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Decidability of Fluted Logic with Identity

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Abstract Fluted logic is the restriction of pure predicate logic to formulas in which variables play no essential role. Although fluted logic is significantly weaker than pure predicate logic, it is of interest because it seems closely to parallel natural logic, the logic that is conducted in natural language. It has been known since 1969 that if conjunction in fluted formulas is restricted to subformulas of equal arity, satisfiability is decidable. However, the decidability of sublogics lying between this restricted (homogeneous) fluted logic and full predicate logic remained unknown. In 1994 it was shown that the satisfiability of fluted formulas without restriction is decidable, thus reducing the unknown region significantly. This paper further reduces the unknown region. It shows that fluted logic with the logical identity is decidable. Since the reflection functor can be defined in fluted logic with identity, it follows that fluted logic with the reflection functor also lies within the region of decidability. Relevance to natural logic is increased since the identity permits definition of singular predicates, which can represent anaphoric pronouns.

1 Introduction Fluted logic is the restriction of pure predicate logic to formulas in which variables play no essential role. Although fluted logic is significantly weaker than pure predicate logic, it is of interest because it seems to closely parallel *natural logic*, the logic that is conducted in natural language.

Historically, fluted logic arose as a byproduct of Quine's Predicate Functor Logic (PFL), a syntactic variant of pure predicate logic. See, for example, Quine [8], [10], [11], [12]. PFL consists of predicate symbols, and alethic and combinatory functors. The alethic functors $\exists$, $\neg$, and $\wedge$ correspond directly to the operations denoted by the same symbols in predicate logic. The combinatory functors *inv*, *Inv*, *pad*, and *ref* replace the variables of predicate logic, and so clearly delineate the roles that variables play in predicate logic. The logical identity relation is sometimes included among the predicate symbols. If the combinatory functors are eliminated, the logic that results is called *fluted logic*.

It has been known since 1969 that if conjunction in fluted formulas is restricted to subformulas of equal arity, satisfiability is decidable (Quine [9], Noah [5]). How-

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ever, the decidability of sublogics lying between this restricted (so-called *homogeneous*) fluted logic and full PFL remained unknown. In 1994 it was shown that the satisfiability of fluted formulas without restriction is decidable (cf. Purdy [7]). This reduced the unknown region significantly. This paper further reduces the unknown region. It shows that fluted logic with the logical identity is decidable. Since the reflection functor *ref* can be defined in fluted logic with identity, it follows that fluted logic with the reflection functor also lies within the region of decidability.

The identity relation greatly increases the relevance of fluted logic to natural logic because it permits definition of unary singular predicates, which can be used to represent anaphoric pronouns. It is wellknown that anaphora play an important role in natural logic. In particular, E-type pronouns provide intersentence coindexing (cf. Purdy [6]) similar to that provided by Skolem constants in clausal logic.

This paper employs Hintikka's theory of constituents (or distributive normal forms). According to this theory, any formula is equivalent to a disjunction of constituents, which can be computed effectively from that formula. Therefore, the question of satisfiability of a formula reduces to the question of satisfiability of a constituent. This reduction is advantageous because constituents are such highly structured formulas. One can prove statements about constituents that would not be feasible to prove about arbitrary formulas. The general proof strategy followed in this paper is that established in [7]. But the details of the proofs are changed and the level of complexity of the proofs is increased significantly by the presence of the identity relation.

2 Preliminaries This paper assumes the usual definition of the pure predicate calculus with the logical identity relation. Typically the set of predicate symbols will be those that occur in some given finite set of formulas or *premises*. The finite set of predicate symbols will be referred to as the *lexicon*. I is the identity relation. Let L be a lexicon and $R \in L$. Then $ar(R)$ denotes the arity of R. Define $ar(L) := max\{ar(R) : R \in L\}$.

A standard result from predicate calculus is the following.

Theorem 2.1 (The Principle of Monotonicity) *Let θ be a subformula, not in the scope of $\neg$, that occurs as a conjunct in formula φ. Then φ' can be inferred from φ, where φ' is obtained from φ by deleting θ.*

Proof: See Andrews [1], Theorem 2105, Substitutivity of Implication. Note that the empty conjunction is defined to be equivalent to $\top$ (verum). □

An *interpretation* I of a lexicon L consists of a set $\mathcal{D}$, the *domain* of I, and a mapping that assigns to I the diagonal relation on $\mathcal{D}$, and to each $R \in L$ a subset of $\mathcal{D}^{ar(R)}$. The notions of satisfaction and truth are the standard ones. If φ is a formula over L with free variables among $\{x_1, \ldots, x_k\}$, and φ is satisfied in I by the assignment to variables $\{x_i \mapsto a_i\}_{1 \le i \le k}$, we write $a_1 \ldots a_k \models \varphi$. If $a_1 = \cdots = a_k = a$, we write $a^k \models \varphi$. If φ is a sentence and φ is true in I, we write $I \models \varphi$.

3 Fluted formulas Let L be a finite set of predicate symbols containing the identity relation I. Let $X_m := \{x_1, \ldots, x_m\}$ be an ordered set of m variables where $m \ge 0$. An *atomic fluted formula* of L over X_m is $Rx_{m-n+1} \ldots x_m$, where $R \in L$ and $ar(R) =$

$n \leq m$. The set of all atomic fluted formulas of L over X_m will be denoted $Af_L(X_m)$. Define $Af_L(X_0) := \{\top\}$.

A *fluted formula* of L over X_m is defined inductively.

1. An atomic fluted formula of L over X_m is a fluted formula of L over X_m.
2. If φ is a fluted formula of L over X_{m+1}, then $\exists x_{m+1}\varphi$ and $\forall x_{m+1}\varphi$ are fluted formulas of L over X_m.
3. If φ and ψ are fluted formulas of L over X_m, then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, and $\neg\varphi$ are fluted formulas of L over X_m.

The fluted formulas just defined will be referred to as *standard* fluted formulas. In addition, any alphabetic variant of a standard fluted formula is defined to be a fluted formula. Two formulas are alphabetic variants of one another if they differ only in an inessential renaming of variables (see Enderton [2], pp. 118–120 for a precise definition). No other formula is a fluted formula.

The fluted formulas of L form a proper subset of the formulas of the pure predicate calculus with predicate symbols L. The semantics of the fluted formulas of L coincides with the usual semantics of the pure predicate calculus. In connection with standard fluted formulas, $abc\ldots \models \varphi$ will always mean that φ is satisfied (in the interpretation given by the context) by the assignment to variables $\{x_1 \mapsto a, x_2 \mapsto b, x_3 \mapsto c, \ldots\}$.

It might be noted in passing that in the predicate calculus restricted to fluted formulas, it is possible to dispense with variables entirely, since the arity and position of a predicate symbol completely determine the sequence of variables that follow the predicate symbol. However, variables will be retained to make the presentation more explicit.

4 Fluted constituents A conjunction in which for each $\rho \in Af_L(X_m)$ either ρ or $\neg\rho$ (but not both) occurs as a conjunct will be called a *minimal conjunction over $Af_L(X_m)$* (because it is an atom in the Boolean lattice generated by $Af_L(X_m)$). The arity $ar(\theta)$ of a minimal conjunction is defined to be the maximum of the arities of the predicate symbols occurring in θ. The set of minimal conjunctions over $Af_L(X_m)$ will be denoted $\Delta Af_L(X_m)$ (cf. Rantala [13]). Note that if $\Delta Af_L(X_m) = \{\theta_1, \ldots, \theta_l\}$, and φ is any quantifier-free formula over $Af_L(X_m)$, then

1. $\neg(\theta_i \wedge \theta_j)$ for $i \neq j$,
2. $\theta_1 \vee \cdots \vee \theta_l$, and
3. either $\theta_i \rightarrow \varphi$, or $\theta_i \rightarrow \neg\varphi$, for $1 \leq i \leq l$,

are tautologies (see [13]).

Let $\mathbf{P}$ be the positive integers, and $\mathbf{P}^*$ the set of finite strings over $\mathbf{P}$. String concatenation is denoted by juxtaposition. The empty string is ε. If $i_1, \ldots, i_n \in \mathbf{P}$, and $\alpha = i_1 \ldots i_n$, then for $k \leq n$, $(k : \alpha) := i_1 \ldots i_k$ is the k-prefix of α.

A subset $\mathcal{T} \subseteq \mathbf{P}^*$ is a *tree domain* if

1. $\varepsilon \in \mathcal{T}$, and
2. if $\alpha i \in \mathcal{T}$, where $\alpha \in \mathbf{P}^*$ and $i \in \mathbf{P}$, then
   (a) $\alpha j \in \mathcal{T}$ for $0 < j < i$, and
   (b) $\alpha \in \mathcal{T}$.

Define the *height of* $\alpha \in \mathcal{T}$, $h(\alpha) :=$ the length of string α. For all $\alpha, \beta \in \mathbf{P}^*$, $i \in \mathbf{P}$, if $\alpha i\beta \in \mathcal{T}$ then $\alpha i\beta$ is a *descendant of* α and α is an *ancestor of* $\alpha i\beta$. Moreover, αi is an *immediate descendant of* α and α is an *immediate ancestor of* αi. Define $w(\alpha) :=$ the number of immediate descendants of α. Thus $\alpha 1, \alpha 2, \ldots, \alpha w(\alpha)$ are the immediate descendants of α. If $w(\alpha) = 0$, then α is *terminal* in $\mathcal{T}$. If all terminal elements of $\mathcal{T}$ have the same height, then $\mathcal{T}$ is *balanced*. In this case, $h(\mathcal{T}) := h(\alpha)$, where α is any terminal element in $\mathcal{T}$. If $0 < h(\alpha) < h(\mathcal{T})$, then α is *internal* in $\mathcal{T}$.

An element α together with all of its descendants is defined to be the *subtree rooted on* α, and is denoted $(\alpha]$. An element α together with all of its immediate descendants will be called the *elementary subtree rooted on* α. An element α together with all of its ancestors is defined to be the *path from* ε to α, and is denoted $[\alpha)$.

Let $\mathcal{T}$ be a balanced tree domain. A *labeled tree domain* $\mathcal{T}_L$ is defined to be $\mathcal{T}$ with a formula $\theta_\alpha \in \Delta Af_L(X_{h(\alpha)})$ associated with each $\alpha \in \mathcal{T}$. The labeled subtree of $\mathcal{T}_L$ rooted on α will be denoted $(\theta_\alpha]$. The labeled path in $\mathcal{T}_L$ from ε to α will be denoted $[\theta_\alpha)$. The subtree $(\theta_\alpha]$ is given the following interpretation.

1. If α is terminal, then $(\theta_\alpha]$ denotes θ_α.
2. If α is nonterminal with height k, then $(\theta_\alpha]$ denotes $\theta_\alpha \wedge \exists x_{k+1}(\theta_{\alpha 1}] \wedge \cdots \wedge \exists x_{k+1}(\theta_{\alpha w(\alpha)}] \wedge \forall x_{k+1}((\theta_{\alpha 1}] \vee \cdots \vee (\theta_{\alpha w(\alpha)}])$.

