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Field on the Notion of Consistency

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Abstract Field's claim that we have a notion of consistency which is neither model-theoretic nor proof-theoretic but primitive, is examined and criticized. His argument is compared to similar examinations by Kreisel and Etchemendy, and Etchemendy's distinction between interpretational and representational semantics is employed to reveal the flaw in Field's argument.

In [4], Field argued that we have at our disposal an informal notion of consistency that is different from either the model-theoretic or the proof-theoretic notion of consistency.[1] Field contends that this notion is a primitive modal notion that is not to be defined or analyzed in terms of mathematical entities such as models and proofs, and that the nominalist thus can freely employ this notion. In general it is very difficult to determine the plausibility of the claim if one says that a certain thing not amenable to analysis exists, and Field's claim is no exception. But I think there is good reason to believe that Field's claim for the primitiveness of the notion is not justified. In this paper, I would like to show this by comparing Field's argument to similar examinations by Kreisel and Etchemendy.

Field begins his argument with a criticism of the model-theoretic definition of consistency:

A is logically consistent if and only if there is a model of A.

This definition is superior to the proof-theoretic definition of consistency, because for the proof-theoretic definition a particular formal system has to be chosen, but it is rather arbitrary which formal system we choose. Field points out, however, that there is something unnatural in the model-theoretic definition, too. For instance, suppose A is the conjunction of all the facts about sets which are statable in the language of set theory. Then since A is true, it must be consistent. But it is not at all apparent that there is a model of A, because the domain of the most natural model would be the set of all sets, yet there is no such set. Actually there is a less natural model if A is stated in a language of first-order logic, but that's only an "accident of first-order

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logic" and it is far from evident that the above biconditional ought to be true. A much more plausible picture, Field contends, is that the meaning of consistency is captured by the following two intuitively plausible principles, the *model-theoretic possibility principle* (MTP) and the *modal soundness principle* (MS):

(MTP) If there is a model of A, then A is consistent.

(MS) If A is consistent, then there is no refutation of A (i.e., proof of $\neg A$) in any intuitively sound formal system F.

According to Field, they together govern our notion of consistency; thus, consistency is neither a purely model-theoretic nor a purely proof-theoretic notion. It is a *primitive* notion. On this view, the significance of the completeness theorem for formal system F, which asserts that if there is no refutation of A in F then there is a model of A, is to tie the loose ends of the circle so that the three notions, viz., primitive consistency, the existence of a model, and the nonexistence of a refutation, extensionally coincide.

What exactly does Field mean when he says that the notion of consistency is primitive? Field explains:

> When I say that we should regard the notion of consistency as primitive, I don't mean that there is nothing we can do to help clarify its meaning. The claim that consistency should be regarded as a primitive notion does involve the claim that we can't clarify its meaning *by giving a definition of it in more basic terms*. Similarly, logical notions like 'and', 'not', and 'there is' are primitive. We don't learn these notions by defining them in more basic terms. Rather, we learn them by learning to use them in accordance with certain rules; and we clarify their meaning by unearthing the rules that govern them. The same holds for consistency and implication. I claim: there are "procedural rules" governing the use of these terms, and it is these rules that give the terms the meaning they have, not some alleged definitions, whether in terms of models or of proofs or of substitution instances. ([4], p. 5)

As for the procedural rules, there are two kinds: " '*C*-rules' for showing that something *is* consistent, or that one thing *does not* imply another, and '*I*-rules' for showing that something *isn't* consistent, or that one thing *does* imply another" ([4], p. 6).

I think that this is a very persuasive argument, and that it does establish the fact that we have an informal notion of consistency. But I do not think it succeeds in showing that the notion in question is primitive. It only shows that the informal notion is different from either the model-theoretic or the proof-theoretic notion, but leaves open the possibility that it can be defined in other terms. Our subsequent examination of Kreisel's claim on the same subject will indeed give support to this point.

Before we get into that, however, a comment on the identity of mathematical notions may be in order. One problem of Field's argument is that Field never makes clear the identity condition of mathematical (or logical) notions; that is, he never says when two possibly distinct mathematical notions are indeed identical. A natural candidate for the condition is *necessary coextensiveness*: two notions are indeed identical just in case they are necessarily coextensive. However, this is too coarse a condition. If this were the condition, then, for instance, given the completeness of the formal system F, the three aforementioned notions of consistency would prove to be one and the same notion, not just three notions with the same extension, since all mathematical truths are necessary truths. This would destroy the whole basis of Field's argument. Then what is the further condition? Field does not answer this question and seems to simply rely on our intuition on this matter. The further condition seems to have something to do with *self-evidence* or *triviality* of identity: crudely put, if two notions are indeed identical, it must be self-evident or trivially true that they are. I suppose that Field holds something like this as part of the identity condition of mathematical notions.

