Notre Dame Journal of Formal Logic
Volume 38, Number 2, Spring 1997

Ontologies for Plane, Polygonal Mereotopology

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Abstract Several authors have suggested that a more parsimonious and conceptually elegant treatment of everyday mereological and topological reasoning can be obtained by adopting a spatial ontology in which regions, not points, are the primitive entities. This paper challenges this suggestion for mereotopological reasoning in two-dimensional space. Our strategy is to define a mereotopological language together with a familiar, point-based interpretation. It is proposed that, to be practically useful, any alternative region-based spatial ontology must support the same sentences in our language as this familiar interpretation. This proposal has the merit of transforming a vague, open-ended question about ontologies for *practical* mereotopological reasoning into a precise question in model theory. We show that (a version of) the familiar interpretation is countable and atomic, and therefore prime. We conclude that useful alternative ontologies of the plane are, if anything, less parsimonious than the one which they are supposed to replace.

1 The problem One of the many achievements of coordinate geometry has been to provide a conceptually elegant and unifying account of the nature of geometrical entities. According to this account, the one primitive spatial entity is the point, and the one primitive geometrical property of points is coordinate position. All other geometrical entities—lines, curves, surfaces, and bodies—are nothing but collections of points; and all properties and relations involving these entities may be defined in terms of the relative positions of the points which make them up. The success and power of this reduction is so great that the identification of spatial regions with the sets of points they contain has come to seem virtually axiomatic.

Yet various authors have sought to reverse this order of rational reconstruction, treating *regions* as primary, and admitting points, if at all, as logical constructions out of them. The best known of these approaches is perhaps Tarski's [35] axiomatization of Euclidean geometry, taking spheres to be the primitive entities. But the

Received February 4, 1997; revised September 26, 1997

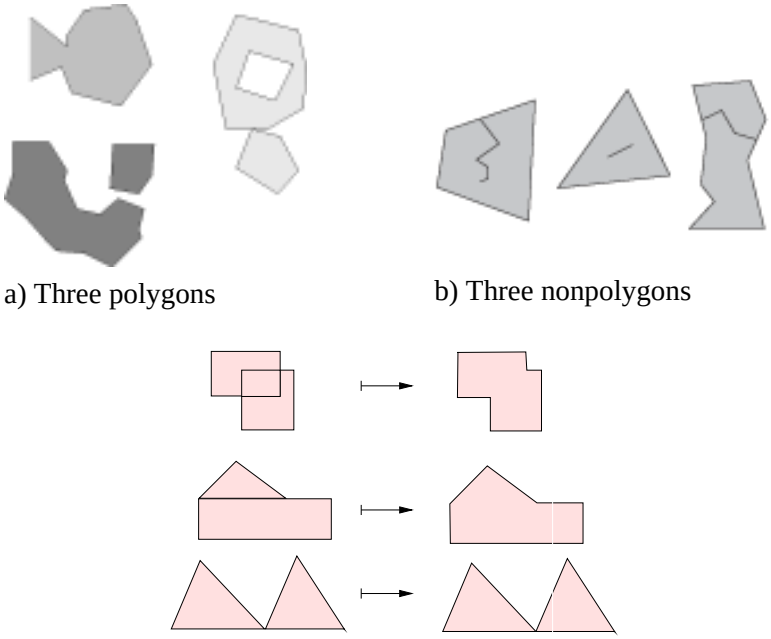
policy of taking regions as primitive is most attractive when considering problems involving mereological (part-whole) and topological notions—that is, where no metric information is to hand. If regions are first-class entities and points are logical constructions based on them, then who knows what interesting new ways of considering spatial entities and relations there might be? Clarke [11], [12], following an idea of Whitehead [39], sought to reconstruct mereotopology in terms of a primitive relation of connection holding between regions. Following this work, Biacino and Gerla [5] have studied models of Clarke's theory. More recently, and partially as a response to debates concerning temporal reasoning and knowledge representation (for example, Allen [1]), Clarke's mereotopology has received attention from several research groups in AI, working with the loosely defined area of *qualitative spatial reasoning*, for example, Gotts, Gooday and Cohn [17], Asher and Vieu [2], and Borgo, Guarino, and Masolo [6]. For a treatment of region-based topology in a general setting, see Roeper [31].

Motivations for these developments vary, and we do not intend to provide a comprehensive account of them here. However, one common recurring theme is the suspicion that the familiar, point-based view of space generates a richer ontology than is needed for mereotopological reasoning in *practical* situations. For example, Euclidean space contains not only the sorts of regions we want to recognize for everyday purposes, but also strange, physically unrealizable regions of the kind that populate point-set topology textbooks. Such regions seem to be mere artifacts of the Euclidean model of space—useless for describing, and reasoning about, the world we inhabit. If, on the other hand, we regard regions as primitive entities, perhaps we can be more selective as to what regions we take to exist and what mereotopological properties we take them to have. Perhaps—so some researchers in mereotopology suggest—treating regions as primary opens up the prospect of simpler and more parsimonious spatial ontologies than the familiar model based on points in the real plane.

The present paper examines this suggestion for the special case of *plane* mereotopology. We show that, under certain reasonable assumptions as to what practical mereotopological reasoning might involve, taking regions rather than points as primitive cannot lead to a more parsimonious spatial ontology.

2 Polygonal mereotopology To get an idea of what practical mereotopological reasoning might involve, consider computer systems specialized for representing plane spatial data, such as Geographic Information Systems (GISs). Virtually all such systems represent regions of space by means of boundaries consisting of finitely many straight lines and straight-line segments. In effect, then, all plane regions recognized by such systems are polygons. Experience has shown that such a spatial ontology, whatever its philosophical shortcomings, is certainly equal to the task of describing everyday planar spatial arrangements such as those found on maps and charts, since any arrangement of regions one is likely to encounter can be approximated by polygons with arbitrarily high accuracy.

Suppose we take as our spatial ontology the set P of polygons in the plane. We discuss the formal construction of P later; for the present, all that matters is that all members of P are plane regions bounded by finitely many straight lines as shown in Figure 1a. (As explained below, we take these regions not to include their boundaries.)



a) Three polygons

b) Three nonpolygons

c) Three pairs of polygons and their respective sums

Figure 1: Polygonal mereotopology

Note that we allow polygons to consist of more than one piece, to be unbounded, and to contain holes, as long as those holes have straight-line boundaries; however, polygons are not allowed to contain "cracks," as shown in Figure 1b. In addition, we consider the empty set and the whole plane to be polygons.

It turns out that P forms a Boolean algebra. In this Boolean algebra, the product of two polygons is their intersection; the complement of a polygon is that part of the plane lying outside it and its boundary; and the sum of two polygons is the polygon formed by taking their union and "rubbing out" any internal boundaries that result. Figure 1c illustrates the sum-operation. Accordingly, our mereotopological language will be equipped with functions-symbols $\cdot$, $-$, and $+$ to denote these operations, as well as the constants 0 and 1 to denote the empty set and the whole plane, respectively. Note that the formula $x \cdot y = x$ states that x is a subset of y. Hence the Boolean functions can express various mereological properties and relations involving polygons.

In point-set topology, it is usual to define an open set as being *connected* if it is not the union of two disjoint, nonempty, open sets. Intuitively, connected sets are just that—they consist of one piece. Accordingly, our language will be equipped with a one-place predicate $c(x)$ to express the property of being a connected polygon. (Incidentally, since we take polygons not to include their boundaries, the right-most polygon in Figure 1a is not connected.) If x and y are disjoint, connected and nonempty, then it is possible to show that the formula $c(x+y)$ is satisfied if and only if x and y share one or more proper straight line segments on their boundaries. In other words, the formula $c(x+y)$ can be used to express the relation of external contact along an edge. Hence, the predicate $c(x)$, together with the Boolean functions, can express various topological properties and relations involving polygons.

Thus, we take our mereotopological language $\mathcal{L}$ to be a first-order language with equality and nonlogical constants $+$, $\cdot$, $-$, 0, 1, and $c(x)$. The set P of polygons will form the domain over which the variables of $\mathcal{L}$ range, and the interpretation of the nonlogical constants of $\mathcal{L}$ given above defines a model $\mathfrak{P}$ on the domain P. The sentences $\mathrm{Th}(\mathfrak{P})$ true in this model represent, as it were, the facts of mereotopology according to the the polygonal ontology employed in most computer systems for representing plane spatial data.