The formula denoted by $(\theta_\alpha]$ is a *fluted constituent of* L of height $h(\mathcal{T}) - h(\alpha)$ *over the variables* $X_{h(\alpha)}$. If $h(\alpha) = 0$, the formula denoted by $(\theta_\alpha]$ is a *constituent sentence*.

The path $[\theta_\alpha)$ denotes $\theta_\varepsilon \wedge \theta_{1:\alpha} \wedge \theta_{2:\alpha} \wedge \cdots \wedge \theta_\alpha$. If $\theta_\varepsilon = \neg\top$, then $\mathcal{T}_L$ is *trivial*. If $\mathcal{T}_L$ is nontrivial, θ_ε can usually be elided. Notice that for paths of nonzero length, $[\theta_\alpha)$ is not a fluted formula, but rather a conjunction of fluted formulas, each over a different set of variables. Nonetheless, it will be possible, and convenient, to consider paths together with fluted formulas.

In the remainder of this paper, all tree domains will be nontrivial labeled balanced tree domains. Moreover, $(\theta_\alpha]$ and $[\theta_\alpha)$ will not be distinguished from the formulas they denote. Constituents and subconstituents will be considered as sets, as contrasted with multisets. Therefore the assumption that there are no occurrences of repeated constituents or subconstituents will be tacit in the discussion that follows. Finally, constituents that differ only in the left-to-right order in an elementary subtree will not be distinguished.

If φ is a constituent or path, then define:

1. $\varphi^{[-k]}$ is φ with the last k variables eliminated;
2. $\varphi_{[-k]}$ is φ with the first k variables eliminated.

Here elimination of a variable is accomplished by removing all atomic formulas in which that variable occurs, as well as the quantifier, if any, associated with that variable.

If φ is a fluted formula (including tree and path), containing only occurrences of variables $x_l, \ldots, x_k$ in that order, then $\varphi^\dagger := \varphi\{x_l \mapsto x_1, \ldots, x_k \mapsto x_{k-l+1}\}$ is the *standardization* of φ.

Fluted constituents are Hintikka constituents of the second kind (cf. [13]) restricted to fluted formulas. The proofs of the main results in [13] are indifferent to the precise nature of the atomic formulas. The proofs go through unchanged if atomic fluted formulas replace atomic formulas of the pure predicate calculus. Therefore, the main results for Hintikka constituents hold for fluted constituents. The following theorems extend the results for atomic constituents given at the beginning of this section to constituents in general.

Theorem 4.1 (The Fundamental Property of Constituents) *(i) If φ and ψ are fluted constituents of L of height k over the variables X_l, and $\varphi \neq \psi$, then $\varphi \wedge \psi$ is inconsistent. (ii) The disjunction of all fluted constituents of L of height k over the variables X_l is logically valid.*

Proof: See [13], Theorem 3.10. □

Theorem 4.2 *Let φ be a standard fluted formula of L containing variables X_m, where variables $X_k \subseteq X_m$ are free. Then φ is logically equivalent to a disjunction of fluted constituents of height $m - k$ over X_k.*

Proof: See [13], Theorem 4.1. □

According to the Fundamental Property of Constituents, constituents of the same height over the same variables, considered as formulas, are either identical or inconsistent. It is also easy to see that paths of the same height over the same variables, considered as formulas, are either identical or inconsistent. This is formalized by the following theorem.

Theorem 4.3 (The Fundamental Property of Paths) *If φ and ψ are paths of the same height from ε of a constituent sentence and $\varphi \neq \psi$, then $\varphi \wedge \psi$ is inconsistent.*

It is a corollary that if $\alpha, \beta \in \mathcal{T}$ at the same height, and $a_1 \ldots a_k \models [\theta_\alpha)$ and $a_1 \ldots a_k \models [\theta_\beta)$, then $[\theta_\alpha) = [\theta_\beta)$.

A weight function on tree domains is defined as follows.

1. If $\theta_{\alpha j}$ contains a positive occurrence of I, then $wgt(\alpha j) := wgt(\alpha) + 1$.
2. Otherwise, $wgt(\alpha j) := 0$.

5 Trivial inconsistency If $\mathcal{T}_L$ is a constituent sentence, there are certain trivial syntactic properties that, if present, suffice to conclude that $\mathcal{T}_L$ is inconsistent. They are specified in the following lemma.

Lemma 5.1 *A constituent sentence $\mathcal{T}_L$ is inconsistent if either of the following conditions fails to hold.*

1. *For $0 \leq k \leq h(\mathcal{T}) : \mathcal{T}_L^{[-k]} = (\mathcal{T}_{L[-k]})^\dagger$*
2. *For all internal $\alpha \in \mathcal{T}$, there exists exactly one j such that $1 \leq j \leq w(\alpha)$ and $wgt(\alpha j) > 0$. Moreover, for this j:*
$$\begin{aligned} ([\theta_{\alpha j})_{[-h(\alpha j)+wgt(\alpha j)]})^\dagger &= ([\theta_\alpha)_{[-h(\alpha)+wgt(\alpha j)]})^\dagger, \text{ and} \\ ((\theta_{\alpha j}]_{[-h(\alpha j)+wgt(\alpha j)]})^\dagger &= ((\theta_\alpha]^{[-1]}_{[-h(\alpha)+wgt(\alpha j)]})^\dagger. \end{aligned}$$
3. *Either for all internal $\alpha \in \mathcal{T}, w(\alpha) > 1$, or for all nontermimal $\alpha \in \mathcal{T}$, $w(\alpha) = 1$.*

Proof: (i) By the Principle of Monotonicity, $\mathcal{T}_L \to \mathcal{T}_L^{[-k]}$ and $\mathcal{T}_L \to (\mathcal{T}_{L[-k]})^\dagger$. Hence $\mathcal{T}_L \to (\mathcal{T}_L^{[-k]} \wedge (\mathcal{T}_{L[-k]})^\dagger)$. Moreover, $\mathcal{T}_L^{[-k]}$ and $(\mathcal{T}_{L[-k]})^\dagger$ are constituent sentences of the same height. It follows from the Fundamental Property of Constituents that either $\mathcal{T}_L^{[-k]}$ and $(\mathcal{T}_{L[-k]})^\dagger$ are identical or $\mathcal{T}_L$ is inconsistent.

(ii) First suppose that for some internal α at height k, for every j such that $1 \leq j \leq w(\alpha)$, $\theta_{\alpha j}$ contains a negative occurrence of I. Then by the Principle of Monotonicity, $\exists x_k \forall x_{k+1} \neg I x_k x_{k+1}$, which contradicts the semantics of the identity relation.

Next suppose that for some internal α at height k, there exist i, j such that $1 \leq i \neq j \leq w(\alpha)$, and $\theta_{\alpha i}$ and $\theta_{\alpha j}$ both contain positive occurrences of I. By the Principle of Monotonicity, $\exists x_1 \ldots \exists x_k (\exists x_{k+1}(\theta_{\alpha i}] \wedge \exists x_{k+1}(\theta_{\alpha j}])$. The semantics of the identity relation dictate that in any model of $\mathcal{T}_L$, for some $a_1, \ldots, a_k \in \mathcal{D} : a_1 \ldots a_k a_k \models (\theta_{\alpha i}] \wedge (\theta_{\alpha j}]$. Then by the Fundamental Property of Constituents, $i = j$, a contradiction.

Finally suppose that $wgt(\alpha j) = l > 0$ and $([\theta_{\alpha j})_{[-k-1+l]})^\dagger \neq ([\theta_\alpha)_{[-k+l]})^\dagger$. The semantics of the identity relation dictate that in any model of $\mathcal{T}_L$, for some $a \in \mathcal{D}$, $a^l \models ([\theta_{\alpha j})_{[-k-1+l]})^\dagger$ and $a^l \models ([\theta_\alpha)_{[-k+l]})^\dagger$. But by the Fundamental Property of Paths, this implies that $([\theta_{\alpha j})_{[-k-1+l]})^\dagger = ([\theta_\alpha)_{[-k+l]})^\dagger$, a contradiction. The proof for subtrees is similar.

(iii) Suppose that $w(\alpha) = 1$ for some internal α at height k. Assume that $wgt(\alpha) > 0$ (otherwise, condition (2) fails). Then by the Principle of Monotonicity, $\exists x_k \forall x_{k+1} I x_k x_{k+1}$. The semantics of the identity relation dictate that in any model of $\mathcal{T}_L$, $card(\mathcal{D}) = 1$. If now $h(\beta) = l < h(\mathcal{T})$ and $w(\beta) > 1$, then by the Principle of Monotonicity, $\exists x_1 \ldots \exists x_l (\exists x_{l+1}(\theta_{\beta 1}] \wedge \exists x_{l+1}(\theta_{\beta 2}])$. But then by the Fundamental Property of Constituents, $(\theta_{\beta 1}] = (\theta_{\beta 2}]$, a contradiction. A constituent in which $w(\alpha) = 1$ for all nonterminal α is a *vine*. This concludes the proof of the lemma. □

A constituent sentence for which one of the conditions of Lemma 5.1 fails is said to be *trivially inconsistent* (cf. Hintikka [3, 4], which deal with trivial inconsistency in predicate logic.). Thus a constituent sentence is inconsistent if it is trivially inconsistent. The principal objective of this paper is to establish the converse of Lemma 5.1, viz., a constituent sentence is inconsistent only if it is trivially inconsistent.

Condition (1) of Lemma 5.1 can be expressed in several equivalent forms.

Lemma 5.2 *Let $\mathcal{T}_L$ be a constituent sentence. Then the following conditions are equivalent.*

1. *For $0 \leq k \leq h(\mathcal{T}) : \mathcal{T}_L^{[-k]} = (\mathcal{T}_{L[-k]})^\dagger$*
2. *For any $\alpha \in \mathcal{T}$, for all k such that $0 \leq k \leq h(\alpha)$, there exists $\gamma \in \mathcal{T}$ such that*

$$\begin{aligned} [\theta_\gamma) &= ([\theta_\alpha)_{[-k]})^\dagger, \text{ and} \\ \{(\theta_{\gamma j}]^{[-k]} : 1 \leq j \leq w(\gamma)\} &= \{((\theta_{\alpha j}]_{[-k]})^\dagger : 1 \leq j \leq w(\alpha)\}. \end{aligned}$$

3. *For any $\alpha \in \mathcal{T}$, there exists $\gamma \in \mathcal{T}$ such that*

$$\begin{aligned} [\theta_\gamma) &= ([\theta_\alpha)_{[-1]})^\dagger, \text{ and} \\ \{(\theta_{\gamma j}]^{[-1]} : 1 \leq j \leq w(\gamma)\} &= \{((\theta_{\alpha j}]_{[-1]})^\dagger : 1 \leq j \leq w(\alpha)\}. \end{aligned}$$

4. *For any nonterminal* $\alpha \in \mathcal{T}$, *for all k such that* $0 \leq k \leq h(\alpha)$, *there exists* $\gamma \in \mathcal{T}$, *such that*

$$\begin{aligned} [\theta_\gamma) &= ([\theta_\alpha)_{[-k]})^\dagger, \text{ and} \\ \{[\theta_{\gamma j}) : 1 \leq j \leq w(\gamma)\} &= \{([\theta_{\alpha j})_{[-k]})^\dagger : 1 \leq j \leq w(\alpha)\}. \end{aligned}$$

Proof: It is easy to see that (1) and (2) are equivalent, and that (2) implies both (3) and (4). Therefore it suffices to prove that (3) implies (2) and (4) implies (2).

