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Wittgenstein on Mathematical Meaningfulness, Decidability, and Application

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Abstract From 1929 through 1944, Wittgenstein endeavors to clarify *mathematical meaningfulness* by showing how (algorithmically decidable) mathematical propositions, which lack contingent "sense," have mathematical sense in contrast to all infinitistic "mathematical" expressions. In the middle period (1929–34), Wittgenstein adopts strong formalism and argues that mathematical calculi are formal *inventions* in which meaningfulness and "truth" are entirely *intra*systemic and epistemological affairs. In his later period (1937–44), Wittgenstein resolves the conflict between his intermediate strong formalism and his criticism of set theory by requiring that a *mathematical* calculus (vs. a "sign-game") must have an *extra*systemic, real world application, thereby returning to the weak formalism of the *Tractatus*.

1 Introduction Wittgenstein's ruminations on mathematics from 1929 to 1944 are at one time or another baffling, paradoxical, counterintuitive, and apparently absurd. The aim of the present paper is to advance our critical understanding of some of the more important themes of his theory of mathematics by focusing on the single most important issue in Wittgenstein's philosophy as a whole: meaningfulness. More than any other issue, the question of meaningfulness dominates Wittgenstein's philosophy from 1918 through 1951. In the *Tractatus*, he provides a criterion for the meaningfulness of contingent propositions, and then shows that although tautologies, contradictions, and mathematical equations do not satisfy this criterion, they are still meaningful in an analytic or intrasystemic sense. When, from 1929 through 1944, Wittgenstein focuses on mathematical meaningfulness, he develops a theory based upon the Tractarian notion that only contingent propositions can be true or false by correspondence to facts. To distinguish meaningful mathematical propositions from expressions that are "senseless" (*sinnlos*) or meaningless, Wittgenstein argues that the math-

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ematician invents mathematics bit by little bit. Mathematical truth is not discovered, it is invented.

In elaborating this theory of *mathematics by invention*, Wittgenstein attempts to describe what we really *have* in mathematics, and what we really *do* in mathematics. What we really have, he says, are purely formal calculi, consisting of finite extensions (e.g., concatenations of signs, sequences, and sets of numerals) and rules. What we really do in mathematics is use these rules to generate, operate on, and, in the case of propositions, decide finite extensions. In the latter case, since we have no algorithmic means of distinguishing propositions of a given calculus from expressions that are undecidable in, or independent of, the calculus, Wittgenstein stipulates that an expression is a meaningful proposition of a calculus if and only if we know of an applicable and effective "method of solution" (i.e., decision procedure).

In what follows, I shall endeavor to show first, that in the middle period Wittgenstein adopts a strong variant of formalism in order to show that mathematical calculi are formal inventions in which meaningfulness and truth are entirely intrasystemic affairs. Second, I shall explicate Wittgenstein's finitism and algorithmic decidability, and show how they give rise to the radical position that conjectures such as Goldbach's Conjecture (GC) and putatively proved theorems, such as Euclid's Prime Number Theorem, are meaningless (sinnlos) expressions. Third, I shall argue that in adopting extrasystemic application as a necessary condition of a meaningful mathematical calculus in Wittgenstein [19], thereby returning to the weak formalism of the *Tractatus*, Wittgenstein resolves the conflict between his intermediate strong formalism and his relentless denigration of Transfinite Set Theory (TST) as a meaningless, nonmathematical calculus. In Section 4, we shall see that Wittgenstein's long standing claim that syntactical contradictions are innocuous is incompatible with extrasystemic application, and hence with the weak formalism of [19].

2 Wittgenstein's intermediate formalism: 1929–34 It is somewhat surprising that Wittgenstein's intermediate formalism has not been sufficiently recognized. Part of the reason for this is that Wittgenstein criticizes many aspects of Hilbert's philosophy of mathematics, both during his middle period and later in [19]. Thus, many commentators and critics, noting these numerous criticisms, erroneously conclude that Wittgenstein cannot be a formalist because he so obviously disagrees with Hilbert, who, after all, is the father of twentieth century formalism (see, e.g., Frascolla [3], pp. 44–55). The mistake made here is the tacit identification of formalism with Hilbert's program (e.g., the possibility and usefulness of metamathematics) and with whatever Hilbert says about mathematics (e.g., contentual number theory). This mistake can be corrected if we distinguish different variants of formalism as distinct philosophies of mathematics. To this end, I offer the following four definitions.

Strong Formalism (SF): A mathematical calculus is defined by its accepted or stipulated propositions (e.g., axioms) and rules of operation. Mathematics is syntactical, not semantical: the meaningfulness of propositions within a calculus is an entirely intrasystemic matter. A mathematical calculus may be invented as an uninterpreted formalism, or it may result from the axiomatization of a "meaningful language." If, however, a mathematical calculus has a semantic interpretation or an extrasystemic application, it is inessential, for a calculus is essentially a "sign-game"—its signs and propositions do not refer to or designate extramathematical objects or truths.

Weak Formalism (WF): A mathematical calculus is a formal calculus in the sense of SF, but a formal calculus is a mathematical calculus only if it has been given an extrasystemic application to a real world domain.

Extreme Formalism (EF): A mathematical calculus is a formal calculus in the sense of SF, except that it cannot have an extrasystemic application or interpretation, because its "axioms" and rules are arbitrarily chosen, and necessarily so, for one cannot "point at anything as a justification" for one's choice of syntax (Waismann [14], p. 105). Thus, the construction of a mathematical calculus cannot be driven by either a primitive semantical interpretation or the aim of applying the calculus, extrasystemically, to a real world domain.

Radical Formalism (RF): A mathematical calculus is a formal calculus in the sense of SF, except that it need not be syntactically consistent.

The core idea of formalism, namely, that mathematics is essentially syntactical, is indeed the core of Wittgenstein's view of mathematics from 1929 through 1944. "We *make* mathematics," Wittgenstein tells us, "mathematics can in a certain sense only be made" ([14], p. 34, n. 1). This view of "making mathematics" is rearticulated throughout Wittgenstein's work when, for example, he says that "we can't describe mathematics, we can only do it" ([17], p. 159; [19], 5:5, 9), when he says we "invent" mathematics ([19], 1:168; 2:38; 5:9, 11; [16], pp. 469–70), and when he asserts that "the mathematician is not a discoverer: he is an inventor" ([19], 2, app. 2).[1]

The first, and perhaps principal, aspect of this view is that, *contra* platonism, the signs and propositions of a mathematical calculus do not refer to anything outside the calculus. As Wittgenstein says at [14], p. 34, n. 1, "numbers are not represented by proxies; numbers *are there*." This means not only that numbers are there in the use (i.e., "[a] calculus . . . works by means of strokes, numerals, etc." [14], p. 106), it means that the numerals *are* the numbers, for "arithmetic doesn't talk about numbers, it works with numbers" ([17], 109). Wittgenstein makes essentially the same point at ([16], p. 333).

> What arithmetic is concerned with is the schema | | | | .—But does arithmetic talk about the lines I draw with pencil on paper?—Arithmetic doesn't talk about the lines, it *operates* with them.

