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The "Relevance" of Intersection and Union Types

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Abstract The aim of this paper is to investigate a Curry-Howard interpretation of the intersection and union type inference system for Combinatory Logic. Types are interpreted as formulas of a Hilbert-style logic L, which turns out to be an extension of the intuitionistic logic with respect to provable disjunctive formulas (because of new equivalence relations on formulas), while the implicational-conjunctive fragment of L is still a fragment of intuitionistic logic. Moreover, typable terms are translated in a typed version, so that $\vee$-$\wedge$-typed combinatory logic terms are proved to completely codify the associated logical proofs.

1 Introduction In the last few years typing has become a crucial aspect in functional programming languages design, as a way of incorporating in the language itself the logic of program properties. This perspective gives an important role to the "Curry-Howard isomorphism," which provides a constructive explanation of type disciplines, by the analogies "types as logical formulas" and "terms as constructive proofs." In this approach the implicational fragment of propositional intuitionistic logic is related, in a natural way, to the basic functional type theory of Curry and Feys [8], whose only type constructor is the arrow for building functional types. Roughly speaking, inhabited arrow-types are interpreted as implicational theorems, since the axioms for implication become the types of atomic combinators, and modus ponens corresponds to well-typed application of terms.

Intersection types were introduced in Coppo, Dezani-Ciancaglini, and Venneri [6] as a generalization of Curry's basic system, mainly with the aim of describing the functional behavior of all terminating programs. In the intersection type discipline the usual $\rightarrow$-based type language is extended by adding a new connective '$\wedge$' for the intersection of two types. With suitable axioms and rules assigning types to terms, the obtained system enjoys the following main properties.

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(i) The set of types given to a term is invariant with respect to reduction (namely, β-reduction for λ-calculus and weak-reduction for combinatory logic).

(ii) The set of all strongly normalizing terms can be characterized very neatly by the types of their members.

Union types were first introduced in MacQueen, Plotkin, and Sethi [16], where the properties of the formal system were not investigated, but only a brief discussion of their use to argue about the *types as ideals* semantics was given. Their interest in computing is discussed for example in Jensen [14], and Coppo and Ferrari [7]. The intersection and union type inference system for λ-calculus is defined and studied in Barbanera and Dezani-Ciancaglini [3] and Barbanera, Dezani-Ciancaglini, and de'Liguoro [4].

The present paper concerns a system $\mathsf{TA}_{\wedge\vee}$ for assigning both intersection and union types to terms of combinatory logic. The most obvious extension of implicational propositional calculus with intuitionistic conjuction and disjunction is the natural candidate for a logical mapping of $\mathsf{TA}_{\wedge\vee}$. Unfortunately, intersection does not correspond to conjunction as already noted in Hindley [12]; a simple counterexample is the intuitionistic theorem $A \to B \to A \wedge B$ which cannot be deduced for any combinator. Similarly, union $\vee$ does not correspond to disjunction; a simple counterexample is given by the intuitionistic theorem $(A \to B) \to (C \to B) \to A \vee C \to B$. All these are consequences of the fact that $\to$ is not the right adjoint of $\wedge$. Instead, there is some "duality" between intersection and union since the formula $(A \to B) \wedge (C \to B) \to A \vee C \to B$ is a type for the identity combinator.

The crucial point in defining a logical mapping for this type discipline is the rules for introducing intersection and for eliminating union, that is,

$$(\wedge I) \qquad \frac{\Gamma \vdash^{\wedge\vee} M : \sigma \quad \Gamma \vdash^{\wedge\vee} M : \tau}{\Gamma \vdash^{\wedge\vee} M : \sigma \wedge \tau}$$

$$(\vee E) \qquad \frac{\Gamma, x : \sigma \vdash^{\wedge\vee} M : \rho \quad \Gamma, x : \tau \vdash^{\wedge\vee} M : \rho \quad \Gamma \vdash^{\wedge\vee} N : \sigma \vee \tau}{\Gamma \vdash^{\wedge\vee} M[N/x] : \rho}$$

which look far from the standard shape of logical rules. In fact if terms are to be identified with proofs, then $(\wedge I)$ says that a proof of a conjunctive formula requires the *same* proof in deriving both the conjuncts, that is, a stronger request than their simple provability. Analogously, rule $(\vee E)$ says that a proof of the formula A can be obtained from $B \vee C$ only provided that the *same* proof of A can be obtained both from B and from C. In other words, rules $(\wedge I)$ and $(\vee E)$ are constrained by a global condition of applicability involving properties of the whole subderivations.

Our goal is to set out a logical system matching $\mathsf{TA}_{\wedge\vee}$ such that the intersection and union type constructors are interpreted as propositional connectives and then their derivability is completely represented by derivability in a logical Hilbert-style axiomatization L. We will do that using essentially the results of Venneri [20], [3], and [4]. In [20] a new formulation of the intersection type inference for combinatory logic is presented, which is equivalent to the original version of the system of Dezani-Ciancaglini and Hindley [9], while the intersection operator is no longer dealt with as a proof-functional connective. As a consequence a Hilbert-style axiomatization is obtained in such a way that inhabited intersection types are all and only the provable formulas in the logical system there defined.

In the present paper we will succeed in modeling such a logical system for the union constructor as well, by mapping each type into a finite set of types without unions and each deduction in $\mathsf{TA}_{\wedge\vee}$ into a set of deductions in the intersection type assignment system. As a consequence, a typed version of combinatory logic with union and intersection types is presented, such that all typed combinators are viewed as proofs in the logical axiomatization L and vice versa.

Let us remark that an important component of the system $\mathsf{TA}_{\wedge\vee}$ is the inclusion ($\leq$) relation between types, which is justified by interpreting $\rightarrow$ as a function space constructor, $\wedge$ as intersection, and $\vee$ as union between types. This ordering relation will be totally mapped into axioms and rules of the Minimal Relevant Logic B^+ as defined in Meyer and Routley [17], plus one clause corresponding to the Extended Disjunction Property (Harrop [11]). The logical axiomatization isomorphic to $\mathsf{TA}_{\wedge\vee}$ will be defined as the pure implicational calculus, increased by the fact that all theorems of the above relevant logic are assumed and used as major premises of modus ponens.

The results of the present paper lead to the main conclusion that the $\wedge$ and $\vee$ type constructors are essentially interpreted as the conjunction and disjunction of a relevant logic, respectively. In other words, proof-theoretic conditions of applicability in $\wedge$-introduction and $\vee$-elimination are logically translated by requirements of *relevant* dependencies between the assumptions and the conclusion of a proof.

In order to obtain these results, the original version of the $\wedge$-$\vee$-type inference will be mapped into progressively more restricted systems, in the sense that proof functional rules are eliminated and they are replaced by equivalent logical rules. To help the reader progress along this tortuous path, we provide the following Table 1 listing the notations and definitions of the various systems.

System Name	Derivability	Axioms and Rules	Definition
$\mathsf{TA}_{\wedge\vee}$	$\vdash^{\wedge\vee}$	$\mathsf{S}, \mathsf{K}, \mathsf{I}$, (VAR), $(\rightarrow E), (\wedge I), (\vee E), (\leq)$	2.5
$\mathsf{TA}^*_{\wedge\vee}$	$\vdash^*$	$\mathsf{S}, \mathsf{K}, \mathsf{I}$, (VAR), $(\rightarrow E), (\wedge I), (\vee E), (\leq^*)$	2.9
$\mathsf{TA}_{\wedge}$	$\vdash^{\wedge}$	$\mathsf{S}, \mathsf{K}, \mathsf{I}$, (VAR), $(\rightarrow E), (\wedge I), (\leq_{\wedge})$	2.10
$\mathsf{TA}^{\#}$	$\vdash^{\#}$	$\mathsf{S}^{\#}, \mathsf{K}^{\#}, \mathsf{I}^{\#}$, (VAR), $(\rightarrow E), (\leq_{\wedge})$	2.26
TA	$\vdash$	$\mathsf{S}^{\#}, \mathsf{K}^{\#}, \mathsf{I}^{\#}$, (VAR), $(\rightarrow E), (\leq^*)$	2.29

Table 1.

Lastly, we want to point out that our approach is in the framework of Curry-Howard isomorphism and so it is completely different from that of Lopez-Escobar [15], Mintz [19], Alessi and Barbanera [2], and Barbanera and Martini [5]. In fact the authors of those papers investigated the intersection as a proof-theoretic operator, in the context of "untyped terms as realizers of logical formulas."

2 Intersection and union types for combinatory logic For the main definitions and notions in combinatory logic we refer to Hindley and Seldin [13], chapter 2. Let us recall some basic notions in order to fix the notations.

Definition 2.1 (Combinatory logic) Assume that an infinite set of variables and the basic combinators S, K, and I are given. The set C of CL-terms is built from variables and S, K, I by application. An *atom* is a variable or a basic combinator, a *combinator* is a CL-term without variables. Each atomic combinator is assumed to have an axiom-scheme for reduction:

$\mathsf{S}fgx \to fx(gx)$	(composition)
$\mathsf{K}xy \to x$	(formation of constant function)
$\mathsf{I}x \to x$	(identity).

Weak reduction ($\to$) of CL-terms is defined as usual.

Let us recall also the following *abstraction algorithm* which is defined in [8] section 6A:

$$
\begin{array}{lcll}
\lambda^*x.x & \equiv & \mathsf{I} & \\
\lambda^*x.M & \equiv & \mathsf{K}\,\mathrm{M} & \text{if } x \notin FV(M) \\
\lambda^*x.Ux & \equiv & U & \text{if } x \notin FV(U) \\
\lambda^*x.UV & \equiv & \mathsf{S}(\lambda^*x.U)(\lambda^*x.V) & \text{otherwise.}
\end{array}
$$

2.1 Type inference We consider the set of types built out of an infinite set of type variables by the function space ('$\to$'), intersection ('$\wedge$'), and union ('$\vee$') type constructors.

Definition 2.2 (Intersection and union types) The set T of intersection and union types is inductively defined by:

1. $\alpha, \beta, \gamma, \delta, \ldots \in T$ (type variables);
2. $\sigma, \tau \in T \Longrightarrow (\sigma \to \tau), (\sigma \wedge \tau), (\sigma \vee \tau) \in T$.

Notation 2.3 We omit parentheses assuming that:

1. $\to$ associates to the right;
2. $\wedge$ and $\vee$ have precedence over $\to$.

$\bigwedge_{i\in I}\sigma_i$ and $\bigvee_{i\in I}\sigma_i$, where $I = \{i_1, \ldots, i_n\}$, are shorts for $\sigma_{i_1} \wedge \cdots \wedge \sigma_{i_n}$ and $\sigma_{i_1} \vee \cdots \vee \sigma_{i_n}$, respectively.

A *preorder* relation on types naturally arises if $\to$ is regarded as the function space constructor, while $\wedge$ and $\vee$ are regarded as intersection and union on sets, respectively.