(3) $\Rightarrow$ (2). Suppose that (3) holds. Inductively assume that for any $\alpha \in \mathcal{T}$, there exists $\gamma \in \mathcal{T}$ such that

$$\begin{aligned} [\theta_\gamma) &= ([\theta_\alpha)_{[-k+1]})^\dagger, \text{ and} \\ \{(\theta_{\gamma j}]^{[-k+1]} : 1 \leq j \leq w(\gamma)\} &= \{((\theta_{\alpha j}]_{[-k+1]})^\dagger : 1 \leq j \leq w(\alpha)\}. \end{aligned}$$

By (3), there exists $\delta \in \mathcal{T}$ such that

$$\begin{aligned} [\theta_\delta) &= ([\theta_\gamma)_{[-1]})^\dagger, \text{ and} \\ \{(\theta_{\delta j}]^{[-1]} : 1 \leq j \leq w(\delta)\} &= \{((\theta_{\gamma j}]_{[-1]})^\dagger : 1 \leq j \leq w(\gamma)\}. \end{aligned}$$

Hence

$$\begin{aligned} [\theta_\delta) = ([\theta_\gamma)_{[-1]})^\dagger &= ((([\theta_\alpha)_{[-k+1]})^\dagger)_{[-1]})^\dagger, \text{ and} \\ \{((\theta_{\delta j}]^{[-1]})^{[-k+1]} : 1 \leq j \leq w(\delta)\} &= \{(((\theta_{\gamma j}]_{[-1]})^\dagger)^{[-k+1]} : 1 \leq j \leq w(\gamma)\} \\ &= \{((((\theta_{\alpha j}]_{[-k+1]})^\dagger)_{[-1]})^\dagger : 1 \leq j \leq w(\alpha)\}, \end{aligned}$$

which yields the desired result.

(4) $\Rightarrow$ (2). Suppose that (4) holds. Then for any α, there exists γ such that

$$\begin{aligned} [\theta_\gamma) &= ([\theta_\alpha)_{[-k]})^\dagger, \text{ and} \\ \{[\theta_{\gamma j}) : 1 \leq j \leq w(\gamma)\} &= \{([\theta_{\alpha j})_{[-k]})^\dagger : 1 \leq j \leq w(\alpha)\}. \end{aligned}$$

It will suffice to prove that for such α and γ,

$$[\theta_\gamma) = ([\theta_\alpha)_{[-k]})^\dagger \text{ implies } (\theta_\gamma]^{[-k]} = ((\theta_\alpha]_{[-k]})^\dagger.$$

Define $d := h(\mathcal{T}) - h(\alpha)$, the *depth of* α, and proceed by induction on d. For the basis, let $d = 0$. Then

$$[\theta_\gamma) = ([\theta_\alpha)_{[-k]})^\dagger \text{ implies } \theta_\gamma = (\theta_{\alpha[-k]})^\dagger \text{ implies } (\theta_\gamma]^{[-k]} = ((\theta_\alpha]_{[-k]})^\dagger.$$

For the induction step, let $d > 0$. By the induction hypothesis,

$$\{(\theta_{\gamma j}]^{[-k]} : 1 \leq j \leq w(\gamma)\} = \{((\theta_{\alpha j}]_{[-k]})^\dagger : 1 \leq j \leq w(\alpha)\}.$$

Moreover, $\theta_\gamma = (\theta_{\alpha[-k]})^\dagger$. Hence $(\theta_\gamma]^{[-k]} = ((\theta_\alpha]_{[-k]})^\dagger$. This completes the proof of the lemma. $\square$

When $wgt(\alpha j) = h(\alpha j) - 1$, condition (2) of Lemma 5.1 takes the following form.

$$\begin{aligned} ([\theta_{\alpha j})_{[-1]})^\dagger &= [\theta_\alpha), \text{ and} \\ ((\theta_{\alpha j}]_{[-1]})^\dagger &= (\theta_\alpha]^{[-1]}. \end{aligned}$$

6 Simple fluted constituents A fluted constituent sentence $\mathcal{T}_L$ is *simple* if for all $\alpha \in \mathcal{T}$,

1. $ar(\theta_\alpha) = h(\alpha)$, and
2. $1 \le i < j \le w(\alpha)$ implies $\theta_{\alpha i} \ne \theta_{\alpha j}$.

A constituent that fails to satisfy (2) will be said to have *occurrences of equal siblings*.

A simple constituent sentence possesses a regularity that eliminates the need for consideration of a number of special cases when reasoning about it. If $\mathcal{T}_L$ is a simple constituent sentence, then it follows that no two distinct paths denote the same formula. Therefore, any two distinct paths of the same height from ε are inconsistent. The objective of this section is to show that it is possible to restrict our attention to simple constituent sentences.

Lemma 6.1 *Let $\mathcal{T}_L$ be a fluted constituent sentence. Then there exists a fluted constituent sentence $\mathcal{T}'_{L'}$, such that*

1. *$L \subseteq L'$*
2. *for all $\alpha \in \mathcal{T}' : ar(\theta_\alpha) = h(\alpha)$*
3. *$\mathcal{T}'_{L'}$ is trivially inconsistent iff $\mathcal{T}_L$ is*
4. *$\mathcal{T}'_{L'} \to \mathcal{T}_L$.*

Proof: The proof is by induction on the number of $\beta \in \mathcal{T}$ such that $ar(\theta_\beta) < h(\beta)$. The basis is vacuous. For the induction step, let $ar(\theta_\beta) < h(\beta)$, and let β have minimal height among such elements. Since $h(\beta)$ is minimal, $ar(\theta_\beta) = h(\beta) - 1$. Let Q be a new predicate symbol of arity $h(\beta)$, and define $L' := L \cup \{Q\}$. $\mathcal{T}'_{L'}$ is obtained from $\mathcal{T}_L$ as follows.

1. If $h(\alpha) < h(\beta)$, then θ_α is unchanged.
2. If $h(\alpha) \ge h(\beta)$, then substitute $\theta_\alpha \wedge Qx_p \cdots x_q$ for θ_α, where $p = h(\alpha) - h(\beta) + 1$ and $q = h(\alpha)$.

Now it is obvious that $\mathcal{T}'_{L'}$ is trivially inconsistent iff $\mathcal{T}_L$ is. Moreover, by the Principle of Monotonicity, $\mathcal{T}'_{L'} \to \mathcal{T}_L$. This completes the proof. □

If $\mathcal{T}_L$ is viewed as a formula over the lexicon L', then $\mathcal{T}'_{L'}$ is a constituent of $\mathcal{T}_L$. If $\mathcal{T}'_{L'}$ is consistent, Q will be interpreted as the universal predicate of arity $h(\beta)$.

Lemma 6.2 *Let $\mathcal{T}_L$ be a fluted constituent sentence that is not trivially inconsistent, and that has occurrences of equal siblings. Let m be the minimum height of such occurrences. Then there exists a fluted constituent sentence $\mathcal{T}'_{L'}$, such that*

1. *$L \subseteq L'$*
2. *the number of occurrences of equal siblings at height m in $\mathcal{T}'_{L'}$ is less than the number of occurrences of equal siblings at height m in $\mathcal{T}_L$*
3. *$\mathcal{T}'_{L'}$ is not trivially inconsistent*
4. *$\mathcal{T}'_{L'} \to \mathcal{T}_L$.*

Proof: In view of Lemma 6.1, it can be assumed that for all $\alpha \in \mathcal{T}$: $ar(\theta_\alpha) = h(\alpha)$. Let $\beta \in \mathcal{T}$ be an element at height $m - 1$ such that $1 \le i < j \le w(\beta)$ and $\theta_{\beta i} = \theta_{\beta j}$. To simplify notation, suppose that $\theta_{\beta 1} = \theta_{\beta 2} = \cdots = \theta_{\beta l}$, where $l \le w(\beta)$. Let $h(\mathcal{T}_L) = h$.

The proof proceeds by constructing $\mathcal{T}'_{L'}$ inductively in the order of height k. First new predicates are introduced to partition $\theta_{\beta 1}$ $(= \theta_{\beta 2} = \cdots = \theta_{\beta l})$ into l disjoint expressions. This is possible since $(\theta_{\beta 1}], \ldots, (\theta_{\beta l}]$ are distinct constituents, and so pairwise inconsistent. Then the remainder of $\mathcal{T}_L$ is modified to yield a constituent of L' that is a constituent of $\mathcal{T}_L$, and moreover is not trivially inconsistent.

When $m > 1$, the construction of [7], Lemma 5 suffices because in this case, $\theta_{\beta 1}, \theta_{\beta 2}, \ldots, \theta_{\beta l}$ must contain an occurrence of $\neg I$. This makes it unnecessary to treat I specially. The construction ensures satisfaction of condition (1) of Lemma 5.1. It is then easy to show that condition (2) is satisfied as well. But when $m = 1$, the construction becomes more complex, requiring introduction of constituents (subtrees) during the construction of $\mathcal{T}_{L'}$ that have no counterpart in $\mathcal{T}_L$. These subtrees will be called *exceptional subtrees*. Since the proof must allow $m \geq 1$, the more complex construction must be used.

Let $Q_1, \ldots, Q_r$ be new predicate symbols of arity m, where $2^{r-1} < l \leq 2^r$, and define $L' := L \cup \{Q_1, \ldots, Q_r\}$. Let $\rho_1, \ldots, \rho_l$ be any distinct minimal conjunctions over $\{Q_1, \ldots, Q_r\}$. If $\rho = \sigma_1 \wedge \cdots \wedge \sigma_r$, where for $1 \leq i \leq r$, $\sigma_i = Q_i$ or $\sigma_i = \neg Q_i$, then let $\rho x_p \ldots x_q$ abbreviate $\sigma_1 x_p \ldots x_q \wedge \cdots \wedge \sigma_r x_p \ldots x_q$. In the construction of $\mathcal{T}'_{L'}$, $\mathcal{T}^{(k)}_{L'}$ will be the result corresponding to height k. $\mathcal{T}^{(h)}_{L'}$ will be the result at the conclusion of the construction.

Let φ be a constituent of the lexicon L'. The following operations are defined.

