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Propositional Quantification in the Topological Semantics for **S4**

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Abstract Fine and Kripke extended **S5**, **S4**, **S4.2** and such to produce propositionally quantified systems **S5**π**+**, **S4**π**+**, **S4.2**π**+**: given a Kripke frame, the quantifiers range over all the sets of possible worlds. **S5**π**+** is decidable and, as Fine and Kripke showed, many of the other systems are recursively isomorphic to second-order logic. In the present paper I consider the propositionally quantified system that arises from the *topological* semantics for **S4**, rather than from the Kripke semantics. The topological system, which I dub **S4**π**t**, is strictly weaker than its Kripkean counterpart. I prove here that second-order arithmetic can be recursively embedded in **S4**π**t**. In the course of the investigation, I also sketch a proof of Fine's and Kripke's results that the Kripkean system **S4**π**+** is recursively isomorphic to second-order logic.

1 Introduction One way to extend a propositional logic to a language with propositional quantifiers is to begin with a semantics for the logic; extract from the semantics a notion of a proposition; and interpret the quantifiers as ranging over the propositions. Thus, Fine [4] extends the Kripke semantics for modal logics to propositionally quantified systems **S5**π**+**, **S4**π**+**, **S4.2**π**+**, and such: given a Kripke frame, the quantifiers range over all sets of possible worlds. **S5**π**+** is decidable ([4] and Kaplan [14]). In later unpublished work, Fine and Kripke independently showed that **S4**π**+**, **S4.2**π**+**, **K4**π**+**, **T**π**+**, **K**π**+**, and **B**π**+** and others are recursively isomorphic to full second-order classical logic.

(Fine informs me that he later proved this stronger result. Kripke informs me that he too proved this stronger result in the early 1970s. A proof of this result occurs in Kaminski and Tiomkin [13], who use techniques similar to those used in Kremer [16] and to those used below. These techniques do not apply to **S4.3**π**+**. But according to Kaminski and Tiomkin, work of Gurevich and Shelah ([9], [10], and [39]) implies that second-order arithmetic is interpretable in **S4.3**π**+** and furthermore that, under

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certain set-theoretic assumptions, **S4.3** π**+** is recursively isomorphic to second-order logic.)

Kripke's semantics for modal logic is the most well known, but is predated by *topological* semantics for **S4** (Tsao-Chen [40], McKinsey [22], McKinsey-Tarski [23], [24], [25], and Rasiowa-Sikorski [34]). In the topological semantics, a *model* is a topological space X together with an assignment of a subset of X to each propositional variable. Conjunction is interpreted as intersection on the subsets of X, disjunction as union, negation as complementation; and $\Box$ is interpreted as *topological interior* (int).

The present paper will extend the topological interpretation of **S4** to a propositionally quantified topological system **S4**π**t**: the quantifiers will range over the subsets of topological spaces. **S4**π**t** is strictly weaker than its Kripkean counterpart **S4**π**+**. The main result is that second-order arithmetic can be recursively embedded in **S4**π**t**. In the course of the investigation, I will sketch a proof of Fine's and Kripke's results that **S4**π**+** is recursively isomorphic to second-order logic. I include this proof since proving of the topological result will rely on the ideas in it, as well as additional ideas specific to the topological framework. I do not know whether **S4**π**t** is recursively isomorphic to second-order logic, but I conjecture that it is.

Just as there are both Kripke and topological semantics for **S4**, there are both Kripke and topological semantics for the intuitionistic logic **H**. In Kremer [17], I began with the Kripke semantics for **H**, and defined a Kripkean propositionally quantified intuitionistic system, **H**π**+**, analogous to **S4**π**+**. I showed that **H**π**+** is recursively isomorphic to second-order logic. The proof is similar to that given below for **S4**π**+**, but additional bells and whistles are needed in the intuitionsitic context, given the expressive weakness of the intuitionistic language.

One can also define a topological propositionally quantified intuitionistic system, **H**π**t**. Given the details of the topological semantics for **H**, the propositional quantifiers range over the open subsets of a topological space in the intuitionistic context. I have recently discovered a proof that second-order arithmetic can be embedded in **H**π**t**. The proof involves a nontrivial extension of the topological ideas in the current paper and the intuitionistic ideas in [17].

Troelstra [41] and Polacik [30] and [31] have already given a topological interpretation of propositional quantifiers in intuitionistic logic, but they restrict their attention to this interpretation's behavior in Cantor space, CS. Note that the propositionally quantified intuitionistic theory of CS is decidable: it can be encoded in S2S, the monadic second-order theory of two successors, proved decidable by Rabin [32]. For details on reproducing the topology of CS in S2S, see Rabin [33].

Semantic approaches are not the only ways to enrich nonclassical propositional logics with propositional quantifiers. Axiomatic approaches have been considered, extending propositional logics by adding new axioms or rules of inference governing the quantifiers. (See Kripke [18], Bull [1], [4], Murungi [27], Dishkant [3], Ghilardi-Zawadowski [8] as well as the classic Lewis-Langford [20] on modal logic; and Gabbay [6] and [7], Löb [21], Sobolev [38], Kreisel [15], Scedrov [35], and Pitts [29] on intuitionistic logic.) Axiomatic approaches are closely related to semi-semantic substitutional interpretations of the quantifiers. (See, for example, the modal systems of Gabbay [5].) Axiomatic systems can often be given a semantics by beginning with a propositional semantics and adapting Henkin's [11] techniques for *secondary* modeling of axiomatizable fragments of second-order logic. (See [1], [4] and [7].)

***2 The modal systems* S4πt *and* S4π+** Our language has a countable set $PV = \{p_1, \ldots, p_n, \ldots\}$ of propositional variables; connectives &, $\neg$ and $\Box$; and a propositional quantifier $\forall$. We use $p, q, \ldots$ for members of PV and $A, B, \ldots$ for formulas. We assume that $\vee$, $\Diamond$, $-\!\!\ni$ and $\exists$ are defined in the usual manner.

Definition 2.1 Given a topological space X, a *proposition* is a subset of X. A *topological model* is an ordered pair $M = (X, V)$ where X is a topological space and where V (the *valuation function*) assigns a proposition to each $p \in PV$. Given a model M, a proposition P, and a propositional variable p, $M[P/p]$ is the model just like M except that it assigns the proposition P to the propositional variable p.

Definition 2.2 A *Kripke frame* is a 3-tuple $F = (W, 0, \leq)$ where W is a nonempty set; $0 \in W$; and $\leq$ is a reflexive and transitive relation on W. Given a Kripke frame, a *proposition* is a subset of W. A Kripke model is a pair $M = (F, V)$ where $F = (W, 0, \leq)$ is a Kripke frame and V assigns a proposition to each $p \in PV$. $M[P/p]$ is defined as above.

Definition 2.3 Given a topological model $M = (X, V)$ and a formula A, we define $M(A)$, the *proposition assigned by* M *to* A : $M(p) = V(p)$; $M(A \,\&\, B) = M(A) \cap M(B)$; $M(\neg A) = X - M(A)$; $M(\Box A) = \mathrm{int}(M(A))$, the topological interior of $M(A)$; $M(\forall p A) = \cap\{M[P/p](A) : P \subseteq X\}$.

Definition 2.4 Given a Kripke model $M = ((W, 0, \leq), V)$ and a formula A, we define $M(A)$, the *proposition assigned by* M *to* A : $M(p) = V(p)$; $M(\neg A) = W - M(A)$; $M(A \,\&\, B) = M(A) \cap M(B)$; $M(\Box A) = \{w \in W : \forall w'(w \leq w' \implies w' \in M(A))\}$; $M(\forall p A) = \cap\{M[P/p](A) : P \subseteq W\}$.

Definition 2.5 Suppose that $M = (X, V)$ is a topological model and A is a formula. M validates $A (M \models A)$ if and only if $M(A) = X$. X validates $A (X \models A)$ if and only if $M \models A$ for every model $M = (X, V)$. A is valid ($\models A$) if and only if A is validated by every topological model (or, equivalently, by every topological space).