Definition 2.4 (Preorder $\leq$ on T)

(i) The *preorder* $\leq$ on T is inductively defined by the following axioms and rules:

(a) Axioms

1. $\sigma \leq \sigma \wedge \sigma, \sigma \vee \sigma \leq \sigma$
2. $\sigma \wedge \tau \leq \sigma, \sigma \wedge \tau \leq \tau$
3. $\sigma \leq \sigma \vee \tau, \tau \leq \sigma \vee \tau$
4. $(\sigma \to \rho) \wedge (\sigma \to \tau) \leq \sigma \to \rho \wedge \tau$
5. $(\sigma \to \rho) \wedge (\tau \to \rho) \leq \sigma \vee \tau \to \rho$
6. $\sigma \wedge (\tau \vee \rho) \leq (\sigma \wedge \tau) \vee (\sigma \wedge \rho)$

(b) Rules

1. $\sigma \leq \sigma', \tau \leq \tau' \Longrightarrow \sigma \wedge \tau \leq \sigma' \wedge \tau'$
2. $\sigma \leq \sigma', \tau \leq \tau' \Longrightarrow \sigma \vee \tau \leq \sigma' \vee \tau'$
3. $\sigma \leq \sigma', \tau \leq \tau' \Longrightarrow \sigma' \to \tau \leq \sigma \to \tau'$
4. $\sigma \leq \tau, \tau \leq \rho \Longrightarrow \sigma \leq \rho$.

(ii) The relation of equivalence $\sim$ on T is defined in the following way:

$$\sigma \sim \tau \text{ if and only if } \sigma \leq \tau \text{ and } \tau \leq \sigma.$$

Many interesting equivalence relations on types are provable: reflexivity of $\sim$, commutativity and associativity of $\wedge$ and $\vee$. We mention here also that

$$(*) \qquad (\sigma \to \rho) \wedge (\tau \to \rho) \sim \sigma \vee \tau \to \rho$$

$$(**) \qquad \sigma \wedge (\tau \vee \rho) \sim (\sigma \wedge \tau) \vee (\sigma \wedge \rho),$$

since these relations will be used in the following. $(*)$ holds by axiom 5 in one direction, and by axioms 1 and 3, and rules 1, 3, and 4 in the other one. $(**)$ holds by axiom 6 in one direction, and by axioms 1 and 2 and rules 1, 2, and 4 in the other one. The following expressions link the notions of types and terms.

1. A *statement* is an expression of the form $M : \sigma$, where M is a term (*subject*) and σ is a type (*predicate*).
2. An *assumption* is a statement whose subject is a term variable.
3. A *basis* is a set of assumptions with distinct term variables.

If Γ is a basis, then $FV(\Gamma)$ will denote the set of term variables which are subjects of some assumption in Γ. Given a basis Γ such that $x \notin FV(\Gamma)$, the basis $\Gamma \cup \{x : \sigma\}$ will be denoted by $\Gamma, x : \sigma$. Now we introduce a type inference system, as a set of axioms and rules deriving intersection and union types for untyped CL-terms.

Definition 2.5 (The type assignment system $\mathsf{TA}_{\wedge\vee}$) A statement $M : \sigma$ is derivable from a basis Γ, notation $\Gamma \vdash^{\wedge\vee} M : \sigma$, if $\Gamma \vdash^{\wedge\vee} M : \sigma$ can be proved using the following axioms and inference rules.

(a) Axioms

1. S:$[(\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma]$,
2. K:$[\alpha \to \beta \to \alpha]$,
3. I:$[\alpha \to \alpha]$,

where $[\sigma]$ denotes any instance of σ, that is, any type obtained by substituting types for type variables in σ.

(b) Rules

$$(VAR) \qquad \Gamma, x : \sigma \vdash^{\wedge\vee} x : \sigma$$

$$(\to E) \qquad \frac{\Gamma \vdash^{\wedge\vee} M : \sigma \to \tau \quad \Gamma \vdash^{\wedge\vee} N : \sigma}{\Gamma \vdash^{\wedge\vee} MN : \tau}$$

$$(\wedge I) \qquad \frac{\Gamma \vdash^{\wedge\vee} M : \sigma \quad \Gamma \vdash^{\wedge\vee} M : \tau}{\Gamma \vdash^{\wedge\vee} M : \sigma \wedge \tau}$$

$$(\vee E) \qquad \frac{\Gamma, x : \sigma \vdash^{\wedge\vee} M : \rho \quad \Gamma, x : \tau \vdash^{\wedge\vee} M : \rho \quad \Gamma \vdash^{\wedge\vee} N : \sigma \vee \tau}{\Gamma \vdash^{\wedge\vee} M[N/x] : \rho}$$

$$(\leq) \qquad \frac{\Gamma \vdash^{\wedge\vee} M : \sigma \qquad \sigma \leq \tau}{\Gamma \vdash^{\wedge\vee} M : \tau}$$

For example, $\vdash^{\wedge\vee} \mathsf{SII} : (\alpha \wedge (\alpha \to \beta) \to \beta) \vee \delta$. Clearly, if $\Gamma \vdash^{\wedge\vee} M : \sigma$ and $\Gamma' \supseteq \Gamma$, then $\Gamma' \vdash^{\wedge\vee} M : \sigma$. The rules

$$(\wedge E) \qquad \frac{\Gamma \vdash^{\wedge\vee} M : \sigma \wedge \tau}{\Gamma \vdash^{\wedge\vee} M : \sigma} \qquad \frac{\Gamma \vdash^{\wedge\vee} M : \sigma \wedge \tau}{\Gamma \vdash^{\wedge\vee} M : \tau}$$

and

$$(\vee I) \qquad \frac{\Gamma \vdash^{\wedge\vee} M : \sigma}{\Gamma \vdash^{\wedge\vee} M : \sigma \vee \tau} \qquad \frac{\Gamma \vdash^{\wedge\vee} M : \tau}{\Gamma \vdash^{\wedge\vee} M : \sigma \vee \tau}$$

are derivable in $\mathsf{TA}_{\wedge\vee}$, as particular cases of rule $(\leq)$. Moreover, it is easy to check that the following rules are admissible:

$$(\leq L) \qquad \frac{\Gamma, x : \sigma \vdash^{\wedge\vee} M : \tau \quad \rho \leq \sigma}{\Gamma, x : \rho \vdash^{\wedge\vee} M : \tau}$$

$$(\vee IL) \qquad \frac{\Gamma, x : \sigma \vdash^{\wedge\vee} M : \rho \qquad \Gamma, x : \tau \vdash^{\wedge\vee} M : \rho}{\Gamma, x : \sigma \vee \tau \vdash^{\wedge\vee} M : \rho}$$

The admissibility of rule $(\leq L)$ is an immediate consequence of using rule $(\leq)$ on premises. For rule $(\vee IL)$ notice that its assumptions imply $\Gamma, y : \sigma \vdash^{\wedge\vee} M[y/x] : \rho$ and $\Gamma, y : \tau \vdash^{\wedge\vee} M[y/x] : \rho$, where y does not occur in Γ and M, since derivations are independent from the names of free term variables. So the following deduction shows the admissibility of $(\vee IL)$:

$$(\vee E) \qquad \frac{\begin{array}{c}\Gamma, x : \sigma \vee \tau, y : \sigma \vdash^{\wedge\vee} M[y/x] : \rho \\ \Gamma, x : \sigma \vee \tau, y : \tau \vdash^{\wedge\vee} M[y/x] : \rho \qquad \Gamma, x : \sigma \vee \tau \vdash^{\wedge\vee} x : \sigma \vee \tau\end{array}}{\Gamma, x : \sigma \vee \tau \vdash^{\wedge\vee} M : \rho}$$

The aim of this paper, as mentioned in the introduction, is to investigate the logical characterization of the system $\mathsf{TA}_{\wedge\vee}$ by means of the Curry-Howard approach.

Intersection and union considered as type forming operators look like the propositional connectives *conjunction* and *disjunction*, respectively. However in [12] it is shown that intersection on types does not correspond to (product) types, that is, to propositional conjunction. After a brief look at the type inference system one can notice immediately that in order to assign an intersection type, say $\sigma \wedge \tau$, to a term M from a basis Γ it is necessary to assign separately both types σ and τ to the term M from the same basis Γ. Thus the term M in the conclusion is the same term in both premises. This is a point where the Curry-Howard isomorphism is lost, since the term remains the same, although the inference (deduction) grows, that is, terms are not encoding deductions anymore. Nevertheless an equivalent type inference system for the restriction of $\mathsf{TA}_{\wedge\vee}$ to the intersection types is conceived in [20], together with the Hilbert-style logical system which corresponds to it by the Curry-Howard isomorphism. The main idea in [20] is to avoid rule $(\wedge I)$ replacing it by "structural" inference rules.

If we analyze the union types in the same framework, we notice that a similar difficulty arises in the case of rule $(\vee E)$, since the term M has to be the same in both premises. So our aim is to consider a system containing both intersection and union types as defined above and then to show that rule $(\vee E)$ can be avoided too, still assigning the same types to the same combinators.

First, let us discuss the consequences of leaving out rule $(\vee E)$. In [4] it is shown that the type inference system $\mathsf{TA}_{\wedge\vee}$ is closed under parallel β-reduction, that is, types are preserved under parallel β-reduction. The following example shows that this property is lost by leaving out rule $(\vee E)$.

Example 2.6 Let $\Gamma = \{x : (\sigma \to \sigma \to \rho) \wedge (\tau \to \tau \to \rho),\ \ y : \varphi \to \sigma \vee \tau,\ \ z : \varphi\}$. We have the following derivation in $\mathsf{TA}_{\wedge\vee}$

$$(\vee E) \qquad \frac{\Gamma,\ t : \sigma \vdash^{\wedge\vee} xtt : \rho \quad \Gamma,\ t : \tau \vdash^{\wedge\vee} xtt : \rho \quad \Gamma \vdash^{\wedge\vee} yz : \sigma \vee \tau}{\Gamma \vdash^{\wedge\vee} x(yz)(yz) : \rho}$$

We obtain the following derivation without rule $(\vee E)$, from the same basis Γ by setting

$$\mathsf{I} : (\sigma \to \sigma) \wedge (\tau \to \tau), \text{ and}$$

$$\mathsf{S} : ((\sigma \to \sigma \to \rho) \to (\sigma \to \sigma) \to \sigma \to \rho) \wedge ((\tau \to \tau \to \rho) \to (\tau \to \tau) \to \tau \to \rho),$$

$$(\to E) \qquad \frac{\dfrac{\Gamma \vdash^{\wedge\vee} \mathsf{S}x\mathsf{I} : (\sigma \to \rho) \wedge (\tau \to \rho)}{\Gamma \vdash^{\wedge\vee} \mathsf{S}x\mathsf{I} : \sigma \vee \tau \to \rho}\ (\leq) \qquad \Gamma \vdash^{\wedge\vee} yz : \sigma \vee \tau}{\Gamma \vdash^{\wedge\vee} \mathsf{S}x\mathsf{I}(yz) : \rho}$$

However, $x(yz)(yz)$ is the normal form of $\mathsf{S}x\mathsf{I}(yz)$. Without rule $(\vee E)$ it is not possible to derive $\Gamma \vdash^{\wedge\vee} x(yz)(yz) : \rho$. This shows that types are not preserved under any notion of reduction. Thus we obtain the same type inhabited, but for an expanded term.