1. $\varphi^{\flat}$ is φ with all occurrences of $Q_1, \ldots, Q_r$ deleted.
2. $\varphi^{\sharp}$ is φ with all occurrences of I deleted.
3. $\varphi^{\natural}$ is φ with all occurrences of exceptional subtrees deleted.
4. $\varphi^{\flat n}$ is φ with all occurrences of $Q_1, \ldots, Q_r$ deleted at height n and above.
5. $\varphi^{\natural n}$ is φ with all occurrences of exceptional subtrees deleted at height n and above.

The proof that $\mathcal{T}^{(h)}_{L'}$ satisfies the lemma is by induction. The induction hypothesis is

1. $\mathcal{T}^{(k)}_{L'}$ is not trivially inconsistent, up to height k. That is, in $\mathcal{T}^{(k)}_{L'}$:
   (a) for each $\alpha \in \mathcal{T}^{(k)}$ such that $h(\alpha) \leq k$, there exist $\gamma \in \mathcal{T}^{(k)}$ such that
$$\begin{aligned} h(\gamma) &= h(\alpha) - 1, \text{ and} \\ [\theta_\gamma) &= ([\theta_\alpha)_{[-1]})^{\dagger} \text{ and} \\ \{(((\theta_{\gamma j}]^{[-1]})^{\flat k})^{\natural k} : 1 \leq j \leq w(\gamma)\} &= \{((\theta_{\alpha j}]_{[-1]})^{\dagger} : 1 \leq j \leq w(\alpha)\}; \end{aligned}$$
   (b) for each $\alpha \in \mathcal{T}^{(k)}$ such that $0 < h(\alpha) < k$, there exists exactly one j such that $1 \leq j \leq w(\alpha)$ and $wgt(\alpha j) > 0$. Moreover, for this j:
$$\begin{aligned} ([\theta_{\alpha j})_{[-h(\alpha j)+wgt(\alpha j)]})^{\dagger} &= ([\theta_\alpha)_{[-h(\alpha)+wgt(\alpha j)]})^{\dagger}, \text{ and} \\ ((\theta_{\alpha j}]_{[-h(\alpha j)+wgt(\alpha j)]})^{\dagger} &= ((((\theta_\alpha]^{[-1]}_{[-h(\alpha)+wgt(\alpha j)]})^{\flat k})^{\natural k})^{\dagger}. \end{aligned}$$
2. $((\mathcal{T}^{(k)}_{L'})^{\flat})^{\natural} = \mathcal{T}_L$.

In $\mathcal{T}_{L'}^{(h)}$,

$$\begin{array}{rcl} (((\theta_{\gamma j}]^{[-1]})^{\flat h})^{\natural h} & = & (\theta_{\gamma j}]^{[-1]} \text{ and} \\ ((((\theta_\alpha]^{[-1]}_{[-h(\alpha)+wgt(\alpha j)]})^{\flat h})^{\natural h})^{\dagger} & = & ((\theta_\alpha]^{[-1]}_{[-h(\alpha)+wgt(\alpha j)]})^{\dagger}, \end{array}$$

and so it follows from (1) of the induction hypothesis that $\mathcal{T}_{L'}^{(h)}$ is not trivially inconsistent. Moreover, since $((\mathcal{T}_{L'}^{(h)})^{\flat})^{\natural}$ is obtained from $\mathcal{T}_{L'}^{(h)}$ by deleting conjuncts of the form $\rho x_p \ldots x_q$, and conjunctive (exceptional) subtrees, then by the Principle of Monotonicity, it follows from (2) of the induction hypothesis that $\mathcal{T}_{L'}^{(h)} \to \mathcal{T}_L$.

For the basis step, let $k = m$. Then $\mathcal{T}_{L'}^{(m)}$ is obtained from $\mathcal{T}_L$ as follows.

1. For $1 \leq i \leq l$, substitute $\theta_{\beta i} \wedge \rho_i x_1 \ldots x_m$ for $\theta_{\beta i}$.
2. For $l < i \leq w(\beta)$, substitute $\theta_{\beta i} \wedge \rho_1 x_1 \ldots x_m$ for $\theta_{\beta i}$.
3. For all other elements αi at height m, substitute $\theta_{\alpha i} \wedge \rho_1 x_1 \ldots x_m$ for $\theta_{\alpha i}$.

The basis step has introduced a partition of $[\theta_{\beta 1})$, making $[\theta_{\beta 1} \wedge \rho_1 x_1 \ldots x_m), \ldots, [\theta_{\beta l} \wedge \rho_l x_1 \ldots x_m)$ distinct in $\mathcal{T}_{L'}^{(m)}$. (1) of the induction hypothesis holds since $\mathcal{T}_L$ is not trivially inconsistent. Obviously, $((\mathcal{T}_{L'}^{(m)})^{\flat})^{\natural} = \mathcal{T}_L$. Therefore the induction hypothesis holds for the basis step.

For the induction step, let $m < k \leq h$. The induction step modifies the tree $\mathcal{T}_{L'}^{(k-1)}$ to yield a tree $\mathcal{T}_{L'}^{(k)}$ that is not trivially inconsistent, up to height k. The construction considers in turn each $\alpha \in \mathcal{T}^{(k-1)}$ such that $h(\alpha) = k-1$. By (1) of the induction hypothesis,

1. for each $\alpha \in \mathcal{T}^{(k-1)}$ such that $h(\alpha) = k-1$, there exist $\gamma \in \mathcal{T}^{(k-1)}$ such that $h(\gamma) = k-2$, and

$$\begin{array}{rcl} [\theta_\gamma) & = & ([\theta_\alpha)_{[-1]})^{\dagger} \text{ and} \\ \{((\theta_{\alpha j}]_{[-1]})^{\dagger} : 1 \leq j \leq w(\alpha)\} & = & \{(((\theta_{\gamma j}]^{[-1]})^{\flat(k-1)})^{\natural(k-1)} : 1 \leq j \leq w(\gamma)\} \\ & = & \{(\theta^{\flat}_{\gamma j}]^{[-1]} : (1 \leq j \leq w(\gamma)) \wedge \\ & & \quad ((\theta_{\gamma j}] \text{ is not exceptional})\} \end{array}$$

2. for each $\delta \in \mathcal{T}^{(k-1)}$ such that $h(\delta) = k-2$, there exists exactly one j such that $1 \leq j \leq w(\delta)$ and $wgt(\delta j) > 0$. Moreover, for this j:

$$\begin{array}{rcl} ([\theta_{\delta j})_{[-h(\delta j)+wgt(\delta j)]})^{\dagger} & = & ([\theta_\delta)_{[-h(\delta)+wgt(\delta j)]})^{\dagger}, \text{ and} \\ ((\theta_{\delta j}]_{[-h(\delta j)+wgt(\delta j)]})^{\dagger} & = & ((((\theta_\delta]^{[-1]}_{[-h(\delta)+wgt(\delta j)]})^{\flat(k-1)})^{\natural(k-1)})^{\dagger}. \end{array}$$

It suffices to prove that:

1. for each $\alpha \in \mathcal{T}^{(k)}$ such that $h(\alpha) = k$, there exist $\gamma \in \mathcal{T}^{(k)}$ such that $h(\gamma) = k-1$, and

$$\begin{array}{rcl} [\theta_\gamma) & = & ([\theta_\alpha)_{[-1]})^{\dagger} \text{ and} \\ \{((\theta_{\alpha j}]_{[-1]})^{\dagger} : 1 \leq j \leq w(\alpha)\} & = & \{(\theta^{\flat}_{\gamma j}]^{[-1]} : (1 \leq j \leq w(\gamma)) \wedge \\ & & \quad ((\theta_{\gamma j}] \text{ is not exceptional})\} \end{array}$$

2. for each $\delta \in \mathcal{T}^{(k)}$ such that $h(\delta) = k-1$, there exists exactly one j such that $1 \leq j \leq w(\delta)$ and $wgt(\delta j) > 0$. Moreover, for this j:

$$\begin{aligned} ([\theta_{\delta j})_{[-h(\delta j)+wgt(\delta j)]})^{\dagger} &= ([\theta_{\delta})_{[-h(\delta)+wgt(\delta j)]})^{\dagger}, \text{ and} \\ ((\theta_{\delta j}]_{[-h(\delta j)+wgt(\delta j)]})^{\dagger} &= ((((\theta_{\delta}]^{[-1]}_{[-h(\delta)+wgt(\delta j)]})^{\flat k})^{\natural k})^{\dagger}. \end{aligned}$$

The induction step will be facilitated if the following cases are considered.

1. $k = m+1$
   (a) $m = 1$
   (b) $m > 1$
2. $k > m+1$

To simplify notation, let $\theta_{\gamma j} = \theta^{\flat}_{\gamma j} \wedge \rho x_{p-1} \dots x_{k-1}$. Let $(\theta_{\alpha i} \wedge \rho x_p \dots x_k]$ denote the subtree obtained from $(\theta_{\alpha i}]$ by substitution of $\theta_{\alpha i} \wedge \rho x_p \dots x_k$ for $\theta_{\alpha i}$, $(\theta^{\flat}_{\gamma j}]$ denote the subtree obtained from $(\theta_{\gamma j}]$ by substitution of $\theta^{\flat}_{\gamma j}$ for $\theta_{\gamma j}$, and $(\theta^{\sharp}_{\alpha i} \wedge \neg I x_{k-1} x_k \wedge \rho x_p \dots x_k]$ denote the subtree obtained from $(\theta_{\alpha i}]$ by substitution of $\theta^{\sharp}_{\alpha i} \wedge \neg I x_{k-1} x_k \wedge \rho x_p \dots x_k$ for $\theta_{\alpha i}$, and similarly for paths.

Case 1: $(k = m+1)$ This case deals with the step immediately following the basis step of the construction. Note that no exceptional subtrees exist at height $\leq m$. Exceptional subtrees exist at height $> m$ only if the induction step immediately following the basis step introduces exceptional subtrees.

Subcase 1: $(m = 1)$ Here $ar(\rho) = 1$, and $\gamma = \varepsilon$. Define $(\alpha i, j)$ for $1 \leq i \leq w(\alpha)$ and $1 \leq j \leq w(\varepsilon)$ as follows.

1. If $(\theta^{\flat}_{j}]^{[-1]} = ((\theta_{\alpha i}]_{[-1]})^{\dagger}$ and $([\theta_{\alpha i} \wedge \rho x_2)_{[-h(\alpha i)+wgt(\alpha i)]})^{\dagger}$ $= ([\theta_{\alpha})_{[-h(\alpha)+wgt(\alpha i)]})^{\dagger}$, then $(\alpha i, j) = 1$.
2. If $(\theta^{\flat}_{j}]^{[-1]} = ((\theta_{\alpha i}]_{[-1]})^{\dagger}$ and $([\theta_{\alpha i} \wedge \rho x_2)_{[-h(\alpha i)+wgt(\alpha i)]})^{\dagger}$ $\neq ([\theta_{\alpha})_{[-h(\alpha)+wgt(\alpha i)]})^{\dagger}$, then $(\alpha i, j) = 2$.
3. Otherwise $(\alpha i, j) := 0$.