We propose to take $\mathrm{Th}(\mathfrak{P})$ to be the facts of practical mereotopological reasoning. After all, the polygonal model $\mathfrak{P}$ is relatively simple, admits no pathological regions, and yet is mereotopologically nontrivial and finds use in many practical applications without apparent loss of useful representational power. Moreover, it will turn out in Section 6 that $\mathfrak{P}$ can be considerably liberalized without changing the resulting theory. We further propose that an alternative spatial ontology for practical mereotopological reasoning is simply an alternative model of $\mathrm{Th}(\mathfrak{P})$—that is, a model $\mathfrak{A}$ such that $\mathfrak{A} \equiv \mathfrak{P}$ but $\mathfrak{A} \not\cong \mathfrak{P}$. The domain A of $\mathfrak{A}$ will form the set of regions of space and the relations and properties needed to interpret the terms in $\mathcal{L}$ will give this space its mereotopological structure.

Note that the domain P contains only polygonal *regions*, and not the points and lines of which they are made up. Thus, we employ a language which can talk, in the first instance, only about regions, in keeping with the spirit of mereotopology. On the other hand, our model $\mathfrak{P}$ is fundamentally Euclidean, in that polygons are objects in the Euclidean plane defined in terms of the points they contain or the lines that bound them. Thus, $\mathfrak{P}$ is, as it were, our *familiar* ontology—one constructed in the familiar way from points in the Euclidean plane. A *general* model of $\mathrm{Th}(\mathfrak{P})$, by contrast, may have any sorts of objects in its domain, either primitive or constructed in some other way. The problem we face in the sequel is to identify such general models, and to determine whether any of them constitute a more elegant and parsimonious spatial ontology than $\mathfrak{P}$.

It may be objected that this strategy is too conservative. After all, *who says* that the facts of practical mereotopological reasoning—the facts that we would want any alternative spatial ontology to support—are the facts that are true of the polygonal data-structures employed in many computer systems? Perhaps we could find a better theory of space by *revising* this "computer-mereotopology." To some extent, this criticism is justified: we have little to say in favor of the polygonal theory, except that it is familiar, easily formalized and seems to be widely and successfully used in a vast range of practical applications. (Proponents of other theories of mereotopological reasoning should be so lucky.) Nevertheless, our strategy does have the virtue of transforming a vague and open-ended question about ontologies for practical mereotopological reasoning into a precise, technical question in model theory. To be sure, we do not regard the solution of this problem to be the last word on the metaphysics of space; but at least we now have a definite question to address.

The plan of the rest of the paper is as follows. Section 3 outlines the construction of the polygonal ontology, Section 4 establishes some preliminary topological features of this ontology and Section 5 uses these results to prove the model-theoretic results which form the core of the paper. Finally, Section 6 generalizes these results beyond the polygonal case.

3 The polygonal models Our first task in formalizing the polygonal ontology is to resolve the issue of whether regions include their boundary points. We adopt an approach, based on *regular open sets*, which has become reasonably standard in discussions of spatial description languages.

Definition 3.1 Let X be a topological space and $x \subseteq X$. Then the set $\bigcup\{y \subseteq X |\ y \text{ open}, y \cap x = \varnothing\}$ is an open set in X called the *pseudocomplement of* x, written x'. We say that $x \subseteq X$ is *regular* if $x = x''$.

Note, for brevity, we use the term "regular" where most authors would use "regular open". We shall never have occasion to refer to regular closed sets.

The following well-known theorem underlies the importance of the regular sets to mereotopology. We state it here without proof.

Theorem 3.2 *Let X be a topological space. Then the set* $\mathrm{RO}(X)$ *of regular sets in X forms a Boolean algebra with top and bottom defined by $1 = X$ and $0 = \varnothing$, and Boolean operations defined by $x \cdot y = x \cap y$, $x + y = (x \cup y)''$ and $-x = x'$.*

In fact, $\mathrm{RO}(X)$ is a complete Boolean algebra, and moreover, every complete Boolean algebra is isomorphic to $\mathrm{RO}(X)$ for some topological space X; however, we will not be concerned with these facts about regular sets. (See, e.g., Koppelberg [22], pp. 26 and 60.) Accordingly, we shall sometimes use the term *regular Boolean algebra* of a topological space X to refer to $\mathrm{RO}(X)$. When dealing with the elements of such a Boolean algebra, we shall write $x \cdot y$, $x + y$, $-x$ and instead of $x \cap y$, $(x \cup y)''$, and x', respectively.

Theorem 3.2 shows that the part-whole relationship, restricted to the regular sets, still obeys the axioms of a Boolean algebra, so that confining our attention to such sets will result in a mathematically manageable theory. Actually, some mereotopologists think it important that the empty set not count as a region, and be eschewed from the domain of quantification of mereotopological theories. We see no reason for such a restriction, but readers who disagree can easily adapt the results below to ontologies from which the empty set is excluded.

If X is a topological space and $y \subseteq X$, we denote interior of y (the largest open set contained in y) by y^0, and the closure of y (the smallest closed set containing y) by $[y]$. (We reserve the more usual notation $\bar{y}$ for n-tuples.) The set $[x] \setminus [x]^0$ is called the *frontier* of x, and is denoted by $\mathcal{F}(x)$. The following facts about regular sets are well known.

Lemma 3.3 *Let X be a topological space and $x \subseteq X$. Then $x' = X \setminus [x]$ and $x'' = [x]^0$.*

Lemma 3.3 shows that restricting attention to regular sets is a sensible means of ignoring boundary points, because no two regular regions differ only with respect to their boundary points. As usual in topology, we say that an open set x is *connected* if there do not exist two nonempty, disjoint open sets whose union is x. The maximal connected subsets of a set x are called the *components* of x. The next two results are again so straightforward we state them without proof.

Lemma 3.4 *Let a_1, a_2 be connected, regular sets with $a_1 \cdot a_2 \neq 0$. Then $a_1 + a_2$ is connected.*

Lemma 3.5 *Any component of a regular set is regular.*

Let X be any topological space and M be any Boolean subalgebra of $\mathrm{RO}(X)$. If A is a finite subset of M and the elements of A are pairwise disjoint, nonempty, and sum to $a \in M$, we call A a *partition of* a in M. If, in addition, every element of A is connected, we call A a *connected partition of* a in M. In the case $a = 1$, we refer to A simply as a (*connected*) *partition* in M. The following (rather technical) lemma will be useful later.

Lemma 3.6 *Let X be a topological space, M a Boolean subalgebra of $\mathrm{RO}(X)$ and $a_1, \ldots, a_n$ a partition in M. Let m be such that $1 \leq m \leq n$. Then*

$$a_1 + \cdots + a_m = a_1 \cup \cdots \cup a_m \cup \{p \mid\ p \in \mathcal{F}(a_i) \text{ for some } i\ (1 \leq i \leq m),\ p \notin \mathcal{F}(a_j) \text{ for any } j\ (m < j \leq n)\}.$$

Proof: Denote the right-hand side of the above equation by x. Suppose $p \in [a_j]$ for some j $(m < j \leq n)$. Then $p \in \mathcal{F}(a_j)$ or $p \in [a_j]^0 = a_j$ by Lemma 3.3. If $p \in a_j$, then $p \notin [a_i]$ for any i $(1 \leq i \leq m)$ by the disjointness of $a_1, \ldots, a_n$. Either way, then, $p \notin x$.

Suppose $p \notin [a_j]$ for any j $(m < j \leq n)$. Then certainly $p \notin \mathcal{F}(a_j)$ for any j $(m < j \leq n)$. Moreover, $a_1, \ldots, a_n$ sum to 1, so $[a_1] \cup \cdots \cup [a_n] = 1$. Hence $p \in [a_i]$ for some i $(1 \leq i \leq m)$, so again, $p \in \mathcal{F}(a_i)$ or $p \in [a_i]^0 = a_i$ by Lemma 3.3. Either way, $p \in x$. Hence $x = (1 \setminus [a_{m+1}]) \cap \cdots \cap (1 \setminus [a_n])$. By the first part of Lemma 3.3, $x = (-a_{m+1}) \cdot \cdots \cdot (-a_n) = a_1 + \cdots + a_m$. □

Having dealt with the very general notions of regular sets and their Boolean algebras, we turn to the definition of polygonal regions. Any line in $\mathbb{R}^2$ cuts $\mathbb{R}^2$ into two residual domains, which we shall call *half-planes*. It is easy to see that these sets are regular, with each being the pseudocomplement of the other. Hence, we can speak about the sums, products, and complements of half-planes in $\mathrm{RO}(\mathbb{R}^2)$.

