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Semantics-based Nonmonotonic Inference

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Abstract In this paper we discuss Gabbay's idea of basing nonmonotonic deduction on semantic consequence in intuitionistic logic extended by a consistency operator and Turner's suggestion of replacing the intuitionistic base system by Kleene's three-valued logic. It is shown that a certain counterintuitive feature of these approaches can be avoided by using Nelson's constructive logic N instead of intuitionistic logic or Kleene's system. Moreover, in N a more general notion of consistency can be defined and nonmonotonic deduction can thus be based on a logical system satisfying the Deduction Theorem.

1 Introduction The aim of this paper is to revive interest in Gabbay's approach to defining nonmonotonic deduction, see his [5]. The strategy consists in: (i) showing that a rather unpleasant property of the original suggestion can easily be circumvented by choosing a particular (monotonic) base logic which is natural and useful in the field of knowledge representation anyway; and, (ii) emphasizing the richness and flexibility of Gabbay's idea.

In order to avoid fixed-point definitions of nonmonotonic inference, in [5] Gabbay suggested basing nonmonotonic deduction on semantic consequence in a logical system extended by a consistency operator M (see also Clarke [2] and Clarke and Gabbay [3]). $\mathsf{M}A$ is to be read as "it is consistent to assume at this stage that A." In this approach a wff A is said to nonmonotonically follow from a set of assumptions $\Delta = \{A_1, \ldots, A_n\}$ ($\Delta |\sim A$) if there are wffs $B_1, \ldots, B_m$ such that

$$\begin{array}{l} A_1, \ldots, A_n \rightsquigarrow B_1 \\ A_1, \ldots, A_n, B_1 \rightsquigarrow B_2 \\ \quad \vdots \\ A_1, \ldots, A_n, B_1, \ldots, B_m \rightsquigarrow A, \end{array}$$

and $\rightsquigarrow$ is defined as follows: $C_1, \ldots, C_k \rightsquigarrow C$ if there exist extra assumptions $D_1, \ldots, D_j$ such that: (i) $\{C_1, \ldots, C_k, \mathsf{M}D_1, \ldots, \mathsf{M}D_j\}$ is consistent; and, (ii) $\{C_1, \ldots, C_k, \mathsf{M}D_1, \ldots, \mathsf{M}D_j\} \models C$. This notion of nonmonotonic inference requires, of course, a clear semantics for consistency assertions $\mathsf{M}A$. Gabbay's idea is to interpret M as possibility with respect to the 'information ordering' $\sqsubseteq$ in Kripke models

for intuitionistic logic, that is, $\mathsf{M}A$ is true at an information state α iff there is a state β such that $\alpha \sqsubseteq \beta$ and A is true at β. Let us refer to the result of extending intuitionistic propositional logic by M as $\mathbf{H}(\mathsf{M})$.

Assuming that nonmonotonic inferences are appropriate only in the presence of incomplete information, Turner [13] suggested using Gabbay's definition of nonmonotonic inference based on a system of *partial*, in effect Kleene's three-valued logic. Turner considers model structures $\langle I, \sqsubseteq\rangle$, where I is the set of all partial interpretations of the atomic sentences and $\sqsubseteq$ is a 'plausibility' relation on I, that is, a reflexive transitive relation such that $\alpha \sqsubseteq \beta$ implies $\alpha \leq \beta$, where $\leq$ is the natural 'information ordering' on partial interpretations. Consistency assertions $\mathsf{M}A$ are evaluated as true at an information state $\alpha \in I$ like in $\mathbf{H}(\mathsf{M})$, $\mathsf{M}A$ is defined to be false at α, if A is false at every information state β which is at least as plausible as α, and $\mathsf{M}A$ is evaluated as undefined at α otherwise. Let us call Turner's system $\mathbf{K}(\mathsf{M})$.