Definition 2.6 Suppose that $F = (W, 0, \leq)$ is a Kripke frame, that $M = (F, V)$ is a Kripke model and that A is a formula. M validates $A (M \models A)$ if and only if $0 \in M(A)$. F validates $A (F \models A)$ if and only if $M \models A$ for every Kripke model $M = (F, V)$. A is valid in the Kripkean sense ($\models_K A$) if and only if A is validated by every Kripke model (or, equivalently, by every Kripke frame).

Theorem 2.7 (McKinsey [22]) *If A is a quantifier-free formula then $A \in$* **S4** *if and only if $\models A$.*

Theorem 2.8 (Kripke [19]) *If A is a quantifier-free formula then $A \in$* **S4** *if and only if $\models_K A$.*

Definition 2.9 **S4πt** $=_{\mathrm{df}} \{A : \models A\}$.

Definition 2.10 **S4**π**+** $=_{\text{df}} \{A : \models_K A\}$.

Theorem 2.11 (Main result) *Second-order arithmetic can be recursively embedded in* **S4**π**t**.

Theorem 2.12 (Fine, Kripke) **S4**π**+** *is recursively isomorphic to second-order logic*.

For the proof of Theorem 2.12 see Section 3, and for the proof of Theorem 2.11 see Section 4 and Section 5 below.

Theorem 2.13 **S4**π**t** $\subsetneq$ **S4**π**+**.

Proof: Suppose that A is invalidated by the Kripke model $M = ((W, 0, \leq), V)$. Let M_T be the topological model (W, V), where a subset of W is open if and only if it is closed under $\leq$. Note that, for every formula B, $M_T(B) = M(B)$. So A is invalidated by M_T. This shows that **S4**π**t** $\subseteq$ **S4**π**+**. Example 6.3, in Section 6 below, provides a topological space that does not validate the Barcan formula $(\forall q \Box B \supset \Box \forall q B)$, where B is the formula $\Box\Diamond q \vee \neg q \vee r$. And Example 6.4, provides a space that does not validate the Barcan formula $(\forall q \Box B \supset \Box \forall q B)$, where B is the formula $\neg q \vee \Box\Diamond q$. But every Barcan formula is validated by every Kripke model, since the intersection of arbitrary sets closed under $\leq$ is also closed under $\leq$. This shows that **S4**π**t** $\neq$ **S4**π**+**. $\Box$

3 Theorem 2.12: S4π+ is recursively isomorphic to second-order logic Here we sketch a proof that second-order logic can be recursively embedded in the Kripkean system **S4**π**+**. The proof is a simplification of the proof in Kremer [17] for the analogous Kripkean intuitionistic system **H**π**+**. We will rely on an idea from Nerode and Shore [28]: they reproduce unpublished considerations of Rabin and Scott, showing how to code arbitrary n-ary relations by *sib* (symmetric irreflexive binary) relations. So second-order logic is recursively isomorphic to second-order logic with second-order quantification restricted to sib relations. Let 2-SIB^2 be the second-order theory of domains with two or more elements, with all second-order quantification over sib relations. Then second-order logic is recursively isomorphic to 2-SIB^2. So our job is reduced to encoding 2-SIB^2 in **S4**π**+**.

To effect this encoding, we focus our attention on a particular class of Kripke frames. First we define a simple Kripke frame to be one satisfying the following condition: for every $w \in W, 0 \leq w$. And we define a simple Kripke model to be one whose underlying frame is simple. Note that **S4**π**+** $= \{A : A$ is validated by every simple Kripke model$\}$. So henceforth we assume that all Kripke frames and models are simple. Among simple Kripke frames, we distinguish *3-tiered* frames. Before we define this notion, we introduce the following notation: $w < w'$ if and only if $w \leq w'$ and $w' \not\leq w$. A (simple) Kripke frame is *3-tiered* if and only if (1) if $w \leq w'$ and $w' \leq w$ then $w = w'$; (2) there exists w, w' such that $0 < w < w'$; and (3) for no w, w' and w'' do we have $0 < w < w' < w''$. A Kripke model is 3-tiered if and only if its underlying frame is. Figures 1 to 6 represent sample 3-tiered frames. Precise definitions of tier$_1$, tier$_2$ and tier$_3$ are easy enough to give.

The idea behind our encoding of 2-SIB^2 in **S4**π**+** is this: suppose we begin with a domain of two or more individuals, and we want to quantify over the individuals and

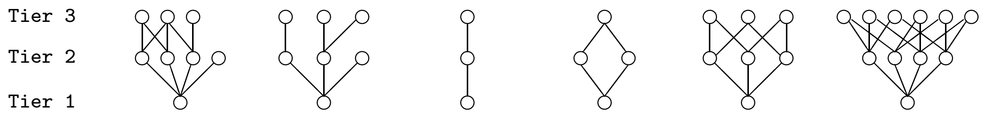

Figure 1 Figure 2 Figure 3 Figure 4 Figure 5 Figure 6

sib relations. We will represent such domains by peculiar 3-tiered frames. We want 3-tiered frames so that the points in $tier_2$ stand in for the *individuals* in the domain; and the points in $tier_3$ stand in for *unordered pairs of distinct individuals* from the domain. The *subsets* of $tier_3$ can then stand in for sib relations on the domain, since a sib relation can be thought of as a set of unordered pairs of distinct individuals. This motivates our definition, given presently, of a special kind of 3-tiered frame, a sib frame. Figures 4, 5, and 6 above represent 3-tiered sib frames representing domains of, respectively, two, three, and four members. Figures 1, 2, and 3 represent less well-behaved 3-tiered frames.

Definition 3.1 A 3-tiered frame is a sib frame if and only if

1. every pair of distinct points in $tier_2$ has a unique upper bound;
2. every point in $tier_3$ is the upper bound of two points in $tier_2$; and
3. no three distinct points in $tier_2$ have an upper bound.

A Kripke model is a sib model if and only if the underlying Kripke frame is a sib frame.

Definition 3.2 **SIB-S4**π**+**$= \{A$: for every 3-tiered sib model M, $M \models A\}$.

Our encoding of 2-SIB^2 in **S4**π**+** will now proceed in two steps. Step 1 is to find a formula *sib* of the propositional language with the following property: for every formula A, $A \in$ SIB-**S4**π**+** if and only if $(sib \supset A) \in$ **S4**π**+**. This shows that SIB-**S4**π**+** can be encoded in **S4**π**+**. Step 2 is to recursively encode 2-SIB^2 in SIB-**S4**π**+**.

For step 1, it suffices for the formula *sib* to express the claim that the model (or frame) under consideration is a sib model (or frame). So the following suffices: for every model M, $M \models sib$ if and only if M is a sib model. We will construct the formula *sib* in stages, keeping the following idea in mind. Given a Kripke model $M = ((W, 0, \leq), V)$, we can think of a formula A as playing two roles: (i) A *names* a subset of W, in particular, $M(A)$; and (ii) A *makes a claim* about the model. For example, $(p \mathrel{-\!\!\ni} q)$ *names* the set $M(p \mathrel{-\!\!\ni} q)$ and $(p \mathrel{-\!\!\ni} q)$ *says that* $M(p) \subseteq M(q)$ since, for every model M, $M \models (p \mathrel{-\!\!\ni} q)$ if and only if $M(p) \subseteq M(q)$. (I appeal to the same considerations in [16] and [17].) Table 1 defines some object language connectives and formulas, and indicates what the definienda say. In particular, Table 1 defines a two-place connective $\in$. If p is a propositional variable and if A and B are formulas, then $(\forall p \in A)B$ is an abbreviation of the formula $\forall p((p \in A) \supset B)$; and $(\exists p \in A)B$ is an abbreviation of the formula $\exists p((p \in A) \,\&\, B)$. Using $\in$, we can mimic quantification over the elements of W by restricting quantification to the singleton subsets of W.