Notice that for each i, there exist j such that $(\alpha i, j) = 1$. This is seen as follows. By (1a) of the induction hypothesis, for each i, $(\theta^{\flat}_{j}]^{[-1]} = ((\theta_{\alpha i}]_{[-1]})^{\dagger}$ for one or more j such that $1 \leq j \leq w(\varepsilon)$. If $wgt(\alpha i) = 0$, it follows immediately that $(\alpha i, j) = 1$ for each such j. Suppose that $wgt(\alpha i) > 0$. Since $\mathcal{T}_L$ is not trivially inconsistent, this i is unique, and moreover, $([\theta_{\alpha i})_{[-1]})^{\dagger} = [\theta^{\flat}_{\alpha})$ and $((\theta_{\alpha i}]_{[-1]})^{\dagger} = (\theta^{\flat}_{\alpha}]^{[-1]}$. Hence if ρx_1 occurs in θ_{α}, then $([\theta_{\alpha i} \wedge \rho x_2)_{[-1]})^{\dagger} = [\theta_{\alpha})$, and so $(\alpha i, j) = 1$ for $j = \alpha$. Also notice that $(\alpha i, j) = 2$ for some j only if $wgt(\alpha i) = 1$.

Replace the subtrees $\{(\theta_{\alpha i}] : 1 \leq i \leq w(\alpha)\}$ with the subtrees $\{(\theta_{\alpha i} \wedge \rho x_2] : (\alpha i, j) = 1\} \cup \{(\theta^{\sharp}_{\alpha i} \wedge \neg I x_1 x_2 \wedge \rho x_2] : (\alpha i, j) = 2\}$. Of these subtrees, those in the second set and only those are defined to be exceptional. Since the number of subtrees lying above α may increase in number as a result of this replacement, it may be necessary to reindex the tree domain.

When all $\alpha \in \mathcal{T}^{(m)}$ at height m have been considered, the result is $\mathcal{T}_{L'}^{(m+1)}$. Now for each αi at height $m+1$, there exist j at height m such that

$$(((\theta_j]^{[-1]})^{\flat(m+1)})^{\natural(m+1)} = ((\theta_{\alpha i}]_{[-1]})^{\dagger}.$$

Hence

1. for each αi at height $m+1$, there exist j at height m such that

$$\begin{aligned} [\theta_j) &= ([\theta_{\alpha i})_{[-1]})^{\dagger} \text{ and} \\ \{((\theta_{\alpha ir}]_{[-1]})^{\dagger} : 1 \le r \le w(\alpha i)\} &= \{(\theta_{jq}^{\flat}]^{[-1]} : (1 \le q \le w(j)) \wedge \\ &\qquad ((\theta_{jq}] \text{ is not exceptional})\} \end{aligned}$$

Further, from the definition of $(\alpha i,\ j)$,

2. for each α at height m, there exists exactly one i such that $1 \le i \le w(\alpha)$ and $wgt(\alpha i) > 0$. Moreover, for this i:

$$\begin{aligned} ([\theta_{\alpha i})_{[-1]})^{\dagger} &= [\theta_{\alpha}), \text{ and} \\ ((\theta_{\alpha i}]_{[-1]})^{\dagger} &= (((\theta_{\alpha}]^{[-1]})^{\flat(m+1)})^{\natural(m+1)}. \end{aligned}$$

Thus the induction hypothesis holds.

Subcase 2: $(m > 1)$ Define $(\alpha i, \gamma j)$ for $1 \le i \le w(\alpha)$ and $1 \le j \le w(\gamma)$ as follows.

1. If $(\theta_{\gamma j}^{\flat}]^{[-1]} = ((\theta_{\alpha i}]_{[-1]})^{\dagger}$, then $(\alpha i, \gamma j) = 1$.
2. Otherwise $(\alpha i, \gamma j) := 0$.

By (1a) of the induction hypothesis, for each i, there exist one or more j such that $(\alpha i, \gamma j) = 1$.

Replace the subtrees $\{(\theta_{\alpha i}] : 1 \le i \le w(\alpha)\}$ with subtrees $\{(\theta_{\alpha i} \wedge \rho x_2 \ldots x_{m+1}] : (\alpha i, \gamma j) = 1\}$. None of the new subtrees is exceptional. The operation $\cdot^{\natural(m+1)}$ is redundant and is retained only for uniformity with the other cases. As before, it may be necessary to reindex the tree domain. When all $\alpha \in \mathcal{T}^{(m)}$ at height m have been considered, the result is $\mathcal{T}_{L'}^{(m+1)}$. Now for each αi at height $m+1$, there exist γj at height m such that $(((\theta_{\gamma j}]^{[-1]})^{\flat(m+1)})^{\natural(m+1)} = ((\theta_{\alpha i}]_{[-1]})^{\dagger}$. Hence

1. for each αi at height $m+1$, there exist γj at height m such that

$$\begin{aligned} [\theta_{\gamma j}) &= ([\theta_{\alpha i})_{[-1]})^{\dagger} \text{ and} \\ \{((\theta_{\alpha ir}]_{[-1]})^{\dagger} : 1 \le r \le w(\alpha i)\} &= \{(\theta_{\gamma jq}^{\flat}]^{[-1]} : (1 \le q \le w(\gamma j)) \wedge \\ &\qquad ((\theta_{\gamma jq}] \text{ is not exceptional})\} \end{aligned}$$

It remains to prove condition (2), that is,

2. for each α at height m, there exists exactly one i such that $1 \le i \le w(\alpha)$ and $wgt(\alpha i) > 0$. Moreover, for this i:

$$\begin{aligned} ([\theta_{\alpha i})_{[-h(\alpha i)+wgt(\alpha i)]})^{\dagger} &= ([\theta_{\alpha})_{[-h(\alpha)+wgt(\alpha i)]})^{\dagger}, \text{ and} \\ ((\theta_{\alpha i}]_{[-h(\alpha i)+wgt(\alpha i)]})^{\dagger} &= ((((\theta_{\alpha}]^{[-1]}_{[-h(\alpha)+wgt(\alpha i)]})^{\flat(m+1)})^{\natural(m+1)})^{\dagger}. \end{aligned}$$

That there exists exactly one i such that $1 \le i \le w(\alpha)$ and $wgt(\alpha i) > 0$ follows from the assumption that $\mathcal{T}_L$ is not trivially inconsistent and the observation that the construction preserves this uniqueness. To complete the proof of condition (2), first suppose that $wgt(\alpha i) = h(\alpha i) - 1$. Then $wgt(\alpha) = h(\alpha) - 1$. Let $\delta = (m-1) : \alpha$. By the induction hypothesis, in $\mathcal{T}_{L'}^{(m)}$,

$$\begin{aligned}([\theta_\alpha)_{[-1]})^\dagger &= [\theta_\delta), \text{ and}\\ ((\theta_\alpha]_{[-1]})^\dagger &= (((\theta_\delta]^{[-1]})^{\flat m})^{\natural m}.\end{aligned}$$

Thus

$$\{((\theta_{\alpha j}]_{[-1]})^\dagger : 1 \le j \le w(\alpha)\} = \{(((\theta_{\delta j}]^{[-1]})^{\flat m})^{\natural m} : 1 \le j \le w(\delta)\}.$$

Therefore, δ is the (or one of the) γ whose existence is asserted by the induction hypothesis. As a result, in $\mathcal{T}_{L'}^{(m+1)}$, for each αi at height $m+1$, there exist δj at height m such that $(((\theta_{\delta j}]^{[-1]})^{\flat(m+1)})^{\natural(m+1)} = ((\theta_{\alpha i}]_{[-1]})^\dagger$. Since $wgt(\alpha i) > 0$, the δj associated with αi must be α. Hence

$$\begin{aligned}([\theta_{\alpha i})_{[-1]})^\dagger &= [\theta_\alpha), \text{ and}\\ ((\theta_{\alpha i}]_{[-1]})^\dagger &= (((\theta_\alpha]^{[-1]})^{\flat(m+1)})^{\natural(m+1)}.\end{aligned}$$

Next suppose that $0 < wgt(\alpha i) < h(\alpha i) - 1$. Under this supposition, $wgt(\gamma j) = wgt(\alpha i)$. Since in $\mathcal{T}_{L'}^{(m+1)}$, $[\theta_{\gamma j}) = ([\theta_{\alpha i})_{[-1]})^\dagger$, it follows that

$$([\theta_{\alpha i})_{[-h(\alpha i)+wgt(\alpha i)]})^\dagger = ([\theta_{\gamma j})_{[-h(\gamma j)+wgt(\gamma j)]})^\dagger.$$

By the induction hypothesis,

$$([\theta_{\gamma j})_{[-h(\gamma j)+wgt(\gamma j)]})^\dagger = ([\theta_{\gamma})_{[-h(\gamma)+wgt(\gamma j)]})^\dagger.$$

Since $[\theta_\gamma) = ([\theta_\alpha)_{[-1]})^\dagger$, it follows that

$$([\theta_{\alpha i})_{[-h(\alpha i)+wgt(\alpha i)]})^\dagger = ([\theta_{\alpha})_{[-h(\alpha)+wgt(\alpha i)]})^\dagger.$$

This is the first equation of condition (2).

Since $\mathcal{T}_L$ is not trivially inconsistent,

$$((((\theta_{\alpha i}]_{[-h(\alpha i)+wgt(\alpha i)]})^\flat)^\natural)^\dagger = ((((\theta_\alpha]^{[-1]}_{[-h(\alpha)+wgt(\alpha i)]})^\flat)^\natural)^\dagger.$$

It follows from the first equation above that

$$(\theta_{\alpha i[-h(\alpha i)+wgt(\alpha i)]})^\dagger = (\theta_{\alpha[-h(\alpha)+wgt(\alpha i)]})^\dagger.$$

Therefore,

$$((\theta_{\alpha i}]_{[-h(\alpha i)+wgt(\alpha i)]})^\dagger = ((((\theta_\alpha]^{[-1]}_{[-h(\alpha)+wgt(\alpha i)]})^{\flat(m+1)})^{\natural(m+1)})^\dagger.$$

This is the second equation of condition (2). This completes the proof of condition (2). Thus the induction hypothesis holds.

Case 2: $(k > m+1)$ This case deals with the induction steps subsequent to the first. Notice that if $\rho x_{p-1} \dots x_{k-1}$ occurs in θ_α, then $\rho x_{p-2} \dots x_{k-2}$ occurs in both $(\theta_{\alpha[-1]})^\dagger$ and θ_γ.

Define $(\alpha i, \gamma j)$ as follows.