2.2 How to avoid the* $(\vee E)$ *rule It is shown in [4] that the type inference system $\mathsf{TA}_{\wedge\vee}$ is not closed under weak reduction. However, a union type inference system is introduced in [3], which is shown to be closed under weak reduction. This system is characterized by a new preorder on types, denoted by $\leq^*$, defined as an extension of the preorder $\leq$ which is given in Definition 2.4. $\leq^*$ uses the predicate $P : T \to \{\mathit{true}, \mathit{false}\}$, which is essentially a syntactical constraint such that P is true for a type τ if $\vee$ occurs in τ only in the left arguments of some arrows.

Definition 2.7 (Preorder relation $\leq^*$ on T)

(i) The predicate $P : T \to \{\mathit{true}, \mathit{false}\}$ is defined by induction on types as follows:

$$\begin{array}{lcl} P(\alpha) & = & \mathit{true} \\ P(\sigma \to \tau) & = & P(\tau) \\ P(\sigma \wedge \tau) & = & P(\sigma) \ \mathit{and} \ P(\tau) \ \text{(Here \textit{and} is the usual logical conjunction.)} \\ P(\sigma \vee \tau) & = & \mathit{false}. \end{array}$$

(ii) The preorder relation $\leq^*$ on T is inductively defined by adding to the axioms and the rules of Definition 2.4 the following axiom.

7. $\sigma \to \rho \vee \tau \leq^* (\sigma \to \rho) \vee (\sigma \to \tau)$ for any σ such that $P(\sigma)$ is true.

(iii) $\sigma \sim^* \tau$ if and only if $\sigma \leq^* \tau$ and $\tau \leq^* \sigma$.

Notice that we can prove the following equivalence relation:

$$(***) \qquad \sigma \to \rho \vee \tau \ \sim^* \ (\sigma \to \rho) \vee (\sigma \to \tau)$$

for any σ such that $P(\sigma)$ is true. $(***)$ holds by axiom 7 in one direction, and by axioms 1 and 3 and rules 2, 3, and 4 in the other one. Using $(*)$, $(**)$, and $(***)$, we can rewrite any type σ as an equivalent type $\sigma_1 \vee \cdots \vee \sigma_n$ $(n > 0)$, such that each σ_i does not contain the union type operator. Namely, a very simple procedure can be devised to pull out all union types from the inside of σ. In fact by $(*)$ we can eliminate union types on the left side of arrows. By $(**)$ we can pull unions out of intersections. Lastly by $(***)$ we pull out unions which occur in the right arguments of arrows. For example, we have

$$(\alpha \to \mu \vee \nu) \to \varphi \sim^* ((\alpha \to \mu) \to \varphi) \wedge ((\alpha \to \nu) \to \varphi).$$

In fact,

$$(\alpha \to \mu \vee \nu) \to \varphi \sim^* (\alpha \to \mu) \vee (\alpha \to \nu) \to \varphi$$

by $(***)$ and

$$(\alpha \to \mu) \vee (\alpha \to \nu) \to \varphi \sim^* ((\alpha \to \mu) \to \varphi) \wedge ((\alpha \to \nu) \to \varphi)$$

by $(*)$. The mapping $\mathbf{m}$ of Definition 2.13 will show this property in a formal way, by associating to each type σ a set of types $\mathbf{m}(\sigma) = \{\sigma_1, \ldots, \sigma_n\}$, such that $\sigma \sim^* \sigma_1 \vee \cdots \vee \sigma_n$ and no σ_i contains $\vee$.

Remark 2.8 In the definition of $\leq^*$, the additional axiom 7 does not naturally arise from the interpretation of $\to$ as function space constructor. However, if types are interpreted as subsets of Scott domains which can model the lambda calculus, then "$P(\sigma)$ is true" implies that the interpretation of σ has a least element; when $P(\sigma)$ is false, either the interpretation of σ does not have a least element, or there exists a τ such that $\sigma \sim^* \tau$ and $P(\tau)$ is true. There is an obvious relation between the present P and the predicate C as defined in Abramsky [1], that is, $P(\sigma) = \mathit{true}$ implies $C(\sigma) = \mathit{true}$ and $C(\sigma) = \mathit{true}$ implies that there is a τ such that $\sigma \sim^* \tau$ and $P(\tau) = \mathit{true}$. Actually in [1] types are interpreted as *compact-open* subsets and the condition C means "to be a *coprime*."

Definition 2.9 (The type assignment system $\mathsf{TA}^*_{\wedge\vee}$) The type inference system $\mathsf{TA}^*_{\wedge\vee}$ is defined by the axioms and rules of Definition 2.5, where rule $(\leq)$ is substituted by rule $(\leq^*)$, obtained by replacing the preorder $\leq$ by $\leq^*$. We write $\Gamma \vdash^* M : \sigma$ for the derivability in this system.

This is the type inference system introduced and investigated in [4]. The derivability in the subsystem of [16] restricted to the type constructors $\to$, $\wedge$, and $\vee$ implies derivability in the system $\mathsf{TA}^*_{\wedge\vee}$.

Let $\mathsf{TA}_\wedge$ denote the system obtained from $\mathsf{TA}^*_{\wedge\vee}$ by restricting types to intersection types only. In order to show that $\mathsf{TA}^*_{\wedge\vee}$ is closed under weak reduction one has to relate it to $\mathsf{TA}_\wedge$. Then the problem is shifted to $\mathsf{TA}_\wedge$, which is known to be closed under weak reduction.

Definition 2.10 (The type assignment system $\mathsf{TA}_\wedge$)

- (i) $T_\wedge$ is the set of *intersection types* built out of type variables using only the type constructors $\to$ and $\wedge$.
- (ii) The preorder $\leq_\wedge$ is the relation defined by restricting all axioms and rules in Definition 2.4 to types in $T_\wedge$.
- (iii) The system $\mathsf{TA}_\wedge$ is the subsystem of $\mathsf{TA}^*_{\wedge\vee}$ where only types from $T_\wedge$ are used.

We write $\Gamma \vdash^\wedge M : \sigma$ for the derivability in this system.

Namely, in $\mathsf{TA}_\wedge$, rule $(\vee E)$ is not included and in rule $(\leq^*)$ the preorder $\leq^*$ is replaced by the preorder $\leq_\wedge$. Rule $(\leq_\wedge)$ will denote this restriction of rule $(\leq^*)$. Theorem 2.20 will show that $\mathsf{TA}^*_{\wedge\vee}$ is conservative over $\mathsf{TA}_\wedge$. The system $\mathsf{TA}_\wedge$ for deriving intersection types for CL-terms has been formulated in [9], where the following property is proved for a number of abstraction algorithms, including the one considered in the present paper.

Lemma 2.11 ([9]) $\quad \Gamma, x : \sigma \vdash^\wedge M : \tau \Longleftrightarrow \Gamma \vdash^\wedge \lambda^* x.M : \sigma \to \tau.$

In order to associate to each type in T a set of intersection types (a subset of $T_\wedge$), we can use some properties of types which are stated in [4]. The mapping of intersection and union type derivability into intersection type derivability, stated in the following Theorem 2.20, means that rule $(\vee E)$ can be avoided in type derivations by essentially using rule $(\leq^*)$. We shall obtain, as a main result, that any derivation in $\mathsf{TA}^*_{\wedge\vee}$ is associated to a set of derivations in $\mathsf{TA}_\wedge$.

First we define a mapping $\mathbf{m}$ between types with $\to$, $\vee$, and $\wedge$, and sets of types without $\vee$. This mapping is extended to bases in a natural way.

Notation 2.12 $\quad \Xi(I_1, \ldots, I_n, J)$ is the finite set of all functions from $I_1 \times \cdots \times I_n$ to J, where $I_1, \ldots, I_n, J$ range over finite sets of indexes.

Definition 2.13 (Mapping between T and $T_\wedge$)

- (i) The mapping $\mathbf{m} : T \to \mathcal{P}(T_\wedge)$ is inductively defined by

$$\begin{array}{lcl} \mathbf{m}(\alpha) & = & \{\alpha\} \\ \mathbf{m}(\rho \to \tau) & = & \{\wedge_{i\in I}(\rho_i \to \tau_{\chi(i)}) \mid \chi \in \Xi(I, J)\} \\ \mathbf{m}(\rho \wedge \tau) & = & \{\rho_i \wedge \tau_j \mid i \in I,\ j \in J\} \\ \mathbf{m}(\rho \vee \tau) & = & \mathbf{m}(\rho) \cup \mathbf{m}(\tau), \end{array}$$

  under the assumptions $\mathbf{m}(\rho) = \{\rho_i \mid i \in I\}$ and $\mathbf{m}(\tau) = \{\tau_j \mid j \in J\}$.
- (ii) If Γ is a basis, then $\mathbf{B}(\Gamma)$ is the set of bases defined by

$$\begin{array}{l} \mathbf{B}(\Gamma) = \{\Gamma' \mid FV(\Gamma) = FV(\Gamma') \text{ and} \\ \qquad\qquad x : \sigma \in \Gamma \Longrightarrow \exists \sigma' \in \mathbf{m}(\sigma) \text{ such that } \quad x : \sigma' \in \Gamma'\}. \end{array}$$

A way of rephrasing 2.13(ii) is: if $x : \sigma$ is a statement in Γ, then each Γ' in $\mathbf{B}(\Gamma)$ will contain one statement $x : \sigma'$ for some $\sigma' \in \mathbf{m}(\sigma)$ (and vice versa). Notice that only types without $\vee$ occur in the bases belonging to $\mathbf{B}(\Gamma)$.