Given the last row of Table 1, step 1 is completed. Although most of the definitions in Table 1 are straightforward, the definitions of the formulas *3-tier* and *sib* are difficult to parse. In the definition of *3-tier*, we are expressing, in the modal object-language, the three conditions placed on 3-tiered frames or models. Similarly, in the

Definiendum	Definiens	What the definiendum says: $M \models$ Definiendum if and only if
T	$\exists pp$	
$(A \mathrel{-\!\!\!\ni} B)$	$\Box(A \supset B)$	$M(A) \subseteq M(B)$
$(A = B)$	$(A \mathrel{-\!\!\!\ni} B)$ & $(B \mathrel{-\!\!\!\ni} A)$	$M(A) = M(B)$
$(A \in B)$	$\Diamond A$ & $(A \mathrel{-\!\!\!\ni} B)$ & $\forall p(\Diamond(A \,\&\, p) \supset (A \mathrel{-\!\!\!\ni} p))$	for some $w \in M(B)$, $M(A) = \{w\}$
$(A \leq B)$	$(A \in T)$ & $(B \in T)$ & $(A \mathrel{-\!\!\!\ni} \Diamond B)$	for some w, w', $M(A) =$ $\{w\}$and $M(B) = \{w'\}$ and $w \leq w'$
$(A < B)$	$(A \leq B)$ & $\neg(B \leq A)$	for some w, w', $M(A) =$ $\{w\}$ and $M(B) = \{w'\}$ and $w < w'$
3-*tier*	$(\forall p \in T)(\forall q \in T)((p \leq q) \,\&\, (q \leq p) \supset$ $(p = q))$ & $(\exists p \in T)(\exists q \in T)(\exists r \in T)$ $(p \,\&\, (p < q) \,\&\, (q < r))$ & $\neg(\exists p \in T)(\exists q \in T)(\exists r \in T)(\exists s \in T)$ $(p \,\&\, (p < q) \,\&\, (q < r) \,\&\, (r < s))$	M is 3-*tiered*
sib	3-*tier* & $\forall p \forall q(\neg p \,\&\, \neg q \,\&\, \exists r(p < r) \,\&\, \exists r(q < r) \supset$ $\exists r((p < r) \,\&\, (q < r) \,\&\, \forall s((p < s) \,\&\, (q < s) \supset (s = r))))$ & $\forall p(\exists q \exists r(q \,\&\, (q < r) \,\&\, (r < p)) \supset$ $\exists q \exists r \exists s(q \,\&\, (q < r) \,\&\, (q < s) \,\&\, (r < p) \,\&\, (s < p)\,\&$ $\forall u((q < u) \,\&\, (u < p) \supset (u = r) \vee (u = s)))$ & $\neg \exists p \exists q \exists r \exists s \exists u(p \,\&\, (p < q) \,\&\, (p < r) \,\&\, (p < s)\,\&$ $(q < u) \,\&\, (r < u) \,\&\, (s < u) \,\&\, \neg(q = r) \,\&\, \neg(q = s)\,\&$ $\neg(r = s))$	M is a sib model

Table 1:

definition of *sib*, we are expressing the various conditions of sib frames.

Now that step 1 is completed, we move to step 2: we want a translation of a second-order language with second-order quantification over sib relations to our propositionally quantified modal language. So we assume that we are working with a second-order classical language with individual variables $x_1, \ldots, x_n, \ldots$; binary relational variables $R_1, \ldots, R_n, \ldots$; parentheses; connectives & and $\neg$; identity, $=$; and first- and second-order universal quantifiers. Shortly we define a recursive 1-1 function, f_1, from second-order formulas to modal formulas. In the definition of f_1, propositional variables with even subscripts stand in for individual variables, and with odd subscripts, binary relational variables. The variable q should be chosen in some systematic way so as not to conflict with quantifiers. Note also that in our definitions of $f_1(\forall x_i A)$ and of $f_1(\forall R_i A)$, we restrict quantification to propositions representing individuals in a classical domain, and sib relations in a classical domain. Here is our definition of f_1:

$$f_1(x_i = x_j) \quad = \quad (p_{2i} = p_{2j})$$

$$
\begin{array}{lcl}
f_1(R_i x_j x_k) & = & (\exists q \in p_{2i-1})((p_{2j} < q) \,\&\, (p_{2k} < q)) \\
f_1(\neg A) & = & \neg f_1(A) \\
f_1(A \,\&\, B) & = & f_1(A) \,\&\, f_1(B) \\
f_1(\forall x_i A) & = & \forall p_{2i}(\neg p_{2i} \,\&\, \exists q(p_{2i} < q) \supset f_1(A)) \\
f_1(\forall R_i A) & = & \forall p_{2i-1}((p_{2i-1} \mathrel{-\!\!\!\ni} \forall q(q \supset \Box q)) \supset f_1(A)).
\end{array}
$$

Note that for any closed second-order formula A, $A \in$ 2-SIB^2 if and only if $f_1(A) \in$ **SIB-S4**π**+**. Now we define a recursive 1-1 function f_2 from second-order formulas to second-order formulas. Suppose that A is a second-order formula and that n is the greatest number such that x_n or R_n appears in A. Let $f_2(A) = \forall x_1, \ldots, \forall x_n \forall R_1, \ldots, \forall R_n A$. Note that for any second-order formula A, $A \in$ 2-SIB^2 if and only if $f_1 f_2(A) \in$ SIB-**S4**π**+**. This suffices for step 2, and for our desired result.

4 Proof of Theorem 2.11: Expressing topological notions in the object language

Before we prove that second-order arithmetic can be recursively embedded in **S4**π**t** (Theorem 2.11), we specify some preliminary topological notions. First, a *pointed* topological space is an ordered pair $Y = (X, b)$ where $b \in X$. A *pointed* topological model is an ordered pair $M = (Y, V)$ where Y is a pointed topological space and V is, as above, a valuation function. A *proposition* will just be a subset of a pointed topological space. Clearly we can give the same definition of $M(A)$ as for unpointed topological models. In the case of a pointed topological model $M = ((X, b), V)$, we say that $M \models A$ if and only if $b \in M(A)$. And we say that $(X, b) \models A$ if and only if, for every pointed model $M = ((X, b), V)$, we have $M \models A$. Note that $\models A$ if and only if A is validated by every pointed topological space if and only if A is validated by every pointed topological model. So we can henceforth restrict our attention to pointed topological spaces and models. The advantage of this is that they behave much more like Kripke frames, each with a privileged world.

We will need a number of other topological notions. These are motivated by considering the expressive resources of the object language, in the context of pointed topological spaces and models. In Section 3, we considered the expressive resources in the context of Kripke frames and models, and summarized some of those considerations in Table 1. Here, we reconsider some of the connectives defined there, in the new context. In our reconsiderations, we assume that $M = ((X, b), V)$ is a pointed topological model, and that $P, Q \subseteq X$.

Reconsider $(A \mathrel{-\!\!\!\ni} B)$. Note that $M \models (A \mathrel{-\!\!\!\ni} B)$ if and only if there is some open set O such that $b \in O$ and $O \cap M(A) \subseteq M(B)$. So henceforth we will say that $P \subseteq_b Q$ if and only if for some open set O, $b \in O$ and $O \cap P \subseteq Q$. Thus $M \models (A \mathrel{-\!\!\!\ni} B)$ if and only if $M(A) \subseteq_b M(B)$. $\subseteq_b$ is the topological analogue of $\subseteq$.

Reconsider $(A = B)$. Note that $M \models (A = B)$ if and only if there is some open set O such that $b \in O$ and $O \cap M(A) = O \cap M(B)$. So henceforth we will say that P and Q are indistinguishable ($P =_b Q$) if and only if for some open set O, $b \in O$ and $O \cap P = O \cap Q$. Thus $M \models (A = B)$ if and only if $M(A) =_b M(B)$. Note that $=_b$ is an equivalence relation: we will call the equivalence classes *indistinguishability* classes, and we will use $\alpha, \beta, \ldots$ to range over them. We will write $|P|$ for the class of propositions indistinguishable from P.