Gabbay's and Turner's approaches both sucessfully deal with various counterintuitive features of McDermott and Doyle's [10] nonmonotonic formalism. In McDermott and Doyle's logic, for instance, $\{\neg\mathsf{M}p\}$ is nonmonotonically inconsistent, since the nonderivability of $\neg p$ forces $\mathsf{M}p$ to be assumed. Moreover,

$$\begin{array}{ll} \{(\mathsf{M}p \supset q), \neg q\} & \text{is inconsistent} \\ \{\mathsf{M}p, \neg p\} & \text{is satisfiable} \\ \{\mathsf{M}(p \wedge q), \neg p\} & \text{is satisfiable} \\ \mathsf{M}(p \wedge q) \not\sim \mathsf{M}p. & \end{array}$$

However, Gabbay's and Turner's nonmonotonic systems suffer from another weakness (see Łukaszewicz [9]), namely, the fact that $\mathsf{M}p \supset p \sim \neg p$, since in $\mathbf{H}(\mathsf{M})$ as well as in $\mathbf{K}(\mathsf{M})$, $\{(\mathsf{M}p \supset p), \mathsf{M}\neg p\} \models \neg p$ and $\{(\mathsf{M}p \supset p), \mathsf{M}\neg p\}$ is consistent. Intuitively $\mathsf{M}p \supset p \sim \neg p$ clearly fails to be sound, no matter that also $\mathsf{M}p \supset p \sim p$: $\neg p$ should simply not be nonmonotonically derivable from the assumption that p is true by default. According to [9] this weakness renders it problematic to apply Gabbay's and Turner's definitions of nonmonotonic inference to formalizing commonsense reasoning.

In the present paper it is observed that if Gabbay's definition of nonmonotonic inference is based on semantic consequence in Kripke models for Nelson's system N of constructive logic with strong negation (see Almukdad and Nelson [1]), then $(\mathsf{M}p \supset p) \wedge \mathsf{M} \sim p \models \sim p$ does not hold, where $\sim$ denotes strong negation. N combines the advantages of: (i) having a constructive and hence a genuine implication satisfying the Deduction Theorem; and, (ii) semantically being based on partial, three-valued interpretations. As we shall see, this indeed suffices to overcome the problem with the approaches of Gabbay and Turner. Moreover, we shall discuss justifying the choice of a suitable base logic, and, in the course of this discussion, suggest evaluating $\mathsf{M}A$ as true at an information state α iff A fails to be false at any possible extension of α. It will turn out that this notion of consistency is *definable* in N itself and, moreover, its definition in N directly expresses a natural and basic constraint on formalizing M.

2 Intuitionistic base The system $\mathbf{H}(\mathsf{M})$ is the theory of the class of all intuitionistic Kripke models in the language $\{\neg, \mathsf{M}, \supset, \wedge, \vee\}$. An intuitionistic *Kripke frame*

is a structure $\langle I, \sqsubseteq\rangle$, where I is a nonempty set and $\sqsubseteq$ is a reflexive transitive relation on I. An intuitionistic *Kripke model* is a structure $\langle I, \sqsubseteq, v\rangle$, where $\langle I, \sqsubseteq\rangle$ is an intuitionistic Kripke frame, v is a total valuation function assigning to each propositional variable a subset of I, and for every propositional variable p and every $\alpha, \beta \in I$:

(persistence$_0$) if $\alpha \sqsubseteq \beta$, then $\alpha \in v(p)$ implies $\beta \in v(p)$.

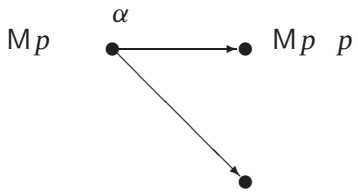
Kripke [8] suggested the following 'informational' reading of frames $\langle I, \sqsubseteq\rangle$: I is a set of information states and $\sqsubseteq$ is the relation of possible expansion of information states over time. Let $\mathcal{M} = \langle I, \sqsubseteq, v\rangle$ be an intuitionistic Kripke model, $\alpha \in I$ and A a wff. $\mathcal{M}, \alpha \models A$ (A is verified at α in $\mathcal{M}$) is inductively defined as follows:

$\mathcal{M}, \alpha \models p$ iff $\alpha \in v(p)$, where p is a propositional variable
$\mathcal{M}, \alpha \models B \wedge C$ iff $\mathcal{M}, \alpha \models B$ and $\mathcal{M}, \alpha \models C$
$\mathcal{M}, \alpha \models B \vee C$ iff $\mathcal{M}, \alpha \models B$ or $\mathcal{M}, \alpha \models C$
$\mathcal{M}, \alpha \models B \supset C$ iff $(\forall \beta \in I)$ if $\alpha \sqsubseteq \beta$, then $\mathcal{M}, \beta \models B$ implies $\mathcal{M}, \beta \models C$
$\mathcal{M}, \alpha \models \neg B$ iff $(\forall \beta \in I)$ if $\alpha \sqsubseteq \beta$, then $\mathcal{M}, \beta \not\models B$
$\mathcal{M}, \alpha \models \mathsf{M}B$ iff $\exists \beta \in I, \alpha \sqsubseteq \beta$ and $\mathcal{M}, \beta \models B$.