Having said this, I now examine Kreisel's notion of consistency. Although Field attributes the basic idea of his view to Kreisel [6] and never mentions any difference between their views, Kreisel's notion of informal consistency is in fact quite different from Field's. Kreisel discusses his view in terms of validity rather than consistency, and distinguishes the informal notion of validity *Val* from the formal notion V as follows:

> The intuitive meaning of *Val* differs from that of V in one particular: $V\alpha$ (merely) asserts that α is true in all structures in the cumulative hierarchy, i.e., in all sets in the precise sense of *set* above, while $Val\alpha$ asserts that α is true in *all* structures .... (Hintikka [5], p. 90)

Analogously, a statement is consistent in the informal sense if and only if it is true in at least one of the structures. Note that this is a *definition* of consistency, a definition in terms of mathematical entities, to boot. So Kreisel's notion of consistency is *not* primitive in Field's sense, and if our informal notion of consistency is as Kreisel describes, the nominalist cannot employ the notion. Moreover, Kreisel's notion resolves in the most natural manner the problem with the model-theoretic notion of consistency cited by Field: the domain of the model in question would be a structure that is not a set. So we cannot but wonder what would justify the claim that our informal notion of consistency is not Kreisel's but Field's.

One may think that such a justification should come from the plausibility of MS. One may argue that although MS should simply follow from the meaning of consistency, it will not if consistency is understood à la Kreisel. It is self-evident from the definitions of V and *Val* that $Val(A) \to V(A)$. By contraposition, it is self-evident that Kreisel's notion of consistency satisfies MTP: if A is true in a model (i.e., a structure in the cumulative hierarchy), then A is Kreisel-consistent (i.e., true in a structure). But it is not at all evident, one may claim, that there cannot be a situation in which A is true in a structure but there is also a proof of $\neg A$ in a sound formal system F. Kreisel's notion is semantic; so there cannot be an obvious relation to the existence or nonexistence of proofs.

I think that whether this argument is persuasive or not depends on how we understand the notion of *necessity* established by the informal proofs. We can see this point clearly by considering Etchemendy's further examination of the issue (Etchemendy [2], chap. 11). Etchemendy claims that we have an intuitive notion of logical truth, LTr, which is distinct from Kreisel's Val. The difference is, in Etchemendy's terms, that Val is an *interpretational* notion while LTr is *representational*: roughly, $Val(A)$ is true if A is true no matter how the nonlogical terms in A are interpreted, while $LTr(A)$ is true if A is true no matter what the state of affairs is (or no matter what possible world we are in). For the former, we keep the world fixed, and consider different interpretations of the words; for the latter, we keep the meaning of the words fixed, and consider different states of affairs. Yet another notation Etchemendy introduces is $D(A)$, which reads "there is a proof of A in a sound formal system F."

Etchemendy maintains that while both

$$D(A) \to LTr(A)$$

and

$$Val(A) \to V(A)$$

are trivially true, the relation between $Val(A)$ and $LTr(A)$, that is, the relation between interpretational and representational semantics, is not a trivial matter. And Etchemendy claims that although

$$D(A) \to Val(A)$$

is indeed true, its truth does not come trivially from the meaning of Val.

Let us reexamine our previous argument in this light. Suppose we determine by a routine examination that every inference rule—counting axioms as rules without premises—of the formal system F is "truth-preserving" (or "sound"); that is, that it is necessary that if the premises Γ of the rule are true, so is the conclusion C. But what exactly does "necessary" mean here? It can be understood in terms of either *interpretation* or *representation*: it can mean "no matter how the nonlogical terms in Γ and C are interpreted," or "no matter what the state of affairs is (or no matter what possible world we are in)."[2] If we employ the first, interpretational understanding, then if there is a proof of $\neg A$ in F, $\neg A$ is true in all structures; so A is true in no structure. Thus, it is trivially true that A is Kreisel-inconsistent. But if we employ the second, representational understanding, then the Kreisel-inconsistency of A does not follow trivially, as it was argued above, because it is not at all obvious that there cannot be a statement A that is false no matter what the actual state of affairs is, but can become true if it is given a different interpretation.