Reconsider $(A \in B)$. In the Kripke semantics, this expresses the claim that $M(A)$ is a *singleton* subset of $M(B)$. The topological analogue of "being a singleton" will be "being singular" in the following sense: we say that P is *singular* if and only if $b \in cl(P)$ and for every Q, if $b \in cl(P \cap Q)$ then $P \subseteq_b Q$. Thus $M \models (A \in B)$ if and only if $M(A)$ is singular and $M(A) \subseteq_b M(B)$. The notion of singularity can also be applied to indistinguishability classes: we say that an indistinguishability class α is *singular* if and only if some $Q \in \alpha$ is singular (equivalently, if and only if *every* $Q \in \alpha$ is singular). We will so often mention singular indistinguishability classes that we henceforth call them *sics*.

Reconsider $(A \leq B)$. Note that $M \models (A \leq B)$ if and only if $M(A)$ and $M(B)$ are both singular, and $M(A) \subseteq_b cl(M(B))$. So henceforth we will say, for any singular propositions P and Q, that $P \leq Q$ if and only if $P \subseteq_b cl(Q)$. And we will say that $P < Q$ if and only if $P \leq Q$ and $Q \nleq P$. We can apply these notions to sics: $\alpha \leq \beta$ if and only if, for some $P \in \alpha$ and some $Q \in \beta$ we have $P \leq Q$ (equivalently, for *every* $P \in \alpha$ and *every* $Q \in \beta$ we have $P \leq Q$). And $\alpha < \beta$ if and only if $\alpha \leq \beta$ and $\beta \nleq \alpha$.

We point out some straightforward facts concerning these notions. $\subseteq_b$ is reflexive and transitive. $P \subseteq_b Q$ and $Q \subseteq_b P$ if and only if $P =_b Q$. If $P \subseteq_b Q$ then $(P - Q) =_b \varnothing$. $P \subseteq_b Q \cap R$ if and only if $P \subseteq_b Q$ and $P \subseteq_b R$. If O is open and $b \in O$, then $O \cap P =_b P$. If P is singular and $Q =_b P$, then Q is singular. If P is singular, then, for each Q, either $P \subseteq_b Q$ or $P \subseteq_b (X - Q)$. $\{b\}$ is singular.

Now for some strategy. With every pointed topological model M we will associate a Kripke model M_K. Since the role of singleton propositions in the Kripke semantics is played by singular propositions in the pointed topological semantics, the worlds of M_K should be the singular propositions. But this is too quick: we want to identify *indistinguishable* singular propositions. So the worlds of the Kripke model M_K will be the sics. With these ideas on the table, we can define M_K.

Definition 4.1 If $M = ((X, b), V)$ is a pointed topological model, we define the associated Kripke model $M_K =_{\mathrm{df}} ((W, 0, \leq), V_K)$ as follows: $W = \{\alpha : \alpha \text{ is a sic}\}$; $0 = |\{b\}|$; $\alpha \leq \beta$ is defined as above; and $V_K(p) = \{\alpha : \text{for some } P \in \alpha,\ P \subseteq_b V(p)\}$ or equivalently $V_K(p) = \{\alpha : \text{for every } P \in \alpha,\ P \subseteq_b V(p)\}$. Note that M_K is a simple Kripke model, as defined in Section 3.

It is, unfortunately, not always the case that M and M_K validate the same formulas. We get something close to this, however, if the underlying pointed topological space satisfies two conditions: *specifiability* and *singularizability*. We say that a pointed topological space (X, b) is *specifiable* if and only if whenever P is singular and $P \subseteq_b cl(Q)$, we can specify a singular R such that $R \subseteq_b Q$ and $P \leq R$. And we say that (X, b) is *singularizable* if and only if there are $P_\alpha \in \alpha$ for each sic α, such that the P_α are pairwise disjoint. We will say that a pointed topological model is singularizable (specifiable) if and only if the underlying pointed topological space is.

If M is both specifiable and singularizable, then M and M_K come pretty close to satisfying the same formulas. In order to state this as a precise theorem, we introduce one more notion. For each modal formula A, we introduce a new formula BARCAN(A), which is so-called because it is the universal closure of the conjunction of the following instances of the Barcan formula, where $\forall qC$ is a subformula of A, and where p is the first variable not occurring in A:

$$\forall q \Box (p \supset C) \supset \Box \forall q (p \supset C).$$

The central lemma of this section is as follows.

Lemma 4.2 *Suppose that M is a specifiable and singularizable pointed topological model and that $M \models \text{BARCAN}(A)$. Then $M_K \models A$ if and only if $M \models A$.*

Remark 4.3 Dougherty's Example 6.3 below, is of a specifiable and singularizable pointed topological model that does not validate every Barcan formula. This is not only helpful in showing that $\mathbf{S4}\pi\mathbf{t} \subsetneq \mathbf{S4}\pi\mathbf{+}$ (Theorem 2.13, above), but it also shows that the clause '$M \models \text{BARCAN}(A)$' is not redundant in the statement of Lemma 4.2. This example and three other examples are quite involved and tangential to our main argument, so we save them for a separate section, Section 6 below.

Lemma 4.2 is a corollary to Lemma 4.7 which we state and prove below. For now, we comment on the significance of Lemma 4.2. Suppose that we could express both specifiability and singularizability in the object language. That is, suppose that we could define formulas, *spec* and *sing*, with the following characteristic: for every pointed topological model M, both $M \models \mathit{spec}$ if and only if M is specifiable and $M \models \mathit{sing}$ if and only if M is singularizable. Then, given Lemma 4.2, we would have an encoding of $\mathbf{S4}\pi\mathbf{+}$ into $\mathbf{S4}\pi\mathbf{t}$, and thus of second-order logic into $\mathbf{S4}\pi\mathbf{t}$: the reason is that we would have, for every formula A, $A \in \mathbf{S4}\pi\mathbf{+}$ if and only if $(\mathit{spec} \,\&\, \mathit{sing} \,\&\, \text{BARCAN}(A) \supset A) \in \mathbf{S4}\pi\mathbf{t}$.

Unfortunately, we were not able to find a suitable formula *sing*, expressing singularizability. There is, however, a formula expressing specifiability:

$$\mathit{spec} =_{\text{df}} \forall q (\forall p \in \Diamond q)(\exists r \in q)(p \leq r).$$

Lemma 4.4 *For every pointed topological model M, $M \models$ spec if and only if M is specifiable.*

Proof: It suffices to consider what is expressed by *spec* in light of the definition of a specifiable model. □

Remark 4.5 We note that $\mathit{spec} \in \mathbf{S4}\pi\mathbf{+}$, since it is validated by every Kripke model: in the context of Kripke semantics, *spec* says that if a proposition Q is possible relative to a world w then there is a world $w' \in Q$ such that $w \leq w'$. In the present context, *spec* expresses a different claim, that M is specifiable. Example 6.1 in Section 6 below, is of a nonspecifiable pointed topological space. This shows that $\mathit{spec} \notin \mathbf{S4}\pi\mathbf{t}$. So we have another proof that $\mathbf{S4}\pi\mathbf{t} \subsetneq \mathbf{S4}\pi\mathbf{+}$ (Theorem 2.13 above). As an added bonus, Example 6.1 will be singularizable, showing that singularizability does not imply specifiability.

As pointed out above, we were not able to find a formula expressing singularizability. Example 6.2 in Section 6 (emailed to me by Dougherty) is of a specifiable but nonsingularizable pointed topological space, showing both that nonsingularizable pointed topological spaces exist and that specifiability does not imply singularizability. Though not all pointed topological spaces are singularizable, a large and useful class of them are.