Example 2.14 If $\sigma \equiv \alpha_1 \vee \alpha_2 \to \beta_1 \vee \beta_2$, and $\mathbf{2} = \{1, 2\}$, then we have

$$\begin{array}{lcl} \mathbf{m}(\alpha_1 \vee \alpha_2) & = & \{\alpha_1, \alpha_2\} \\ \mathbf{m}(\beta_1 \vee \beta_2) & = & \{\beta_1, \beta_2\} \\ \mathbf{m}(\sigma) & = & \{\wedge_{1 \le i \le 2}(\alpha_i \to \beta_{\chi(i)}) \mid \chi \in \Xi(\mathbf{2}, \mathbf{2})\} \\ & = & \{(\alpha_1 \to \beta_1) \wedge (\alpha_1 \to \beta_2) \wedge (\alpha_2 \to \beta_1) \wedge (\alpha_2 \to \beta_2)\}. \end{array}$$

If $\Gamma = \{x : \alpha_1 \vee \alpha_2, y : \beta_1 \vee \beta_2\}$, then $\mathbf{B}(\Gamma)$ contains exactly the following four bases:

$$\begin{array}{lcl} \Gamma_1 & = & \{x : \alpha_1, y : \beta_1\} \\ \Gamma_2 & = & \{x : \alpha_1, y : \beta_2\} \\ \Gamma_3 & = & \{x : \alpha_2, y : \beta_1\} \\ \Gamma_4 & = & \{x : \alpha_2, y : \beta_2\}. \end{array}$$

Lemma 2.15 *For all $\sigma \in T$, if $\mathbf{m}(\sigma) = \{\sigma_1, \ldots, \sigma_n\}$, then $\sigma \sim^* \sigma_1 \vee \cdots \vee \sigma_n$.*

Proof: By induction on σ. All cases follow easily from Definition 2.13(i) except $\sigma \equiv \rho \to \tau$. Let $\mathbf{m}(\rho) = \{\rho_i \mid i \in I\}$ and $\mathbf{m}(\tau) = \{\tau_j \mid j \in J\}$, then $\mathbf{m}(\sigma) = \{\wedge_{i \in I}(\rho_i \to \tau_{\chi(i)}) \mid \chi \in \Xi(I, J)\}$. By induction we have $\rho \sim^* \vee_{i \in I} \rho_i$. This implies $\sigma \sim^* \vee_{i \in I} \rho_i \to \tau$. By iterated applications of the equivalence $(*)$ to $\vee_{i \in I} \rho_i \to \tau$ we can derive

$$\sigma \sim^* \wedge_{i \in I}(\rho_i \to \tau) \tag{1}$$

Again by induction $\tau \sim^* \vee_{j \in J} \tau_j$. It is easy to verify that $\mathbf{P}(\mu)$ is true for all types $\mu \in T_\wedge$, so in particular we have $\mathbf{P}(\rho_i)$ true for all $i \in I$. By iterated applications of the equivalence $(***)$ to $\rho_i \to \vee_{j \in J} \tau_j$ we can derive

$$\rho_i \to \vee_{j \in J} \tau_j \sim^* \vee_{j \in J}(\rho_i \to \tau_j) \tag{2}$$

From (1) and (2) we have

$$\sigma \sim^* \wedge_{i \in I} \vee_{j \in J} (\rho_i \to \tau_j),$$

and by repeatedly applying the equivalence $(**)$ we conclude

$$\sigma \sim^* \vee_{\chi \in \Xi(I,J)} \wedge_{i \in I} (\rho_i \to \tau_{\chi(i)}). \qquad \square$$

The following property of the mapping **m** was stated and proved in Lemma 4.4 of [4].

Lemma 2.16 *For all $\sigma, \tau \in T$, $\sigma \le^* \tau$ if and only if for all $\sigma' \in \mathbf{m}(\sigma)$ there is $\tau' \in \mathbf{m}(\tau)$ such that $\sigma' \le_\wedge \tau'$.*

Now we can prove the main result, that is, we can associate to each derivation in $\mathsf{TA}^*_{\wedge\vee}$ a set of derivations in $\mathsf{TA}_\wedge$, one for each basis in $\mathbf{B}(\Gamma)$, where Γ is the basis which occurs in the conclusion of the current derivation. The same property has been proved in [4] for λ-calculus instead of combinatory logic. Notice that the difference between the present systems and those given in [4] is the replacement of axioms **S**, **K**, and **I** for the standard rule of $\to$-introduction. For this reason, we have to prove the property for the axioms **S**, **K**, and **I** only, while for rules we refer to the proof of Theorem 4.6 in [4].

Lemma 2.17 *If* $\vdash^* C : \zeta$ *is an axiom, then there is* $\zeta' \in \mathbf{m}(\zeta)$ *such that* $\vdash^\wedge C : \zeta'$.

Proof: We distinguish the cases $C \equiv \mathbf{S}, \mathbf{K}$, or $\mathbf{I}$.

Case 1 $C \equiv \mathbf{I}$*:* In this case, then $\zeta \equiv \sigma \to \sigma$. By Definition 2.13(i) if $\mathbf{m}(\sigma) = \{\sigma_i \mid i \in I\}$, then $\mathbf{m}(\sigma \to \sigma) = \{\wedge_{i\in I}(\sigma_i \to \sigma_{\chi(i)}) \mid \chi \in \Xi(I, I)\}$. Let us choose $\chi(i)$ as the identity, that is, $\chi(i) = i$, so that $\zeta' \equiv \wedge_{i\in I}(\sigma_i \to \sigma_i) \in \mathbf{m}(\sigma \to \sigma)$. By axiom $\mathbf{I}$, $\vdash^\wedge \mathbf{I} : \sigma_i \to \sigma_i$ for all $i \in I$, since $\sigma_i \in T_\wedge$ by construction. Therefore $\vdash^\wedge \mathbf{I} : \wedge_{i\in I}(\sigma_i \to \sigma_i)$ by using $(\wedge I)$.

Case 2 $C \equiv \mathbf{K}$*:* In this case, then $\zeta \equiv \sigma \to \tau \to \sigma$. By Definition 2.13(i) if $\mathbf{m}(\sigma) = \{\sigma_i \mid i \in I\}$ and $\mathbf{m}(\tau) = \{\tau_j \mid j \in J\}$, then $\mathbf{m}(\sigma \to \tau \to \sigma) = \{\wedge_{i\in I} \wedge_{j\in J} (\sigma_i \to \tau_j \to \sigma_{\chi(i,j)}) \mid \chi \in \Xi(I, J, I)\}$. Let us choose $\chi(i, j)$ as the first projection, that is, $\chi(i, j) = i$, so that $\zeta' \equiv \wedge_{i\in I} \wedge_{j\in J} (\sigma_i \to \tau_j \to \sigma_i) \in \mathbf{m}(\sigma \to \tau \to \sigma)$. By axiom $\mathbf{K}$, $\vdash^\wedge \mathbf{K} : \sigma_i \to \tau_j \to \sigma_i$ for all $i \in I$ and $j \in J$, since $\sigma_i, \tau_j \in T_\wedge$ by construction. Therefore $\vdash^\wedge \mathbf{K} : \wedge_{i\in I} \wedge_{j\in J} (\sigma_i \to \tau_j \to \sigma_i)$ by using $(\wedge I)$.

Case 3 $C \equiv \mathbf{S}$*:* In this case, then $\zeta \equiv (\sigma \to \tau \to \rho) \to (\sigma \to \tau) \to \sigma \to \rho$. Let $\Xi(\Xi(I, J, L), \Xi(I, J), I, L)$ denote the finite set of functions from $\Xi(I, J, L) \times \Xi(I, J) \times I$ to L. By Definition 2.13(i) if $\mathbf{m}(\sigma) = \{\sigma_i \mid i \in I\}$, $\mathbf{m}(\tau) = \{\tau_j \mid j \in J\}$, and $\mathbf{m}(\rho) = \{\rho_l \mid l \in L\}$, then $\mathbf{m}((\sigma \to \tau \to \rho) \to (\sigma \to \tau) \to \sigma \to \rho) = \{\wedge_{\chi_1\in\Xi(I,J,L)} \wedge_{\chi_2\in\Xi(I,J)} \wedge_{i\in I}[\varphi^{(\chi_1)} \to \psi^{(\chi_2)} \to \sigma_i \to \rho_{\chi_3(\chi_1,\chi_2,i)}] \mid \chi_3 \in \Xi(\Xi(I, J, L), \Xi(I, J), I, L)\}$, where $\varphi^{(\chi_1)} \equiv \wedge_{i'\in I} \wedge_{j'\in J} (\sigma_{i'} \to \tau_{j'} \to \rho_{\chi_1(i',j')})$, and $\psi^{(\chi_2)} \equiv \wedge_{i''\in I}(\sigma_{i''} \to \tau_{\chi_2(i'')})$. Let us choose $\chi_3(\chi_1, \chi_2, i) = \chi_1(i, \chi_2(i))$, so that $\zeta' \equiv \wedge_{\chi_1\in\Xi(I,J,L)} \wedge_{\chi_2\in\Xi(I,J)} \wedge_{i\in I}[\varphi^{(\chi_1)} \to \psi^{(\chi_2)} \to \sigma_i \to \rho_{\chi_1(i,\chi_2(i))}] \in \mathbf{m}(\zeta)$.

By axiom $\mathbf{S}$, $\vdash^\wedge \mathbf{S} : (\sigma_i \to \tau_j \to \rho_l) \to (\sigma_i \to \tau_j) \to \sigma_i \to \rho_l$ for all $i \in I$, $j \in J$, $l \in L$ since $\sigma_i, \tau_j, \rho_l \in T_\wedge$ by construction. Therefore $\vdash^\wedge \mathbf{S} : \wedge_{l\in L} \wedge_{j\in J} \wedge_{i\in I}[(\sigma_i \to \tau_j \to \rho_l) \to (\sigma_i \to \tau_j) \to \sigma_i \to \rho_l]$ by using $(\wedge I)$.

Now we conclude the proof by showing that

$$(\clubsuit) \qquad \wedge_{l\in L} \wedge_{j\in J} \wedge_{i\in I}[(\sigma_i \to \tau_j \to \rho_l) \to (\sigma_i \to \tau_j) \to \sigma_i \to \rho_l] \leq_\wedge \zeta',$$

that is, ζ' can be assigned to $\mathbf{S}$ by using $(\leq_\wedge)$. To this aim let us define the type

$$\mu \equiv \wedge_{\chi_1\in\Xi(I,J,L)} \wedge_{\chi_2\in\Xi(I,J)} \wedge_{i\in I}[(\sigma_i \to \tau_{\chi_2(i)} \to \rho_{\chi_1(i,\chi_2(i))}) \\ \to (\sigma_i \to \tau_{\chi_2(i)}) \to \sigma_i \to \rho_{\chi_1(i,\chi_2(i))}],$$

then we prove $(\clubsuit)$ in two steps. First we show that $\mu \leq_\wedge \zeta'$. Second, we verify that $\wedge_{l\in L} \wedge_{j\in J} \wedge_{i\in I}[(\sigma_i \to \tau_j \to \rho_l) \to (\sigma_i \to \tau_j) \to \sigma_i \to \rho_l] \equiv \mu$. First notice that $\wedge_{i'\in I} \wedge_{j'\in J} (\sigma_{i'} \to \tau_{j'} \to \rho_{\chi_1(i',j')}) \leq_\wedge \sigma_{i'} \to \tau_{j'} \to \rho_{\chi_1(i',j')}$ for all $i' \in I$, $j' \in J$, so in particular when $i' = i$ and $j' = \chi_2(i)$. Analogously $\wedge_{i''\in I}(\sigma_{i''} \to \tau_{\chi_2(i'')}) \leq_\wedge \sigma_{i''} \to \tau_{\chi_2(i'')}$ for all $i'' \in I$, so in particular when $i'' = i$. Therefore $\mu \leq_\wedge \zeta'$ because of the contravariance of the arrow in the definition of $\leq_\wedge$.