1. If $(\theta_{\gamma j}]$ is not exceptional and $(\theta^\flat_{\gamma j}]^{[-1]} = ((\theta_{\alpha i}]_{[-1]})^\dagger$, then $(\alpha i, \gamma j) = 1$.
2. If $(\theta_{\gamma j}]$ is exceptional and $(\theta^{\flat\sharp}_{\gamma j}]^{[-1]} = ((\theta^\sharp_{\alpha i}]_{[-1]})^\dagger$, then $(\alpha i, \gamma j) = 2$.
3. Otherwise $(\alpha i, \gamma j) := 0$.

By (1a) of the induction hypothesis, for each i, there exist one or more j such that $(\alpha i, \gamma j) = 1$.

Replace the subtrees $\{(\theta_{\alpha i}] : 1 \le i \le w(\alpha)\}$ with the subtrees $\{(\theta_{\alpha i} \wedge \rho x_p \dots x_k] : (\alpha i, \gamma j) = 1\} \cup \{(\theta^\sharp_{\alpha i} \wedge \neg I x_{k-1} x_k \wedge \rho x_p \dots x_k] : (\alpha i, \gamma j) = 2\}$. Of these subtrees, those in the second set and only those are defined to be exceptional. As before, it may be necessary to reindex the tree domain.

When all $\alpha \in \mathcal{T}^{(k-1)}$ at height $k-1$ have been considered, the result is $\mathcal{T}^{(k)}_{L'}$. Now for each αi at height k, there exist γj at height $k-1$ such that $(((\theta_{\gamma j}]^{[-1]})^{\flat k})^{\natural k} = ((\theta_{\alpha i}]_{[-1]})^\dagger$. Hence

1. for each αi at height k, there exist γj at height $k-1$ such that

$$\begin{aligned}
[\theta_{\gamma j}) &= ([\theta_{\alpha i})_{[-1]})^\dagger \text{ and} \\
\{((\theta_{\alpha i r}]_{[-1]})^\dagger : 1 \le r \le w(\alpha i)\} &= \{(\theta^\flat_{\gamma j q}]^{[-1]} : (1 \le q \le w(\gamma j)) \wedge \\
&\qquad ((\theta_{\gamma j q}] \text{ is not exceptional})\}
\end{aligned}$$

It remains to prove condition (2), that is,

2. for each α at height $k-1$, there exists exactly one i such that $1 \le i \le w(\alpha)$ and $wgt(\alpha i) > 0$. Moreover, for this i:

$$\begin{aligned}
([\theta_{\alpha i})_{[-h(\alpha i)+wgt(\alpha i)]})^\dagger &= ([\theta_\alpha)_{[-h(\alpha)+wgt(\alpha i)]})^\dagger, \text{ and} \\
((\theta_{\alpha i}]_{[-h(\alpha i)+wgt(\alpha i)]})^\dagger &= ((((\theta_\alpha]^{[-1]}_{[-h(\alpha)+wgt(\alpha i)]})^{\flat k})^{\natural k})^\dagger.
\end{aligned}$$

The remainder of the proof for Case 2 is similar to that for Subcase 2.

Observe that in every case, $((\mathcal{T}^{(k)}_{L'})^\flat)^\natural = \mathcal{T}_L$. Therefore the induction hypothesis holds for the induction step. Finally, define $\mathcal{T}'_{L'} := \mathcal{T}^{(h)}_{L'}$. This completes the proof. □

If $\mathcal{T}_L$ is viewed as a formula over the lexicon L', then $\mathcal{T}'_{L'}$ is a constituent of $\mathcal{T}_L$. In an interpretation of $\mathcal{T}'_{L'}$, the $\rho_1, \dots, \rho_l$ will be interpreted as subsets of $\mathcal{D}^m$ that separate the subset that interprets $\theta_{\beta 1}$ into l disjoint parts such that each part satisfies one of the existential claims on $\theta_{\beta 1}$. Such separation is always possible since $(\theta_{\beta 1}], \dots, (\theta_{\beta l}]$ are distinct constituents, and so pairwise not simultaneously satisfiable. That is, for any assignment to the free variables of these constituents, no element of the domain can bear witness for more than one of them.

Together these lemmas yield the following theorem.

Theorem 6.3 *Let $\mathcal{T}_L$ be a fluted constituent sentence that is not trivially inconsistent. Then there exists a simple fluted constituent sentence $\mathcal{T}'_{L'}$, such that*

1. *$L \subseteq L'$*
2. *$\mathcal{T}'_{L'}$ is not trivially inconsistent*
3. *$\mathcal{T}'_{L'} \to \mathcal{T}_L$.*

Proof: In view of Lemma 6.1, it can be assumed that for all $\alpha \in \mathcal{T}$: $ar(\theta_\alpha) = h(\alpha)$. If $\mathcal{T}_L$ is a vine, there is nothing to prove, so suppose that $\mathcal{T}_L$ is not a vine. $\mathcal{T}'_{L'}$ is constructed inductively. The construction begins with $\mathcal{T}_L$. Each step employs the construction of Lemma 6.2. Inductively, suppose that n steps have been performed, and that m is the minimum height at which there are occurrences of equal siblings. Then after step $(n+1)$, there are fewer occurrences of equal siblings at height m than after step n. Each step reduces the number of occurrences of equal siblings at the minimum height of such occurrences. When this number reaches zero, it increases the minimum height of such occurrences. Although some steps may increase the total number of occurrences of equal siblings, the construction acts to restrict these occurrences to greater and greater heights, until they only can occur at height h, where they are eliminated entirely by the assumption that $\mathcal{T}'_{L'}$ contains no occurrences of repeated constituents. This completes the proof of the theorem. □

7 Satisfiability of fluted constituents According to Theorem 4.2, every fluted formula is equivalent to a disjunction of fluted constituents of the lexicon of that formula providing they are of sufficient height. Therefore, the question of satisfiability of a fluted formula reduces to the question of satisfiability of a fluted constituent. This involves construction of interpretations of constituents. First some general facts relevant to interpretations of constituents will be established.

Let $\mathcal{T}_L$ be a fluted constituent sentence of L with the identity relation. Suppose that $\mathcal{T}_L$ is not trivially inconsistent. Define the *domain associated with $\mathcal{T}$* to be

$$\mathcal{D} := \{a_\alpha : (\alpha \in \mathcal{T}) \wedge (\alpha \neq \varepsilon)\}.$$

$\mathcal{T}$ itself without the root element ε would serve as well, but $\mathcal{D}$ will be used to enhance readability.

Define $\sim$ initially (it will later be extended) to be the least equivalence relation on $\mathcal{D}$ such that

$$a_\alpha \sim a_{\alpha j} \text{ if } wgt(\alpha j) > 0.$$

$\sim$ is extended to sequences of elements of $\mathcal{D}$ as follows.

$$a_{\alpha_1} \ldots a_{\alpha_l} \sim a_{\gamma_1} \ldots a_{\gamma_m} \text{ iff } l = m \text{ and for } 1 \leq i \leq l : a_{\alpha_i} \sim a_{\gamma_i}$$

The following lemma gives some properties of $\sim$ as defined initially.

Lemma 7.1 *Let $\mathcal{T}_L$ be a fluted constituent sentence of L with the identity relation such that $\mathcal{T}_L$ is not trivially inconsistent. Let $\alpha \in \mathcal{T}$.*

1. *$a_{\alpha i} \sim a_{\alpha j}$ implies $i = j$.*
2. *$a_\alpha \sim a_\delta$ for some unique terminal element $\delta \in \mathcal{T}$.*
3. *$a_\alpha \sim a_\gamma$ implies that α and γ are* lineally related, *i.e., either α is a prefix of γ or γ is a prefix of α.*

4. *If* $a_{l:\alpha}\dots a_\alpha \sim a_{m:\gamma}\dots a_\gamma$ *and* α *is a prefix of* γ*, then* $a_{l:\alpha}\sim\cdots\sim a_\alpha\sim\cdots\sim a_\gamma$*,* $wgt(\gamma)\geq h(\gamma)-l$*,* $wgt(\alpha)\geq h(\alpha)-l$*, and* $([\theta_\gamma)_{[-m+1]})^\dagger = ([\theta_\alpha)_{[-l+1]})^\dagger$*.*

Proof: (1), (2), and (3) follow directly from the definitions of trivial inconsistency, wgt, and $\sim$. The first three assertions of (4), viz., that $a_{l:\alpha}\sim\cdots\sim a_\alpha\sim\cdots\sim a_\gamma$, $wgt(\gamma)\geq h(\gamma)-l$, and $wgt(\alpha)\geq h(\alpha)-l$, follow from the definitions of wgt and $\sim$. For the last assertion of (4), observe that in general $([\theta_\zeta)_{[-q]})^\dagger = ([\theta_\xi)_{[-r]})^\dagger$ implies $([\theta_\zeta)_{[-q-t]})^\dagger = ([\theta_\xi)_{[-r-t]})^\dagger$ for $0\leq t\leq h(\zeta)-q$. This, together with the assumption that $\mathcal{T}_L$ is not trivially inconsistent, yields $([\theta_\gamma)_{[-h(\gamma)+wgt(\alpha)+1]})^\dagger = ([\theta_\alpha)_{[-h(\alpha)+wgt(\alpha)+1]})^\dagger$. But $h(\gamma)-wgt(\alpha)\leq m$ and $h(\alpha)-wgt(\alpha)\leq l$, so $([\theta_\gamma)_{[-m+1]})^\dagger = ([\theta_\alpha)_{[-l+1]})^\dagger$. This completes the proof of the lemma. □

An interpretation with domain $\mathcal{D}/\sim$ satisfying $\mathcal{T}_L$ requires definition of a mapping such that

1. $\top\mapsto\{()\}$
2. $I\mapsto\{(a,a) : a\in\mathcal{D}/\sim\}$
3. $R\mapsto R^I\subseteq(\mathcal{D}/\sim)^{ar(R)}$ for each $R\in L$.

It will suffice to define a mapping of $[\theta_\alpha)$ for each $\alpha\in\mathcal{T}$ that satisfies the following properties.

1. The image of $[\theta_\alpha)$ is a subset of $\mathcal{D}^{h(\alpha)}$.
2. If $a_{\beta_1}\dots a_{\beta_k}\models[\theta_\alpha)$ and $[\theta_\gamma) = [\theta_\alpha)^{[-l]}$, where $1\leq l\leq h(\alpha)$, then $a_{\beta_1}\dots a_{\beta_{k-l}}\models[\theta_\gamma)$.
3. If $a_{\beta_1}\dots a_{\beta_k}\models[\theta_\alpha)$ and $[\theta_\gamma) = ([\theta_\alpha)_{[-l]})^\dagger$, where $1\leq l\leq h(\alpha)$, then $a_{\beta_{l+1}}\dots a_{\beta_k}\models[\theta_\gamma)$.
4. If $a_{\beta_1}\dots a_{\beta_k}\models[\theta_\alpha)$ and $a_{\beta_1}\dots a_{\beta_k}\sim a_{\delta_1}\dots a_{\delta_k}$, then $a_{\delta_1}\dots a_{\delta_k}\models[\theta_\alpha)$.
5. If $a_{\beta_1}\dots a_{\beta_k}\models[\theta_\alpha)$ and $a_{\beta_1}\dots a_{\beta_k}\models[\theta_\gamma)$, then $\alpha=\gamma$.