Lemma 4.6 *Every pointed topological model with countably many sics is singularizable.*

Proof: Suppose that (X, b) is a pointed topological space with countably many sics: $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ with the α_i distinct. Choose any $P_1 \in \alpha_1$. Suppose that $P_1 \in \alpha_1, \ldots, P_n \in \alpha_n$ have been chosen so that they are pairwise disjoint. Choose any $Q \in \alpha_{n+1}$. For $i = 1, \ldots, n$, we have $Q \neq_b P_i$, since $|Q| \neq |P_i|$. So, since Q and the P_i are singular, there are open sets $O_i (i = 1, \ldots, n)$ with $b \in O_i$ and $O_i \cap Q \cap P_i = \varnothing$. Let $P_{n+1} = Q \cap O_1 \cap \cdots \cap O_n$. Then $P_{n+1} =_b Q$ so $P_{n+1} \in \alpha_{n+1}$. Also, P_{n+1} is disjoint from each of $P_1, \ldots, P_n$, as desired. $\square$

Countability plays an important role in Lemma 4.6, and will be the focus of Section 5 below. In Section 5, we will bring Lemmas 4.2 and 4.6 together with some considerations of countability, in order to show that second-order logic over countably infinite domains can be encoded in **S4πt**. This will suffice for our claim that second-order arithmetic can be recursively embedded in **S4πt**.

In the rest of this section, we state and prove Lemma 4.7, to which Lemma 4.2 is a corollary.

Lemma 4.7 *Suppose that $M = ((X, b), V)$ is a specifiable topological model and that $P_\alpha \in \alpha$ have been chosen for each sic α so that the P_α are pairwise disjoint. Also suppose that $M \models \text{BARCAN}(A)$. Then, for each subformula B of A, and for each sic α, we have $\alpha \in M_K(B)$ if and only if $P_\alpha \subseteq_b M(B)$. Here $M_K = ((W, 0, \leq), V_K)$ is defined as in Definition 4.1.*

Proof: By induction on the complexity of B. Here, α and β range over sics.

Case 1 (B atomic): Then, by definition, $\alpha \in M_K(B)$ if and only if, for some $P \in \alpha$, $P \subseteq_b M(B)$. But this is true if and only if $P_\alpha \subseteq_b M(B)$ since, for every $P \in \alpha$ we have $P =_b P_\alpha$.

Case 2 ($B = (C \,\&\, D)$): $\alpha \in M_K(C \,\&\, D)$ if and only if $\alpha \in M_K(C)$ and $\alpha \in M_K(D)$ if and only if $P_\alpha \subseteq_b M(C)$ and $P_\alpha \subseteq_b M(D)$ (by IH) if and only if $P_\alpha \subseteq_b M(C) \cap M(D)$ if and only if $P_\alpha \subseteq_b M(C \,\&\, D)$.

Case 3 ($B = \neg C$): $\alpha \in M_K(\neg C)$ if and only if $\alpha \notin M_K(C)$ if and only if $P_\alpha \subseteq_b M(C)$ (by IH) if and only if $P_\alpha \subseteq_b (X - M(C))$ (since P_α is singular) if and only if $P_\alpha \subseteq_b M(\neg C)$.

Case 4 ($B = \Box C$): We consider both directions of the biconditional separately.

$(\Longrightarrow)$ Suppose that $P_\alpha \nsubseteq_b M(\Box C)$. Then $P_\alpha \nsubseteq_b \text{int}(M(C))$. Since P_α is singular, $P_\alpha \subseteq_b (X - \text{int}(M(C))) = cl(X - M(C))$. By the specifiability of the pointed topological space, there is a singular $R \subseteq X$ such that $P_\alpha \leq R$ and $R \subseteq_b (X - M(C))$. Let $\beta = |R|$. So β is a sic with $\alpha \leq \beta$. Since $P_\beta =_b R$, we have $P_\beta \subseteq_b (X - M(C))$. So $P_\beta \nsubseteq M(C)$. So, by IH, $\beta \notin M_K(C)$. So $\alpha \notin M_K(\Box C)$, as desired.

$(\Longleftarrow)$ Suppose that $\alpha \notin M_K(\Box C)$. Then there is a sic β with $\alpha \leq \beta$ and $\beta \notin M_K(C)$. We want to show that $P_\alpha \nsubseteq \text{int}(M(C))$. By IH, $P_\beta \nsubseteq M(C)$, in which case $P_\beta \subseteq_b (X - M(C))$, since P_β is singular. Since $\alpha \leq \beta$, we have $P_\alpha \leq P_\beta$, that is,

$P_\alpha \subseteq_b cl(P_\beta)$. So $P_\alpha \subseteq_b cl(X - M(C)) = X - \mathrm{int}(M(C))$. So $P_\alpha \not\subseteq \mathrm{int}(M(C))$, as desired.

Before we do the inductive step for $B = \forall qC$, we introduce some new notions. Recall that $M_K = ((W, 0, \leq), V_K)$ as in Definition 4.1. Now for each $Q \subseteq X$, define $Q_K = \{\alpha : P_\alpha \subseteq_b Q\}$. And for each $Q \subseteq W$, define $Q_T = \cup\{P_\alpha : \alpha \in Q\}$. Note that, for $Q \subseteq X$, $M_K[Q_K/q] = M[Q/q]_K$, as can be seen by unpacking the definitions of M_K and of Q_K. Now we show that, for $Q \subseteq W$, $(Q_T)_K = Q$. To see that $Q \subseteq (Q_T)_K$, note that $\beta \in Q \Longrightarrow P_\beta \subseteq Q_T \Longrightarrow P_\beta \subseteq_b Q_T \Longrightarrow \beta \in (Q_T)_K$. To see that $(Q_T)_K \subseteq Q$, note that $\beta \notin Q \Longrightarrow P_\beta \notin \{P_\alpha : \alpha \in Q\} \Longrightarrow P_\beta$ is disjoint from every member of $\{P_\alpha : \alpha \in Q\} \Longrightarrow P_\beta \cap Q_T = \varnothing \Longrightarrow P_\beta \not\subseteq_b Q_T \Longrightarrow \beta \notin (Q_T)_K$. (We could also use the specifiability of M and the disjointness of the P_α to show, for $Q \subseteq X$, that $(Q_K)_T =_b Q$. But we do not need this fact here.)

Now we do our induction for $B = \forall qC$. We consider both directions of the desired biconditional separately.

($\Longrightarrow$) Suppose that $\alpha \in M_K(B) = M_K(\forall qC)$. We want to show that $P_\alpha \subseteq_b M(\forall qC)$. First, we show $(*)$: for each $Q \subseteq X$, $P_\alpha \subseteq_b M[Q/q](C)$. So choose $Q \subseteq X$. Since $\alpha \in M_K(\forall qC)$, we have $\alpha \in M_K[Q_K/q](C) = M[Q/q]_K(C)$. So, by IH, $P_\alpha \subseteq_b M[Q/q](C)$. So $(*)$ is proved. Let p be the first variable not occurring in A. From $(*)$ we have, for each $Q \subseteq X$, $M[P_\alpha/p][Q/q](p) \subseteq_b M[P_\alpha/p][Q/q](C)$. So, for each $Q \subseteq X$, $M[P_\alpha/p][Q/q] \models \Box(p \supset C)$. So $M[P_\alpha/p] \models \forall q\Box(p \supset C)$. So, since $M \models \textsc{barcan}(A)$, we have $M[P_\alpha/p] \models \Box\forall q(p \supset C)$. So $M[P_\alpha/p] \models \Box(p \supset \forall qC)$. So $P_\alpha \subseteq_b M(\forall qC)$, as desired.

($\Longleftarrow$) Suppose that $\alpha \notin M_K(\forall qC)$. Then for some $Q \subseteq W$, $\alpha \notin M_K[Q/q](C) = M_K[(Q_T)_K/q](C) = M[Q_T/q]_K(C)$. (We just used the fact that $(Q_T)_K = Q$). So, by IH, $P_\alpha \not\subseteq_b M[Q_T/q](C)$. So $P_\alpha \not\subseteq_b M(\forall qC)$, as desired. $\square$

5 Proof of Theorem 2.11: Considerations of countability. Before we consider countability, it will be useful to have the following general definition of extensions of **S4**π**+** and **S4**π**t**.

