., $M = (2^N, -, \cap, l)$, where $l(X) = \{a : \forall b(aRb \Rightarrow b \in X)\}$ $(X \subseteq N)$. We are going to consider the subalgebra B of M generated from the set $C = \{C_i : i \geq 0\}$, where $C_i = N - \{i\}$ $(i \geq 0)$. Note that $a \in B$ iff either $a = \bigcap Y$ or $a = -\bigcap Y$ for some $Y \subseteq_f C$.

For any $\varnothing \neq Y \subseteq_f C$ the symbol $i(Y)$ will denote the greatest natural number in $\{i : C_i \in Y\}$. Moreover for any $Y \subseteq_f C$ we define $\bar{Y}$ as follows: $\bar{\varnothing} = \varnothing$ and $\bar{Y} = \{C_i : i \leq i(Y) + 1\}$ if $Y \neq \varnothing$. Observe that for any $Y \subseteq_f C$ we have $l(-\bigcap Y) = \varnothing$ and $l(\bigcap Y) = \bigcap \bar{Y}$.

Remark 4.1 We could also say that B is the set of finite and cofinite subsets of N, and l is defined thus: $l(X) = \{i : \text{if } j \in -X \text{ then } i > j\}$ if X is infinite and $l(X) = \varnothing$ if X is finite.

Now we define the following sequence of formulas: $\gamma_0 = p$ and $\gamma_i = \Box^{i-1} p \to \Box^i p$ $(i \geq 1)$. Further for any $a \in B$ we define the formula ψ_a as follows:

- $\psi_a = \bigwedge\{\gamma_i : C_i \in Y\}$ if $a = \bigcap Y$, $Y \subseteq_f C$
- $\psi_a = \neg \bigwedge\{\gamma_i : C_i \in Y\}$ if $a = -\bigcap Y$, $Y \subseteq_f C$

Lemma 4.2 *The logic $E(B)$ does not have the fmp.*

Proof: By the results in [4] and the fact that B is a subalgebra of M we have that if $E(B) \subseteq E(A)$ and $\gamma_2 \notin E(A)$ then A is infinite. On the other hand γ_2 is refuted in B by a valuation v such that $v(p) = C_0$.

It is not difficult to verify that the following formulas are valid in B.

(0) $\Box\alpha \to \alpha$

(I) $\Diamond\Box\alpha \to (\Box(\alpha \to \Box\alpha) \to \Box^2\alpha)$

(II) $\Diamond\Box\alpha \to \Diamond\Box^2\alpha$

(III) $\Diamond(\alpha \to \Box\alpha)$

(IV) $\Diamond\Box\alpha \to \Box\Diamond\alpha$

(V) $\Box(\alpha \vee \beta) \to \Diamond(\Box\alpha \vee \Box\beta)$

Theorem 4.3 *For any $\alpha \in FOR$ we have $\alpha \notin E(B)$ iff there is a substitution e such that $\Box e\alpha \to (\Box^{k+1}\Diamond\Box p \to p) \in T'$, where $k = m(\alpha)$ and $T' = T + \{I, II, III, IV, V\}$.*

Proof: $(\leftarrow)$ Since $\Box^{k+1}\Diamond\Box p \to p \notin E(B)$.

$(\to)$ Assume that $\alpha \notin E(B)$. Then $v(\alpha) \neq 1_B = N$ for some valuation v on B. Let e be a substitution such that $e(u) = \psi_{vu}$ $(u \in VAR)$. We show that for any $\beta \in SUB(\alpha)$

$$\Box^k\Diamond\Box p \to \Box^{k(\beta)}(e\beta \equiv \psi_{v\beta}) \in T'.$$

We proceed by induction on the complexity of β.

(1) $\beta \in VAR$. Simple.