As I have said, some commentators, such as Frascolla, claim that the intermediate Wittgenstein is not a formalist, at least partly on the grounds that Wittgenstein explicitly criticizes Hilbertian formalism.[2] The first thing to note in this connection is that, contrary to popular opinion, Hilbert's formalism is weaker than both EF and SF. True, Hilbert [6] does occasionally come close to EF, as, for instance, when he speaks of "arbitrarily given axioms" and claims that so long as an axiomatic formal system is consistent, its axioms and theorems are true in the sense that its "theorems hold true for any interpretation of the primitive notions and fundamental relationships for which the axioms are satisfied" (Reid [11], p. 60). For the most part, however, Hilbert seems closer to SF, for already in [6], Section 1, he says that "the axioms of geometry are divided into five groups; every one of these groups expresses closely related fundamental facts of our *intuition*" [italics mine]. Twenty-six years later in [7], Hilbert still maintains the crucial importance of Kantian a priori intuition in mathematics, this time claiming that Frege's and Dedekind's attempts "to make pure logic provide for arithmetic a foundation that would be independent of all intuition and experience" "were bound to fail" because as "Kant [correctly] taught . . . mathematics has at its disposal a content secured independently of all logic and hence can never be provided with a foundation by means of logic alone" ([7], p. 376).

> As a condition for the use of logical inferences and the performance of logical operations, something must already be given to our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought. ([8], p. 464)

Given Hilbert's lifelong Kantianism, his formalism seems principally motivated by the desire to emphasize the possibility of, and benefit from, the formalization of mathematical systems.[3] Indeed, if a mathematical calculus must express primitive synthetic a priori (or essentially mathematical) intuitions, Hilbert's formalism can at most be characterized as a weak variant of SF—almost WF, in fact, except that Hilbert requires only applicability (i.e., consistency), not actual extrasystemic application.

The second thing to note about Wittgenstein's criticisms of Hilbert is that Wittgenstein *misconstrues* Hilbertian formalism, and thus thinks that in avoiding his misinterpretation of Hilbertian formalism, he avoids formalism altogether. This is particularly evident when he says that "Frege was right in objecting to the conception that the numbers of arithmetic are signs. The sign '0', after all, does not have the property of yielding the sign '1' when it is added to the sign '1' " ([14], p. 105).[4]

> Frege was right in this criticism [of formalism]. Only he did not see the other, justified side of formalism, that the symbols of mathematics, although they are not signs, lack a meaning. For Frege the alternative was this: either we deal with strokes of ink on paper or these strokes of ink are signs of *something* and their meaning is what they go proxy for. . . . There is still a third possibility, the signs can be used the way they are in the game. ([14], p. 105)

The most striking thing about this passage is that Wittgenstein seems to believe that formalists like Hilbert think that a mathematical calculus is defined only as concatenations of meaningless signs. But nothing could be further from the truth, for Hilbert always maintains that rules of operation, together with the syntax of well-formed formulas, constitute the distinctive (grammatical) syntax of a formal, mathematical calculus.[5] Thus, in saying that mathematical symbols lack a meaning (i.e., *Bedeutung*), that they do not go proxy for things which are "their meaning[s]," Wittgenstein is really agreeing with Hilbert by saying that "the signs can be used the way they are in a game." On the other hand, in saying this Wittgenstein is really disagreeing with Frege about the signs '0' and '1', for Wittgenstein means that the sign '0' by itself "does not have the property of yielding the sign '1', whereas Frege's point is that even with rules, formal arithmetic does not "differ from a mere game," since its "theorems only *seem* to say something about the figures" [italics mine], where in reality they "throw light upon the properties of the rules of the game" ([4], p. 169). For Frege, if the rules are arbitrary, then the calculus is a mere game, and if the rules are not arbitrary, then the calculus has a semantical interpretation, in which case it is not a mere game, or not just a sign-game (e.g., it may be a meaningful, mathematical calculus). Put differently, Wittgenstein's third possibility is really the very type of formal mathematics (e.g., formal arithmetic) that Frege is attacking!

An immediate consequence of Wittgenstein's view that mathematical symbols do not "go proxy for" things that are their meanings, is that in doing mathematics, we do not discover preexisting truths that were "already there without one knowing" ([16], p. 481)—we invent mathematics, bit by little bit. With respect to application, Wittgenstein makes this point by saying that "mathematics is its own application" ([14], p. 34, n. 1) and that "arithmetic is its own application" ([17], p. 109). What is meant here is internal or intrasystemic application, whereby:

> You apply a calculus in such a way that it yields the grammar of a language. In grammar, then, the words 'sense' and 'senseless' correspond to what a rule permits and prohibits. ([14], p. 126)

Given that "the truth in formalism is that every syntax can be conceived of as a system of rules of a game" ([14], p. 103), it follows that the rules of a calculus, together with its primitive propositions, determine what is a meaningful proposition of the calculus and what is not.

> You could say arithmetic is a kind of geometry; i.e. what in geometry are constructions on paper, in arithmetic are calculations (on paper).—You could say it is a more general kind of geometry. ([17], 109)[6]

> The point of the remark that arithmetic is a kind of geometry is simply that arithmetical constructions are autonomous like geometrical ones, and hence so to speak themselves guarantee their applicability. For it must be possible to say of geometry, too, that it is its own application. ([17], p. 111)

This, then, is the core of Wittgenstein's life long formalism. As Wittgenstein puts it in [14], p. 106, "Mathematics is always a machine. The calculus does not describe anything." When we prove a theorem or decide a proposition, we operate in a purely formal manner. "If you want to know what $2+2=4$ means," says Wittgenstein, "you have to ask how we work it out. That means that we consider the process of calculation as the essential thing" ([16], p. 333). Hence, the only meaning that a mathematical proposition has is intrasystemic meaning, which is entirely determined by its syntactical relations to other propositions of the calculus.

2.1 Wittgenstein's intermediate strong formalism In the middle period, however, Wittgenstein goes beyond the core idea of formalism (i.e., that mathematics is essentially syntactical) and adopts strong formalism, primarily to show that extrasystemic application is not a necessary condition of a mathematical calculus.[7] In [17], p. 109, for instance, Wittgenstein argues that mathematical calculi do not require extrasystemic applications.

> One always has an aversion to giving arithmetic a foundation by saying something about its application. It appears firmly enough grounded on itself. And that of course derives from the fact that arithmetic is its own application.
>
> Every mathematical calculation is an application of itself and only as such does it have a sense.

> It seems to me that you can develop arithmetic completely autonomously and its application takes care of itself since wherever it's applicable we may also apply it. ([17], p. 109; see also [16], p. 308)

In this passage, Wittgenstein denies that mathematics needs a foundation, a denial which he maintains right through [19], as is evidenced by 3:43, 85; and especially 7:16. Wittgenstein objects to foundationalism in [17], 109, by stressing internal or intrasystemic application, which is the strong formalist idea that the meaningfulness and the truth or falsity of propositions within a calculus are determined entirely by the axioms and rules of operation of that calculus, without any necessary reference to an extrasystemic application. The principal point here, viz. SF, is also made at [14], p. 104, where Wittgenstein says that "all you can say is that syntax can be applied only to what it can be applied to."[8]

In [16], Wittgenstein reiterates his claim that mathematical calculi do not need extrasystemic applications.