If Δ is a set of wffs, then $\mathcal{M}, \alpha \models \Delta$ iff $\mathcal{M}, \alpha \models A$ for every $A \in \Delta$. A wff A is said to be entailed by Δ ($\Delta \models A$) iff for every intuitionistic Kripke model $\mathcal{M} = \langle I, \sqsubseteq, v\rangle$ and every $\alpha \in I$: $\mathcal{M}, \alpha \models \Delta$ implies $\mathcal{M}, \alpha \models A$. Whereas formulas A in $\{\neg, \supset, \wedge, \vee\}$ satisfy

(persistence) if $\alpha \sqsubseteq \beta$, then $\alpha \models A$ implies $\beta \models A$,

(persistence) fails to hold for arbitrary formulas. Also the Deduction Theorem does not hold. Although $\{\mathsf{M}p, \mathsf{M}p \supset q\} \models q$, in the following model $\alpha \models \mathsf{M}p$ and $\alpha \not\models (\mathsf{M}p \supset q) \supset q$:



3 Kleene 3-valued base In his [13] Turner suggested basing Gabbay's definition of nonmonotonic inference on Kleene's three-valued logic. In Kleene's logic implication is defined by $A \supset B =_{\text{def}} \neg A \vee B$, resulting in the following truth table for $\supset$:

$A \supset B$	1	u	0
1	1	u	0
u	1	u	u
0	1	1	1

where the third truth value u is to be read as "undecided." Since there are no tautologies, the Deduction Theorem does not hold in Kleene's logic.

Turner's semantics for $\mathbf{K}(\mathsf{M})$ makes use of *partial interpretations.* A partial interpretation is a mapping from the set of propositional variables into the set of truth values $\{1, 0, u\}$. The natural 'definedness ordering' $\lhd$ on the set of truth values is given by $u \lhd u$, $1 \lhd 1$, $0 \lhd 0$, $u \lhd 1$, and $u \lhd 0$; it gives rise to an 'information ordering' $\leq$ on partial interpretations: $v \leq v'$ iff for every propositional variable p, $v(p) \lhd v'(p)$. A binary relation $\sqsubseteq$ on a set of partial interpretations I is called a *plausibility relation* on I, if $\sqsubseteq$ is a reflexive transitive relation and for every v, $v' \in I$: $v \sqsubseteq v'$ implies $v \leq v'$. A *model structure* is a pair $\langle I, \sqsubseteq\rangle$, where I is the set of all partial interpretations, and $\sqsubseteq$ is a plausibility relation on I. Obviously, every model structure is an intuitionistic Kripke frame. Let $\mathcal{M} = \langle I, \sqsubseteq\rangle$ be a model structure, $\alpha \in I$ and A a wff in the language with M. The notions $\mathcal{M}, \alpha \models^+ A$ (A is verified at α in $\mathcal{M}$) and $\mathcal{M}, \alpha \models^- A$ (A is falsified at α in $\mathcal{M}$) are inductively defined as follows:

$$
\begin{array}{lll}
\mathcal{M}, \alpha \models^+ p & \text{iff} & \alpha(p) = 1, \text{ where } p \text{ is a propositional variable} \\
\mathcal{M}, \alpha \models^- p & \text{iff} & \alpha(p) = 0, \text{ where } p \text{ is a propositional variable} \\
\mathcal{M}, \alpha \models^+ B \wedge C & \text{iff} & \mathcal{M}, \alpha \models^+ B \text{ and } \mathcal{M}, \alpha \models^+ C \\
\mathcal{M}, \alpha \models^- B \wedge C & \text{iff} & \mathcal{M}, \alpha \models^- B \text{ or } \mathcal{M}, \alpha \models^- C \\
\mathcal{M}, \alpha \models^+ B \vee C & \text{iff} & \mathcal{M}, \alpha \models^+ B \text{ or } \mathcal{M}, \alpha \models^+ C \\
\mathcal{M}, \alpha \models^- B \vee C & \text{iff} & \mathcal{M}, \alpha \models^- B \text{ and } \mathcal{M}, \alpha \models^- C \\
\mathcal{M}, \alpha \models^+ B \supset C & \text{iff} & \mathcal{M}, \alpha \models^- B \text{ or } \mathcal{M}, \alpha \models^+ C \\
\mathcal{M}, \alpha \models^- B \supset C & \text{iff} & \mathcal{M}, \alpha \models^+ B \text{ and } \mathcal{M}, \alpha \models^- C \\
\mathcal{M}, \alpha \models^+ \neg B & \text{iff} & \mathcal{M}, \alpha \models^- B \\
\mathcal{M}, \alpha \models^- \neg B & \text{iff} & \mathcal{M}, \alpha \models^+ B \\
\mathcal{M}, \alpha \models^+ \mathsf{M}B & \text{iff} & \exists \beta \in I, \alpha \sqsubseteq \beta \text{ and } \mathcal{M}, \beta \models^+ B \\
\mathcal{M}, \alpha \models^- \mathsf{M}B & \text{iff} & \forall \beta \in I, \alpha \sqsubseteq \beta \text{ implies } \mathcal{M}, \beta \models^- B.
\end{array}
$$