(2a) $\beta = \gamma \wedge \delta$. Then $e\beta = e\gamma \wedge e\delta$. Let $\Box^k\Diamond\Box p = \lambda$. Then $\lambda \to \Box^{k(\beta)}(e\beta \equiv e\gamma \wedge e\delta) \in T'$. By the inductive hypothesis we have $\lambda \to \Box^{k(\gamma)}(e\gamma \equiv \psi_{v\gamma}) \in T'$. Also $k(\beta) = k(\gamma)$. Hence

$$\lambda \to \Box^{k(\beta)}(e\beta \equiv \psi_{v\gamma} \wedge \psi_{v\delta}) \in T'.$$

Now we have the following cases:

Case 1: $v\gamma = \bigcap Y_1$, $v\delta = \bigcap Y_2$. Then $v\beta = \bigcap(Y_1 \cup Y_2)$. Hence $\psi_{v\gamma} \wedge \psi_{v\delta} = \psi_{v\beta}$. Thus $\lambda \to \Box^{k(\beta)}(e\beta \equiv \psi_{v\beta}) \in T'$.

Case 2: Say $v\gamma = \bigcap Y_1$, $v\delta = -\bigcap Y_2$. Then $v\beta = -\bigcap(Y_2 - Y_1)$. Since $\gamma_i \wedge \neg\gamma_j \equiv \neg\gamma_j \in T$ $(i \neq j)$ we have $\psi_{v\gamma} \wedge \psi_{v\delta} \equiv \psi_{v\beta} \in T$. Thus $\lambda \to \Box^{k(\beta)}$ $(e\beta \equiv \psi_{v\beta}) \in T'$.

Case 3: $v\gamma = -\bigcap Y_1$, $v\delta = -\bigcap Y_2$. Then $v\beta = -\bigcap(Y_1 \cap Y_2)$.
Since $\neg\gamma_i \wedge \gamma_j \equiv \neg(p \to p) \in T$ $(i \neq j)$ we have $\psi_{v\gamma} \wedge \psi_{v\delta} \equiv \psi_{v\beta} \in T$.
Thus $\lambda \to \Box^{k(\beta)}(e\beta \equiv \psi_{v\beta}) \in T'$.

(2b) $\beta = \neg\gamma$. Simple.

(2c) $\beta = \Box\gamma$. Then $e\beta = \Box e\gamma$, so $\lambda \to \Box^{k(\beta)}(e\beta \equiv \Box e\gamma) \in T'$. By the inductive hypothesis $\lambda \to \Box^{k(\gamma)}(e\gamma \equiv \psi_{v\gamma}) \in T'$. Also $k(\beta) + 1 = k(\gamma)$. Hence

$$\lambda \to \Box^{k(\beta)}(e\beta \equiv \Box\psi_{v\gamma}) \in T'.$$

We have the following cases:

Case 1: $v\gamma = \bigcap Y$. Then $v\beta = \bigcap \bar{Y}$. Now for every $i \geq 0$ we have

$$\Diamond\Box p \to (\Box\gamma_i \equiv \Box^{i+1} p) \in T + \{I, II\}$$

and

$$\gamma_0 \wedge \ldots \wedge \gamma_i \equiv \Box^i p \in T$$

so

$$\Diamond\Box p \to (\Box\psi_{v\gamma} \equiv \psi_{v\beta}) \in T'$$
$$\Box^{k(\beta)}\Diamond\Box p \to \Box^{k(\beta)}(\Box\psi_{v\gamma} \equiv \psi_{v\beta}) \in T'.$$

Further $k \geq k(\beta)$, so

$$\Box^k\Diamond\Box p \to \Box^{k(\beta)}(\Box\psi_{v\gamma} \equiv \psi_{v\beta}) \in T'$$

and

$$\lambda \to \Box^{k(\beta)}(e\beta \equiv \psi_{v\beta}) \in T'.$$

Case 2: $v\gamma = -\bigcap Y$. Then $v\beta = \varnothing$, $\psi_{v\beta} = \neg(p \to p)$.
Since for every $i \geq 1$

$$\Box\neg\gamma_i \equiv \neg(p \to p) \in T + III$$

it is easy to show that

$$\Box \bigvee\{\neg\gamma_i : C_i \in Y, i \geq 1\} \equiv \neg(p \to p) \in T + \{III, V\}.$$