> But (as we all know) arithmetic *isn't at all concerned* about [extrasystemic] application. Its applicability takes care of itself. [italics mine]
>
> . . . .
>
> The equation 4 apples + 4 apples = 8 apples is a substitution rule which I use if instead of substituting the sign "8" for the sign "4 + 4", I substitute the sign "8 apples" for the sign "4 + 4 apples". ([16], p. 308)

This restatement of SF is given a new argument in the following two pages.

> But what does the application add to the calculation? Does it introduce a new calculus? In that case it isn't any longer the same calculation. Or does it give it substance in some sense which is essential to mathematics (logic)? If so, how can we abstract from the application at all, even only temporarily? ([16], pp. 309–10)

In this passage Wittgenstein articulates SF by saying that an extrasystemic application *cannot* give a calculus "substance in some sense which is essential to mathematics (logic)," arguing that if it could, we would not be able to "abstract from the application at all, even only temporarily." Though this is clearly SF, it should be noted that this reason, or argument, is at best weak, for it may well be that an extrasystemic application makes a formal calculus mathematically meaningful, and yet we can look at, speak of, and examine the formal calculus without reference to its application.[9]

A second reason that the middle Wittgenstein is drawn to SF seems to be a new concern with questions of decidability. Probably influenced by the philosophical writings of Brouwer and Hilbert, Wittgenstein uses SF to forge a new connection between mathematical meaningfulness and decidability.[10]

> An equation is a rule of syntax. Doesn't that explain why we cannot have questions in mathematics that are in principle unanswerable? For if the rules of syntax cannot be grasped, they're of no use at all. . . . [This] makes intelligible the attempts of the formalist to see mathematics as a game with signs. ([17], 121)

Here Wittgenstein connects his formalism with decidability by saying that "an equation is a rule of syntax . . . explains why we cannot have" unanswerable mathematical questions or undecidable mathematical propositions. At [17], 202 Wittgenstein explicates this notion by stressing that "a mathematical proposition can only be either a stipulation, or a result worked out from stipulations in accordance with a definite method. And this must hold for '9 is divisible by 3' or '9 is not divisible by 3'." In Section 3, we shall see in detail how Wittgenstein goes beyond both Hilbert and Brouwer by maintaining the Law of the Excluded Middle in a way that restricts mathematically meaningful propositions to expressions that are algorithmically decidable.

It must be admitted, however, that my interpretation of the middle Wittgenstein as a strong formalist faces the objection that Wittgenstein occasionally seems to go beyond SF and embrace EF. For instance, in [14], p. 105 Wittgenstein says that "if you ask me me why I use a syntax, I cannot point at anything as a justification. You cannot give reasons for syntax. Hence it is arbitrary." Similarly, in [16], 310 Wittgenstein claims that "in arithmetic we cannot make preparations for a grammatical application," suggesting, as he does in [14], p. 105 that we cannot create or construct a calculus with an extrasystemic application or interpretation in mind (i.e., as our goal). One page later, Wittgenstein goes even further by denying the very possibility of extrasystemic application, saying that "what is incorrect is the idea that the application of a calculus in the grammar of real language correlates it to a reality or gives it a reality that it did not have before" ([16], 311).

Wittgenstein's apparent EF denial of the possibility of extrasystemic application is, I believe, best understood as an anomalous overstatement, given that he admits extrasystemic application in both [14] and [16]. Consider first, his second answer to the question "What does it mean to apply a calculus?"

> A calculus can be applied in such a way that true and false propositions correspond to the configurations of the calculus. In the case the calculus yields a theory that describes something. . . . Geometry too can be understood in this way by taking it as a description of the results of actual measurements.
>
> Then it is statements that we have before us, and statements can indeed contradict each other.
>
> Whether a theory can describe anything depends on whether the logical product of its axioms is a contradiction. ([14], pp. 125–26)

This passage is somewhat startling in light of Wittgenstein's first answer to this question, namely, "syntax cannot be justified." What is even more perplexing is that a mere eight pages after his strong denial of the possibility of extrasystemic application ([16], 311), Wittgenstein appears once again to admit the possibility of extrasystemic application.

> Geometry isn't the science (natural science) of geometric planes, lines and points, as opposed to some other science of gross physical lines, stripes and surfaces and *their* properties. The relation between geometry and propositions of practical life, about stripes, colour boundaries, edges and corners, etc. isn't that the things geometry speaks of, though *ideal* edges and corners, resemble those spoken of in practical propositions; it is the relation between those propositions and their grammar. Applied geometry is the grammar of statements

> about spatial objects. The relation between what is called a geometrical line and boundary between two colours isn't like the relation between something fine and something coarse, but like the relation between possibility and actuality. (Think of the notion of possibility as a shadow of actuality.) ([16], 319)

In saying that "the relation between geometry and propositions of practical life . . . is the relation between those propositions and their grammar," Wittgenstein clearly allows that there can be such a thing as "applied geometry," which (like [14], pp. 125–26) seems to flatly contradict his claim that a mathematical calculus cannot be "correlated with" (applied to) a reality outside itself ([16], 311). If "a calculus can be applied in such a way that true and false propositions correspond to the configurations of the calculus," then "$3 + 4 = 7$" *can* be applied to plums and men ([14], p. 34, n. 1). When a purely formal calculus is so applied to a real world domain, it yields a descriptive theory consisting of propositions such as "3 plums + 4 plums = 7 plums," which can then be used to make an inference from, for instance, two nonmathematical (contingent) propositions (e.g., "There are three plums on the left side of this table" and "There are four plums on the right side of this table") to another nonmathematical proposition (i.e., "There are seven plums on this table"). This shows that "the application of a calculus in the grammar of real language correlates it to a reality," from which it follows that Wittgenstein is also wrong to say that "I cannot point at anything as a justification" (for "why I use a syntax"), for if we construct a calculus so that it can be applied to a real world domain, then the formulation of the calculus and its syntax may very well be guided by the nature of that domain ([14], p. 105).

Wittgenstein, however, also goes beyond SF in making various RF claims suggesting that a mathematical calculus need not be syntactically consistent. For our purposes his following remarks are most pertinent.

> My list of rules, then, is in order. I see no contradiction. I now ask, "Does any danger still remain?" Out of the question! For what are we afraid of? Of a contradiction, of all things? As long as a contradiction is not there it does not concern me. I can accordingly stay quite calm and do my calculations. Would the discovery of a contradiction in mathematics, then, make all the calculations cease to exist that have been established by mathematicians in the course of several hundred years? Should we say that they were not calculations? Absolutely not. If a contradiction is going to occur, we shall manage. Now, however, we need not worry about it. ([14], pp. 195–96)

There are a number of important points made in this passage. First, if we have not proved a contradiction, or do not see one that has been proved, it is a "hidden contradiction," which really "is not there" (i.e., does not exist). Wittgenstein claims that "[a] contradiction is a contradiction only *if it is there*" ([14], p. 120). If there is "no procedure for finding a contradiction . . . , there is no sense wondering if our inferences might not *eventually* lead to a contradiction" ([14], p. 120; cf. [16], 305). We need not worry about this possible eventuality, because if one day a contradiction arises, "it would be the easiest thing in the world to find a remedy" ([14], p. 120).