It can be shown that every M-free wff A satisfies the following persistence properties:

(persistence$^+$) if $\alpha \sqsubseteq \beta$, then $\alpha \models^+ A$ implies $\beta \models^+ A$
(persistence$^-$) if $\alpha \sqsubseteq \beta$, then $\alpha \models^- A$ implies $\beta \models^- A$.

4 The problematic case When applied to the set $\{\mathsf{M}p \supset p\}$, the nonmonotonic consequence relations based on $\mathbf{H}(\mathsf{M})$ and $\mathbf{K}(\mathsf{M})$ turn out to be problematic, since $\mathsf{M}p \supset p \mid\!\sim \neg p$, by virtue of: (i) $\{(\mathsf{M}p \supset p), \mathsf{M}\neg p\}$ being consistent; and, (ii) $\{(\mathsf{M}p \supset p), \mathsf{M}\neg p\} \models \neg p$. We shall now take a closer look at why (ii) holds true.

Intuitionistic base: Suppose $\alpha \models \mathsf{M}p \supset p$, $\alpha \models \mathsf{M}\neg p$. The second assumption means that $\neg p$ is verified at some possible expansion of α, and hence, due to (persistence), p cannot be verified at α. If $\alpha \not\models \neg p$, then there is a state γ such that $\alpha \sqsubseteq \gamma$ and $\gamma \models p$. In other words, $\alpha \models \mathsf{M}p$. With the first assumption, $\alpha \models p$, *quod non.* Hence, using the semantic clause for intuitionistic negation, $\neg p$ is verified at α. If, as in partial logic, we use partial valuations and define the negation of p to be verified at a state iff p is falsified at that very state, then, of course, if $\mathsf{M}p \supset p$ is verified at α and $\neg p$ is not, this does not imply that p is verified at α.

Kleene 3-valued base: Suppose that $\alpha \models^+ (\mathsf{M}p \supset p) \wedge \mathsf{M}\neg p$ and $\alpha \not\models^+ \neg p$. Consider the following two cases (as in [9]):

(a) $\alpha \models^- \neg p$, i.e., $\alpha \models^+ p$. By (persistence$^+$), $\beta \models^+ p$ for every β such that $\alpha \sqsubseteq \beta$. But since $\alpha \models^+ \mathsf{M}\neg p$, there is a state β such that $\alpha \sqsubseteq \beta$ and $\beta \models^- p$, *quod non*.

(b) Neither $\alpha \models^- \neg p$ nor $\alpha \models^+ \neg p$. By the verification conditions for implications, $\alpha \models^- \mathsf{M}p$, since $\alpha \models^+ \mathsf{M}p \supset p$. Hence $\beta \models^- p$ for every β such that $\alpha \sqsubseteq \beta$. In particular, $\alpha \models^- p$, *quod non*. Clearly, in this case the problem can be seen to arise by defining $A \supset B$ as $\neg A \vee B$.

5 Constructive base Nelson's constructive logic (see [1]) combines the virtues of intuitionistic logic (namely its constructive implication) and partial logic (namely its suitability for representing incomplete information). From a philosophical point of view, one main advantage of Nelson's logic is that it allows to falsify formulas on the spot in intuitionistic Kripke frames.[1]

A *Nelson model* is a structure $\langle I, \sqsubseteq, v\rangle$, where $\langle I, \sqsubseteq\rangle$ is an intuitionistic Kripke frame and v a mapping that assigns to each $\gamma \in I$ a partial interpretation v_γ such that for every propositional variable p and every $\alpha, \beta \in I$:

(persistence$_0^+$) if $\alpha \sqsubseteq \beta$, then $v_\alpha(p) = 1$ implies $v_\beta(p) = 1$

(persistence$_0^-$) if $\alpha \sqsubseteq \beta$, then $v_\alpha(p) = 0$ implies $v_\beta(p) = 0$.