(1)–(3) ensure that the mapping respects the definition of path from ε; (4) imposes the semantics of the identity relation, and (5) ensures that the mapping respects the Fundamental Property of Paths and the Fundamental Property of Constituents. A mapping satisfying (1)–(5) is well-defined. Moreover, it induces a well-defined mapping from L into the structure with domain $\mathcal{D}/\sim$. This latter mapping defines an interpretation in $\mathcal{D}/\sim$.

The following theorem, which provides a decision procedure for the question of satisfiability of constituents, is the main result of the paper.

Theorem 7.2 *A fluted constituent sentence with the identity relation is unsatisfiable iff it is trivially inconsistent.*

Proof: The 'if' direction is given by Lemma 5.1. The 'only-if' direction will be proved in its contrapositive form. Let $\mathcal{T}_L$ be a fluted constituent sentence of height h that is not trivially inconsistent. In view of Theorem 6.3, it can be assumed without loss of generality that $\mathcal{T}_L$ is simple. It will be shown that $\mathcal{T}_L$ is satisfiable in an interpretation I with domain $\mathcal{D}$, and whose mapping satisfies the five conditions enumerated above. Two claims will be proved.

Claim 7.3 *An interpretation* I *of* L *can be constructed with the property that if* α *is nonterminal at height* k*, and* $a_{\beta_1}\dots a_{\beta_k}\models[\theta_\alpha)$*, then*

1. *for* $1 \leq j \leq w(\alpha) : \exists a_\beta \in \mathcal{D} : a_{\beta_1} \ldots a_{\beta_k} a_\beta \models [\theta_{\alpha j})$, *and*
2. $\forall a_\beta \in \mathcal{D} : a_{\beta_1} \ldots a_{\beta_k} a_\beta \models [\theta_{\alpha 1}) \vee \cdots \vee [\theta_{\alpha w(\alpha)})$.

Claim 7.4 *In an interpretation I of L with the property of Claim 7.3, if $h(\alpha) = k$ and $a_{\beta_1} \ldots a_{\beta_k} \models [\theta_\alpha)$, then $a_{\beta_1} \ldots a_{\beta_k} \models (\theta_\alpha]$.*

The theorem follows from these claims since, letting I be the interpretation of Claim 7.3, we have $I \models [\theta_\varepsilon)$, because $\mathcal{T}_L$ is nontrivial, and so by Claim 7.4, $I \models (\theta_\varepsilon]$, i.e., $I \models \mathcal{T}_L$. Proofs of the claims are now given.

Proof of Claim 7.3: For the proof of this claim, it will be helpful to invoke geometric intuition by viewing $[\theta_\alpha)$, where $h(\alpha) = k$, as a subspace in the k-dimensional space with coordinate axes $x_1, x_2, \ldots, x_k$. On this view, the tuple $(a_{\beta_1}, \ldots, a_{\beta_k})$, which will be written $a_{\beta_1} \ldots a_{\beta_k}$, is a point in the k-dimensional space. The statement $a_{\beta_1} \ldots a_{\beta_k} \in [\theta_\alpha)$ is defined to be equivalent to $a_{\beta_1} \ldots a_{\beta_k} \models [\theta_\alpha)$. $a_{1:\alpha} \ldots a_\alpha$ also is a point in the k-dimensional space. Points of this latter form, as well as points of the form $a_{i:\alpha} \ldots a_\alpha$ $(1 \leq i \leq k)$, and points $\sim$-equivalent to them, will be called *standard points*. In the usual way, a subspace of k-dimensional space becomes a subspace of $(k+1)$-dimensional space by cylindrification or ringing along the $(k+1)$-st coordinate.

The mapping of I is defined in three parts. Each part is ordered by height. The first part of the mapping is given as follows. For each $\alpha \in \mathcal{T}$, define

1. $a_{1:\alpha} \ldots a_\alpha \models [\theta_\alpha)$, and for $1 < i \leq h(\alpha) : a_{i:\alpha} \ldots a_\alpha \models [\theta_\gamma)$, where $[\theta_\gamma) = ([\theta_\alpha)_{[-i+1]})^\dagger$.

This definition is then extended just so that it is closed under $\sim$. It is easy to see that the definition satisfies conditions (1)–(3). Also condition (4) is satisfied by the closure under $\sim$. It remains to show that condition (5) is satisfied. Suppose that $a_{l:\alpha} \ldots a_\alpha \models [\theta_\gamma)$, where $[\theta_\gamma) = ([\theta_\alpha)_{[-l+1]})^\dagger$, and $a_{l:\alpha} \ldots a_\alpha \models [\theta_\zeta)$. Then there exists δ and m such that $a_{l:\alpha} \ldots a_\alpha \sim a_{m:\delta} \ldots a_\delta$, $a_{m:\delta} \ldots a_\delta \models [\theta_\zeta)$, and $[\theta_\zeta) = ([\theta_\delta)_{[-m+1]})^\dagger$. By Lemma 7.1.4, $([\theta_\delta)_{[-m+1]})^\dagger = ([\theta_\alpha)_{[-l+1]})^\dagger$. Hence $[\theta_\zeta) = [\theta_\gamma)$, and since $\mathcal{T}_L$ is simple, $\zeta = \gamma$. Thus condition (5) is satisfied. This concludes the first part of the mapping. Following this part, every standard point is committed.

The second part of the mapping is defined next, ordered by height. Let $h(\alpha) = k > 0$. We extend the interpretation of the $[\theta_{\alpha j})$ as follows. For each $\beta \in \mathcal{T}$, if

1. $a_{i:\delta} \ldots a_\delta \models [\theta_\alpha)$,
2. $a_{i:\delta} \ldots a_\delta a_\beta \models [\theta_{\alpha j})_{[-1]}$, and
3. it is not the case that $1 \leq l \neq j \leq w(\alpha)$, $a_{m:\zeta} \ldots a_\zeta a_\gamma \sim a_{i:\delta} \ldots a_\delta a_\beta$, and $a_{m:\zeta} \ldots a_\zeta a_\gamma \models [\theta_{\alpha l})$,

then define

1. $a_{i:\delta} \ldots a_\delta a_\beta \models [\theta_{\alpha j})$, and
2. for $1 \leq l \leq h(\delta) - i : a_{(i+l):\delta} \ldots a_\delta a_\beta \models [\theta_{\gamma s})$, where $[\theta_{\gamma s}) = ([\theta_{\alpha j})_{[-l]})^\dagger$.

If $wgt(\alpha j) > 0$, then extend the equivalence relation $\sim$ just so that $a_\delta \sim a_\beta$. Then extend this definition of the second part as well as the definition of the first part just so it is closed under $\sim$. The definition ensures that conditions (1)–(3) are satisfied, and closure under $\sim$ ensures that condition (4) is satisfied. It remains to show that condition (5) is satisfied. Suppose that $a_{(i+l):\delta} \ldots a_\delta a_\beta \models [\theta_{\gamma s})$, where $[\theta_{\gamma s}) = ([\theta_{\alpha j})_{[-l]})^\dagger$, and further that $a_{(i+l):\delta} \ldots a_\delta a_\beta \models [\theta_{\zeta t})$. It must be shown that $\zeta t = \gamma s$. According to the definition of the second part of the mapping, there must exist μ, ν, ρ, r, m, and n such that $a_{(m+n):\mu} \ldots a_\mu a_\nu \models [\theta_{\zeta t})$, where $[\theta_{\zeta t}) = ([\theta_{\rho r})_{[-n]})^\dagger$, and $a_{(m+n):\mu} \ldots a_\mu a_\nu \sim a_{(i+l):\delta} \ldots a_\delta a_\beta$. The reasoning relative to the first part of the mapping yields $([\theta_\rho)_{[-n]})^\dagger = ([\theta_\alpha)_{[-l]})^\dagger$. That is, $[\theta_\zeta) = [\theta_\gamma)$. Since $\mathcal{T}_L$ is simple, $\zeta = \gamma$. But $a_\nu \sim a_\beta$, and so by the definition of the second part of the mapping, $t = s$. That is, $\zeta t = \gamma s$. Thus condition (5) is satisfied.

This concludes the second part of the mapping. Now every point $\sim$-equivalent to one of the form $a_{i:\alpha} \ldots a_\alpha a_\beta$ is committed, where α is nonterminal and $1 \leq i \leq h(\alpha)$.

The intent of the first two parts of the mapping is to ensure that at every point $a_{i:\alpha} \ldots a_\alpha$, if $a_{i:\alpha} \ldots a_\alpha \models [\theta_\gamma)$, then: (i) for every $[\theta_{\gamma j})$ there is some a_β such that $a_{i:\alpha} \ldots a_\alpha a_\beta \models [\theta_{\gamma j})$, and (ii) for every a_β, there is some $[\theta_{\gamma j})$ such that $a_{i:\alpha} \ldots a_\alpha a_\beta \models [\theta_{\gamma j})$.

The third and final part of the mapping is now defined, ordered by height. Let $h(\alpha) = k > 1$. The interpretation of the $[\theta_{\alpha j})$ is extended as follows. Let $a_{\beta_1} \ldots a_{\beta_k}$ be a nonstandard point such that $a_{\beta_1} \ldots a_{\beta_k} \models [\theta_\alpha)$. For each $\beta \in \mathcal{T}$,

1. if $a_\beta \sim a_{\beta k}$ and $wgt(\alpha j) > 0$, then define $a_{\beta_1} \ldots a_{\beta_k} a_\beta \models [\theta_{\alpha j})$;
2. if $\neg(a_\beta \sim a_{\beta k})$ and $wgt(\alpha j) = 0$, then
   (a) if $a_{1:\alpha} \ldots a_\alpha a_\beta \models [\theta_{\alpha j})$, then define $a_{\beta_1} \ldots a_{\beta_k} a_\beta \models [\theta_{\alpha j})$;
   (b) if $a_{1:\alpha} \ldots a_\alpha a_{\beta k} \models [\theta_{\alpha j})$, then define $a_{\beta_1} \ldots a_{\beta_k} a_\alpha \models [\theta_{\alpha j})$.

This definition is extended just so that it is $\sim$-closed. Satisfaction of conditions (1)–(3) is inherited from the first and second parts. Since the definition is $\sim$-closed, condition (4) is satisfied. That condition (5) is satisfied is easy to prove by an inductive argument based on the order of the definition.

The intent is that the third part inherit from the first and second parts the property that for every $[\theta_{\alpha j})$ there is some a_β such that $a_{\beta_1} \ldots a_{\beta_k} a_\beta \models [\theta_{\alpha j})$, and also that for every a_β, there is some $[\theta_{\alpha j})$ such that $a_{\beta_1} \ldots a_{\beta_k} a_\beta \models [\theta_{\alpha j})$. Following this final part, every point is committed. This concludes the definition of the mapping.