The main point of these RF statements seems to be that we need not worry about a contradiction arising, and that even if one does arise, this does not make our past work with the calculus nonmathematical activity. What is hard to discern is whether

Wittgenstein means that a calculus with a derived and *seen* contradiction is still a mathematical calculus, or whether the contradiction must then be remedied if the calculus is to regain (or retain) its mathematical status. In Section 4, we shall see that extrasystemic application, which is rightly admitted by Wittgenstein in [14], p. 125–26 and [16], 319, and required in [19], presupposes syntactical consistency. Given that Wittgenstein's statements of EF are unargued and, moreover, incompatible with his correct claims in [14], pp. 125–26 and his admissions of applicability ([16], 319), SF is the most coherent construal of his intermediate position. When we add to this the fact that extrasystemic application presupposes syntactical consistency (contra RF), the best conclusion to draw is that SF is the most coherent and most defensible construal of Wittgenstein's intermediate formalism. The main thrust of his intermediate remarks on application is the SF claim that neither extrasystemic application nor applicability is a necessary condition of a meaningful mathematical calculus.

2.2 Cardinal numbers in the middle years The picture drawn in the previous section of Wittgenstein's SF faces the possibly incompatible claim, made by Frascolla, that Wittgenstein's intermediate ruminations on cardinal numbers constitute an "acknowledgement, absent in the *Tractatus*, of the central role of the concept of cardinal of a class" ([3], p. 45). In support of this claim, Frascolla suggests that Wittgenstein offers his stroke-notational symbol '| | |' "as a paradigmatic representation of the class property of having three elements," ([3], p. 46). which in turn suggests an analogy with Frege's definitions of cardinal numbers based upon 1 to 1 correspondence and paradigmatic definitions of '0' and '1'.

Initially, it must be admitted that Wittgenstein came to have strong misgivings about his Tractarian conception of number, for he says that "a nebulous introduction of the concept of number by means of the general form of operation—such as I gave [in the *Tractatus*]—can't be what's needed" ([17], 109). As regards, however, Frascolla's claim that Wittgenstein acknowledges "the central role of the concept of cardinal of a class," though Wittgenstein certainly wrestles with the term 'cardinal number', he in no way acknowledges that it plays a central role, either in mathematics or in the philosophy of mathematics. Frascolla is right to say that "it is nonsense to say of an extension that it has such and such a number, since the number is an internal property of the extension" ([17], 119) means that a cardinal number is an internal property of a finite set in extension (i.e., "a list"). As Wittgenstein says,

> The sign for the extension of a concept is a list. We might say, as an approximation, that a number is an external property of a concept and an internal property of its extension (the list of objects that fall under it). A number is a schema for the extension of a concept. That is, as Frege said, a statement of number is a statement about a concept (a predicate). ([16], 332)

The problem, however, is that this is a far cry from saying that Wittgenstein's stroke-notational symbol '| | |' is "a paradigmatic representation of the class property of having three elements." There is nothing in [16], 333–35 that can be reasonably construed as a definition of a cardinal number, as for instance, Frege gives by means of paradigmatic definitions of '0' and '1' and equinumerosity. Frascolla, I think, is partially misled by the reference to Frege in [16], 332, but he is not completely misled, for he explicitly notes Wittgenstein's assertion that "what we are looking for is *not* a definition of the concept of number, but an exposition of the grammar of the word 'number' and of the numerals" ([16], 321) [italics mine].[11] Indeed, when Wittgenstein asks "What are numbers?," he emphatically answers "What numerals signify" ([16], 321). Frascolla seemingly understands all of this, as he quotes it piecemeal, but he interprets Wittgenstein's crucial elucidation, "an investigation of what [numerals] signify is an investigation of the grammar of numerals" ([16], 321) to mean that the analysis in question is "the analysis of the meaning of the general term 'number'." Frascolla's mistake is that he fails to understand that what Wittgenstein means by "an exposition of the grammar of the *word* "number" *and of the numerals*" [italics mine] is that the word 'number' is understood by means of *its* grammar, and the numerals are understood by means of *their* grammar. But these are two very distinct undertakings! The grammar of the word 'number' is discussed by Wittgenstein in [17], 105, 119 and in [16], 332, which Frascolla interprets correctly: a number is an internal property of a list and an external property of a concept; but unlike Frege, in the latter case this is not, as Wittgenstein clearly says, a definition of 'cardinal number' or 'number'. The grammar of numerals, however, is not discussed in those [17] and [16] passages. Rather, it is discussed throughout Wittgenstein's middle discussions of the logical grammar and logical syntax of mathematics. The grammar of numerals consists in the syntactical relations among numerals as determined by purely syntactical, arithmetical rules.

I conclude, therefore, that Wittgenstein's intermediate ruminations on cardinal number in no way make him any less of a formalist. In fact, just the opposite is the case. If numbers are "what numerals signify," and if what numerals signify is determined by their grammar, the intermediate Wittgenstein is, as I have argued, a strong formalist.

3 Mathematical conjectures, meaningfulness, and decidability In examining Wittgenstein's views on meaningfulness and decidability, it should be noted at the outset that Wittgenstein's intermediate concern with mathematical meaningfulness is not a departure from his Tractarian claim that mathematical propositions, *qua* equations, are pseudopropositions ([20], 6.2). The point of this claim is that mathematical propositions, like tautologies and contradictions, lack sense in that they say nothing about the world ([20], 4.461). Given that only contingent propositions can be true by correspondence to facts in the world, only they have sense, and so tautologies, contradictions, and mathematical propositions are sinnlos (senseless). Despite this fact, tautologies, contradictions, and mathematical propositions are *not* nonsensical, for "they are part of the symbolism" ([20], 4.4611), they *show* "the logic of the world" ([20], 6.22), and thereby enable us to infer one contingent proposition from one or more other contingent propositions ([20], 5.12, 6.1201, 6.211).

In the middle period, Wittgenstein maintains precisely the same distinction between contingent propositions, which have *sense* in that they say something about the world, and mathematical propositions, which do not have sense because they say nothing about the world. The principal difference is that the intermediate Wittgenstein wishes to distinguish between mathematical propositions *of* a given calculus, which have mathematical sense in *that* calculus, and expressions that are not propositions of that calculus, and which, therefore, are sinnlos (senseless, meaningless) with respect to that calculus. All mathematical propositions are still sinnlos in contrast to contingent propositions, but when we speak of a given mathematical calculus, some expressions (i.e., propositions) have intrasystemic sense, while others are sinnlos, viz., that calculus.

The critical question is: Why does Wittgenstein wish to distinguish between meaningful mathematical propositions and meaningless expressions? There are a number of related answers to this question, but the short answer is that Wittgenstein is constrained by his view of mathematics by invention. If we do not discover preexisting truths that were "already there without one knowing" ([16], 481), does this mean that an expression is only a meaningful mathematical proposition when it has actually been proved? Wittgenstein rejects this position on the grounds that it would "wipe out the existence of mathematical problems."