Let $\mathcal{M} = \langle I, \sqsubseteq, v\rangle$ be a Nelson model, $\alpha \in I$ and let A be a wff in the language $\{\sim, \mathsf{M}, \supset, \wedge, \vee\}$. The notions $\mathcal{M}, \alpha \models^+ A$ (A is verified at α in $\mathcal{M}$) and $\mathcal{M}, \alpha \models^- A$ (A is falsified at α in $\mathcal{M}$) are inductively defined as follows:

$\mathcal{M}, \alpha \models^+ p$	iff	$v_\alpha(p) = 1$, where p is a propositional variable
$\mathcal{M}, \alpha \models^- p$	iff	$v_\alpha(p) = 0$, where p is a propositional variable
$\mathcal{M}, \alpha \models^+ B \wedge C$	iff	$\mathcal{M}, \alpha \models^+ B$ and $\mathcal{M}, \alpha \models^+ C$
$\mathcal{M}, \alpha \models^- B \wedge C$	iff	$\mathcal{M}, \alpha \models^- B$ or $\mathcal{M}, \alpha \models^- C$
$\mathcal{M}, \alpha \models^+ B \vee C$	iff	$\mathcal{M}, \alpha \models^+ B$ or $\mathcal{M}, \alpha \models^+ C$
$\mathcal{M}, \alpha \models^- B \vee C$	iff	$\mathcal{M}, \alpha \models^- B$ and $\mathcal{M}, \alpha \models^- C$
$\mathcal{M}, \alpha \models^+ B \supset C$	iff	$(\forall \beta \in I)$ if $\alpha \sqsubseteq \beta$, then $\mathcal{M}, \beta \models^+ B$ implies $\mathcal{M}, \beta \models^+ C$
$\mathcal{M}, \alpha \models^- B \supset C$	iff	$\mathcal{M}, \alpha \models^+ B$ and $\mathcal{M}, \alpha \models^- C$
$\mathcal{M}, \alpha \models^+ \sim B$	iff	$\mathcal{M}, \alpha \models^- B$
$\mathcal{M}, \alpha \models^- \sim B$	iff	$\mathcal{M}, \alpha \models^+ B$
$\mathcal{M}, \alpha \models^+ \mathsf{M}B$	iff	$\exists \beta \in I, \alpha \sqsubseteq \beta$ and $\mathcal{M}, \beta \models^+ B$
$\mathcal{M}, \alpha \models^- \mathsf{M}B$	iff	$\forall \beta \in I, \alpha \sqsubseteq \beta$ implies $\mathcal{M}, \beta \models^- B$.

Nelson's system N is the theory of the class of all Nelson models in the language $\{\sim, \supset, \wedge, \vee\}$. It can easily be shown that every wff A in this language satisfies (persistence$^+$) and (persistence$^-$). Moreover, for every model $\mathcal{M} = \langle I, \sqsubseteq, v\rangle$ and every $\alpha \in I$ we have

$$\alpha \models^- A \text{ implies } \alpha \not\models^+ A.$$

The law of contraposition does not hold in N. Moreover, provable equivalence fails to be a congruence relation on the set of wffs. If wffs A and B are defined to be *strongly equivalent* iff both A and B and their strong negations $\sim A$ and $\sim B$ are provably equivalent, then it can be shown that strong equivalence is a congruence relation in N. In Wansing [16] it is argued that in the context of abstract information structures these are desirable properties.[2]

Obviously, N is closely related to three-valued logic. If one permits only one-state Nelson models, one obtains a three-valued logic that differs from the well known systems of Kleene, Łukasiewicz, and Bochvar insofar as it has the following truth table for $\supset$:

$A \supset B$	1	u	0
1	1	u	0
u	1	1	1
0	1	1	1

This system can be axiomatized by adding the axiom scheme $((A \supset \sim A) \supset \sim (A \supset \sim A)) \supset A$ to the Hilbert-style axiomatization of N given in the Appendix, see Gurevich [6].