Definition 3.7 A *basic polygon* is the intersection of finitely many half-planes in $\mathbb{R}^2$. A *polygon* is the sum, in $\mathrm{RO}(\mathbb{R}^2)$, of any finite set of basic polygons.

We denote the set of polygons by R, and will sometimes refer to it as the *polygonal domain*. Thus, the elements of R are simply polygons as introduced in the previous section.

Of course, R is not the only well-behaved spatial domain we might choose. If a line is defined by an equation $ax + by + c = 0$, where a, b, and c are rational numbers, we call it a *rational line*; and if a half-plane is bounded by a rational line, we call it a *rational half-plane*.

Definition 3.8 A *rational basic polygon* is the intersection of finitely many rational half-planes in $\mathbb{R}^2$. A *rational polygon* is the sum, in $\mathrm{RO}(\mathbb{R}^2)$, of any finite set of rational basic polygons.

We denote the set of rational polygons by Q, and will sometimes refer to it as the *rational polygonal domain*. Thus, R, or perhaps, more modestly, Q, is the spatial ontology recognized by computer systems such as GISs—both domains provide a simple view of space from which any remotely pathological behavior has been excluded. Clearly, R is uncountable, whereas Q is countable; so R and Q are differ-

ent structures. Nevertheless, these ontologies are very similar, and share many basic properties. For brevity, we use the symbol P to denote either R or Q.

Theorem 3.9 *P is a Boolean subalgebra of* $\mathrm{RO}(\mathbb{R}^2)$.

Proof: We need only show that P is closed under the Boolean operations. But this is obvious given the distribution laws for $\mathrm{RO}(\mathbb{R}^2)$ and the fact that the pseudocomplement of a half-plane is a half-plane. □

Now that we have defined the polygonal domain (or more precisely, domains) of quantification, P, we introduce our mereotopological language $\mathcal{L}$. Let $\mathcal{L}$ be the first-order language with signature $\langle c(x), +, \cdot, -, 0, 1\rangle$, where $c(x)$ is a 1-place predicate, $+$ and $\cdot$ are binary function symbols, $-$ is a unary function symbol, and 0 and 1 are individual constants. Informally, $c(x)$ denotes the property of connectedness (in the usual topological sense), the function-symbols $+$, $\cdot$, and $-$ denote the obvious operations in the Boolean algebra $\mathrm{RO}(\mathbb{R}^2)$, and 0 and 1 denote the empty set and $\mathbb{R}^2$, respectively. Thus, $\mathcal{L}$ has a mereological component in the form of Boolean connectives representing operations on regular sets, and a topological component in the form of a connectedness predicate.

Formally, we give $\mathcal{L}$ two "familiar" interpretations, $\mathfrak{R}$ and $\mathfrak{Q}$, corresponding to the domains R and Q, respectively.

Definition 3.10 We define the *polygonal model* $\mathfrak{R}$ to have the domain R and the following interpretations of the predicate, constant, and function symbols in $\mathcal{L}$:

1. $c(x)^{\mathfrak{R}} = \{a \in R | a \text{ is connected}\}$
2. $0^{\mathfrak{R}} = \varnothing$; $1^{\mathfrak{R}} = \mathbb{R}^2$
3. For all $a \in R$: $-^{\mathfrak{R}}(a) = -a$
4. For all $a, b \in R$: $+^{\mathfrak{R}}(a, b) = a + b$ and $\cdot^{\mathfrak{R}}(a, b) = a \cdot b$

We define the *rational polygonal model* $\mathfrak{Q}$ exactly as for $\mathfrak{R}$ but with $\mathfrak{R}$ and R replaced throughout by $\mathfrak{Q}$ and Q, respectively.

Again, in view of the similarities between $\mathfrak{R}$ and $\mathfrak{Q}$, we write $\mathfrak{P}$ to refer indeterminately to either. Thus, the domain of $\mathfrak{P}$ is P. Anticipating a result of the next section, it turns out—unsurprisingly—that $\mathfrak{R}$ and $\mathfrak{Q}$ make the same sentences of $\mathcal{L}$ true. That is, the ontologies $\mathfrak{Q}$ and $\mathfrak{R}$ are indistinguishable for the mereotopological language $\mathcal{L}$. Hence we may write $\mathrm{Th}(\mathfrak{P})$ to denote $\mathrm{Th}(\mathfrak{R}) = \mathrm{Th}(\mathfrak{Q})$. Our main task in this paper is to find alternative models of $\mathrm{Th}(\mathfrak{P})$.

We finish this section on the familiar models for $\mathcal{L}$ with an example to show that the pains we took to define our domain of interpretation were not in vain. Consider the following formula of $\mathcal{L}$:

$$\forall x_1 \forall x_2 \forall x_3 \left(\left(\bigwedge_{1 \le i \le 3} c(x_i) \wedge c(x_1 + x_2 + x_3) \right) \longrightarrow (c(x_1 + x_2) \vee c(x_1 + c_3)) \right).$$

This formula asserts that, if the sum of three connected regions is connected, then the first must be connected to at least one of the other two. It is true in the model $\mathfrak{P}$; but it would be false in a model whose domain extended to all regular sets of the plane.

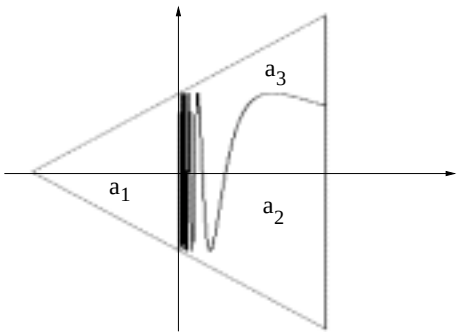


Figure 2: Three regular sets in the plane

For consider the regions a_1, a_2, and a_3 defined by

$$\begin{aligned} a_1 &= \{(x,y) | -1 < x < 0\ ;\ -1-x < y < 1+x\} \\ a_2 &= \{(x,y) | 0 < x < 1\ ;\ -1-x < y < \sin(1/x)\} \\ a_3 &= \{(x,y) | 0 < x < 1\ ;\ \sin(1/x) < y < 1+x\}\ , \end{aligned}$$

and depicted in Figure 2. (Note: in this figure, the x-axis has been dilated.) It is not difficult to show that a_1, a_2, and a_3 are regular, that $a_1 + a_2 + a_3$ is the interior of the large triangle in Figure 2 and so is connected, but that neither $a_1 + a_2$ nor $a_1 + a_3$ is connected. This example demonstrates the importance of having a precise characterization of the regions our mereotopological language talks about. When looking for models elementarily equivalent to $\mathfrak{P}$ as alternative ontologies for practical mereotopological reasoning, we are making some very specific choices about the facts of mereotopology that we want to support.

4 Topological analysis Our next step is to establish some basic topological properties of P. All the results in this section are routine and, in one form or another, well known. The development is in some places perhaps more explicit than is necessary; however, this will prove useful when we generalize our results in Section 6. The most important results for our purposes are Theorem 4.9 and Lemmas 4.13 and 4.14; the rest are ancillary. We begin with a lemma on which much of the subsequent analysis depends.

Lemma 4.1 *Any element of P is the sum of finitely many connected elements of P.*

Proof: Since half-planes are convex, basic polygons are convex, and so are certainly connected. □

It easy to see that this property does not hold for all Boolean subalgebras of $\mathrm{RO}(\mathbb{R}^2)$, even where the elements are relatively well behaved. For example, even when x and y are Jordan domains, the intersection $x \cdot y$ may have infinitely many disconnected parts. It is precisely to prevent this possibility that we shall restrict attention to the domains R and Q. To be sure, the polygons are not the only regions satisfying Lemma 4.1, as we shall see in Section 6, but they are the simplest.