> That is to say, it isn't as if it were only certain that a mathematical proposition made sense when it (or its opposite) had been proved. (This would mean that its opposite would never have a sense (Weyl).) ([17], 148)[12]

This view is absurd, according to Wittgenstein, for "it obviously makes sense to say 'I know how you check [$36 \times 47 = 128$]', even before you've done so" ([17], 153). The problem with this obvious fact, however, is that it suggests that perhaps an expression is a meaningful proposition only if we know how to decide it, which immediately raises the question: "How can there be conjectures in mathematics?" ([17], 161)

> We might also ask: What is it that goes on when, while we've as yet no idea how a certain proposition is to be proved, we still ask 'Can it be proved or not?' and proceed to look for a proof? If we 'try to prove it', what do we do? Is this a search which is essentially unsystematic, and therefore strictly speaking not a search at all, or can there be some plan involved? How we answer this question is a pointer as to whether the as yet unproved—or as yet unprovable—proposition is senseless or not. For, in a very important sense, every significant proposition must teach us through its sense how we are to convince ourselves whether it is true or false. 'Every proposition says what is the case if it is true.' And with a mathematical proposition this 'what is the case' must refer to the way in which it is to be proved. Whereas—and this is the point—you cannot have a logical plan of search for a sense you don't know. ([17], 148)

Though it is not immediately apparent from the foregoing quotations, Wittgenstein claims that so-called mathematical conjectures are meaningless expressions, not meaningful mathematical propositions or meaningful mathematical questions. Consider, for example, [17], 150.

> Fermat's [Last Theorem] makes no *sense* until I can *search* for a solution to the equation in cardinal numbers.
>
> And 'search' must always mean: search systematically. Meandering about in infinite space on the look-out for a gold ring is no kind of search.

As Wittgenstein puts it, "the so-called 'Fermat's Last Theorem' isn't a proposition. (Not even in the sense of a proposition of arithmetic.)" ([17], 189)[13] What is particularly controversial here is that these circumstances are precisely what makes, for most mathematicians, a putatively meaningful proposition a conjecture. In Wittgenstein's own words, we are faced with a conjecture when "we've as yet no idea how a certain proposition is to be proved, [and yet we] ask 'Can it be proved or not?' and proceed to look for a proof." For Wittgenstein, however, the crucial point is that the "sort of thing . . . that looks like a conjecture in mathematics" ([17], 161) is not a mathematical proposition because we do not know how to decide it systematically.

3.1 Algorithmic decidability As I have said, Wittgenstein adopts this radical position for a number of reasons. Like Brouwer, Wittgenstein holds that there are no "unknown truth[s]" in mathematics ([2], p. 90). But unlike Brouwer he denies the existence of "undecidable propositions" on the grounds that such propositions would have no sense, "and the consequence of this is precisely that the propositions of logic lose their validity for it" ([17], 173). In particular, if there are no "unknown mathematical truths," but there are undecidable mathematical propositions, then at least some mathematical propositions are neither true nor false. For Wittgenstein, however, it is a defining feature of a mathematical proposition that it is either true or false, and if it is true (or false), its negation is false (or true). "Where the law of the excluded middle doesn't apply," Wittgenstein asserts, "no other law of logic applies either, because in that case we aren't dealing with propositions of mathematics. (Against Weyl and Brouwer)" ([17], 151).

To maintain this position, Wittgenstein distinguishes between meaningful mathematical propositions and meaningless (sinnlos) expressions by stipulating that an expression is a meaningful proposition of a calculus if and only if we *know* of an applicable decision procedure. "Where there's no logical method for finding a solution," states Wittgenstein, "the question doesn't make sense either" ([17], 149). "We may only put a question in mathematics (or make a conjecture[14])," he adds, "where the answer runs: 'I must work it out' " ([17], 151).[15] That an applicable decision procedure must be known is stressed in [17], 151 where Wittgenstein says that "the question 'How many solutions *are* there to this equation?' is the holding in readiness of the general method for solving it." Similarly, when Wittgenstein asserts that "if there is no method provided for deciding whether the proposition is true or false, then it is pointless, and that means senseless," ([16], 452) he says that the relevant decision procedure must be "provided."[16] Wittgenstein emphasizes the importance of algorithmic decidability clearly and emphatically: "In mathematics *everything* is algorithm and *nothing* is meaning; even when it doesn't look like that because we seem to be using *words* to talk *about* mathematical things. Even these words are used to construct an algorithm" ([16], 468). When, therefore, Wittgenstein says that if "[the law of the excluded middle] is supposed not to hold, we have altered the concept of proposition" ([16], 368), he means that an expression is only a meaningful proposition if we know of an applicable decision procedure for deciding it.[17] If a proposition is undecided, the law of the excluded middle holds in the sense that we know that we will make the proposition true or false by applying an applicable decision procedure.

There is, however, a problem facing this epistemological interpretation of algorithmic decidability, insofar as Wittgenstein seems to deny that the "holding in readiness" of a decision procedure is a *psychological* matter.

> We call it a problem, when we are asked "how many are 25×16", but also when we are asked: what is $\int \sin^2 x \, dx$. We regard the first as much easier than the second, but we don't see that they are "problems" in different senses. *Of course*, the distinction is not a psychological one; it isn't a question of whether

> the pupil can solve the problem, but whether the calculus can solve it, or which calculus can solve it. ([16], 379)

Wittgenstein repeats this denial explicitly when he says that "whether a pupil *knows a rule* for ensuring a solution to $\int \sin^2 x\, dx$ is of no interest; what does interest us is whether the *calculus* we have before us (and that he happens to be using) contains such a rule" ([16], 379). Wittgenstein similarly states:

> Euclid doesn't show us how to look for the solutions to his problems; he gives them to us and then proves that they are solutions. And this isn't a psychological or pedagogical matter, but a mathematical one. That is, the *calculus* (the one he gives us) doesn't enable us to look for the construction. A calculus which does enable us to do that is a *different* one. ([16], 387)

The problem is simply: How can Wittgenstein maintain that "the distinction is not a psychological one," given that 'I know how you check "$36 \times 47 = 128$"' seems to require that one psychologically knows how to decide the proposition? The answer to this problem resides in ([16], 379) where Wittgenstein says that "it isn't a question of whether the pupil can solve the problem, but whether the calculus can solve it." Wittgenstein realizes that if the requisite knowledge of a decision procedure is relative to an individual, then even if two individuals are operating with the same axioms and rules of operation, if one knows of a decision procedure that the other does not, then they are operating with different calculi. To preclude this possibility, Wittgenstein emphasizes that it is not whether an individual knows of a decision procedure, but whether "the *calculus* we have before us (and that he happens to be using) contains such a rule" ([16], 379). What this means, I believe, is the very point that Wittgenstein hammers away at in [19], namely, that a rule is only part of a calculus if a community of individuals share certain conventions and have decided to accept (or ratify) the rule as part of their calculus.[18] A decision procedure must be known, but individual psychological states of knowledge are not sufficient, for a community must know of the decision procedure and, moreover, agree to accept it as part of their calculus.

From this we can see that Wittgenstein introduces the epistemological criterion of algorithmic decidability as a means of maintaining the law of the excluded middle and avoiding predeterminateness in mathematics, which is essential to his view of mathematics by invention. Specifically, he wishes to deny both that an expression is meaningful before we have a relevant decision procedure in hand, and also that a meaningful proposition is true or false before we have actually decided it. Half of this goal is achieved by adopting SF, and thereby restricting mathematically meaningful propositions to the syntax and purely syntactical rules of a calculus. The second half of the goal is achieved by further restricting meaningful mathematical propositions to expressions that we know how to decide algorithmically.