I do not intend to decide here which of the two[3] is our actual intuitive understanding of truth-preservation (or soundness). I am not as confident as Etchemendy that the second is. To me, our intuitive understanding does not seem so clear-cut, and seems to fluctuate between the two. However, I think I still can make the following trichotomous claims against Field's view. First, if the interpretational understanding is our actual understanding, then Kreisel-consistency trivially satisfies both MTP and MS; so there is at least as good reason to believe that Kreisel-consistency is our informal notion of consistency. In fact, I think there is reason to prefer Kreisel's notion to Field's: Field's notion is totally *obscure*; he does not give any substantial explanation of what his notion of consistency is like. Kreisel-consistency also fits well with our procedures for determining that something is or is not consistent, so that does not favor Field's consistency over Kreisel's. In fact, for lack of explanation, it seems quite natural for us to understand Field's modal notion '$\diamond_L$' ("it is consistent that") à la Kreisel. In that case, '$\diamond_L$' would not be a primitive modal notion, but a notion to be analyzed in terms of mathematical entities. So the nominalist cannot employ the notion.

Second, if Etchemendy is correct and the representational understanding is our actual understanding of truth-preservation, then the view that we indeed have two distinct informal notions of implication, Val and LTr, and two distinct notions of consistency, say, Kreisel- and Etchemendy-consistency, is better than Field's view that we have only one notion. The problem for Field's view is essentially the same as the problem for Kreisel-consistency on the representational understanding of truth-preservation: Field's notion of consistency is supposed to satisfy both MTP and MS trivially, simply by virtue of its meaning, as MTP and MS govern the meaning of consistency. But then, by transitivity, the claim that if there is a model of A, then there is no refutation of A in F, must be trivially true. However, it is not at all evident that there cannot be a situation in which there is a model of A and there is also a proof of $\neg A$. On the representational understanding, the latter implies that A is false no matter what the state of affairs is, but it is not obvious that A cannot have a model in such a situation. So there cannot be one notion that satisfies both MTP and MS trivially.[4] It is more natural to assume that we indeed have two distinct notions of consistency, even at the intuitive level, which are connected to the C- and I-rules respectively. In that case, Field's '$\diamond_L$' would be taken as equivocal between Kreisel- and Etchemendy-consistency. Etchemendy-consistency is defined also in terms of mathematical (or if not mathematical, abstract) entities, viz., states of affairs, possible worlds, and so on. So the nominalist cannot use this notion either.

Finally, one may say that we have only one notion of consistency and it is Etchemendy-consistency. I think this is a fairly plausible view, because it conforms to our common practice: to claim that statement A is consistent, we often try to come up with a situation in which A is true. And Etchemendy-consistency also handles the problem with the model-theoretic notion of consistency quite nicely: if the statement in question is true, then of course it is consistent (i.e., true in at least one possible world). For this view, it is natural to take the representational understanding of truth-preservation; then MS will hold trivially. However, MTP does not hold trivially, because there is no guarantee that if statement A has a model (i.e., can be interpreted as true), it (without a change of meaning) will be true in at least one possible world. In fact, this conditional is probably not even true, because an analytically but not logically false statement A, such as "Some bachelors are married," has a model but is not true in any possible world. So if this view is correct, we have to give up the basic principle MTP, on which Field's argument is based.[5]

Therefore, no matter how we understand the relevant notions, we have good reason to doubt that our informal notion of consistency is primitive. Or at least, Field failed to show that it is. I must admit that the above argument stands on a somewhat delicate ground, in two respects: one, the identity condition of mathematical notions is left undetermined, and I also have relied somewhat on our intuition; and two, we do not *really* know what Field has in mind when he talks about the primitive notion of consistency. But the suspicion has been expressed (in, e.g., Shapiro [7]) that Field might not have avoided but only have suppressed mathematical entities when he claimed that the notion of consistency was primitive. The above examination goes toward confirming this suspicion.[6]

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NOTES

1. See also Field [3].
2. Yet another understanding, not discussed here, is substitutional. The conclusions of my

present examination remain true even if we replace the term “the representational understanding” with “the substitutional understanding” in the following.

3. Or three, if you count the substitutional understanding.

4. From Field’s account it is not entirely clear what the relation is between the MTP and MS principles, on the one hand, and the C- and I-rules, on the other, and there is a possibility that Field would say that it is not the MTP and MS principles but the C- and I-rules that determine the meaning of consistency. But that would not essentially affect the present argument. In that case, the question to be asked would be: what sort of procedures are those sanctioned by the I-rules? If they are the procedures to determine that A cannot have a model, then what I said in the last paragraph will apply. If they are not, then it will not be a trivial matter that the C- and I-rules do not give conflicting answers to the question of A’s consistency.

5. In the second half of his paper, Field criticizes MTP and MS from the nominalist point of view and replaces them with their nominalist counterparts. But the latter are also based on our intuitive understanding of consistency (that is, the formulations of them involve ‘$\diamond_L$’). Thus, if Field’s argument in the first half of the paper fails, so does the one in the second half.

6. It also should substantiate the opening remarks of Akiba [1].

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