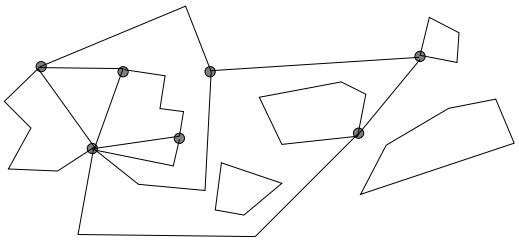


Figure 3: A graph* with two nodeless edges

Lemma 4.2 *Let $a \in P$ and let c be a component of a. Then $c \in P$. Moreover, a equals the sum of its components.*

Proof: By Lemma 4.1, let $c_1, \ldots, c_n$ be connected elements of P such that $a = c_1 + \cdots + c_n$. For all i ($1 \leq i \leq n$), if $c \cdot c_i \neq 0$ then, by Lemma 3.4, $c_i + c$ is connected. If, in addition, $(-c) \cdot c_i \neq 0$, then $c < c + c_i$, contradicting the maximality of c. Thus, if $c \cdot c_i \neq 0$, then $(-c) \cdot c_i = 0$. Hence c can be expressed as the sum of various c_i ($1 \leq i \leq n$), and $c \in P$. The remainder of the lemma is trivial. □

Lemma 4.3 *Let A be a finite subset of P. Then there exists a connected partition C in P such that each $a \in A$ is expressible as a sum of zero or more elements of C.*

Proof: If $A = \{a_1, \ldots, a_n\}$, let C be the set of all components of all nonzero products of the form $\pm a_1 \cdot \cdots \cdot \pm a_n$. By Lemma 4.2, these components are elements of P, and form a connected partition such that every a_i can be expressed as a sum of zero or more elements of C. □

Furthermore, it should come as no surprise that we can picture connected partitions in P by thinking in terms of plane graphs.

Definition 4.4 A *graph* G* is a plane graph in the closed real plane having no nodes of degree 0, together with a (possibly empty) set of *nodeless edges*. These nodeless edges are all Jordan curves intersecting no other edge of G (nodeless or otherwise). A graph* is *piecewise linear* if all of its edges lie on finitely many straight lines; a graph* is *rational piecewise linear* if all of its edges lie on finitely many rational straight lines. A graph* is said to *have an isthmus* if there is one edge whose removal increases the number of its connected components.

Figure 3 shows a piecewise linear graph* (where the page represents the whole closed plane) with two nodeless edges. This specimen also has no isthmuses and no nodes of degree 2. We note also that Euler's formula for a k-component graph, namely $n - e + f = k + 1$, applies also to a k-component graph*, where nodeless edges do not count as components.

If G is a graph*, we denote by $|G|$ the set of points in the edges and vertices of G, ignoring the point at infinity. It makes sense to talk about the faces of G in the open plane—that is, the components of $\mathbb{R}^2 \setminus |G|$. Henceforth, if G is a graph*, when we speak of 'the faces of G', we mean 'the faces of G in the open plane.'

The following basic theorem establishes the importance of piecewise linear and rational piecewise linear graphs.

Theorem 4.5 *Let A be a connected partition in R; then there exists a finite piecewise linear graph* with no isthmuses whose faces are precisely A. Conversely, let G be a finite piecewise linear graph* with no isthmuses; then the faces of G form a finite connected partition in R. The above equivalence also holds if "R" is replaced by "Q" and "piecewise linear" by "rational piecewise linear".*

Proof (The symbol $\star$ is for later reference): Suppose that $a_1, \ldots, a_n$ form a connected partition in R. Consider all the half-planes involved in the construction of elements $a_1, \ldots, a_n$. The lines bounding these half-planes form a finite graph* G^* in the obvious way, and the faces of G^* must form a connected partition consisting of basic polygons, say, $b_1, \ldots, b_N$. Moreover, each a_i $(1 \le i \le n)$ can certainly be expressed as a sum of various b_j $(1 \le j \le N)$.

$\star$ By renumbering if necessary, let $a_1 = b_1 + \cdots + b_m$ for some m $(1 \le m \le N)$. Now remove from G all nodes p such that $p \notin \bigcup\{\mathcal{F}(b_k) | m < k \le N\}$ and all edges e such that $e \not\subseteq \bigcup\{\mathcal{F}(b_k) | m < k \le N\}$. The result will be a graph* G_1 in which the faces $b_1, \ldots, b_m$ are merged into a number of faces $f_1, \ldots, f_{n'}$ for some n' $(1 \le n' \le m)$. The union of these faces will then be the set

$$b_1 \cup \cdots \cup b_m \cup \{p \in |G| : p \in \mathcal{F}(b_i) \text{ for some } i \ (1 \le i \le m), \\ p \notin \mathcal{F}(b_j) \text{ for any } j \ (m < j \le N)\}.$$

By Lemma 3.6 this set is just $b_1 + \cdots + b_m = a_1$. Since a_1 is connected, $n' = 1$ and G_1 contains the face $f_1 = a_1$. Proceeding in the same way for $a_2, \ldots, a_n$ yields a graph* $G = G_n$ with faces $a_1, \ldots, a_n$. That G has no isthmuses follows from the fact that each face of G is regular.

Conversely, suppose that G is a finite piecewise linear graph*; then the edges of G lie on finitely many straight lines. Consider the graph G^* made up of all of these lines (extended in both directions). Each face of G^* is a basic polygon; hence each face f_i of G will be divided into a finite number of basic polygons, say, $b_{i,1}, \ldots, b_{i,m_i}$ by a finite number of straight lines. Since G has no isthmuses, f is a regular set, and it is easy to check that no smaller regular set contains $b_{i,1}, \ldots, b_{i,m_i}$. In other words, $f = b_{i,1} + \cdots + b_{i,m_i} \in R$. The corresponding proof for Q is identical except for the obvious changes. □

Next we come to some topological results concerning P which, as we shall see in the next section, will have a significant effect on possible alternative models of Th($\mathfrak{P}$). We say that an *end-cut* in an open set x is a Jordan arc lying in x except for one endpoint, which lies on $\mathcal{F}(x)$.

Lemma 4.6 *Let $a \in R$ and $p \in \mathcal{F}(a)$. Then there is a piecewise linear end-cut α in a with endpoint p. If $a \in Q$, and p has rational coordinates, then α may be chosen so as to be rational piecewise linear.*

Proof: Obvious. □

Lemma 4.7 *There exists a function $e : \mathbb{N} \to \mathbb{N}$ such that, for all $n > 0$, if G is a graph* whose n faces form a connected partition in P, then there exist at most $e(n)$ points lying on the boundaries of more than two of these faces.*

Proof: We suppose that p_1, p_2, p_3 are distinct points all lying on the boundaries of distinct faces a_1, a_2, a_3 of G, and we derive a contradiction. The result then follows by putting $e(n) = n(n-1)(n-2)/3$. Let p_1, p_2, p_3 and a_1, a_2, a_3 be as described. Choose points q_1, q_2, q_3 such that $q_i \in a_i$ $(i = 1, 2, 3)$. By Lemma 4.6, draw three end-cuts in a_i, say $\alpha_{i,1}$, $\alpha_{i,2}$, and $\alpha_{i,3}$ from the point q_i to the points p_1, p_2, and p_3, respectively. Since we can easily choose $\alpha_{i,1}$, $\alpha_{i,2}$ and $\alpha_{i,3}$ so that they intersect only at q_i, this gives us a planar embedding of the graph $K_{3,3}$, which is well known to be nonplanar. □

Lemma 4.8 *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that, for all $n > 0$, if G is a graph* with n faces forming a connected partition in P, and G has no isthmuses and no nodes of degree 2, then the size of G is bounded by $f(n)$.*

Proof: It is easy to show that, in a plane graph* with no isthmuses, any node of degree greater than two must lie on the boundary of at least three faces. Then, by Lemma 4.7, the number of nodes in G is bounded by a function of n. The result then follows from Euler's formula. □

We then have the following theorem.