Second, let us replace $\wedge_{\chi_2\in\Xi(I,J)}$ by $\wedge_{j\in J}$ in μ. This leaves μ unchanged, since $\chi_2(i)$ is used only when the argument i is fixed. So we can rewrite μ as follows.

$$\wedge_{\chi_1\in\Xi(I,J,L)} \wedge_{j\in J} \wedge_{i\in I}[(\sigma_i \to \tau_j \to \rho_{\chi_1(i,j)}) \to (\sigma_i \to \tau_j) \to \sigma_i \to \rho_{\chi_1(i,j)}].$$

Analogously we can replace $\wedge_{\chi_1 \in \Xi(I,J,L)}$ by $\wedge_{l \in L}$ in the last type. So we can rewrite μ as follows:

$$\wedge_{l \in L} \wedge_{j \in J} \wedge_{i \in I}[(\sigma_i \to \tau_j \to \rho_l) \to (\sigma_i \to \tau_j) \to \sigma_i \to \rho_l]. \qquad \Box$$

Example 2.18 If $\sigma \equiv (\alpha_1 \vee \alpha_2 \to \beta \to \gamma) \to (\alpha_1 \vee \alpha_2 \to \beta) \to \alpha_1 \vee \alpha_2 \to \gamma$, then

$$\begin{array}{rcl} \mathbf{m}(\sigma) & = & \{(\alpha_1 \to \beta \to \gamma) \wedge (\alpha_2 \to \beta \to \gamma) \to \\ & & (\alpha_1 \to \beta) \wedge (\alpha_2 \to \beta) \to (\alpha_1 \to \gamma) \wedge (\alpha_2 \to \gamma)\}. \end{array}$$

We have $\vdash^* \ \mathbf{S} : \sigma$, and also

$$\vdash^\wedge \ \mathbf{S} : (\alpha_1 \to \beta \to \gamma) \wedge (\alpha_2 \to \beta \to \gamma) \to (\alpha_1 \to \beta) \wedge (\alpha_2 \to \beta) \to (\alpha_1 \to \gamma) \wedge (\alpha_2 \to \gamma),$$

by axiom ($\mathbf{S}$) and rule $(\leq_\wedge)$.

Example 2.19 If $\sigma \equiv (\alpha \to \beta \to \gamma_1 \vee \gamma_2) \to (\alpha \to \beta) \to \alpha \to \gamma_1 \vee \gamma_2$, then

$$\begin{array}{rcl} \mathbf{m}(\sigma) & = & \{[(\alpha \to \beta \to \gamma_1) \to (\alpha \to \beta) \to \alpha \to \gamma_1] \wedge \\ & & [(\alpha \to \beta \to \gamma_2) \to (\alpha \to \beta) \to \alpha \to \gamma_1], \\ & & [(\alpha \to \beta \to \gamma_1) \to (\alpha \to \beta) \to \alpha \to \gamma_1] \wedge \\ & & [(\alpha \to \beta \to \gamma_2) \to (\alpha \to \beta) \to \alpha \to \gamma_2], \\ & & [(\alpha \to \beta \to \gamma_1) \to (\alpha \to \beta) \to \alpha \to \gamma_2] \wedge \\ & & [(\alpha \to \beta \to \gamma_2) \to (\alpha \to \beta) \to \alpha \to \gamma_1], \\ & & [(\alpha \to \beta \to \gamma_1) \to (\alpha \to \beta) \to \alpha \to \gamma_2] \wedge \\ & & [(\alpha \to \beta \to \gamma_2) \to (\alpha \to \beta) \to \alpha \to \gamma_2]\}. \end{array}$$

We have $\vdash^* \ \mathbf{S} : \sigma$, and also

$$\vdash^\wedge \ \mathbf{S} : [(\alpha \to \beta \to \gamma_1) \to (\alpha \to \beta) \to \alpha \to \gamma_1] \wedge [(\alpha \to \beta \to \gamma_2) \to (\alpha \to \beta) \to \alpha \to \gamma_2],$$

by axiom ($\mathbf{S}$) and rule $(\wedge I)$.

Theorem 2.20 (Relations between $\mathsf{TA}^*_{\wedge\vee}$ and $\mathsf{TA}_\wedge$)

$$\Gamma \ \vdash^* \ M : \sigma \Longleftrightarrow \forall \Gamma' \in \mathbf{B}(\Gamma) \ \ \exists \sigma' \in \mathbf{m}(\sigma). \Gamma' \ \vdash^\wedge \ M : \sigma'.$$

Proof: By induction on derivations. The first step is proved in Lemma 2.17. For the induction step, we can use the proof of Theorem 4.6 in [4]. $\Box$

Theorem 2.20 allows us to extend the most important properties of intersection type disciplines to union types as well, in particular the invariance of types under weak reduction of subjects.

Theorem 2.21 (Invariance of types under weak reduction)

(i) $\Gamma \vdash^\wedge M : \sigma$ *and* $M \to N \Longrightarrow \Gamma \vdash^\wedge N : \sigma$.
(ii) $\Gamma \vdash^* M : \sigma$ *and* $M \to N \Longrightarrow \Gamma \vdash^* N : \sigma$.

Proof:

(i) This property is the invariance of types under weak reduction proved in [9].

(ii) Immediately from (i) and Theorem 2.20. □

The meaning of Theorem 2.20 is that one associates to each deduction in $\mathsf{TA}^*_{\wedge\vee}$ a finite set of deductions in $\mathsf{TA}_{\wedge}$, in a unique way, for the same CL-term, mainly by eliminating rule $(\vee E)$ and by replacing rule $(\leq_{\wedge})$ for rule $(\leq^*)$. It is worth pointing out that axioms 5, 6, and 7 of the definition of the preorder $\leq^*$ are significant for the existence of the mapping $\mathbf{m}$. Hence their importance for the type invariance of $\mathsf{TA}^*_{\wedge\vee}$ is obvious.

We can also associate to each deduction in $\mathsf{TA}^*_{\wedge\vee}$ exactly one deduction in $\mathsf{TA}_{\wedge}$. The price that must be paid is that of considering the abstraction λ^* of the given CL-term with respect to all variables which have union types as predicates in the current basis. The remaining part of the present section is devoted to this proof.

The following lemma allows to eliminate the unions occurring in Γ by taking advantage of $\mathbf{B}(\Gamma)$.

Lemma 2.22

(i) If $\sigma_i \sim^* \vee_{j\in J_i}\sigma_i^{(j)}$ $(1 \leq i \leq n)$, *then*

$$\wedge_{j_1\in J_1}\cdots\wedge_{j_n\in J_n}(\sigma_1^{(j_1)} \to \cdots \to \sigma_n^{(j_n)} \to \tau) \sim^* \sigma_1 \to \cdots \to \sigma_n \to \tau.$$

(ii) Let $\mathbf{B}(\Gamma) = \{\Gamma_h\}_{1\leq h\leq m}$ *where* $\Gamma_h = \{x_i : \sigma_i^{(h)}\}_{1\leq i\leq n}$. *If* $\sigma_i \sim^* \vee_{1\leq h\leq m}\sigma_i^{(h)}$, *then*

$$\wedge_{1\leq h\leq m}(\sigma_1^{(h)} \to \cdots \to \sigma_n^{(h)} \to \tau) \sim^* \sigma_1 \to \cdots \to \sigma_n \to \tau.$$

Proof:

(i) First notice that we have

$$(\rho \to \mu) \wedge (\rho \to \nu) \sim^* (\rho \to \mu \wedge \nu)$$

by axiom 4 in one direction, and by axioms 1 and 2 and rules 1, 3, and 4 in the other one. This implies

$$\begin{aligned}\wedge_{j_n\in J_n}(\sigma_1 \to \cdots \to \sigma_{n-1} \to \sigma_n^{(j_n)} \to \tau) \sim^* \sigma_1 \to \cdots \\ \to \sigma_{n-1} \to \wedge_{j_n\in J_n}(\sigma_n^{(j_n)} \to \tau)\end{aligned} \tag{3}$$

By iterated applications of equivalence $(*)$ to $\sigma_n \sim^* \vee_{j\in J_n}\sigma_n^{(j)}$, we have

$$\wedge_{j_n\in J_n}(\sigma_n^{(j_n)} \to \tau) \sim^* \sigma_n \to \tau \tag{4}$$

From (3) and (4) we derive:

$$\wedge_{j_n\in J_n}(\sigma_1 \to \cdots \to \sigma_{n-1} \to \sigma_n^{(j_n)} \to \tau) \sim^* \sigma_1 \to \cdots \to \sigma_{n-1} \to \sigma_n \to \tau.$$

Analogously, by repeating this argument in turn to $\sigma_{n-1}, \ldots, \sigma_1$, we prove the lemma.

(ii) Notice that, by construction, if $\mathbf{m}(\sigma_i) = \{\sigma_i^{(j)} \mid j \in J_i\}$, then

$$\wedge_{j_1 \in J_1} \cdots \wedge_{j_n \in J_n} (\sigma_1^{(j_1)} \to \cdots \to \sigma_n^{(j_n)} \to \tau) \sim^* \wedge_{1 \leq h \leq m}(\sigma_1^{(h)} \to \cdots \to \sigma_n^{(h)} \to \tau)$$

for all τ. Therefore (i) implies

$$\wedge_{1 \leq h \leq m}(\sigma_1^{(h)} \to \cdots \to \sigma_n^{(h)} \to \tau) \sim^* \sigma_1 \to \cdots \to \sigma_n \to \tau. \qquad \square$$

Obviously, any basis Γ can be split into a basis $\Gamma_\wedge$, containing the assumptions whose predicates can be written without $\vee$, and a basis $\Gamma_\vee$, containing all the predicates which require the $\vee$-operator.

Definition 2.23 For any basis Γ, let $\Gamma_\wedge$ and $\Gamma_\vee$ denote the following related bases:

1. $\Gamma_\wedge = \{x : \sigma \mid x : \sigma \in \Gamma$ and $\mathbf{m}(\sigma)$ contains only one type$\}$, and
2. $\Gamma_\vee = \Gamma - \Gamma_\wedge$.

Theorem 2.24 (Mapping of $\mathsf{TA}^*_{\wedge\vee}$ into $\mathsf{TA}_\wedge$) *Let Γ be a basis and $\mathbf{B}(\Gamma_\vee) = \{\Gamma_h\}_{1 \leq h \leq m}$ where $\Gamma_h = \{x_i : \sigma_i^{(h)}\}_{1 \leq i \leq n}$. Then*

$$\Gamma \vdash^* M : \tau \Longleftrightarrow \Gamma_\wedge \vdash^\wedge \lambda^* x_1, \ldots, x_n.M : \wedge_{1 \leq h \leq m}(\sigma_1^{(h)} \to \cdots \to \sigma_n^{(h)} \to \tau^{(h)})$$

for a suitable $\tau^{(h)} \in \mathbf{m}(\tau)$.

Proof:

$(\Longrightarrow)$ $\Gamma \vdash^* M : \tau$

$\Longleftrightarrow$ $\forall \Gamma_h \exists \tau^{(h)} \in \mathbf{m}(\tau).\ \Gamma_\wedge \cup \Gamma_h \vdash^\wedge M : \tau^{(h)}$ (by Theorem 2.20, since $\mathbf{B}(\Gamma) = \{\Gamma_\wedge \cup \Gamma_h\}_{1 \leq h \leq m}$ by construction)

$\Longleftrightarrow$ $\forall \Gamma_h \exists \tau^{(h)} \in \mathbf{m}(\tau)$.