Definition 5.1 If A is a formula and T is a set of formulas, then $T + A =_{\mathrm{df}} \{B : A \supset B \in T\}$.

Lemma 5.2 **S4**π**t** $+ A_1 + \cdots + A_n = \{B : A_1 \& \cdots \& A_n \supset B \in$ **S4**π**t**$\} = \{B : B$ *is validated by every pointed topological model validating* $A_1, \ldots, A_n\}$. **S4**π**+** $+ A_1 + \cdots + A_n = \{B : A_1 \& \cdots \& A_n \supset B \in$ **S4**π**+**$\} = \{B : B$ *is validated by every Kripke model validating* $A_1, \ldots, A_n\}$.

Remark 5.3 The theory SIB-**S4**π**+** defined in Section 3 is, in this new terminology, **S4**π**+** $+ \textit{sib}$.

Now we can outline our strategy for encoding second-order arithmetic in **S4**π**t**. First, it suffices to encode second-order logic over countably infinite domains. Secondly, Nerode and Shore's [28] strategy for encoding arbitrary relations as sib relations (see Section 3 above) applies in countably infinite domains. So it suffices to encode second-order logic over countably infinite domains, with all second-order quantification over sib relations. Furthermore, it will be convenient (though unnecessary) to enrich the second-order language of Section 3 above, with standardly interpreted unary predicate variables, $X_1, X_2, \ldots$. Let us call the resulting second-order theory ω-SIB^2. So we will encode ω-SIB^2 in **S4πt**.

Before we indicate how to effect this encoding, we note that there is a second-order formula, COUNT, which is true in and only in countably infinite domains (with R ranging over sib relations):

$$\begin{aligned}\text{COUNT} \quad =_{\text{df}} \quad & \exists R(\forall x\forall y\forall z\forall w(Rxy \mathbin{\&} Rxz \mathbin{\&} Rxw \supset y = z \vee y = w \vee z = w) \mathbin{\&} \\ & \exists! x\exists! yRxy \mathbin{\&} \forall x\exists yRxy \mathbin{\&} \forall X(\exists xXx \supset \exists x(Xx \mathbin{\&} \forall y\forall z(Xy \mathbin{\&} Xz \mathbin{\&} \\ & Rxy \mathbin{\&} Rxz \supset y = z)))).\end{aligned}$$

Now adjust the definition of the translation function f_1, from Section 3, to get a translation function g_1 from the enriched second-order language (with unary predicate constants) as follows:

$$\begin{array}{lcl} g_1(x_i = x_j) & = & (p_{3i} = p_{3j}) \\ g_1(R_i x_j x_k) & = & (\exists q \in p_{3i-1})((p_{3j} < q) \mathbin{\&} (p_{3k} < q)) \\ g_1(\neg A) & = & \neg g_1(A) \\ g_1(A \mathbin{\&} B) & = & g_1(A) \mathbin{\&} g_1(B) \\ g_1(\forall x_i A) & = & \forall p_{3i}(\neg p_{3i} \mathbin{\&} \exists q(p_{3i} < q) \supset f_1(A)) \\ g_1(\forall R_i A) & = & \forall p_{3i-1}((p_{3i-1} \multimap \forall q(q \supset \Box q)) \supset g_1(A)) \\ g_1(\forall X_i A) & = & \forall p_{3i-2}(\neg p_{3i-2} \mathbin{\&} (\forall q \in p_{3i-2})\exists r(q < r) \supset g_1(A)) \end{array}$$

And define closed modal formula *count* $=_{\text{df}} g_1(\text{COUNT})$. We will now consider what is expressed by *count* in both the Kripke and the topological semantics.

Actually, in the context of all Kripke models, it is unclear, and not very interesting, what *count* expresses. We do, however, have the following theorem.

Lemma 5.4 *For every sib Kripke model M, $M \models$ count if and only if M is countably infinite.*

Proof: This can be seen by considering the constraints put on the size of tier$_2$ by the fact that the model validates *count*. These constraints are the same as are put on a classical domain, if the second order formula COUNT is true in that domain. □

So, in the context of sib Kripke models, *count* expresses the claim that the model is countably infinite, and in particular that its second tier is countably infinite. So, among all Kripke models, the formula (*sib* & *count*) expresses the claim that the model is a countably infinite sib model: for every Kripke model $M, M \models$ (*sib* & *count*) if and only if M is *sib* and countably infinite. If we define the function g_2 analogously to f_2 in Section 3, we then find that for any second-order formula A, $A \in \omega$-SIB^2 if and only if $g_1 g_2(A) \in$ **S4π+** + *sib* + *count*. This gives us the following.

Lemma 5.5 *Second-order arithmetic can be recursively embedded in* **S4π+** + *sib* + *count*.

Of course, we are not primarily interested in Kripke models and extensions of **S4**π**+**, but in pointed topological models and extensions of **S4**π**t**. But our strategy will rely on Lemma 5.5: we will show that **S4**π**+** $+$ *sib* $+$ *count* can be recursively embedded in **S4**π**t** (Corollary 5.11 below), and this will suffice for our main result. In order to show this, we must consider what is expressed by *sib* and *count* in the context of pointed topological models.

For this discussion, we assume that $M = ((W, b), V)$ is a pointed topological model. First we consider what is expressed by the formula *3-tier*, defined in Table 1, in the context of pointed topological models. Recall that 3-tier $=_{\text{df}}$ $(\forall p \in T)(\forall q \in T)((p \leq q) \,\&\, (q \leq p) \supset (p = q)) \,\&\, (\exists p \in T)(\exists q \in T)(\exists r \in T)(p \,\&\, (p < q) \,\&\, (q < r)) \,\&\, \neg(\exists p \in T)(\exists q \in T)(\exists r \in T)(\exists s \in T)(p \,\&\, (p < q) \,\&\, (q < r) \,\&\, (r < s))$. *3-tier* says two things.

1. For singular propositions P and Q, $P \leq Q$ and $Q \leq P$ if and only if $P =_b Q$.
2. Singular propositions come in three varieties:
   (a) *first tier singular propositions* that are indistinguishable from $\{\mathrm{b}\}$;
   (b) *second tier singular propositions* P such that, for some singular proposition Q, we have $\{b\} < P < Q$, and for no singular proposition Q do we have $\{b\} < Q < P$;
   (c) *third tier singular propositions* P such that for some singular proposition Q, we have $\{b\} < Q < P$, and for no singular proposition Q do we have $P < Q$.

   These claims can also be expressed as claims about sics. This motivates the following definitions and lemma.

Definition 5.6 A pointed topological space (X, b) is *3-tiered* if and only if

1. for sics α and β we have if $\alpha \leq \beta$ and $\beta \leq \alpha$ then $\alpha = \beta$;
2. there exist sics α and β such that $|\{b\}| < \alpha < \beta$; and
3. for sics α, β, and γ we never have $|\{b\}| < \alpha < \beta < \gamma$. (Compare this to the definition of 3-tiered Kripke frames in Section 2.)

A 3-tiered pointed topological *model* is one whose underlying space is 3-tiered. Given a 3-tiered topological space, we define $\text{tier}_1 =_{\text{df}} \{|\{b\}|\}$; $\text{tier}_2 =_{\text{df}} \{\alpha : \alpha$ is a sic and $|\{b\}| < \alpha$ and for some sic $\beta, \alpha < \beta\}$; and $\text{tier}_3 = \{\alpha : \alpha$ is a sic and for some sic β, $|\{b\}| < \beta < \alpha\}$.