Hence

$$\Box \bigvee\{\neg\gamma_i : C_i \in Y\} \to \Diamond\Box\neg p \in T + \{III, V\}.$$

Also

$$\Diamond\Box p \wedge \Diamond\Box\neg p \to \neg(p \to p) \in T + IV$$

so

$$\Diamond\Box p \wedge \Box \bigvee\{\neg\gamma_i : C_i \in Y\} \to \neg(p \to p) \in T'$$

and

$$\Diamond\Box p \to (\Box\psi_{v\gamma} \equiv \neg(p \to p)) \in T'$$
$$\Diamond\Box p \to (\Box\psi_{v\gamma} \equiv \psi_{v\beta}) \in T'$$
$$\lambda \to \Box^{k(\beta)}(\Box\psi_{v\gamma} \equiv \psi_{v\beta}) \in T'$$
$$\lambda \to \Box^{k(\beta)}(e\beta \equiv \psi_{v\beta}) \in T'.$$

Therefore $\lambda \to (e\alpha \equiv \psi_{v\alpha}) \in T'$, where $\lambda = \Box^k\Diamond\Box p$.

Now we have the following cases:

Case 1: $v\alpha = \bigcap Y$: Since $v\alpha \neq 1$, $\gamma_i \in Y$ for some $i \geq 0$. Hence $e\alpha \to (\lambda \to \gamma_i) \in T'$ and $\Box e\alpha \to (\Box\lambda \to \Box\gamma_i) \in T'$. Further

$$\Diamond\Box p \to (\Box\gamma_i \to \Box^{i+1} p) \in T'$$

so

$$\Box\lambda \to (\Box\gamma_i \to \Box^{i+1} p) \in T'.$$

Thus

$$\Box e\alpha \to (\Box\lambda \to \Box^{i+1} p) \in T'$$

and

$$\Box e\alpha \to (\Box\lambda \to p) \in T'.$$

Case 2: $v\alpha = -\bigcap Y$: Then $e\alpha \to (\lambda \to \bigvee\{\neg\gamma_i : C_i \in Y\}) \in T'$, so $\Box e\alpha \to (\Box\lambda \to \Box\bigvee\{\neg\gamma_i : C_i \in Y\}) \in T'$. We have already shown that

$$\Diamond\Box p \to (\Box \bigvee\{\neg\gamma_i : C_i \in Y\} \to \neg(p \to p)) \in T'$$

so

$$\Box\lambda \to (\Box \bigvee\{\neg\gamma_i : C_i \in Y\} \to \neg(p \to p)) \in T'.$$

Thus

$$\Box e\alpha \to (\Box\lambda \to \neg(p \to p)) \in T'$$

and

$$\Box e\alpha \to (\Box\lambda \to p) \in T'$$

which completes the proof.

Now we define the sequential refutation system $\Sigma = (P, \{\Diamond\Box p/p\})$, where P is the set of the following sequents:

$\varnothing/\varphi$ (φ is an axiom of T)
$\varnothing/\Diamond\Box p \to (\Box(p \to \Box p) \to \Box^2 p)$
$\varnothing/\Diamond\Box p \to \Diamond\Box^2 p$
$\varnothing/\Diamond(p \to \Box p)$
$\varnothing/\Diamond\Box p \to \Box\Diamond p$
$\varnothing/\Box(p \vee q) \to \Diamond(\Box p \vee \Box q)$
$p, p \to q/q$
$p/\Box p.$

Note that Σ is finite.

Theorem 4.4 *For any $\alpha \in FOR$ we have $\alpha \notin E(B)$ iff α is refutable in Σ.*

Proof: ($\leftarrow$) Since $\Diamond\Box p/p$ is not valid in B and all sequents in P are valid in B.
($\rightarrow$) By Theorem 4.3.

Therefore there is a normal modal logic without the fmp that has a very simple refutation system. Now we show that this logic is decidable.