It remains to show that this interpretation has the property claimed for it, viz., if α is nonterminal at height k, and $a_{\beta_1} \ldots a_{\beta_k} \models [\theta_\alpha)$, then

1. for $1 \leq j \leq w(\alpha) : \exists a_\beta \in \mathcal{D} : a_{\beta_1} \ldots a_{\beta_k} a_\beta \models [\theta_{\alpha j})$, and
2. $\forall a_\beta \in \mathcal{D} : a_{\beta_1} \ldots a_{\beta_k} a_\beta \models [\theta_{\alpha 1}) \vee \cdots \vee [\theta_{\alpha w(\alpha)})$.

The proof is by induction on k.

For the basis step, $k = 0$. By the first part of the definition, for $1 \leq j \leq w(\varepsilon)$, $a_j \models [\theta_j)$. Therefore, item (1) of Claim 7.3 holds. Since $\mathcal{T}_L$ is not trivially inconsistent, for all $\beta \in \mathcal{T}$, there is some j such that $[\theta_j) = ([\theta_\beta)_{[-h(\beta)+1]})^\dagger$. Hence by the first part of the definition of the mapping, $a_\beta \models [\theta_j)$. Thus item (2) of Claim 7.3 holds.

For the induction step, $k > 0$. The proof is subdivided into three cases.

Case 1: $a_{\beta_1} \ldots a_{\beta_k} \sim a_{1:\alpha} \ldots a_\alpha$. By the first part of the definition, for $1 \leq j \leq w(\alpha)$,

$$a_{1:\alpha} \ldots a_\alpha a_{\alpha j} \models [\theta_{\alpha j}).$$

Therefore, item (1) of Claim 7.3 holds. From the definition of the second part of the mapping, if

$$a_{2:\alpha} \dots a_\alpha a_\beta \models ([\theta_{\alpha j})_{[-1]})^\dagger,$$

then either already

$$a_{1:\alpha} \dots a_\alpha a_\beta \models [\theta_{\alpha l}),$$

for some l such that $1 \leq l \leq w(\alpha)$, or we define

$$a_{1:\alpha} \dots a_\alpha a_\beta \models [\theta_{\alpha j}).$$

By the induction hypothesis, item (2) holds for $a_{2:\alpha} \dots a_\alpha \models ([\theta_\alpha)_{[-1]})^\dagger$, and hence item (2) also holds for $a_{1:\alpha} \dots a_\alpha \models [\theta_\alpha)$.

Case 2: $a_{\beta_1} \dots a_{\beta_k} \sim a_{i:\delta} \dots a_\delta$ for some $\delta \in \mathcal{T}$. From Case 1, Claim 7.3 holds for $a_{1:\delta} \dots a_\delta \models [\theta_\delta)$. Since $\mathcal{T}_L$ is not trivially inconsistent, there exists $\gamma \in \mathcal{T}$ such that

1. $[\theta_\gamma) = ([\theta_\delta)_{[-i+1]})^\dagger$, and
2. $\{[\theta_{\gamma j}) : 1 \leq j \leq w(\gamma)\} = \{([\theta_{\delta j})_{[-i+1]})^\dagger : 1 \leq j \leq w(\delta)\}$.

Since $\mathcal{T}_L$ is simple, γ is unique. Hence $\gamma = \alpha$. By the first and second parts of the definition, therefore, Claim 7.3 holds for $a_{i:\delta} \dots a_\delta \models \theta_\alpha$ also.

Case 3: $a_{\beta_1} \dots a_{\beta_k}$ is nonstandard. Claim 7.3 follows from the third part of the definition and Case 1.

In every case, then, Claim 7.3 holds. This concludes the proof of Claim 7.3. □

Proof of Claim 7.4: This proof is by induction on the depth $d = h - k$, where k is the height of $\alpha \in \mathcal{T}$. The induction hypothesis is that Claim 7.4 holds for all elements with depth $< d$.

For the basis step, $d = 0$, θ_α is at height h. Here $(\theta_\alpha] = \theta_\alpha$ by definition, and so the induction hypothesis is trivially true.

For the induction step, $d > 0$, θ_α is at height $k = h - d$. Suppose $a_{\beta_1} \dots a_{\beta_k} \models [\theta_\alpha)$. Since I is assumed to have the property of Claim 7.3,

1. for $1 \leq j \leq w(\alpha) : \exists a_\beta \in \mathcal{D} : a_{\beta_1} \dots a_{\beta_k} a_\beta \models [\theta_{\alpha j})$, and
2. $\forall a_\beta \in \mathcal{D} : a_{\beta_1} \dots a_{\beta_k} a_\beta \models [\theta_{\alpha 1}) \vee \dots \vee [\theta_{\alpha w(\alpha)})$.

By the induction hypothesis, if $a_{\beta_1} \dots a_{\beta_k} a_\beta \models [\theta_{\alpha j})$, then $a_{\beta_1} \dots a_{\beta_k} a_\beta \models (\theta_{\alpha j}]$. Therefore,

1. for $1 \leq j \leq w(\alpha) : \exists a_\beta \in \mathcal{D} : a_{\beta_1} \dots a_{\beta_k} a_\beta \models (\theta_{\alpha j}]$, and
2. $\forall a_\beta \in \mathcal{D} : a_{\beta_1} \dots a_{\beta_k} a_\beta \models (\theta_{\alpha 1}] \vee \dots \vee (\theta_{\alpha w(\alpha)}]$.

Thus $a_{\beta_1} \dots a_{\beta_k} \models (\theta_\alpha]$, and the induction hypothesis holds at height k. This concludes the proof of Claim 7.4, and of the theorem. □

□

If φ is a fluted formula, Theorem 4.2 states that φ is equivalent to the disjunction of its constituents. Moreover, the proof of Theorem 4.2 provides an effective method of transforming φ into the disjunction of its constituents. Obviously φ is satisfiable iff one of its constituents is satisfiable. Theorem 7.2 states that a constituent is satisfiable iff it is not trivially inconsistent. Trivial inconsistency can be decided by a finite number of tests on the syntax of the constituent. Theorems 4.2 and 7.2 therefore yield the following conclusion.

Theorem 7.5 *The satisfiability of a fluted formula with the identity relation is decidable.*

8 Discussion The sublogics lying between fluted logic (FL) and first-order logic with identity (FOLI) can be represented by a lattice isomorphic to the Boolean lattice with five generators. It is wellknown that the upper bound (FOLI) lies in the undecidable region. In [7] the lower bound (FL) is shown to lie in the decidable region. This paper is part of a larger effort to establish the exact boundary between decidable and undecidable in the interior of the lattice. It shows that fluted logic extended by adding the identity relation is decidable. There are two corollaries to this result.

First, addition of the reflection functor to fluted logic with identity is conservative, hence decidable. The reflection functor *ref* can be defined in FL with identity as follows. If θ is a fluted formula over X_{k+1}, then $(ref\ \theta)$ is a fluted formula over X_k. Let Q be a predicate symbol of arity k having no previous occurrence, and let

$$\varphi = \forall x_1 \dots \forall x_k (Qx_1 \dots x_k \leftrightarrow \exists x_{k+1}(Ix_k x_{k+1} \wedge \theta)).$$

Then any interpretation I satisfying φ will interpret Q and θ such that

$$a_1 \dots a_k \in (ref\ \theta)^I \text{ iff } a_1 \dots a_k a_k \in \theta^I \text{ iff } a_1 \dots a_k \in Q^I.$$

Thus Q, as defined by the fluted sentence φ, is logically equivalent to $(ref\ \theta)$. Therefore FL with identity and the reflection functor is a conservative extension of FL with identity, hence decidable. It follows that FL with the reflection functor is also decidable.

Second, addition of unary singular predicates to fluted logic with identity is conservative, hence decidable. Unary singular predicates can be defined in fluted logic with identity as follows. Let

$$\varphi = \exists x_1 (Sx_1 \wedge \forall x_2 (Sx_2 \rightarrow Ix_1 x_2)).$$

Then in any interpretation satisfying φ, the interpretation of S is a singleton set. It follows that FL with unary singular predicates is also decidable.

Anaphoric pronouns can be represented by unary singular predicates. As observed in Section 1, anaphoric pronouns play an important role in natural logic, similar to that of Skolem constants in clausal logic. But this topic will not be pursued further here. Rather it will be deferred to a subsequent paper.

REFERENCES

[1] Andrews, P. B., *An Introduction to Mathematical Logic and Type Theory*, Academic Press, Orlando, 1986. Zbl 0617.03001 MR 88g:03001 2

[2] Enderton, H. B., *A Mathematical Introduction to Logic*, Academic Press, New York, 1972. Zbl 0298.02002 MR 49:2239 3

[3] Hintikka, J., "Surface information and depth information," pp. 263–297 in *Information and Inference*, edited by J. Hintikka and P. Suppes, Reidel, Dordrecht, 1970. Zbl 0265.02005 MR 43:4625 5

[4] Hintikka, J., *Logic, Language-Games and Information*, Clarendon Press, Oxford, 1973. Zbl 0253.02005 MR 54:4911 5

[5] Noah, A., "Predicate-functors and the limits of decidability in logic," *Notre Dame Journal of Formal Logic*, vol. 21 (1980), pp. 701–707. Zbl 0416.03014 MR 82b:03042 1

[6] Purdy, W. C., "A variable-free logic for anaphora," pp. 41–70 in *Patrick Suppes: Scientific Philosopher*, edited by P. Humphreys, vol. 3, Kluwer Academic Publishers, Dordrecht, 1994. MR 96e:03036 1

[7] Purdy, W. C., "Fluted formulas and the limits of decidability," *The Journal of Symbolic Logic*, forthcoming. Zbl 0858.03012 MR 97b:03019 1, 1, 6, 8

[8] Quine, W. V., "Variables explained away," *Proceedings of the American Philosophical Society*, vol. 104 (1960), pp. 343–347. 1

[9] Quine, W. V., "On the limits of decision," *Proceedings of the 14th International Congress of Philosophy*, vol. III, University of Vienna, 1969. 1

[10] Quine, W. V., *The Ways of Paradox and Other Essays*, Enlarged Edition, Harvard University Press, Cambridge, 1976. 1

[11] Quine, W. V., "Predicate functors revisited," *The Journal of Symbolic Logic*, vol. 46 (1981), pp. 649–652. Zbl 0472.03007 MR 83e:03028 1

[12] Quine, W. V., *Methods of Logic*, Fourth Edition, Harvard University Press, Cambridge, 1982. 1

[13] Rantala, V., "Constituents," pp. 43–76 in *Jaakko Hintikka*, edited by R. J. Bogdan, Reidel, Dordrecht, 1987. 4, 4, 4, 4, 4, 4

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