Definition 5.7 A 3-tiered pointed topological space (X, b) is a *sib* pointed topological space if and only if

1. every pair of distinct sics in tier_2 has a unique upper bound (with the ordering $\leq$ defined on sics);
2. every sic in tier_3 is the upper bound of two sics in tier_2; and
3. no three distinct sics in tier_2 have an upper bound.

A pointed topological model is a *sib* model if and only if the underlying space is a sib space.

Lemma 5.8 *If M is a pointed topological model then $M \models$ 3-tier if and only if M is 3-tiered, and $M \models$ sib if and only if M is a sib model.*

Proof: This simply requires a careful reading of the formulas *3-tier* and *sib*, defined on Table 1. Such a reading should reveal how the relevant conditions are expressed in the object language. $\square$

Lemma 5.9 *Suppose that the pointed topological model M is sib. Then $M \models$ count if and only if M has countably many sics.*

Proof: See the remarks proving Lemma 5.4. $\square$

Now we bring the pieces of the puzzle together.

Lemma 5.10 *For every formula A, $A \in$ **S4πt** $+$ spec $+$ sib $+$ count $+$ $\textsc{barcan}(A) + \textsc{barcan}(sib) + \textsc{barcan}(count)$ if and only if $A \in$ **S4π+** $+$ sib $+$ count.*

Proof: Suppose that $A \notin$ **S4πt** $+$ *spec* $+$ *sib* $+$ *count* $+$ $\textsc{barcan}(A) + \textsc{barcan}(sib) + \textsc{barcan}(count)$. Then there is a pointed topological model M of *spec* & *sib* & *count* & $\textsc{barcan}$(A) & $\textsc{barcan}(sib)$ & $\textsc{barcan}(count)$ such that $M \not\models A$. M is sib (Lemma 5.8), and M has countably many sics (Lemma 5.9). So M is singularizable (Lemma 4.6). Furthermore, M is specifiable (Lemma 4.4). So, since $M \models \textsc{barcan}(A)$ & $\textsc{barcan}(sib)$ & $\textsc{barcan}(count)$ we have, by Lemma 4.2: $M \models A$ if and only if $M_K \models A$; $M \models sib$ if and only if $M_K \models sib$; and $M \models count$ if and only if $M_K \models count$. So $M_K \not\models A$, although $M_K \models (sib \,\&\, count)$. So $A \notin$ **S4π+** $+ sib + count$.

On the other hand, suppose that $A \notin$ **S4π+** $+ sib + count$. Then there is a Kripke model $M = ((W, 0, \leq), V)$ of *sib* & *count* such that $M \not\models A$. Since all Kripke models validate *spec*, and all Kripke models validate all formulas of the form $(\forall q \Box B \supset \Box \forall q B)$, M validates *spec* & $\textsc{barcan}(A)$ & $\textsc{barcan}(sib)$ & $\textsc{barcan}(count)$. Let X be the topological space W, where the open sets are those closed under $\leq$, and consider the pointed topological model $M' = ((X, 0), V)$. M and M' validate the same formulas. In particular, $M' \models$ (*spec* & *sib* & *count* & $\textsc{barcan}(A)$ & $\textsc{barcan}(sib)$ & $\textsc{barcan}(count)$) and $M' \not\models A$. So $A \notin$ **S4πt** $+ spec + sib + count + \textsc{barcan}(A) + \textsc{barcan}(sib) + \textsc{barcan}(count)$. $\square$

Corollary 5.11 ***S4π+** $+$ sib $+$ count can be recursively embedded in **S4πt**.*

Proof: By Lemma 5.10, for any formula A, we have $A \in$ **S4π+** $+ sib + count$ if and only if (*spec* & *sib* & *count* & $\textsc{barcan}$(A) & $\textsc{barcan}(sib)$ & $\textsc{barcan}(count)$ $\supset A) \in$ **S4πt**. $\square$

Our main result, Theorem 2.11 which says that second-order arithmetic can be recursively embedded in **S4πt**, is a corollary to Corollary 5.11 and Lemma 5.5.

6 *Slightly pathological examples*

Example 6.1 For a nonspecifiable but singularizable pointed topological space, let $X = \mathbb{R}$ with the standard topology, and consider the pointed topological space $(X, 0)$. First we will show $(*)$: no subset of $\mathbb{R} - \{0\}$ is singular. So suppose that P is a singular subset of $\mathbb{R} - \{0\}$. Since $0 \in cl(P) - P$, there is an $S \subseteq P$ such that $0 \in cl(S)$ and $0 \in cl(P - S)$. Since P is singular and since $0 \in cl(P \cap S)$ and $0 \in cl(P \cap (P - S))$, we have $P \subseteq_0 S$ and $P \subseteq_0 (P - S)$. So there are open sets O and O' such that $0 \in O$ and $0 \in O'$ and $O \cap P \subseteq S$ and $O' \cap P \subseteq (P - S)$. Let $O'' = O \cap O'$. So $0 \in O''$ and $O'' \cap P = \varnothing$, contradicting $0 \in cl(P)$ and proving $(*)$. Given $(*)$, any singular subset of X contains 0. So, since $\{0\}$ is singular, $(X, 0)$ has exactly one singular indistinguishability class: $|\{0\}|$. So $(X, 0)$ is trivially singularizable. But it is not specifiable: $\{0\} \subseteq_0 cl(X - \{0\})$, but there is no singular P such that $P \subseteq_0 X - \{0\}$.

Example 6.2 (Almost verbatim from Dougherty's email.) For a specifiable nonsingularizable pointed topological space, let X be the natural numbers $\mathbb{N}$, together with an extra point b. Shortly, we define the open sets. First let Z be an uncountable family of almost disjoint infinite subsets of $\mathbb{N}$ (as given, for example, by Jech [12], Lemma 23.9). Let $\mathcal{F}_0$ be the filter of cofinite subsets of $\mathbb{N}$. Note that, for every P in Z, $(\mathbb{N} - P) \notin \mathcal{F}_0$. By Zorn's Lemma, extend $\mathcal{F}_0$ to a filter $\mathcal{F}$ over $\mathbb{N}$, which is maximal subject to the following condition: for every P in Z, $(\mathbb{N} - P) \notin \mathcal{F}$. The open sets of X are as follows: any subset of $\mathbb{N}$; and any set of the form $\{b\} \cup S$, where $S \in \mathcal{F}$.

First we claim $(*)$: every $P \in Z$ is singular.

Proof: To show $(*)$, we must show that (i) $b \in cl(P)$ and (ii) for any $Q \subseteq X$, either $P \subseteq_b Q$ or $P \cap Q =_b \varnothing$. For (i), note that, since $\mathbb{N} - P$ is not in $\mathcal{F}$, every member of $\mathcal{F}$ meets P. For (ii), let $Q \subseteq X$, and consider three cases.

Case 1: $(P \cap Q) \in \mathcal{F}$. Then $P \subseteq_b Q$, since $\{b\} \cup (P \cap Q)$ is open.

Case 2: $\mathbb{N} - (P \cap Q) \in \mathcal{F}$. Then $(P \cap Q) =_b \varnothing$, since $\{b\} \cup (\mathbb{N} - (P \cap Q))$ is open.

Case 3: $(P \cap Q) \notin \mathcal{F}$ and $\mathbb{N} - (P \cap Q) \notin \mathcal{F}$. By the maximality of $\mathcal{F}$ relative to the above given condition, the filter $\mathcal{F}_1$ generated by $\mathcal{F} \cup \{\mathbb{N} - (P \cap Q)\}$ violates that condition so that there is a $P' \in Z$ with $\mathbb{N} - P' \in \mathcal{F}_1$. So there is an $S \in \mathcal{F}$ such that $S \cap (\mathbb{N} - (P \cap Q)) \subseteq \mathbb{N} - P'$. But then $S \cap P' \subseteq P \cap Q$. Now we will show that $P = P'$. If not, then $P \cap P'$ is finite, since $P, P' \in Z$. Now $S \cap P' \subseteq P \cap P'$, so $S \cap P'$ is finite. So $\mathbb{N} - (S \cap P') \in \mathcal{F}_0 \subseteq \mathcal{F}$. So $S - P' = S \cap (\mathbb{N} - (S \cap P')) \in \mathcal{F}$. So $\mathbb{N} - P' \in \mathcal{F}$, contradicting the condition that, for every $P \in Z, \mathbb{N} - P \notin \mathcal{F}$. So we have shown that $P = P'$. Thus $S \cap P \subseteq Q$. Let $O = \{b\} \cup S$, which is open. Note that $O \cap P \subseteq Q$. So $P \subseteq_b Q$ as desired, proving $(*)$. □

Given $(*)$ and the fact that the topology is trivial away from b, (X, b) is specifiable. All of the sets P in Z are singular and distinguishable from one another, so there are uncountably many sics. There is no way to choose disjoint representatives for uncountably many sics since X is countable. So X is nonsingularizable.