First let us introduce a few definitions. For any $\alpha \in FOR$ and any valuation v on B we define the set

$$K(\alpha, v) = \{i(Y) + 1 : v(\beta) = \bigcap Y, \varnothing \neq Y \subseteq_f C, \beta \in SUB(\alpha)\}.$$

Note that $|K(\alpha, v)| \leq |SUB(\alpha)|$. Moreover for any valuation v on B and any $k \geq 0$ we define the valuation v_k thus:

$$v_k(u) = \begin{cases} \bigcap Y^k & if \quad v(u) = \bigcap Y \\ -\bigcap Y^k & if \quad v(u) = -\bigcap Y \end{cases} \qquad (u \in VAR)$$

where $Y^k = \{C_i : C_i \in Y, i < k\} \cup \{C_{i-1} : C_i \in Y, i > k\} \quad (Y \subseteq_f C)$.

Lemma 4.5 *For any valuation v on B and any $\alpha \in FOR$ we have if $k \neq 0, k \notin K(\alpha, v)$ then*

$$v_k(\beta) = \begin{cases} \bigcap Y^k & if \quad v(\beta) = \bigcap Y \\ -\bigcap Y^k & if \quad v(\beta) = -\bigcap Y \end{cases} \qquad (\beta \in SUB(\alpha))$$

Proof: By induction on the complexity of β.

(1) $\beta \in VAR$. Obvious.

(2a) $\beta = \gamma \wedge \delta$. Then we have the following cases:

Case 1: $v\gamma = \bigcap Y_1, v\delta = \bigcap Y_2$. Then $v\beta = \bigcap(Y_1 \cup Y_2)$. By the inductive hypothesis $v_k\gamma = \bigcap Y_1^k$. Hence $v_k\beta = v_k\gamma \cap v_k\delta = \bigcap Y_1^k \cap \bigcap Y_2^k = \bigcap(Y_1^k \cup Y_2^k) = \bigcap((Y_1 \cup Y_2)^k)$.

Case 2: $v\gamma = -\bigcap Y_1, v\delta = -\bigcap Y_2$. Then $v\beta = -\bigcap(Y_1 \cap Y_2)$. By the inductive hypothesis $v_k\gamma = -\bigcap Y_1^k$. Hence $v_k\beta = -\bigcap Y_1^k \cap -\bigcap Y_2^k = -\bigcap(Y_1^k \cap Y_2^k) = -\bigcap((Y_1 \cap Y_2)^k)$.

Case 3: Say $v\gamma = \bigcap Y_1, v\delta = -\bigcap Y_2$. Then $v\beta = -\bigcap(Y_2 - Y_1)$. By the inductive hypothesis $v_k\gamma = \bigcap Y_1^k, v_k\delta = -\bigcap Y_2^k$. Hence $v_k\beta = \bigcap Y_1^k \cap -\bigcap Y_2^k = -\bigcap(Y_2^k - Y_1^k) = -\bigcap((Y_2 - Y_1)^k)$.

(2b) $\beta = \neg\gamma$. Simple.

(2c) $\beta = \Box\gamma$. Then we have the following cases:

Case 1: $v\gamma = \bigcap Y$. Then $v\beta = \bigcap \bar{Y}$. By the inductive hypothesis $v_k\gamma = \bigcap Y^k$. Since $k \notin K(\alpha, v)$, $k \neq i(Y) + 1$. Now if $0 < k \leq i(Y)$ then $i(Y^k) = i(Y) - 1$, so $v_k(\beta) = lv_k\gamma = l\bigcap Y^k = \bigcap\{C_i : 0 \leq i \leq i(Y)\} = \bigcap \bar{Y}^k$. And if $k > i(Y) + 1$ then $Y^k = Y$ and $\bar{Y} = \bar{Y}^k$, so $v_k\beta = lv_k\gamma = l\bigcap Y^k = l\bigcap Y = \bigcap \bar{Y} = \bigcap \bar{Y}^k$.

Case 2: $v\gamma = -\bigcap Y$. Then $v\beta = \varnothing = -\bigcap \varnothing$. By the inductive hypothesis $v_k\gamma = -\bigcap Y^k$. Hence $v_k\beta = lv_k\gamma = l - \bigcap Y^k = \varnothing = -\bigcap \varnothing = -\bigcap \varnothing^k$.