Let us refer to the above extension of N as $N(\mathsf{M}d)$. In $N(\mathsf{M}d)$ we have adopted Turner's 'dynamic' falsification conditions for consistency assertions $\mathsf{M}A$. This, however, is a deviation from the remaining $\models^-$-clauses, which exemplify falsification "on the spot." If we want to stick to the idea of direct falsification, then Turner's $\models^-$-clause for wffs $\mathsf{M}A$ should be replaced by the less general

$$\mathcal{M}, \alpha \models^- \mathsf{M}B \text{ iff } \mathcal{M}, \alpha \models^+ \sim B.$$

The resulting system, $N(\mathsf{M})$, like $N(\mathsf{M}d)$, is not only void of the problematic features of McDermott and Doyle's approach, we also have

$$\{(\mathsf{M}p \supset p), \mathsf{M} \sim p\} \not\models^+_{N(\mathsf{M}d)} \sim p,$$

$$\{(\mathsf{M}p \supset p), \mathsf{M} \sim p\} \not\models^+_{N(\mathsf{M})} \sim p.$$

6 Justifying the choice of the base system Apart from: (i) (implicitly) claiming that the semantic clause for M in $\mathbf{H}(\mathsf{M})$ captures the notion of consistency in a plausible way; and, (ii) considerably improving on McDermott and Doyle's approach, Gabbay gives little further justification of using an *intuitionistic* base system. Additional evidence for the suitableness of working with intuitionistic logic is, according to Gabbay, given by a certain approximation of the "main formal equation for M" (for consistent wffs A):

$(*1)$ $A \nvdash \neg B$ iff $A \vdash \mathsf{M}B$

or rather its semantic counterpart

$(*2)$ $A \not\models \neg B$ iff $A \models \mathsf{M}B$.

Gabbay points out that in a certain naturally defined intuitionistic Kripke model $\mathcal{M}$ one can show that

$(*3)$ $\mathcal{M}, A \not\models \neg B$ iff $\mathcal{M}, A \models \mathsf{M}B$,

where A and B are formulas in $\{\neg, \wedge, \supset\}$ and the information states are the wffs in this fragment themselves. The justificatory power of this approximation is not quite clear. Whereas $(*2)$ asserts an equivalence between the existence of certain countermodels and a property of *all* models, $(*3)$ is a claim about *one particular* model. Moreover, the intuition behind $(*1)$ rather seems to be this:

$(*4)$ $A \nvdash \neg B$ iff $A \not\sim B$.

If one wants to confine the meaning of M in the logical base system by a general "main formal equation," then the following is a natural and fundamental equivalence (again for consistent wffs A):

$(*5)$ $A, \neg B \models \bot$ iff $A \models \mathsf{M}B$.

In case $\neg$ in $(*5)$ is taken to be intuitionistic negation and $\bot$ intuitionistic falsity (that is, $(\forall \alpha \in I)\, \alpha \not\models \bot$), one obtains a slightly stronger consistency operator than Gabbay's, namely intuitionistic double negation $\neg\neg$.[3] The verification conditions for $\neg\neg A$ imply those for Gabbay's $\mathsf{M}A$:

$$\begin{array}{lll} \alpha \models \neg\neg A & \text{iff} & (\forall \beta \in I)\, \alpha \sqsubseteq \beta \text{ implies } \beta \not\models \neg A \\ & \text{iff} & (\forall \beta \in I)\, \alpha \sqsubseteq \beta \text{ implies } ((\exists \gamma \in I)\, \beta \sqsubseteq \gamma \text{ and } \gamma \models A) \\ & \text{only if} & (\exists \beta \in I)\, \alpha \sqsubseteq \beta \text{ and } \beta \models A. \end{array}$$

Since in intuitionistic logic $\neg\neg\neg A$ is equivalent to $\neg A$, one still has $\{(\mathsf{M}p \supset p), \mathsf{M}\neg p\} \models \neg p$ and hence $\{(\mathsf{M}p \supset p)\} \not\sim \neg p$.