Example 6.3 (Almost verbatim from Dougherty's email.) For a specifiable and singularizable pointed topological model that does not validate every Barcan formula, first let X be the natural numbers $\mathbb{N}$, together with an extra point b; and let

$\mathcal{U}$ be a nonprincipal ultrafilter over $\mathbb{N}$. Let the open subsets of X be $\varnothing$, all the sets in $\mathcal{U}$, and any set of the form $\{b\} \cup S$, for $S \in \mathcal{U}$. Note that the pointed topological space (X, b) is specifiable and singularizable. To see this, note that (X, b) has only four indistinguishability classes: $|\{b\}|$ and $|\mathbb{N}|$, which are singular; and $|X|$ and $|\varnothing|$, which are not singular. Now let M be the model $((X, b), V)$, where $V(r) = \{b\}$, and let B be the formula $\Box\Diamond q \vee \neg q \vee r$. We claim that $M \not\models (\forall q \Box B \supset \Box \forall q B)$, that is, that $M \models \forall q \Box(\Box\Diamond q \vee \neg q \vee r)$, but $M \not\models \Box \forall q(\Box\Diamond q \vee \neg q \vee r)$. For this it suffices to show that (1) for every $Q \subseteq X, b \in \text{int}(\text{int}(cl(Q)) \cup (X - Q) \cup \{b\})$ and that (2) $b \notin \text{int}(\cap\{\text{int}(cl(Q)) \cup (X - Q) \cup \{b\} : Q \subseteq X\})$. For (1), if $Q \cap \mathbb{N} \notin \mathcal{U}$ then $\text{int}(cl(Q)) = X$; and if $Q \cap \mathbb{N} \in \mathcal{U}$ then $(\mathbb{N} - Q) \cup \{b\}$ is an open set containing b. For (2), it suffices to point out that $\cap\{\text{int}(cl(Q)) \cup (X - Q) \cup \{b\} : Q \subseteq X\} \subseteq \{b\}$. For suppose that $n \in X - \{b\} = \mathbb{N}$. Note that $\{n\}$ is not open, and indeed that any other point can be surrounded by an open set disjoint from $\{n\}$. So $\text{int}(cl(\{n\})) = \varnothing$. But $n \notin (X - \{n\}) \cup \{b\}$. So $n \notin \text{int}(cl(\{n\})) \cup (X - \{n\}) \cup \{b\}$. So $n \notin \cap\{\text{int}(cl(Q)) \cup (X - Q) \cup \{b\} : Q \subseteq X\}$.

Example 6.4 For a simpler rejection of a Barcan formula, let X be the natural numbers with an extra point b, and let the open sets be $\varnothing$ and the sets of the form $S \cup \{b\}$ where S is a cofinite subset of $\mathbb{N}$. Let M be any model $((X, b), V)$ and let B be the formula $(\neg q \vee \Box\Diamond q)$. Note the following, where $Q \subseteq X$: (Q is infinite or $b \in Q$) $\Longrightarrow M[Q/q](\Diamond q) = X \Longrightarrow M[Q/q](\Box\Diamond q) = X \Longrightarrow M[Q/q](\neg q \vee \Box\Diamond q) = X \Longrightarrow M[Q/q](\Box(\neg q \vee \Box\Diamond q)) = X \Longrightarrow M[Q/q](\Box B) = M[Q/q](B) = X$; and ($Q$ is finite and $b \in Q$) $\Longrightarrow M[Q/q](\Diamond q) = \varnothing \Longrightarrow M[Q/q](\Box\Diamond q) = \varnothing \Longrightarrow M[Q/q](\neg q \vee \Box\Diamond q) = X - Q \Longrightarrow M[Q/q](\Box(\neg q \vee \Box\Diamond q)) = X - Q \Longrightarrow M[Q/q](\Box B) = M[Q/q](B) = X - Q$. Therefore $M(\forall q \Box B) = M(\forall q B) = \{b\}$ and $M(\Box \forall q B) = \varnothing$. Therefore $M \not\models (\forall q \Box B \supset \Box \forall q B)$.

7 Concluding remarks One extension of the work in this paper would be to consider propositional quantification in the topological semantics for logics stronger than **S4**. Given any propositional modal logic **L** stronger than **S4**, define **L**πt=$_{\text{df}}$ the set of propositionally quantified formulas validated by every topological space that validates all the formulas of **L**. Just as the argument in Section 4 can be adapted to show that the Kripkean system **S4.2**π**+** is recursively isomorphic to second-order logic, the arguments in Section 5 and Section 6 can be adapted to show that second-order arithmetic is recursively embeddable in the topological system **S4.2**π**t**, which is weaker than **S4.2**π**+**. It is worth noting that **S5**π**t** = **S5**π**+**. Clearly **S5**π**t** $\subseteq$ **S5**π**+**. To see that **S5**π**+** $\subseteq$ **S5**π**t**, suppose that $A \notin$ **S5**π**t**. Then there is some topological space X validating every theorem of **S5**, with $X \not\models A$. Since $X \models (p \supset \Box\Diamond p)$, we have $(\forall x \in X)(\forall S \subseteq X)(S \subseteq_x \text{int}(cl(S)))$. And so $x \in S \Longrightarrow (\exists O \subseteq X)(x \in O$ and O is open and $O \cap S \subseteq \text{int}(cl(S))) \Longrightarrow x \in \text{int}(cl(S))$. Thus, $(\forall S \subseteq X)(S \subseteq \text{int}(cl(S)))$. This means that every open set is closed and vice versa, so that, for every $x \in X$, there is a smallest open set O_x containing x. Now since $X \not\models A$, there is some model $M = (X, V)$ and some point $x \in X$ with $x \notin M(A)$. Let M' be the Kripke model $((O_x, x, \leq), V')$ where $\leq$ is the universal relation $O_x \times O_x$, and where, for each $p \in PV$, $V'(p) = V(p) \cap O_x$. Note that $M'(B) = M(B) \cap O_x$, for every formula B. So $M' \not\models A$. So $A \notin$ **S5**π**+**, as desired. Given that **S5**π**t**=**S5**π**+**, the following question arises: is there a natural characterization of the logics **L**, intermediate between **S4** and **S5**, such that **L**π**t**=**L**π**+**?

A second extension of the work in this paper would be to consider issues in propositional quantification in the neighborhood semantics for modal logics, a generalization of the topological semantics. See Montague [26], Scott [36], Segerberg [37], and Chellas [2].

Our work leaves us with a number of open questions. First, is **S4**π**+** recursively isomorphic to second-order logic? Second, is there some way to express singularizability in the object language? As pointed out after the statement of Lemma 4.2, this would give us a way to encode second order logic, and not just second-order arithmetic, in **S4**π**t**. Third, what is the relationship between **S4**π**t** and **S4**π**+**? For example, is there a formula A such that **S4**π**+** $=$ **S4**π**t** $+ A$?

More generally, this work underscores the fact that although the Kripke and the topological semantics agree on which unquantified propositional arguments are valid, they deliver different theories of propositions, differences that can be brought out in an object language with propositional quantifiers.

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