Theorem 4.9 *There exists a function $g : \mathbb{N} \to \mathbb{N}$ such that, for all $n > 0$, there exist at most $g(n)$ n-element connected partitions in P up to homeomorphism.*

Proof: By Theorem 4.5, any such partition is the set of faces of some finite piecewise linear graph* with no isthmuses, hence of some graph* with no isthmuses and no nodes of degree 2, since the nodes of degree 2 can be removed without changing the faces of G. By Lemma 4.8, all such graphs* are of size bounded by $f(n)$. Assuming the result that every abstract graph can be embedded in the closed plane in only finitely many homeomorphically distinct ways, the result follows immediately. □

Note that Theorem 4.9 is limited to partitions in P. The corresponding result fails to hold, for example, for arbitrary partitions in the Boolean algebra $\mathrm{RO}(\mathbb{R}^2)$. (It is easy to find counterexamples using constructions such as that illustrated in Figure 2.) Moreover, the result also fails for partitions in $\mathrm{RO}(\mathbb{R}^3)$, even when we confine ourselves to polyhedral objects.

The following lemmas are concerned with showing that P is, in a sense that will become clear below, topologically homogeneous. It is standard to show, given the first part of Lemma 4.6, that for every finite plane graph* G, there is a homeomorphism of the open plane onto itself taking G to a piecewise linear plane graph* G'. Moreover, the homeomorphism can be chosen so that points in faces bounded only by piecewise linear edges in G are unaffected. In effect, finite plane graphs* can have their curved edges "straightened out" by a homeomorphism. If ν is a homeomorphism of the open plane onto itself and a a subset of the open plane, we write $\nu|_a$ to denote the restriction of ν to a.

Lemma 4.10 *Let a, b be connected elements of R such that there is a homeomorphism μ of the open plane onto itself taking a to b. Let $a_1, \ldots, a_n$ be a connected partition of a in R. Then there exists a connected partition $b_1, \ldots, b_n$ of b in R and a*

homeomorphism ν of the open plane onto itself such that $\nu|_{-a} = \mu|_{-a}$ and $\nu(a_i) = b_i$ for all i ($1 \le i \le n$).

Proof: Let the components of $-a$ be $t_1, \ldots, t_m$. Since $t_1, \ldots, t_m, a_1, \ldots, a_n$ is a connected partition, Theorem 4.5 guarantees that we can find a piecewise linear graph* G with no isthmuses having these elements as faces. Now μ maps a to b, hence the components of $-a$ to the components to $-b$, hence G to a graph* G' with faces $u_1, \ldots, u_m, f_1, \ldots, f_n$, say, where $f_1 + \cdots + f_m = b$. But then we can find a a homeomorphism μ' of the closed plane onto itself which takes G' to a piecewise linear graph* G'' without affecting any points in $-b$ or its frontier. Hence, the faces of G'' will be $u_1, \ldots, u_m, b_1, \ldots, b_n$, say. Since G'' clearly contains no isthmuses, Theorem 4.5 guarantees that the faces of G'' will be in R, so that $\nu = \mu' \circ \mu$ is the required homeomorphism. □

Lemma 4.11 *Let a, b be connected elements of R such that there is a homeomorphism μ of the open plane onto itself taking a to b. Let $a' \in R$ satisfying $a' \le a$. Then there exists $b' \in R$ and a homeomorphism ν of the open plane onto itself such that $\nu|_{-a} = \mu|_{-a}$ and $\nu(a') = b'$.*

Proof: By Lemma 4.3, we can find a connected partition of a in R some of whose elements sum to a'. The result then follows from Lemma 4.10. □

Definition 4.12 Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be elements of P. We say that $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are *similarly situated*, written $a_1, \ldots, a_n \sim b_1, \ldots, b_n$, if there is a homeomorphism μ mapping the open plane onto itself such that $\mu(a_i) = b_i$ for all i ($1 \le i \le n$).

Now we can state the lemma guaranteeing homogeneity of P.

Lemma 4.13 *Let $a_1, \ldots, a_n, b_1, \ldots, b_n, a \in P$ such that $a_1, \ldots, a_n \sim b_1, \ldots, b_n$. Then there exists $b \in P$ such that $a_1, \ldots, a_n, a \sim b_1, \ldots b_n, b$.*

Proof: Assume first that P is R. Let μ be a homeomorphism of the closed plane onto itself mapping $a_1, \ldots, a_n$ to $b_1, \ldots, b_n$. Let $c_1, \ldots, c_N$ be all the components of all nonzero products of the form $\pm a_1 \cdot \cdots \cdot \pm a_n$ and let $d_1, \ldots, d_N$ be all the components of all products of the form $\pm b_1 \cdot \cdots \cdot \pm b_n$. Then, by Lemma 4.2, $c_1, \ldots, c_N$ and $d_1, \ldots, d_N$ are connected partitions in R, and by renumbering if necessary, μ maps $c_1, \ldots, c_N$ to $d_1, \ldots, d_N$. It suffices to find a $b \in R$ such that $c_1, \ldots, c_N, a \sim d_1, \ldots, d_N, b$.

For all j ($1 \le j \le N$), let $c'_j = a \cdot c_j$. By Lemma 4.11, there exists a $d'_j \in R$ and a homeomorphism ν_j mapping c'_j to d'_j and equal to μ outside c_j. Then the mapping

$$\nu = \bigcup \{\nu_j|_{c_j} : 1 \le j \le N\} \cup \mu|_{\mathcal{F}(c_1) \cup \cdots \cup \mathcal{F}(c_N)}$$

is a homeomorphism of the open plane onto itself mapping c_j to d_j for all j ($1 \le j \le N$) and mapping $a = c'_1 + \cdots + c'_N$ to $b = d'_1 + \cdots + d'_N \in R$.

Finally if P is Q, we note that it is possible to show, using the second part of Lemma 4.6, that any piecewise linear graph can be homeomorphically "tweaked" into a rational piecewise linear graph, without affecting any nodes with rational coefficients or the edges which join them (considered as sets). This easily guarantees the existence of the required element b of Q. □

Finally, we address the embedding of Q in R. Similar considerations to those above yield the following lemma.

Lemma 4.14 *Let $a_1, \ldots, a_n \in Q$ and $b \in R$. Then there exists $a \in Q$ such that $a_1, \ldots, a_n, b \sim a_1, \ldots, a_n, a$.*

The details are routine and we omit them.

5 Model-theoretic analysis This section contains the main technical result of this paper, Theorem 5.11. As we shall see, this theorem has negative consequences for the search for alternative spatial ontologies. Throughout this section, we use the notation $\bar{a}$ to denote an ordered n-tuple $a_1, \ldots, a_n$.

Let us begin by establishing the promised elementary equivalence of $\mathfrak{Q}$ and $\mathfrak{R}$. First, a reminder from model theory. A *type* $\Gamma(\bar{x})$ in variables $\bar{x}$ is a maximal consistent set of formulas having $\bar{x}$ as their only free variables. Given a model $\mathfrak{A}$, we say that a tuple $\bar{a}$ (of the right arity) belongs to type $\Gamma(\bar{x})$ if $\mathfrak{A} \models \varphi[\bar{a}]$ for every $\varphi(\bar{x}) \in \Gamma(\bar{x})$.

Lemma 5.1 *Let $\bar{a}$ and $\bar{b}$ be tuples in P such that $\bar{a} \sim \bar{b}$. Then $\bar{a}$ and $\bar{b}$ are of the same type in $\mathfrak{P}$.*

Proof: It is straightforward to show that $\mathfrak{P} \models \varphi[\bar{a}]$ if and only if $\mathfrak{P} \models \varphi[\bar{b}]$ by induction on the complexity of φ, and using Lemma 4.13. □

Lemma 5.2 $\mathfrak{Q} \preceq \mathfrak{R}$.

Proof: According to the Tarski-Vaught Lemma (Hodges [20], p. 55), if $\mathfrak{Q} \subseteq \mathfrak{R}$ and, for any n-tuple $\bar{a}$ of $\mathfrak{Q}$ and any formula $\varphi(\bar{x})$ of the form $\exists y \psi(\bar{x}, y)$ such that $\mathfrak{R} \models \varphi[\bar{a}]$, there exists $b \in Q$ such that $\mathfrak{R} \models \psi[\bar{a}, b]$, then $\mathfrak{Q} \preceq \mathfrak{R}$.