$\Gamma_\wedge \vdash^\wedge \lambda^* x_1, \ldots, x_n.M : \sigma_1^{(h)} \to \cdots \to \sigma_n^{(h)} \to \tau^{(h)}$ (by Lemma 2.11)

$\Longrightarrow$ $\Gamma_\wedge \vdash^\wedge \lambda^* x_1, \ldots, x_n.M : \wedge_{1 \leq h \leq m}(\sigma_1^{(h)} \to \cdots \to \sigma_n^{(h)} \to \tau^{(h)})$ by the rule $(\wedge I)$.

$(\Longleftarrow)$ Lemma 2.22(ii) implies

$\wedge_{1 \leq h \leq m}(\sigma_1^{(h)} \to \cdots \to \sigma_n^{(h)} \to \tau^{(h)}) \leq^* \sigma_1 \to \cdots \to \sigma_n \to \tau$,

recalling that $\tau^{(h)} \in \mathbf{m}(\tau)$ implies $\tau^{(h)} \leq^* \tau$.

$\Gamma_\wedge \vdash^\wedge \lambda^* x_1, \ldots, x_n.M : \wedge_{1 \leq h \leq m}(\sigma_1^{(h)} \to \cdots \to \sigma_n^{(h)} \to \tau^{(h)})$

$\Longrightarrow$ $\Gamma_\wedge \vdash^* \lambda^* x_1, \ldots, x_n.M : \sigma_1 \to \cdots \to \sigma_n \to \tau$ (by above and rule $(\leq^*)$)

$\Longrightarrow$ $\Gamma \vdash^* (\lambda^* x_1, \ldots, x_n.M) x_1, \ldots, x_n : \tau$ (by iterated applications of rule $(\to E)$)

$\Longrightarrow$ $\Gamma \vdash^* M : \tau$ by Theorem 2.21(ii). $\square$

Example 2.25 Let $\Gamma = \{x : (\sigma \to \sigma \to \rho) \wedge (\tau \to \tau \to \rho),\ y : \sigma \vee \tau\}$, then $\Gamma_\wedge = \{x : (\sigma \to \sigma \to \rho) \wedge (\tau \to \tau \to \rho)\}$ and $\mathbf{B}(\Gamma_\vee) = \{\Gamma_1, \Gamma_2\}$, where $\Gamma_1 = \{y : \sigma\}$ and $\Gamma_2 = \{y : \tau\}$. We have $\Gamma \vdash^* xyy : \rho$ and $\Gamma_\wedge \vdash^\wedge \lambda^* y.xyy : (\sigma \to \rho) \wedge (\tau \to \rho)$.

2.3 How to avoid the $(\wedge I)$ ***rule*** Rule $(\wedge I)$ has to be eliminated as well as $(\vee E)$, because of its proof-functional shape. This has already been done in [20], where a new formulation of the system $\mathsf{TA}_\wedge$ has been defined. This new formulation denoted $\mathsf{TA}^\#$ here (and $\mathsf{TA}^\star_\wedge$ in [20]) has been proved to be equivalent to the original system while avoiding rule $(\wedge I)$. The system $\mathsf{TA}^\#$ can be defined from the system $\mathsf{TA}_\wedge$ by

(i) replacing in the axioms "one instance" by "any finite intersection of instances,"
(ii) eliminating rule $(\wedge I)$.

Definition 2.26 (The type assignment system $\mathsf{TA}^\#$) Let us consider only types belonging to $\mathsf{T}_\wedge$. For any type $\sigma \in \mathsf{T}_\wedge$, let $[\sigma]^\#$ denote an arbitrary intersection of instances of σ, that is, any $\sigma_1 \wedge \cdots \wedge \sigma_n$ $(n \geq 1)$ such that each σ_i is an instance of $\sigma (1 \leq i \leq n)$. A statement $M : \sigma$ is derivable from a basis Γ, notation $\Gamma \vdash^\# M : \sigma$, if $\Gamma \vdash^\# M : \sigma$ can be obtained using the following axioms and rules.

(a) Axioms

$(\mathsf{S}^\#)$	$\mathsf{S}: [(\alpha \to \beta \to \delta) \to (\alpha \to \beta) \to \alpha \to \delta]^\#$
$(\mathsf{K}^\#)$	$\mathsf{K}: [\alpha \to \beta \to \alpha]^\#$
$(\mathsf{I}^\#)$	$\mathsf{I}: [\alpha \to \alpha]^\#$.

(b) Rules (VAR), $(\to E)$, $(\leq_\wedge)$.

Theorem 2.27 (Equivalence between $\mathsf{TA}_\wedge$ and $\mathsf{TA}^\#$ [20]) $\Gamma \vdash^\wedge M : \sigma \Longleftrightarrow \Gamma \vdash^\# M : \sigma$.

Hence, we can map any derivation of $\mathsf{TA}^*_{\wedge\vee}$ into a derivation of $\mathsf{TA}^\#$, by using first the mapping from $\mathsf{TA}^*_{\wedge\vee}$ to $\mathsf{TA}_\wedge$, and then the above equivalence between $\mathsf{TA}_\wedge$ and $\mathsf{TA}^\#$. Again we have to apply the mapping $\mathbf{m}$ to types and consider the abstraction of the given CL-term with respect to some term variables occurring in the current basis.

Corollary 2.28 (Mapping $\mathsf{TA}^*_{\wedge\vee}$ into $\mathsf{TA}^\#$) *Let* Γ *be a basis and* $\mathbf{B}(\Gamma_\vee) = \{\Gamma^{(h)}\}_{1\leq h\leq m}$, *where* $\Gamma^{(h)} = \{x_i : \sigma_i^{(h)}\}_{1\leq i\leq n}$. *It follows that*

$$\Gamma \vdash^* M : \tau \Longleftrightarrow \Gamma_\wedge \vdash^\# \lambda^* x_1, \ldots, x_n.M : \wedge_{1\leq h\leq m}(\sigma_1^{(h)} \to \cdots \to \sigma_n^{(h)} \to \tau^{(h)})$$

for a suitable $\tau^{(h)} \in \mathbf{m}(\tau)$.

Proof: $\Gamma \vdash^* M : \tau$

$$\Longleftrightarrow \quad \Gamma_\wedge \vdash^\wedge \lambda^* x_1, \ldots, x_n.M : \wedge_{1\leq h\leq m}(\sigma_1^{(h)} \to \cdots \to \sigma_n^{(h)} \to \tau^{(h)})$$
(by Theorem 2.24)
$$\Longleftrightarrow \quad \Gamma_\wedge \vdash^\# \lambda^* x_1, \ldots, x_n.M : \wedge_{1\leq h\leq m}(\sigma_1^{(h)} \to \cdots \to \sigma_n^{(h)} \to \tau^{(h)})$$
(by Theorem 2.27). □

A natural way of generalizing the system $\mathsf{TA}^\#$ is to extend the set of types from $T_\wedge$ to T and then to replace rule $(\leq_\wedge)$ by rule $(\leq^*)$ in order to assign intersection and union types to CL-terms.

Definition 2.29 (The type assignment system TA) Let σ and all the types of Γ belong to T. A statement $M : \sigma$ is derivable from a basis Γ, notation $\Gamma \vdash M : \sigma$, if $\Gamma \vdash M : \sigma$ can be obtained by using the following axioms and rules.

(a) Axioms $(\mathsf{S}^{\#})$, $(\mathsf{K}^{\#})$, $(\mathsf{I}^{\#})$

(b) Rules (VAR), $(\rightarrow E)$, $(\leq^*)$

TA is the final system allowing the assignment of intersection and union types to CL-terms, while avoiding rules $(\wedge I)$ and $(\vee E)$. So, we are interested in the relation between $\mathsf{TA}^*_{\wedge\vee}$ and TA. Since $(\mathsf{S}^{\#})$, $(\mathsf{K}^{\#})$, and $(\mathsf{I}^{\#})$ are derived rules in $\mathsf{TA}^*_{\wedge\vee}$, then $\mathsf{TA}^*_{\wedge\vee}$ is an extension of TA. But unfortunately, it is a proper extension. In other words, the equivalence between $\mathsf{TA}_{\wedge}$ and $\mathsf{TA}^{\#}$ cannot be generalized to union types. For example, $\{x : (\sigma \rightarrow \sigma \rightarrow \rho) \wedge (\tau \rightarrow \tau \rightarrow \rho), y : \sigma \vee \tau\} \vdash^* xyy : \rho$, but this does not hold for $\vdash$. Our result is that any derivation in $\mathsf{TA}^*_{\wedge\vee}$ having subject M and predicate σ can be translated into a derivation in TA assigning the same type σ to a CL-term M' weakly reducing to M. This translation leaves the basis unchanged. We have, for example, $\{x : (\sigma \rightarrow \sigma \rightarrow \rho) \wedge (\tau \rightarrow \tau \rightarrow \rho), y : \sigma \vee \tau\} \vdash (\lambda^* t.xtt)y : \rho$. However, if M is a combinator, then the type σ can be assigned exactly to M in TA.

Theorem 2.30 (Mapping $\mathsf{TA}^*_{\wedge\vee}$ into TA)

(i) Let Γ *be a basis and* $\Gamma_{\vee} = \{x_1 : \sigma_1, \ldots, x_n : \sigma_n\}$. *Then* $\Gamma \vdash^* M : \tau \Longleftrightarrow \Gamma \vdash (\lambda^* x_1, \ldots, x_n.M)x_1, \ldots, x_n : \tau$.

(ii) $\vdash^* M : \tau \Longleftrightarrow \vdash M : \tau$.

Proof:

(i) $(\Longrightarrow)$ Let $\mathbf{B}(\Gamma_{\vee}) = \{\Gamma^{(h)}\}_{1 \leq h \leq m}$, where $\Gamma^{(h)} = \{x_i : \sigma_i^{(h)}\}_{1 \leq i \leq n}$. Then

$\Gamma \vdash^* M : \tau$

$\Longrightarrow$ $\Gamma_{\wedge} \vdash^{\#} \lambda^* x_1, \ldots, x_n.M : \wedge_{1 \leq h \leq m}(\sigma_1^{(h)} \rightarrow \cdots \rightarrow \sigma_n^{(h)} \rightarrow \tau^{(h)})$
where $\tau^{(h)} \in \mathbf{m}(\tau)$, by Corollary 2.28.

$\Longrightarrow$ $\Gamma_{\wedge} \vdash \lambda^* x_1, \ldots, x_n.M : \wedge_{1 \leq h \leq m}(\sigma_1^{(h)} \rightarrow \cdots \rightarrow \sigma_n^{(h)} \rightarrow \tau^{(h)})$
since TA is an extension of $\mathsf{TA}^{\#}$.