Without doubt, both interpreting M as possibility with respect to the information ordering in intuitionistic Kripke models and interpreting it as intuitionistic double negation prima facie captures interesting and plausible notions of consistency. If one, however, agrees with Turner insofar as nonmonotonic inferences are justified only in the presence of incomplete information, then these notions of consistency are not general enough. For, if there is no possible extension of α at which A is decided, that is, is either true or false, it should still be consistent to assume at α that A. The more general intuition, appropriate also in the three-valued setting, therefore is this:

$$\begin{array}{lll} \mathcal{M}, \alpha \models^{+} \mathsf{M}A & \text{iff} & (\forall \beta \in I)\, \alpha \sqsubseteq \beta \text{ implies } \beta \not\models^{-} A \\ \mathcal{M}, \alpha \models^{-} \mathsf{M}A & \text{iff} & \alpha \models^{+} \sim A. \end{array}$$

In N intuitionistic negation $\neg$ can be defined by $\neg A =_{\text{def}} A \supset \sim A$, and the more general consistency operator M turns out to be definable by $\mathsf{M}A =_{\text{def}} \neg \sim A$:

$$\begin{array}{ll} & \alpha \models^{+} \sim A \supset \sim\sim A \\ \text{iff} & (\forall \beta \in I) \text{ if } \alpha \sqsubseteq \beta, \text{ then } \beta \models^{+} \sim A \text{ implies } \beta \models^{+} A \\ \text{iff} & \text{not } (\exists \beta \in I)\, \alpha \sqsubseteq \beta \text{ and } \beta \models^{+} \sim A \text{ and } \beta \not\models^{+} A \\ \text{iff} & \text{not } (\exists \beta \in I)\, \alpha \sqsubseteq \beta \text{ and } \beta \models^{+} \sim A \\ \text{iff} & (\forall \beta \in I)\, \alpha \sqsubseteq \beta \text{ implies } \beta \not\models^{-} A; \end{array}$$

$$\begin{array}{ll} & \alpha \models^{-} \sim A \supset \sim\sim A \\ \text{iff} & \alpha \models^{+} \sim A \text{ and } \alpha \models^{-} \sim\sim A \\ \text{iff} & \alpha \models^{+} \sim A. \end{array}$$

Note that, due to (persistence$^-$) and the reflexivity of $\sqsubseteq$, for the defined M we have

$$\mathcal{M}, \alpha \models^- \mathsf{M}A \text{ iff } \mathcal{M}, \alpha \models^+ {\sim} A \text{ iff } \forall \beta \in I, \alpha \sqsubseteq \beta \text{ implies } \mathcal{M}, \beta \models^- A,$$

that is, Turner's falsification conditions amount to falsification on the spot.

Obviously, the defined M satisfies $(*5)$:

$$\begin{array}{ll} & A, {\sim} B \models^+ \bot \\ \text{iff} & A \models^+ {\sim} B \supset \bot \\ \text{iff} & A \models^+ \neg {\sim} B \\ \text{iff} & A \models^+ \mathsf{M}B. \end{array}$$

Also in N the counterintuitive results of McDermott and Doyle and the problem with Gabbay's and Turner's systems do not arise: the sets of assumptions $\{\mathsf{M}p \supset q, {\sim} q\}$ and $\{{\sim}\,\mathsf{M}p\}$ are satisfiable, $\{\mathsf{M}p, {\sim} p\}$ and $\{\mathsf{M}(p \wedge q), {\sim} p\}$ are not satisfiable, $\mathsf{M}(p \wedge q) |\!\sim \mathsf{M}p$, and $\{(\mathsf{M}p \supset p), \mathsf{M} {\sim} p\} \not\models^+_N {\sim} p$.

According to Clarke and Gabbay (see page 177 ff. of their [3]), "[i]t could be viewed as a justifiable criticism of the intuitionistic system that $\mathsf{M}C \supset C$ is equivalent to $C \vee \neg C$, i.e., neutral with respect to C or $\neg C$. This," they continue, "is not the usual intention behind defaults." Note that neither in $N(\mathsf{M}d)$ nor in $N(\mathsf{M})$ nor in N we have that $\mathsf{M}A \supset A$ is equivalent to $A \vee {\sim} A$.

7 Summary and outlook It seems as if the fact that on the basis of $\mathbf{H}(\mathsf{M})$ and $\mathbf{K}(\mathsf{M})$ we have $\mathsf{M}p \supset p \not\sim \neg p$ has been considered the main obstacle to working with Gabbay's definition of nonmonotonic inference instead of nonconstructive fixed-point definitions. We have seen that this obstacle can easily be removed by using Nelson's constructive logic N as the underlying base system, that is, by working with $N(\mathsf{M}d)$ or $N(\mathsf{M})$. Moreover, we have seen that in Nelson's N a notion of consistency appropriate for three-valued interpretations can be defined, and nonmonotonic inference thus can be based on a system of partial logic: (i) having a clear and intuitive semantics; and, (ii) still satisfying the Deduction Theorem, which is nice, because the Deduction Theorem expresses the central idea of interaction between syntactical consequence and implication.