By construction, $\mathfrak{Q} \subseteq \mathfrak{R}$. Let $\bar{a}$ be an n-tuple of elements of $\mathfrak{Q}$, and let $\varphi(\bar{x})$ be any formula of $\mathcal{L}$ of the form $\exists y \psi(\bar{x}, y)$ such that $\mathfrak{R} \models \varphi[\bar{a}]$. Then there exists $a \in R$ such that $\mathfrak{R} \models \psi[\bar{a}, a]$. By Lemma 4.14, there exists $b \in Q$ such that $\bar{a}, a \sim \bar{a}, b$. By Lemma 5.1 applied to $\mathfrak{R}$, $\mathfrak{R} \models \psi[\bar{a}, b]$. □

It follows from Lemma 5.2 that $\mathrm{Th}(\mathfrak{Q}) = \mathrm{Th}(\mathfrak{R})$. We have already agreed to denote this set of formulas by $\mathrm{Th}(\mathfrak{P})$.

Clearly, $\mathcal{L}$ contains a formula $\mu_N(\bar{x})$ expressing the notion of being an N-element connected partition.

Lemma 5.3 *For all $N > 0$, let $\mu_N(z_1, \ldots, z_N)$ be the formula*

$$\bigwedge_{1 \leq i \leq N} (c(z_i) \wedge z_i \neq 0) \wedge \bigwedge_{1 \leq i < j \leq N} z_i . z_j = 0 \wedge \sum_{1 \leq i \leq N} z_i = 1 \, .$$

For any N-tuple $\bar{c}$ in P, $\mathfrak{P} \models \mu_N[\bar{c}]$ if and only if $\bar{c}$ is a connected partition in P.

We will continue to use the abbreviation μ_N in the sequel. In addition, when $M = 0$, we interpret the expression $\sum_{1 \leq i \leq M} z_i$ as the $\mathcal{L}$-term 0.

Lemma 5.4 *Let $\bar{a}$ be any n-tuple of elements of P; and let $\bar{x} = x_1, \ldots, x_n$ be an n-tuple of variables. Then $\mathfrak{P} \models \psi[\bar{a}]$ for some formula ψ of the form $\exists z_1 \cdots \exists z_N (\mu_N(\bar{z}) \wedge \pi(\bar{x}, \bar{z}))$, where $\bar{z} = z_1, \ldots, z_N$ and $\pi(\bar{x}, \bar{z})$ is of the form*

$\bigwedge_{1\leq i\leq n}(x_i = \sum_{1\leq j\leq N_i} z_{i,j})$ such that, for all i ($1 \leq i \leq n$), $N_i \geq 0$, and the $z_{i,j}$ ($1 \leq j \leq N_i$) are chosen from among the variables $\bar{z}$.

Proof: Immediate from Lemmas 4.3 and 5.3. □

Having set up our mereotopological language and its countable familiar interpretation $\mathfrak{Q}$, the proof that $\mathfrak{Q}$ constitutes a "minimal" ontology proceeds quite simply using standard techniques from model theory. First, we must make more precise the claim that $\mathfrak{Q}$ is minimal. The relevant concept here is that of a *prime* model.

Definition 5.5 A model $\mathfrak{A}$ is said to be *prime* if, for any model $\mathfrak{B}$, $\mathfrak{A} \equiv \mathfrak{B}$ implies that $\mathfrak{A}$ can be elementarily embedded in $\mathfrak{B}$.

We show that $\mathfrak{Q}$ is prime. It follows that any alternative spatial ontology making the same sentences of our mereotopological language true must contain a copy of $\mathfrak{Q}$, together with some additional elements which make no difference to the formulas satisfied by the elements in that copy of $\mathfrak{Q}$. The technique we use employs the notion of an *atomic* model.

Definition 5.6 A formula $\varphi(\bar{x})$ is said to be *complete* in a theory T if, for all formulas $\theta(\bar{x})$, exactly one of $T \models \varphi \longrightarrow \theta$ and $T \models \varphi \longrightarrow \neg\theta$ hold. A model $\mathfrak{A}$ is said to be *atomic* if any n-tuple $\bar{a}$ in A satisfies a formula $\varphi(\bar{x})$ in $\mathfrak{A}$ such that $\varphi(\bar{x})$ is complete in $\mathrm{Th}(\mathfrak{A})$.

Then we have the following standard result from Chang and Keisler [10].

Theorem 5.7 ([10], 2.3.4) *A model is countable atomic if and only if it is prime.*

Our task, then, is to show that $\mathfrak{Q}$ is atomic. The following results will also feature in the sequel.

Theorem 5.8 ([10], 2.3.3) *If $\mathfrak{A}$ and $\mathfrak{B}$ are countable atomic models and $\mathfrak{A} \equiv \mathfrak{B}$, then $\mathfrak{A} \simeq \mathfrak{B}$.*

Theorem 5.9 ([10], 2.3.13) *Let T be a complete theory. Then T is ω-categorical if and only if, for each n, T has only finitely many types in $x_1, \ldots, x_n$.*

The following lemma contains the main idea of the proof of Theorem 5.11.

Lemma 5.10 *Every finite connected partition in P satisfies a complete formula in* $\mathrm{Th}(\mathfrak{P})$. *In fact, for each N, there exist complete formulas $\gamma_1(\bar{z}), \ldots, \gamma_k(\bar{z})$ (k depending in N) such that* $\mathrm{Th}(\mathfrak{P}) \models \forall z(\mu_N(\bar{z}) \longleftrightarrow (\gamma_1(\bar{z}) \vee \cdots \vee \gamma_k(\bar{z})))$.

Proof: Any connected partition satisfies $\mu_N(\bar{z})$ in $\mathfrak{P}$ for some $N > 0$ by Lemma 5.3. And, conversely, any N-tuple in P satisfying $\mu_N(\bar{z})$ is an N-element connected partition. By Theorem 4.9, there are only finitely many of these up to homeomorphism. Moreover, by Lemma 5.1, any two similarly situated N-tuples belong to the same type $\Gamma(\bar{z})$. Hence, any N-tuple satisfying $\mu_N(\bar{z})$ must belong to one of a finite number of types, $\Gamma_1(\bar{z}), \ldots, \Gamma_k(\bar{z})$ in the variables $\bar{z}$. Now simply select pairwise inconsistent $\delta_i(\bar{z})$ from each Γ_i ($1 \leq i \leq k$) and set $\gamma_i = \mu_N \wedge \delta_i$. □

Theorem 5.11 *$\mathfrak{P}$ is an atomic model.*

Proof: By Lemma 5.4, every n-tuple $\bar{a}$ in P satisfies a formula of the form:

$$\exists z_1 \cdots \exists z_N (\mu_N(\bar{z}) \wedge \pi(\bar{x}, \bar{z})) \ .$$

So let $\bar{c}$ be an N-tuple such that $\bar{a}, \bar{c}$ satisfies π and $\bar{c}$ satisfies μ_N. Thus, $\bar{c}$ form a finite connected partition in P. By Lemma 5.10, let γ be a complete formula in $\mathrm{Th}(\mathfrak{Q})$ satisfied by $\bar{c}$. Then $\bar{a}$ satisfies $\exists z_1 \cdots \exists z_N (\gamma(\bar{z}) \wedge \pi(\bar{x}, \bar{z}))$, which is visibly complete. □

Hence, the familiar model constitutes a "minimal" ontology for practical mereotopology in the following sense.

Corollary 5.12 *If* $\mathfrak{A} \models \mathrm{Th}(\mathfrak{P})$*, then* $\mathfrak{Q}$ *can be elementarily embedded in* $\mathfrak{A}$.

The question of course arises as to whether the familiar model $\mathfrak{Q}$ is *strictly* minimal among countable models of $\mathrm{Th}(\mathfrak{P})$, in that there are countable models of $\mathrm{Th}(\mathfrak{P})$ not isomorphic to $\mathfrak{Q}$. The answer is: yes and no.