This implies, by Lemma 2.22(ii) and by applying rule $(\leq^*)$
$\Gamma_{\wedge} \vdash \lambda^* x_1, \ldots, x_n.M : \sigma_1 \rightarrow \cdots \sigma_n \rightarrow \tau$,
from which the result follows by $(\rightarrow E)$.

$(\Longleftarrow)$ $\Gamma \vdash (\lambda^* x_1, \ldots, x_n.M)x_1, \ldots, x_n : \tau$

$\Longrightarrow$ $\Gamma \vdash^* (\lambda^* x_1, \ldots, x_n.M)x_1, \ldots, x_n : \tau$
since $\mathsf{TA}^*_{\wedge\vee}$ is an extension of TA

$\Longrightarrow$ $\Gamma \vdash^* M : \tau$ by Theorem 2.21(ii).

(ii) Immediate from (i). □

3 Typed** CL**-terms The system TA does not involve any proof-functional rule, such as $(\wedge I)$ and $(\vee E)$, while preserving derivability of union and intersection types. So it allows us to define the set CT of typed CL-terms which corresponds in the standard way to the set of deductions in TA. Namely, by erasing the type information in a typed CL-term of type σ we will obtain a CL-term which has the type σ in TA (and vice versa).

Definition 3.1 (The set CT of typed CL-terms)

(i) The set of preterms is generated by

$$M := x \mid \mathsf{S} \mid \mathsf{K} \mid \mathsf{I} \mid MM \mid \sigma M \mid M\sigma,$$

where x ranges over term variables and σ over the types of T.

(ii) A preterm M is a typed CL-term if and only if there are a basis Γ and a type σ such that $\Gamma \vdash^T M : \sigma$ can be obtained by using the following axioms and rules.

(a) Axioms

$\mathsf{S}\sigma_1\tau_1\rho_1, \ldots, \sigma_n\tau_n\rho_n : \wedge_{1\leq i\leq n}((\sigma_i \to \tau_i \to \rho_i) \to (\sigma_i \to \tau_i) \to \sigma_i \to \rho_i)$
$\mathsf{K}\sigma_1\tau_1, \ldots, \sigma_n\tau_n : \wedge_{1\leq i\leq n}(\sigma_i \to \tau_i \to \sigma_i)$
$\mathsf{I}\sigma_1, \ldots, \sigma_n : \wedge_{1\leq i\leq n}(\sigma_i \to \sigma_i)$.

(b) Rules

(VAR) $$\Gamma, x : \sigma \vdash^T x : \sigma$$

$(\to \mathsf{E})$ $$\frac{\Gamma \vdash^T M : \sigma \to \tau \qquad \Gamma \vdash^T N : \sigma}{\Gamma \vdash^T MN : \tau}$$

$(\leq^*)$ $$\frac{\Gamma \vdash^T M : \sigma \qquad \sigma \leq^* \tau}{\Gamma \vdash^T (\sigma \to \tau)M : \tau}$$

(iii) CT is the set of all typed CL-terms.

Let us remark that the only terms of the shape $M\sigma$ we allow are those generated by the axioms. In contrast the terms of the shape σM are used to represent applications of the subsumption rule $(\leq^*)$.

Definition 3.2 (Forgetful map) There is a trivial forgetful map $|\ |$ from preterms to CL-terms defined as follows:

$$\begin{array}{rcl} |x| & = & x \\ |\mathsf{S}| & = & \mathsf{S} \\ |\mathsf{K}| & = & \mathsf{K} \\ |\mathsf{I}| & = & \mathsf{I} \\ |\sigma M| & = & |M| \\ |M\sigma| & = & |M|\ . \end{array}$$

We can easily prove, by induction on derivations, the desired correspondence between CT and TA.

Theorem 3.3 (CT is the typed version of TA)

(i) Let M be a typed CL-term. Then $\Gamma \vdash^T M : \sigma$ implies $\Gamma \vdash |M| : \sigma$.

(ii) Let M be a CL-term. Then $\Gamma \vdash M : \sigma$ implies that there is a typed CL-term M' such that $\Gamma \vdash^T M' : \sigma$ and $|M'| = M$.

4 Relevant logic corresponding to $\leq^*$ Types of Definition 2.2 can be considered as logical formulas of a propositional language, such that the type variables, $\rightarrow$, $\wedge$, and $\vee$ correspond to propositional variables, implication, conjunction, and disjunction, respectively.

Definition 4.1 (The set F of logical formulas) The set F of logical formulas is inductively defined by

1. propositional letters are formulas;
2. $A, B \in \mathsf{F} \Longrightarrow (A \rightarrow B), (A \wedge B), (A \vee B) \in \mathsf{F}$.

Clearly, we can identify the set T of types with the set F of logical formulas. In this approach, Definition 2.4 and 2.7 of $\leq^*$ can be viewed as a system allowing us to prove theorems of the shape $A \leq^* B$ by using axioms 1 – 7 and rules 1 – 4 as a set of axioms and rules, respectively.

The logical system representing $\leq^*$ will be showed to be a minimal relevant logic, in which the usual contraction and weakening laws do not hold. Namely this logic is the system B^+, defined in [17] and Meyer and Routley [18], without the "Church constant," plus an axiom corresponding to axiom 7 of Definition 2.7.

A similar correspondence has been proved in [20] between the definition of $\leq_\wedge$ and the system B^+ restricted to axioms and rules involving only implicational and conjunctive formulas. In the present paper this result is extended to disjunctive formulas. The logical meaning of axiom 7 is discussed in Remark 5.5.

Definition 4.2 (The minimal relevant logic RL) RL is the logic on the language F defined by the following axioms and rules.

(a) Axioms

- (a1) $A \rightarrow A$
- (a2) $A \wedge B \rightarrow A$, $A \wedge B \rightarrow B$
- (a3) $A \rightarrow A \vee B$, $B \rightarrow A \vee B$
- (a4) $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$
- (a5) $(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$
- (a6) $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$
- (a7) $(A \rightarrow B \vee C) \rightarrow (A \rightarrow B) \vee (A \rightarrow C)$ for any A such that $P(A)$ is true, where P is given in Definition 2.7. (Harrop)

(b) Rules

$A, A \rightarrow B$	$\Longrightarrow B$	(modus ponens)
A, B	$\Longrightarrow A \wedge B$	(adjunction)
$A \rightarrow B$	$\Longrightarrow (B \rightarrow C) \rightarrow A \rightarrow C$	(suffixing)
$B \rightarrow C$	$\Longrightarrow (A \rightarrow B) \rightarrow A \rightarrow C$	(prefixing).

$Th(\mathsf{RL})$ will denote the set of theorems of RL. For example, $A \rightarrow A \wedge A$, $A \vee A \rightarrow A \in Th(\mathsf{RL})$. These theorems can be proved from $(A \rightarrow A) \wedge (A \rightarrow A)$—which belongs to $Th(\mathsf{RL})$ by axiom $(a1)$ and rule $(adjunction)$—by using axioms $(a4)$ and $(a5)$, respectively. It is easy to verify that, if we erase $(a7)$ from the definition of RL, we obtain the B^+ logic, without the "Church constant." A property corresponding to the Extended Disjunction Property of Harrop formulas (see [11]) holds in B^+, that is "for any formula A such that $P(A)$, if $A \to B \vee C$ is provable then either $A \to B$ or $A \to C$ is provable". Analogously, the extended disjunction property is an admissible rule, but not a derived rule, in intuitionistic logic.

Let $\mathsf{RL}_\wedge$ denote the restriction of RL to formulas which do not contain $\vee$. $Th(\mathsf{RL}_\wedge)$ will denote the set of its theorems. We want to prove that the logical connectives of implication, conjunction, and disjunction of RL are the type constructors $\to$, $\wedge$, and $\vee$, when the $\leq^*$-relation is mapped into the minimal relevant implication. In order to do this we extend the equivalence which has been proved in [20] between $\mathsf{TA}_\wedge$ and $\mathsf{RL}_\wedge$ and we use the correspondence between RL and $\mathsf{RL}_\wedge$ proved in Lemma 4.4.

Theorem 4.3 (Equivalence between $\leq_\wedge$ and $\mathsf{RL}_\wedge$ [20])

$$A \leq_\wedge B \Longleftrightarrow A \to B \in Th(\mathsf{RL}_\wedge).$$

Notice that, by reading types as formulas, the mapping $\mathbf{m}$ becomes a way of associating to each formula in RL a set of formulas in $\mathsf{RL}_\wedge$.

Lemma 4.4 (Relation between $\mathsf{RL}_\wedge$ and RL) *If $A \in Th(\mathsf{RL})$, then $\exists A' \in \mathbf{m}(A)$ such that $A' \in Th(\mathsf{RL}_\wedge)$.*

Proof: By induction on a proof of A.

First step: Notice that all axioms of RL are of the shape $B \to C$ where $B \leq^* C$. Then by Lemma 2.16, $\forall B' \in \mathbf{m}(B)\ \exists C' \in \mathbf{m}(C)$ such that $B' \leq_\wedge C'$, which implies $B' \to C' \in Th(\mathsf{RL}_\wedge)$ by Theorem 4.3.

Induction step: The only interesting case is when the last applied rule is modus ponens:

$$B,\ B \to C \Longrightarrow C.$$

Let $\mathbf{m}(B) = \{B_1, \ldots, B_n\}$. By the induction hypothesis $\exists k\ (1 \leq k \leq n)$ such that $B_k \in Th(\mathsf{RL}_\wedge)$ and $\exists D \in \mathbf{m}(B \to C)$ such that $D \in Th(\mathsf{RL}_\wedge)$. By definition $D \equiv \wedge_{1 \leq i \leq n}(B_i \to C_i)$, for some $C_i \in \mathbf{m}(C)$. Then the result follows by modus ponens from B_k and $B_k \to C_k$. □

Corollary 4.5 *If $A \to B \in Th(\mathsf{RL})$ then $\forall A_i \in \mathbf{m}(A) \exists B_i \in \mathbf{m}(B) \quad A_i \to B_i \in Th(\mathsf{RL}_\wedge)$.*

Proof: By Lemma 4.4 there exists $D \in \mathbf{m}(A \to B)$ such that $D \in Th(\mathsf{RL}_\wedge)$. Let $\mathbf{m}(A) = \{A_1, \ldots, A_n\}$. Then D is of the form $\wedge_{1 \leq i \leq n}(A_i \to B_i)$, for some $B_i \in \mathbf{m}(B)$, and the thesis follows easily. □

Theorem 4.6 (Relation between $\leq^*$ and RL)

$$A \leq^* B \Longleftrightarrow A \to B \in Th(\mathsf{RL}).$$

Proof: ($\Longrightarrow$) Straightforward by induction on the definition of $\leq^*$, using the extended disjunction property.