3.2 Algorithmic decidability, finitism, and the problems they engender An initial, contemporary response to algorithmic decidability is to say that it is simply refuted by the fact that there do not exist decision procedures for numerous mathematical calculi, such as Elementary Number Theory and TST (not to mention elementary logical calculi, such as the first-order predicate calculus). That is, since we have proved, for instance, that there is no solution to the *Entscheidungsproblem* for elementary number theory, it follows that there are well-formed formulas of elementary number theory that are not algorithmically decidable, but which may be (unsystematically) decidable. On this account, we can recognize (or decide by means of a decision procedure) that an expression (e.g., GC) is a (meaningful) well-formed formula of elementary number theory, and this well-formed formula may be provable or refutable despite the fact that there does not exist in principle a decision procedure applicable to it. It follows that Wittgenstein's account is not only explicitly revisionistic, it provides no explanation for why a mathematician would bother to look for a decision of a meaningless expression such as GC (i.e., why s/he would let GC "stimulate" her to make a new, mathematical construction and calculus).

In the following three sections I shall first show how Wittgenstein's algorithmic decidability is closely connected with his finitism and his unorthodox construal of mathematical induction, and then I will address the question(s) of revisionism.

3.2.1 Wittgenstein's finitism In his attempt to *describe* mathematics and mathematical activity,[19] Wittgenstein finds that mathematical calculi are inventions, consisting of extensions (e.g., symbols, propositions, sets, sequences) and intensions (e.g., rules, 'laws'). Wittgenstein embraces finitism primarily because a mathematical extension, *qua* concatenation of symbols, is necessarily finite: "The symbol for a class is a list" ([16], 461). We mistakenly believe in the actual infinite ([16], 471), according to Wittgenstein, because we conflate extensions with intensions, erroneously thinking that there is "a dualism" of "the law and the infinite series obeying it" ([17], 180). We think, for instance, that because a real number "endlessly yields the places of a decimal fraction" ([17], 186), a real number (e.g., an irrational) is "a totality" ([14], pp. 81–82, n. 1). But "an irrational number isn't the extension of an infinite decimal fraction, . . . it's a law" ([17], 181) which "yields extensions" ([17], 186).[20] Similarly, when "people speak of a line as composed of points," they make the same mistake, for "[a] line is a law and isn't composed of anything at all" ([17], 173).[21] "The straight line isn't composed of points" ([17], 172) —"those mathematical rules [e.g., $\sqrt{2}$] are the points" ([16], 484). Wittgenstein elucidates this strongly constructivist theme by saying that "a curve is not composed of points, it is a law that points obey, or again, a law according to which points can be constructed" ([16], 463).

What we call an "infinite class" is represented by a recursive rule, or what Wittgenstein sometimes calls "an induction." But though "an induction has a great deal in common with the multiplicity of a class (a finite class, of course), . . . it isn't one, and now it is called an infinite class" ([17], 158). An inductive rule is used to generate finite extensions—it is only "infinite" in the sense that it represents an infinite, or unlimited, possibility. In mathematics, we use the word 'infinite' to refer to unlimited techniques ([15], p. 31), but "to say that a technique is unlimited does not mean that it goes on without ever stopping—that it increases immeasurably; but that it lacks the institution of the end, that it is not finished off" ([19], 2:45). Thus, when we speak of the "infinite set of natural numbers," this indicates "only the infinite possibility of finite series of numbers," for "it is senseless to speak of the whole infinite number series, as if it, too, were an extension" ([17], 144).

We mistakenly think that there are infinite sets and sequences in extension not only because we conflate intensions and extensions, but also because we are not careful to distinguish between the grammar of the words that we use to apply to each. When we are careful, we see that "the words 'finite' and 'infinite' . . . are not adjectives" which "signify a supplementary determination regarding 'class' " ([14], p. 102). " 'Infinite' is not a quantity," Wittgenstein states, "the word 'infinite' has a different syntax from a number word" ([14], p. 228). "We mistakenly treat the word "infinite" as if it were a number word, because in everyday speech both are given as answers to the question 'how many?' " ([16], 463; cf. [17], 142). But since an infinite class is only an inductive rule for generating finite extensions, we cannot apply a cardinal number to an infinite class, because a *rule* simply cannot be counted. Conversely, it is nonsense to say "we cannot enumerate all the numbers of a set, but we can give a description," for one "cannot give a description instead of an enumeration" because "the one is not a substitute for the other" ([14], p. 102).[22] As Wittgenstein puts it in [16], 461, "the mistake in the set-theoretical approach consists time and again in treating laws and enumerations (lists) as essentially the same kind of thing." Wittgenstein consistently elaborates this criticism of set theory right through [19], arguing, for instance, that since the irrationals are not enumerable, "there is no system [i.e., infinite set] of irrational numbers—but also no super-system, no 'set of irrational numbers' of higher order infinity" ([19], 2:33).[23]

The upshot of Wittgenstein's finitism for meaningfulness and decidability is that expressions that purport to quantify over an infinite mathematical domain are not meaningful mathematical propositions. At [17], 127, Wittgenstein argues that an existentially quantified proposition, such as '$(\exists n)4 + n = 7$', cannot be an infinite disjunction.

> What is the meaning of such a mathematical proposition as '$(\exists n)4 + n = 7$'? It might be a disjunction—$(4 + 0 = 7) \vee (4 + 1 = 7) \vee$ etc. *ad infinitum*. But what does that mean? I can understand a proposition with a beginning and an end. But can one also understand a proposition with no end? . . . If no finite product makes a proposition true, that means no product makes it true. And so it isn't a logical product.

Indeed, Wittgenstein goes much further, arguing that even if we have proved, for example, '$\varphi(13)$', we cannot state or infer '$(\exists x)\varphi x$', for "the expression '$(\exists x)\varphi x$' can't be taken to presuppose the totality of numbers" ([17], 173).

> The error arises from regarding an extension as a totality. For it makes good sense to say: If 7 occurs at the 25th place, then 7 occurs between the 20th and the 30th place. But it does not make sense to say: 7 occurs, full-stop. This is not a statement at all. ([14], pp. 81–82, n. 1)

Wittgenstein similarly rejects universal quantification over an infinite domain, saying that "you can't say '$(n)\varphi n$', precisely because 'all natural numbers' isn't a bounded concept" ([17], 126). Later, at [17], 145, Wittgenstein presents a different argument, and then emphatically asserts that he has given "a *reductio ad absurdum* of the concept of an infinite *totality*."

From this brief exposition, we can see that Wittgenstein embraces finitism for two reasons. First, his examination and description of mathematics reveals only finite extensions—no infinite sets, decimal expansions, logical sums, or logical products. Second, expressions that quantify over an infinite domain are meaningless, first because there are no infinite mathematical domains (i.e., sets), and second, because such expressions are not algorithmically decidable.