Theorem 5.13 $\mathrm{Th}(\mathfrak{P})$ *is not* ω*-categorical.*

Proof: By Theorem 5.9, it suffices to prove that $\mathrm{Th}(\mathfrak{P})$ has countably many types in the single variable x. It is easy to see that, for every positive integer m, the formula $\varphi_m(x)$

$$\exists z_1 \cdots \exists z_m \left(\bigwedge_{1 \le i \le m} (c(z_i) \wedge z_i \neq 0) \wedge \bigwedge_{1 \le i < j \le m} \neg c(z_i + z_j) \wedge x = \sum_{1 \le i \le m} z_i \right)$$

is satisfied in $\mathfrak{P}$ by all and only those regions having exactly m components. Hence, the $\varphi_m(x)$ are all satisfied in $\mathfrak{P}$; so each can be extended to a type $\Gamma_m(x)$ of $\mathrm{Th}(\mathfrak{P})$. But the $\varphi_m(x)$ are also pairwise mutually exclusive in $\mathrm{Th}(\mathfrak{P})$; so no two of them can be extended to the same type in $\mathrm{Th}(\mathfrak{P})$. Hence, $\mathrm{Th}(\mathfrak{P})$ has countably many types in x. □

Thus, there exist countable models of $\mathrm{Th}(\mathfrak{P})$ nonisomorphic to $\mathfrak{Q}$. By Theorem 5.8, these models cannot be atomic, and so cannot be prime. Thus $\mathfrak{Q}$ is, in a strong sense, strictly minimal. However, it turns out that $\mathrm{Th}(\mathfrak{P})$ satisfies a weakened form of of ω-categoricity. The next theorem shows that the only alternative models to $\mathfrak{P}$ are those containing regions comprising, as we might put it, infinitely many pieces.

Theorem 5.14 *Any two countable models of* $\mathrm{Th}(\mathfrak{P})$ *omitting the set of formulas*

$$\Sigma(x) = \left\{ \neg \exists z_1 \cdots \exists z_N \left(\bigwedge_{1 \le i \le N} c(z_i) \wedge x = \sum_{1 \le i \le N} z_i \right) \middle| \ N \ge 1 \right\}$$

are isomorphic.

Proof: Let $\mathfrak{A}$ be countable such that $\mathfrak{A} \models \mathrm{Th}(\mathfrak{P})$ and $\mathfrak{A}$ omits $\Sigma(x)$. Since $\mathfrak{A}$ omits $\Sigma(x)$, for every n-tuple $\bar{a}$ in A, there exists an N-tuple $\bar{c}$ satisfying μ_N in $\mathfrak{A}$ such that the elements of $\bar{a}$ are expressible according to $\mathfrak{A}$ as sums of various elements of $\bar{c}$. (To see this, simply take all nonzero products of the form $\pm a_1 \cdot \cdots \cdot \pm a_n$ and, using the fact that $\mathfrak{A}$ omits $\Sigma(x)$, express each such atom as a sum of elements of A, each

of which satisfies $c(x)$ in $\mathfrak{A}$. By Lemma 3.4, $\mathrm{Th}(\mathfrak{P}) \models \forall x \forall y((c(x) \wedge c(y) \wedge x \cdot y \neq 0) \longrightarrow c(x+y))$, so we may sum together any nondisjoint pairs of these elements until we have elements $c_1, \ldots, c_N$ satisfying μ_N in $\mathfrak{A}$.) By Lemma 5.10, any tuple in A realizing $\mu_N(\bar{z})$ realizes a complete formula, whence $\mathfrak{A}$ is clearly atomic, by identical reasoning to that of Theorem 5.11. By Theorem 5.8, $\mathfrak{A} \simeq \mathfrak{Q}$. □

6 Liberalizing the polygonal ontology The purpose of this section is to show that the polygonal ontology with which we have been working can be significantly liberalized without changing the set of truths expressible in $\mathcal{L}$. As in the polygonal case, the technical details in this section are routine and, in one form or another, well known in studies of semialgebraic sets. The reader is referred to Pillay and Steinhorn [26], Knight, Pillay, and Steinhorn [21] and the works cited there.

Definition 6.1 Let $\mathcal{L}'$ be the language with signature $\langle <, +, \cdot, 0, 1\rangle$, interpreted in $\mathbb{R}$ in the usual way (i.e., with $+$ and $\cdot$ denoting addition and multiplication). A set $A \subseteq \mathbb{R}^n$ is said to be *definable (without parameters)* if there exists an $\mathcal{L}'$-formula $\varphi(x_1, \ldots, x_n)$ such that $A = \{\langle a_1, \ldots, a_n\rangle \in \mathbb{R}^n | \mathbb{R} \models \varphi[a_1, \ldots, a_n]\}$.

We extend the use of this term in the obvious way: a point $\bar{a} \in \mathbb{R}^n$ is definable if the set $\{\bar{a}\}$ is definable; a partial function $f : \mathbb{R} \longrightarrow \mathbb{R}^n$ is definable if it is definable considered as a subset $\mathbb{R}^{n+1}$; and plane graph in $\mathbb{R}^2$ is definable if all its nodes are definable points and its edges definable arcs.

Theorem 6.2 *Let T be the set of definable, regular sets in $\mathbb{R}^2$. Then T forms a Boolean subalgebra of* $\mathrm{RO}(\mathbb{R}^2)$.

Proof: It is easy to see that if x is definable, $-x = \mathbb{R}^2 \setminus [x]$ is also definable, and that if y is also definable, $x \cdot y = x \cap y$ is definable. □

Given this theorem, we define the *countable definable model* $\mathfrak{T}$ for our mereotopological language $\mathcal{L}$ by interpreting the primitives of $\mathcal{L}$ over the domain of quantification T in the obvious way. The main result of this section is:

Theorem 6.3 $\mathfrak{Q} \preceq \mathfrak{T}$. *In fact,* $\mathfrak{Q} \simeq \mathfrak{T}$.

It follows of course that $\mathrm{Th}(\mathfrak{T}) = \mathrm{Th}(\mathfrak{P})$, so that the liberalization of the ontology arising from allowing regions to be described by any formula of $\mathcal{L}'$ makes no difference to the set of truths expressible in $\mathcal{L}$. We note in passing that the real polygonal domain R is not a subset of T, since R is uncountable. A corresponding liberalization of R would involve the use of parameters from $\mathbb{R}$ in the defining formulas. We also note that the results of this section might possibly be generalized to apply to regular definable sets in two dimensions over any real closed field. However, it is unclear that such generalizations would have any significance for our current concerns. The remainder of this section is devoted to proving Theorem 6.3.

To see where the difficulties lie, recall our treatment of the polygonal case. Since basic polygons are convex, it was trivial to show that every element of P is the sum of finitely many connected elements of P, and hence that any component of an element of P is an element of P. But it is not immediately obvious that corresponding facts apply to T; and that is what we must show. Once we have done this, the development

parallels that of the polygonal case. The following result is well known (see, e.g., Hodges [20], p. 92 for an explanation).

Theorem 6.4 *Any definable subset of $\mathbb{R}$ is a finite union of points and open intervals (possibly unbounded); moreover, the endpoints of these intervals are all definable.*

Definition 6.5 (Adapted from Knight, Pillay, and Steinhorn) A 0-cell is a point in $\mathbb{R}^2$; a 1-cell is the graph $\{(\xi_1,\ f(\xi_1)) \in \mathbb{R}^2 | \xi_1 \in I\}$, of a definable, continuous function $f : I \longrightarrow \mathbb{R}$ where I is a definable open interval of $\mathbb{R}$ (possibly unbounded); a 2-cell is a set $\{(\xi_1, \xi_2) \in \mathbb{R}^2 | \xi_1 \in I;\ f(\xi_1) < \xi_2 < g(\xi_1)\}$, where I is a definable open interval of $\mathbb{R}$ (possibly unbounded) and f and g are definable, continuous functions from I to $\mathbb{R} \cup \{\pm\infty\}$ such that $f < g$ over I.

Note that since 2-cells are visibly regular, they are elements of T. The critical theorem for us is as follows. (See Knight, Pillay, and Steinhorn [21], §3–5 for a proof of this theorem.)