($\Longleftarrow$) Let $\mathbf{m}(A) = \{A_1, \ldots, A_n\}$. If $A \to B \in Th(\mathsf{RL})$, then, by Corollary 4.5, $\forall A_i \in \mathbf{m}(A)\ \exists B_i \in \mathbf{m}(B)$ such that $A_i \to B_i \in Th(\mathsf{RT}_\wedge)$, that is, $\forall i (1 \leq i \leq n)\ A_i \leq_\wedge B_i$ by Theorem 4.3. Then by rule 2, $\vee_{1 \leq i \leq n} A_i \leq \vee_{1 \leq i \leq n} B_i$, so we can conclude $A \leq^* B$ since $\vee_{1 \leq i \leq n} A_i \sim^* A$ and $\vee_{1 \leq i \leq n} B_i \leq^* B$. □

Let us notice the absence of the theorem $A \to B \to A \wedge B$ as well as the absence of the exportation law

$$(exp) \qquad (A \wedge B \to C) \to A \to B \to C$$

in RL. On the other hand, the following formula

$$(A \to B \to C) \to A \wedge B \to A \wedge B \to C$$

is a theorem of RL. This will allow the converse formula of (exp), that is, the importation law, to be provable in the logical system (presented in the next section) whose derivations will parallel typed CL-terms.

The disjunction is the dual notion of the relevant conjunction. So the axiom for the elimination of disjunction is $(A \to C) \wedge (B \to C) \to A \vee B \to C$, which is not equivalent to the intuitionistic axiom for the disjunction elimination, that is, $(A \to C) \to (B \to C) \to (A \vee B \to C)$.

5 Typed CL-terms as logical proofs Now we shall present a logical system whose deductions correspond, in a Curry-Howard isomorphism, to the typed CL-terms of Definition 3.1. This logic contains a Hilbert-style version of the implicative fragment of the intuitionistic propositional logic. Moreover, conjunctive and disjunctive formulas are derived by means of the following features.

(i) The standard notion of axiom-schemes is extended to include as axioms not only any instance but also any conjunction of instances of the axiom scheme.

(ii) A Relevant Modus Ponens is added as an inference rule, using theorems of the relevant logic RL as major premises.

Definition 5.1 (The logic L) Let $[D]^\#$ denote a conjunction of instances of D, that is, any $D_1 \wedge \cdots \wedge D_n (n \geq 1)$ such that each D_i is an instance of D $(1 \leq i \leq n)$. F is the language of propositional formulas involving $\to$, $\wedge$, and $\vee$ as connectives. L is the logic on the language F, given by the following axioms and rules

(a) Axioms

$$[(A \to B \to C) \to (A \to B) \to A \to C]^\#$$
$$[A \to B \to A]^\#$$
$$[A \to A]^\#.$$

(b) Rules

MP	(Modus Ponens)	$A \to B, A \Longrightarrow B$
RMP	(Relevant Modus Ponens)	$A \to B \in Th(\mathsf{RL}), A \Longrightarrow B.$

Δ will denote a set of assumptions. We will write $\Delta \vdash^L A$ if and only if there is a deduction of A from Δ.

Our main result is that typed CL-terms codify $\vdash^L$-derivations. Let us keep on reading types as formulas, so $\vdash^T$-derivability may be viewed as derivability of formulas for CL-terms.

Theorem 5.2 (Curry-Howard isomorphism)

$$\Delta \vdash^L A \Longleftrightarrow \Gamma \vdash^T M : A \ \textit{for some}\ \ M \in \mathsf{CT},$$

where $B \in \Delta$ *if and only if* $x : B \in \Gamma$ *for some variable* x.

Proof: By induction on the deductions. The proof is trivial by associating:

(i) the typed atomic combinators with axioms of L,
(ii) the rule $(\rightarrow E)$ with the inference rule *MP*,
(iii) the rule $(\leq^*)$ with the inference rule *RMP*, taking into account that $A \leq^* B$ if and only if $A \rightarrow B \in Th(\mathsf{RL})$, by Theorem 4.6. □

Example 5.3 (Contraction law) $\vdash^L (A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$. A proof of the previous theorem is codified by the typed CL-term MN, where $M \equiv (\mathsf{S}(A \rightarrow A \rightarrow B)(A \rightarrow A)(A \rightarrow B))(\mathsf{S}AAB)$, and $N \equiv (\mathsf{K}(A \rightarrow A)(A \rightarrow A \rightarrow B))(\mathsf{I}A)$. Moreover, by using the contraction law and the theorem $(A \rightarrow B \rightarrow C) \rightarrow A \wedge B \rightarrow A \wedge B \rightarrow C$, one proves

$$\vdash^L \quad (A \rightarrow B \rightarrow C) \rightarrow A \wedge B \rightarrow C \qquad \text{(importation law)}$$

and

$$(\diamondsuit) \qquad \vdash^L \quad (A \rightarrow B) \wedge (C \rightarrow B) \rightarrow A \vee C \rightarrow B \qquad \text{(disjunction elimination).}$$

A proof of $(\diamondsuit)$ is codified by the typed CL-term $D(\mathsf{I}(A \rightarrow B) \wedge (C \rightarrow B))$, where $D \equiv ((A \rightarrow B) \wedge (C \rightarrow B) \rightarrow (A \rightarrow B) \wedge (C \rightarrow B)) \rightarrow (A \rightarrow B) \wedge (C \rightarrow B) \rightarrow A \vee C \rightarrow B$ is a theorem of RL.

Let us consider the subsystem $L_\wedge$ defined from Definition 5.1 by considering the language of implicational conjunctive formulas, and by replacing $Th(\mathsf{RL}_\wedge)$ to $Th(\mathsf{RL})$ in the *RMP*-rule. In [20], $L_\wedge$ has been proved to be a logical system corresponding to the intersection type inference. Then the following corollary characterizes derivability in L by means of derivability in $L_\wedge$. Roughly speaking, it says that any proof $\mathcal{D}$ of a theorem in L can be transformed into a proof $\mathcal{D}^*$ of an equivalent formula (modulo $\sim^*$) such that $\mathcal{D}^*$ is built up of a proof in $L_\wedge$ plus one application of rule $(\leq^*)$.

Corollary 5.4

(i) $\vdash^\wedge M : A \Longleftrightarrow \vdash^{L_\wedge} A$.
(ii) $\vdash^L A \Longrightarrow \exists B$ *such that* $B \leq^* A$ *and* $\vdash^{L_\wedge} B$, *where* B *is an implicational conjunctive formula.*

Proof:

(i) Lemma 4.8 in [20].
(ii) Let $\mathbf{m}(A) = \{A_1, \ldots, A_n\}$, then by 2.15 $A \sim^* A_1 \vee \cdots \vee A_n$.

$$\begin{array}{llll}
\vdash^L A & \Longrightarrow & \vdash^T M : A & \text{for some typed CL-term } M \text{ by } 5.2 \\
 & \Longrightarrow & \vdash^* |M| : A & \text{by } 3.3\text{(i) and } 2.30\text{(ii)} \\
 & \Longrightarrow & \vdash^\wedge |M| : A_j \text{ for some } j \leq n & \text{by } 2.20 \\
 & \Longleftrightarrow & \vdash^{L_\wedge} A_j & \text{from point (i),}
\end{array}$$

so we can choose $B \equiv A_j$. □

Remark 5.5 (L versus intuitionistic and Dummet's logic) It is interesting to look at the system L just from the logical point of view, thus comparing it with the intuitionistic and Dummet's logics [10]. To this end let us define L as in Definition 5.1, but erasing the relevant modus ponens, while adding all theorems of RL as axioms for simplicity. This is clearly an equivalent formulation, since if $A \to B \in Th(\mathsf{RL})$, then we can deduce $\vdash^L A \to B$ by *RMP* from $(A \to A) \to A \to B \in Th(\mathsf{RL})$ $(prefixing)$ and the axiom $A \to A$.

As far as implicational-conjunctive formulas are concerned, $L_\wedge$ turns out to be a subsystem of intuitionistic logic, since the intersection is a "relevant" restriction of intuitionistic conjuction as proved in [20]. The handling of disjunction is more difficult, since all axioms of RL are intuitionistic theorems, but the Harrop axiom (a7) is not so.

The logic known as Dummet's logic (DL) can be defined by adding to the intuitionistic axiomatization the axiom scheme

$$(D\text{-}Ax) \qquad (A \to B \vee C) \to (A \to B) \vee (A \to C).$$

$(D\text{-}Ax)$ is valid in L only when A is a Harrop formula, and it is not valid in intuitionistic logic. In DL one gets two different results. The first one is that all the disjunctions are pulled out from the inside of the formulas, by using both (D-Ax) and the law

$$(\circ) \qquad (A \vee B \to C) \to (A \to C) \wedge (B \to C).$$

The second result is that the pure implicational fragment turns out to be also stronger than intuitionistic logic. Namely, in DL one can prove the following theorem, which is not intuitionistically valid.

$$(\circ\circ) \qquad ((A \to B) \to C) \to ((B \to A) \to C) \to C.$$

Thus a question arises about the Harrop-Axiom of L, which looks very similar to (D-Ax), except for restricting the validity of the axiom to the case in which A is a Harrop formula: By adding the (Harrop) axiom, do we get in L the same effect as in Dummet's logic?

The answer is positive with respect to the first result (the elimination of disjunction inside formulas shown by the mapping **m**) and negative with respect to the second one (the extension of intuitionistic implication). This last result shows the "constructive" meaning of the union-type constructor.

Let us notice that in L both the restricted shape of the (Harrop) axiom versus the (D-Ax) and the lack of the exportation law $(A \wedge B \to C) \to (A \to B \to C)$ avoid to prove the above formula $(\circ\circ)$. On the other hand, Corollary 5.4(ii) and the fact that $L_\wedge$ is a subsystem of the implicational conjunctive intuitionistic logic mean that any disjunctive formula provable in L must be considered equivalent to an intuitionistic theorem. Namely, the last one is provable from some implicational conjunctive theorem by using essentially only the $\vee$-introduction rule. Let us consider, for example, the Harrop formula:

$$(A \to B \vee C) \to (A \to B) \vee (A \to C),$$

which is a theorem itself in the case in which A is a propositional variable. In L it is proved to be equivalent to an intuitionistic disjunction of implicational conjunctive formulas in the following way:

$$
\begin{aligned}
&(A \to B \vee C) \to (A \to B) \vee (A \to C) \\
&\sim^* \\
&(A \to B) \vee (A \to C) \to (A \to B) \vee (A \to C) && \text{by (Harrop)} \\
&\sim^* \\
&((A \to B) \to (A \to B) \vee (A \to C)) \wedge ((A \to C) \to \\
&\qquad (A \to B) \vee (A \to C)) && \text{by } (\circ) \\
&\sim^* \\
&(((A \to B) \to A \to B) \vee ((A \to B) \to A \to C)) \wedge \\
&\qquad (((A \to C) \to A \to B) \vee ((A \to C) \to A \to C)) && \text{by (Harrop)} \\
&\sim^* \\
&(((A \to B) \to A \to B) \wedge ((A \to C) \to A \to C)) \vee \cdots && \text{by distributivity}
\end{aligned}
$$

where $((A \to B) \to A \to B) \wedge ((A \to C) \to A \to C)$ is an intuitionistic theorem.

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