3.2.2 Mathematical induction Insofar as Wittgenstein's finitism explains algorithmic decidability as a criterion of meaningfulness, it also explains, to some extent, his radical claim that mathematical conjectures such as GC and Fermat's Last Theorem are meaningless expressions.[24] For if one "can't say '$(n)\varphi n$,' precisely because 'all natural numbers' isn't a bounded concept . . . then neither should one say a general proposition follows from a proposition about the nature of number" ([17], 126). The strongly revisionistic tenor of this assertion consists in the fact that, not only do most mathematicians interpret expressions that quantify over an infinite domain as meaningful, most mathematicians take it as obvious that some such expressions (e.g., Euclid's Prime Number Theorem) have been proved by putatively constructive means (e.g., mathematical induction). But Wittgenstein will have none of this: 'propositions' that universally quantify over an infinite domain, such as GC, cannot be 'proved' by mathematical induction.

> It is first of all striking that the proposition to be proved does not occur in the proof itself at all. Thus the proof does not actually prove the proposition. That is to say, induction is not a procedure leading to a proposition. Rather, induction allows us to see an infinite possibility, and in this alone does the nature of proof by induction consist.
>
> Afterwards we articulate what we have shown by the inductive proof as a proposition, and here we use the word 'all'. But this proposition adds something to the proof, or better, the proposition is related to the proof as a sign is to the thing signified. The proposition is a name for the induction. The former goes proxy for the latter; *it does not follow from it*. ([14], p. 135) [italics mine]

When Wittgenstein says that a proof by mathematical induction "does not actually prove the proposition" (e.g., $(\forall n)(En)$), he means that the so-called conclusion of an inductive proof is a pseudoproposition that stands proxy for a proved inductive base and inductive step (i.e., the induction).

> We are not saying that when $f(1)$ holds and when $f(c+1)$ follows from $f(c)$, the proposition $f(x)$ is *therefore* true of all cardinal numbers: but: "the proposition $f(x)$ holds for all cardinal numbers" means "it holds for $x = 1$, and $f(c+1)$ follows from $f(c)$." ([16], 406)

Thus, a proof by mathematical induction cannot prove, for example, that Euclid's Prime Number Theorem is true; we come to see an infinite possibility, we do not prove a proposition true for an infinite domain (because there is no such thing as an infinite domain in mathematics). Secondly, and more importantly, conjectures such as GC are meaningless because "prior to the [inductive] proof asking about the general proposition made no sense at all, and so wasn't even a question, because the question would only have made sense if a general method of decision had been known before the particular proof was discovered" ([16], 402). As Wittgenstein puts it, if one executes a proof by mathematical induction, we may say, "So he has seen an *induction*!"

> But was he *looking for* an induction? He didn't have any method for looking for one. And if he hadn't discovered one, would he *ipso facto* have found a number which does not satisfy the condition?—The rule for checking can't be: let's see

> where there is an induction or a case for which the law doesn't hold.—If the law of excluded middle doesn't hold, that can only mean that our expression isn't comparable to a proposition. ([16], 400)

The point here is that even an unproved inductive step is meaningless (i.e., "the law of excluded middle doesn't hold"), because we have no decision procedure by which to decide *any* inductive step of *any* mathematical induction proof. If we succeed in proving GC by mathematical induction, we will then and only then have a proof of the inductive step, but since the inductive step was not algorithmically decidable beforehand, in executing the proof "we shall again have a calculus in front of [us], only a different one from the calculus [we] have been using so far" ([14], pp. 174–75). As Wittgenstein says, "it's unintelligible that I should admit, when I've got the proof [of Euclid's Prime Number Theorem], that it's a proof of precisely *this* proposition, or of the induction meant by this proposition" ([17], 155). On Wittgenstein's view, any *successful* proof by mathematical induction necessarily creates a new calculus.

Wittgenstein's account of mathematical induction raises a number of questions, only two of which I will briefly consider here:

1. If we succeed in proving a proposition unsystematically (e.g., by mathematical induction), since the proof was executed without altering the axioms or rules of the calculus, does this not show that the proposition was all along meaningful in the calculus (and, derivatively, that we have not created a *new* calculus)?
2. If we take Wittgenstein on his own terms, what possible reason would one have for attempting to "decide" a meaningless conjecture or expression?

Wittgenstein endeavors to answer the first question in a number of interrelated ways. First and perhaps foremost, Wittgenstein asserts that it is a mistake to think that we can say that "a statement is relevant [decidable] when it is constructed from certain basic formulae with the help of seven principles of combination (among which are 'all' and 'there is'). . . . A statement is relevant," insists Wittgenstein, "if it belongs to a *certain system*" and "what is not visibly relevant, is not relevant at all" ([14], p. 37). What is meant here is that if we wish to speak of meaningful or decidable propositions of a system, as we do, we can neither lay down rules for well-formedness, nor rely upon a belief that, for example, GC is decidable in elementary number theory. Such beliefs are only hunches or hopes, which in no way guarantee that GC is provable or refutable in our calculus. Nor can we invoke the law of the excluded middle, plug in GC to get "GC $\vee\neg$ GC," and then conclude that GC must be decidable since it is either true or false, for this simply begs the question at issue ([19], 5:10, 12). The law of the excluded middle, however, does hold for all meaningful propositions, according to Wittgenstein, but only in the sense that if we know of an applicable decision procedure, then we know that we will make the undecided proposition either true or false. Thus the only way to distinguish between meaningful propositions and expressions that may not be provable or refutable from our axioms (e.g., independent expressions) is to require that meaningful mathematical propositions be algorithmically decidable.

This brings us to the far more difficult question two: What possible reason would one have for attempting to "decide" a meaningless conjecture or expression? Wittgenstein says that there is nothing "wrong or illegitimate" if one lets a formula, such as Fermat's Last Theorem, stimulate one to search for a mathematical construction ([14], p. 144). "A mathematician is of course guided by associations, by certain analogies with the previous system," ([14], p. 144) Wittgenstein says, s/he is "not completely blank and helpless when . . . confronted by [Fermat's Last Theorem]" ([19], 6:13). These admissions, however, despite their obvious veracity, hardly explain why the mathematician would let her/himself be stimulated by Fermat's Last Theorem if it really were a meaningless expression.

The best answer to the question that Wittgenstein can offer, I believe, rests upon a tripartite distinction between two types of *extended* calculi and *entirely new* calculi.[25] A calculus is extended minimally when we decide a proposition by means of a known decision procedure. A calculus is extended moderately when we decide an expression unsystematically, as we do when we successfully construct an inductive proof. In the maximal case, we create an entirely new calculus either by modifying our axiom set or rules of operation, or by creating a new calculus from scratch. An inductive proof thus creates a (moderately) new calculus—a new "calculating machine" ([14], p. 106)—in an epistemological sense, for by means of such a proof we "learn something *completely new*, and not just the way leading to the goal with which I'm already familiar" ([17], 155). The reason we may let a formula stimulate us is, therefore, that we may wish to determine whether we can extend our calculus without altering the axioms or rules, and thereby arrive at a new "machine-part" ([19], 6:13) in a new calculating machine. There is no prohibition on such an attempt, but, on Wittgenstein's view, it is crucial to bear in mind that such a proof creates a new calculus, for we now have "the capacity to *make use of* the proposition" ([19], 6:13). [italics mine]

These are but two of the more important questions raised by Wittgenstein's construal of mathematical induction. Perhaps the most difficult problem for Wittgenstein's view of mathematics by invention is whether his adherence to truth-functionality is compatible with inductive proofs that enable us to directly prove disjunctions where no disjunct has yet been proved.