Now we define the following finite subsets of B:

$$A_n = \{X : X = \bigcap Y \text{ or } X = -\bigcap Y, Y \subseteq \{C_i : 0 \leq i \leq n\}\} \quad (n \geq 0).$$

Lemma 4.6 *For any $\alpha \in FOR$ and any valuation v on B we have if $v(\alpha) \neq 1_B$, $m > |SUB(\alpha)| + 3$ then there is a valuation v' on B such that $v'(SUB(\alpha)) \subseteq A_{m-1}$ and $v'(\alpha) \neq 1_B$.*

Proof: Assume that $v\alpha \neq 1_B, m >| SUB(\alpha) | +3$. Then either $v\alpha = \bigcap Y, C_j \in Y$ for some $j \geq 0$ or $v\alpha = -\bigcap Y$.

Case 1: $v\alpha = \bigcap Y, C_j \in Y$: Since $m > |SUB(\alpha)| + 3$, there is $k \leq m$ such that $k \notin K(\alpha, v), k \neq j, k \neq 0$. Let $v' = v_k$. By Lemma 4.5 we have $v_k\alpha = \bigcap Y^k$. If $k > j$, then $C_j \in Y^k$ and $v_k\alpha \neq 1_B$. If $k < j$, then $C_{j-1} \in Y^k$ and $v_k\alpha \neq 1_B$. Therefore $v_k\alpha \neq 1_B$ and $v'\alpha \neq 1_B$. Also $v'(SUB(\alpha)) \subseteq A_{m-1}$ by Lemma 4.5.

Case 2: $v\alpha = -\bigcap Y$: Let j be the smallest natural number in $\{i : C_i \notin Y\}$. Then there is $k \leq m$ such that $k \neq 0, k \notin K(\alpha, v), k \neq j$. Hence by Lemma 4.5, $v_k\alpha = -\bigcap Y^k$, so $v_k\alpha \neq 1_B$. Let $v' = v_k$. Then $v'\alpha \neq 1_B$ and $v'(SUB(\alpha)) \subseteq A_{m-1}$.

Lemma 4.7 *For any $\alpha \in FOR$ and any valuation v on B we have if $v(\alpha) \neq 1_B$ then there is a valuation v' on B such that $v'(\alpha) \neq 1_B$ and $v'(SUB(\alpha)) \subseteq A_n$, where $n = |SUB(\alpha)| + 3$.*

Proof: By Lemma 4.6.

Finally we can prove that the logic $E(B)$ is decidable.

Theorem 4.8 *For any $\alpha \in FOR$ we have $\alpha \in E(B)$ iff $v(\alpha) = 1_B$ for every valuation v on B such that $v(VAR(\alpha)) \subseteq A_n$, where $n = |SUB(\alpha)| + 3$.*

Proof: By Lemma 4.7.

Since $E(B)$ is decidable, it has a recursive axiomatization. Whether it has a simple and finite one is another problem and will not be considered here.

We end this section with some general remarks about refutation systems as a decision method. It is clear that if a logic has a recursive refutation system then the set of its nontheorems is recursively enumerable so that it is decidable as long as the set of its theorems is also recursively enumerable. Hence refutation systems provide a *theoretical* method of obtaining decidability results. Some general definitions and theorems on that topic can be found in [7]. Moreover a more *practical* decision method using refutation rules of a certain kind is possible. Such rules have the following properties. First they are of the form: if $\dashv \alpha_1, \ldots, \dashv \alpha_n$ then $\dashv \beta$, where β is a formula in normal form and all α_i, after some simple reductions, are simpler (i.e., shorter or having fewer variables) and are also in normal form. Second such rules are justified by theorems of the following kind: $\vdash \beta$ iff for some $1 \leq i \leq n, \vdash \alpha_i$. Using such rules for every formula we can construct either a proof or a disproof of it. This method was in fact introduced by Scott in [6]. A system of this kind for $S4$ is presented in [10].

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