Theorem 6.6 ([21]) *Every definable set in $\mathbb{R}^2$ is a finite union of cells.*

Lemma 6.7 *Any element of T is the sum of finitely many connected elements of T.*

Proof: Let $t = c_1 \cup \cdots \cup c_n$ where the c_i are 0- 1- and 2-cells. Since t is regular, $t = t'' = (c_1 \cup \cdots \cup c_n)''$, and it is routine to show that, for any sets $c_1, \ldots, c_n$, c_i'' is always regular ($1 \leq i \leq n$) and $(c_1 \cup \cdots \cup c_n)'' = c_1'' + \cdots + c_n''$. If c_i is a 0- or 1-cell, then $c_i'' = 0$. If c_i is a 2-cell $c_i'' = c_i$. □

The development now parallels that of the polygonal case.

Lemma 6.8 *Let $a \in T$ and let c be a component of a. Then $c \in T$. Moreover, a equals the sum of its components.*

Lemma 6.9 *Let A be a finite subset of T. Then there exists a connected partition C in T such that each $a \in A$ is expressible as a sum of zero or more elements of C.*

The proofs are identical to those for Lemmas 4.2 and 4.3.

Theorem 6.10 *Let A be a connected partition in T; then there exists a finite definable graph* with no isthmuses whose faces are precisely A. Conversely, let G be a finite definable graph* with no isthmuses; then the faces of G form a finite connected partition in T.*

Proof: Suppose that $a_1, \ldots, a_n$ form a connected partition in T. Take a finite collection C of 2-cells such that each of the a_i can be expressed as a sum of various elements of C. By Theorem 6.4 the boundary of each 2-cell in the closed plane can certainly be drawn using finitely many definable arcs (possibly passing through the point at infinity), and together, these arcs form a definable graph* G^* in the closed plane in the obvious way. Again by Theorem 6.4, G^* has finitely many nodes and edges, and hence finitely many faces, say $b_1, \ldots, b_N$. These faces are the components of finite intersections of 2-cells and so are regular, whence each a_i ($1 \leq i \leq n$) can certainly be expressed as a sum of various b_j ($1 \leq j \leq N$). The proof now proceeds exactly as for Theorem 4.5 from the point marked $\star$.

Conversely, suppose that G is a finite definable graph*. For each edge of G^*, lying on the arc $(\xi_1(t), \xi_2(t))$, say, the set of reals t corresponding to local maxima and minima of $\xi_1(t)$ (including endpoints) is definable and therefore finite by Theorem 6.4. Similarly, the set of intervals over which the function $\xi_1(t)$ is constant is finite. Now add to G by drawing vertical lines at all these (obviously definable) values of $\xi_1(t)$, for each arc in G. The result must be a finite definable graph* G^* each face of which is a 2-cell. If G has no isthmuses, then each face f of G is regular, and is in fact the smallest regular set containing all the faces of G^* (i.e., 2-cells) into which it is divided. Hence f is the sum of these 2-cells, so $f \in T$. □

Lemma 6.11 *Let $a \in R$ and $p \in \mathcal{F}(a)$ be a definable point. Then there is a definable end-cut α in a with endpoint p.*

Proof: This is visibly true if a is a 2-cell. If $a = c_1 + \cdots + c_n$ where c_i is a 2-cell $(1 \leq i \leq n)$, then $\mathcal{F}(a) \subseteq \bigcup_{1 \leq i \leq n} \mathcal{F}(c_i)$, whence the result follows immediately. □

Theorem 6.12 *There exists a function $g : \mathbb{N} \longrightarrow \mathbb{N}$ such that, for all $n > 0$, there exist at most $g(n)$ n-element connected partitions in T up to homeomorphism.*

The proof proceeds, given lemma 6.11, as for Theorem 4.9 and the preceeding lemmas.

Finally, we come to the homogeneity results. The critical observation here is Lemma 6.11. Using this lemma, is it standard to show that, given any finite graph* G, we can find a homeomorphism of the open plane onto itself taking G to a definable graph*, while fixing (setwise) all the definable faces of G. The following lemmas can then be proved as for the polygonal case (with minor changes).

Lemma 6.13 *Let $a_1, \ldots, a_n, b_1, \ldots, b_n, a \in T$ such that $a_1, \ldots, a_n \sim b_1, \ldots, b_n$. Then there exists $b \in T$ such that $a_1, \ldots, a_n, a \sim b_1, \ldots b_n, b$.*

Lemma 6.14 *Let $a_1, \ldots, a_n \in Q$ and $b \in T$. Then there exists $a \in Q$ such that $a_1, \ldots, a_n, b \sim a_1, \ldots, a_n, a$.*

At this point, the conclusion $\mathfrak{Q} \preceq \mathfrak{T}$ follows as for the proof of Lemma 5.2. Thus, $\mathfrak{T} \models \mathrm{Th}(\mathfrak{P})$ and by Lemma 6.9, $\mathfrak{T}$ omits $\Sigma(x)$ (as defined in the statement of Theorem 5.14). Moreover, $\mathfrak{T}$ is countable, so by Theorem 5.14, $\mathfrak{Q} \simeq \mathfrak{T}$.

7 Related work The results presented here not only have ramifications for mereotopological theories (Casati and Varzi [8], [9], Varzi [36] and references in Section 1), but they have connections with more practical disciplines. Various logicians have sought to give deductive theories of space and space-time (Basri [4], Carnap [7], Goldblatt [16], Henkin, Suppes, and Tarski [19]), many in terms of modal logics (Balbiani et al. [3], Rescher and Garson [29], Rescher and Urquhart [30], Segerberg [32], Shehtman [34], von Wright [38]). Recent interest in the analysis of visual languages, such as maps and diagrams (Haarslev [18], Lemon [24], Lemon and Pratt [25], Pratt [27]) has led to the exploration of planar mereotopology in relation to qualitative spatial reasoning, since it is theorized that an important aspect of the semantics of such representations may be given by analysis of spatial relations between representational tokens in the plane. Another more practical area in which ontological issues

about the plane are raised is in the construction of computational spatial representations for robots, and in Geographical Information Systems (Davis [13], Vieu [37]). As we have mentioned, GISs use planar polygonal regions to represent geographic objects. In mobile robotics too, it is common to represent a robot's information about its environment as a planar arrangement of places together with their connection relations (Davis [14], Dudek, Freedman, and Hadjres [15], Kuipers [23], Shanahan [33]). A complete axiomatization of $\mathrm{Th}(\mathfrak{P})$ has been developed in Pratt and Schoop [28].

8 Conclusion In this paper, we have investigated the possibility of alternative spatial ontologies for practical mereotopological reasoning. In order to constrain the problem, we insisted that any such ontology provide a model elementarily equivalent to the "familiar" polygonal model $\mathfrak{P}$. Our motivation for taking $\mathfrak{P}$ as our point of departure was that many computer packages designed to manipulate spatial data, such as GISs, restrict themselves to piecewise linear objects, without any apparent loss of useful representational power.

We identified rational and real "versions" of $\mathfrak{P}$, namely $\mathfrak{Q}$ and $\mathfrak{R}$, with the former being countable. The main technical results of this paper state that, although $\mathfrak{Q}$ is not the only countable countable model of $\mathrm{Th}(\mathfrak{P})$, it is, in the sense of elementary embedding, the minimal such model. Thus, the countable alternatives to $\mathfrak{Q}$ all contain a copy of $\mathfrak{Q}$—the "familiar" regions, plus some "nonfamiliar" regions which make no difference to any properties of the familiar regions expressible in $\mathcal{L}$. Thus, in a strong sense, they are less parsimonious. Moreover, we found a simple condition on models of $\mathrm{Th}(\mathfrak{P})$ which determines $\mathfrak{Q}$ up to isomorphism, and provides a useful characterization of the other models of $\mathrm{Th}(\mathfrak{P})$. Finally, we showed how $\mathfrak{P}$ could be considerably liberalized without affecting the truths expressible in $\mathcal{L}$. Apparently, revisions to our ontology of the plane which do not violate the facts of polygonal mereotopology—to the extent they exist at all—must be less parsimonious than the one we started with.

Acknowledgments With thanks to: Anthony Galton, David Rydeheard, Reg Wood, Dominik Schoop, and the anonymous referee. The authors gratefully acknowledge the support of the Leverhulme Trust (Grant number F/120/AQ), the British Council's British-German ARC Programme (project 720), and the European Commission's TMR Programme (contract number ERBFMBICT972035).

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