The beauty of Gabbay's definition of nonmonotonic deduction resides in the flexibility provided by the choice of the underlying base logic. In principle any logic given by a class of 'information models' will do. What is needed is some kind of information ordering $\sqsubseteq$ to interpret the consistency operator M and, if necessary, some successful strategy to avoid undesirable results like the one discussed at length in this paper. To be more specific, the various persistence conditions and the presence or absence of properties of $\sqsubseteq$ (like reflexivity, seriality, transitivity, etc.) give rise to a semantics-driven landscape of subsystems of $N(\mathsf{M}d)$, $N(\mathsf{M})$ and N. This route to weaker systems is plausible for $\mathbf{H}(\mathsf{M})$ and $\mathbf{K}(\mathsf{M})$, too, since in the absence of (persistence) the derivation of $\mathsf{M}p \supset p \not\sim \neg p$ is blocked for $\mathbf{H}(\mathsf{M})$ and in the absence of either (persistence$^+$) or reflexivity of $\sqsubseteq$ the derivation is blocked for $\mathbf{K}(\mathsf{M})$. There exists thus a large variety of different notions of consistency and hence notions of nonmonotonic deduction, which may be compared, tested against benchmark

problems, and applied in knowledge representation. It should also be pointed out that in semantics-based nonmonotonic reasoning there is no need for the underlying base logic to be monotonic. It is well known that in intuitionistic logic the persistence property corresponds with the validity of the monotonicity axiom scheme $A \supset (B \supset A)$. If thus $\mathsf{M}p \supset p \mid\sim \neg p$ is avoided by giving up the persistence requirement in $\mathbf{H}(\mathsf{M})$, one obtains a relevance logical base system. There is nothing wrong with semantically basing nonmonotonic inference on a nonmonotonic logic, since nonmonotonicity *as such* is only a symptom, comparable to the absence of contraction or permutation of premise occurrences in certain substructural logics. What *is* important is the naturalness of the nonmonotonic inference mechanism. I daresay that Gabbay's definition describes such a simple and natural mechanism for nonmonotonic inference.

There is an open problem, namely to axiomatize $N(\mathsf{M}d)$ and $N(\mathsf{M})$.

Appendix Let $A \equiv B$ abbreviate $(A \supset B) \wedge (B \supset A)$. The propositional system N can be axiomatized by modus ponens and the following axiom schemes:

1. $A \supset (B \supset A)$
2. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
3. $A \supset (B \supset (A \wedge B))$
4. $(A \wedge B) \supset A$
5. $(A \wedge B) \supset B$
6. $A \supset (A \vee B)$
7. $B \supset (A \vee B)$
8. $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
9. $\sim (A \supset B) \equiv (A \wedge \sim B)$
10. $\sim (A \wedge B) \equiv (\sim A \vee \sim B)$
11. $\sim (A \vee B) \equiv (\sim A \wedge \sim B)$
12. $A \equiv \sim\sim A$
13. $A \supset (\sim A \supset B)$.

An axiomatization of intuitionistic propositional logic is given by modus ponens, 1–8, and

14. $A \supset (\neg A \supset B)$
15. $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$.

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NOTES

1. A more detailed motivation of Nelson's (systems of) constructive logic with strong negation and additional references may, for instance, be found in Wansing [16] and Jaspars [7]. Recently, strong negation has become rather prominent in extensions of logic programming, see, for example, Pearce and Wagner [12], Wagner [14], Pearce [11], and Wagner [15].

2. Nelson models have intensively been studied by Jaspars in his [7]. His logic **ud** for updating and downdating does not contain constructive implication as a primitive connective. However, the truth conditions for B updated by A ($[A]_u B$) coincide with the truth conditions for $A \supset B$ (while the falsity conditions for $\sim (A \wedge \sim B)$ still coincide with those for $A \supset B$).

3. A semantic treatment of intuitionistic double negation as a modal operator can be found in Došen [4]. Note that Došen regards $\neg\neg$ as a necessity operator $\Box$. However, he notes that one "can prove $\Box A \leftrightarrow \neg\Box\neg A$, which goes some way towards explaining why intuitively $\Box \ldots$ has some features of possibility," see page 16 of [4].

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