3.2.3 Wittgenstein's revisionism Even if we grant that the foregoing is internally coherent (and this is granting a great deal), the critic will no doubt rejoin that Wittgenstein's account is revisionistic. Surely, s/he will argue, mathematical conjectures are meaningful mathematical propositions, Euclid's Prime Number Theorem is a meaningful proposition, and TST is a mathematical calculus. This accusation of revisionism is particularly threatening in the case of Wittgenstein, for he repeatedly claims, from 1929 through 1944, that he is not meddling with the practice of mathematicians, but only clarifying what they do by examining what they *say* about what they prove (i.e., the concomitant prose of a formal proof).

> It is a strange mistake of some mathematicians to believe that something inside mathematics might be dropped because of a critique of the foundations. . . . What is caused to disappear by such a critique are names and allusions that occur in the calculus, hence what I wish to call prose. ([14], p. 149)

> In mathematics there can only be mathematical troubles, there can't be philosophical ones. The philosopher only marks what the mathematician casually throws off about his activities. ([16], 369)

> The difference between my point of view and that of contemporary writers on the foundations of arithmetic is that I am not obliged to despise particular calculi like the decimal system. For me one calculus is as good as another. ([16], 334)[26]

Wrigley argues that, true to his word, Wittgenstein is *not* a revisionist; he is merely describing and clarifying mathematical practice. According to Wrigley, Wittgenstein's point about consistency proofs "in no way affects the status of such proofs as perfectly respectable mathematics. They can be called 'consistency proofs' on the grounds that the word 'consistent' appears in the calculus, but Wittgenstein wishes to make clear that the words 'consistent' and 'inconsistent' are just signs in the calculus like any others, and this does not mean that those calculi have anything to do with *consistency*" ([22], p. 189). Shanker also seems to want to deny that Wittgenstein is a revisionist, in part, like Wrigley, because Wittgenstein says so frequently that he is not a revisionist. However, in trying to demonstrate that Wittgenstein succeeds in walking a fine line between, on the one hand, philosophical description and clarification, and on the other hand, revisionism, Shanker cannot help but admit that Wittgenstein occasionally overstates his case, especially in connection with TST, where Shanker asks, "how could Wittgenstein seriously hope to maintain the posture that only the prose of Cantor's interpretation had been affected by his critique while the 'calculus' had remained untouched?" ([13], p. 215).[27] What Shanker is referring to in the latter case are Wittgenstein's repeated attacks on TST during his middle period. To cite only a few examples, Wittgenstein says that Dedekind's definition of an infinite class is "laughable" ([16], 464) and "ridiculous" ([16], 465–66), that set theory is "nonsense" because "it builds on a fictitious symbolism [which cannot possibly exist]" ([17], 174), and that the fact that "we can't describe mathematics, we can only do it" in and "of itself abolishes every 'set theory' " ([17], 159). But by far the most interesting of Wittgenstein's remarks on TST are as follows.

> When set theory appeals to the human impossibility of a direct symbolization of the infinite it brings in the crudest imaginable misinterpretation of its own calculus. It is of course this very misinterpretation that is responsible for the invention of the calculus. But of course that doesn't show the calculus in itself to be something incorrect (it would be at worst uninteresting) and it is odd to believe that this part of mathematics is imperilled by any kind of philosophical (or mathematical) investigations. ([16], 469–70)[28]

The last sentence in this passage is the crucial and telling one, for Wittgenstein here says that TST (a "part of mathematics") is not incorrect as a calculus, but rather it is "uninteresting." I believe that this passage, more than any other in the middle period, indicates that at this time Wittgenstein is torn in two opposing directions. On the one hand, he wishes to define a mathematical calculus in a purely formal way and he dearly wishes not to question the status of any putative mathematical calculus. On the other hand, his philosophical criticism of TST, its concepts, and its interpretation clearly question the mathematical status of TST, that is, its status as a meaningful mathematical calculus. Not only does this explictly conflict with Wittgenstein's SF (an issue that will be addressed in Section 5.1), it strongly suggests revisionism with respect to TST.

In my view, if we are to determine whether or not Wittgenstein is a revisionist, we ought not to be swayed by the *non*revisionistic pronouncements of [14], [16], [19], and [18], but instead ascertain whether Wittgenstein's attack on TST, his finitism, and algorithmic decidability are compatible with what mathematicians *do*, including how they interpret their propositions, calculi, and activities. It will not do to say, with Wrigley (and, to a certain extent, Shanker), that in the final analysis, Wittgenstein has no quarrel with mathematicians carrying on with TST or any other calculus. This rejoinder rings all too hollow when Wittgenstein repeatedly calls the fundamental notions of TST "laughable," "nonsense," and "ridiculous," and especially when he says that set theory "brings in the crudest imaginable misinterpretation" which is the "very misinterpretation that is responsible for the invention of the calculus." The very things that Wittgenstein calls "mistakes" and "nonsense" ([16], 461) are not only responsible for the construction of the calculus, they are also responsible for its development, a development that starts with Dedekind's "nonsensical" and "ridiculous" definition, claims to prove the existence of nondenumerable sets, and then constructs upon this a new arithmetic for these higher order, transfinite numbers. But if mistakes have led to the construction of the calculus, including formal components that were only devised because of these mistakes, then clearly Wittgenstein is saying the calculus is in some sense mistaken nonsense.

We may conclude, therefore, that Wittgenstein is not merely "mark[ing] what the mathematician casually throws off about his activities"—he certainly does "despise" a particular calculus (i.e., TST), and so, for him, "one calculus is [not] as good as another." It follows that Wittgenstein is a revisionist in the minimal sense, because his accounts of elementary number theory and TST disagree with mathematicians' interpretations of these calculi, and in the maximal sense, because he legislates what is proper mathematics and what is not proper mathematics (e.g., what is mathematically meaningful, and what is not). Wittgenstein thinks that he is not a revisionist because he thinks that he is only showing us what we really have in mathematics (i.e., symbols, extensions, rules) and what we really do in mathematics (i.e., decide propositions, invent/construct new proofs and calculi). It should be borne in mind, however, that if Wittgenstein is right about what we really have and do, then he is not a revisionist in anything but the minimal sense.

4 Syntactical consistency as a necessary condition of extrasystemic application

As we saw in Section 2.1, the middle Wittgenstein offers a number of different negative views on the need for consistency. The later Wittgenstein makes many of the same points, suggesting, for example, that a "hidden contradiction" is not really "there as long as it is hidden" ([15], p. 217). What this means, however, is just as unclear in the later period as it was in [14], pp. 195–96. Is a contradiction "hidden" if it has not been proved, but is provable? Or "is it hidden," as Wittgenstein suggests, "as long as it hasn't been *noticed* [i.e., it has been proved, but not seen]?" ([15], p. 219). In the latter case, Wittgenstein is unequivocal: "As long as it's hidden, . . . it's as good as gold"; . . . and when it comes out in the open it can do no harm" ([15], p. 219). The question we must answer here is: If extrasystemic application is allowed, as it is in [14] and [16] and indeed required as it is in [19] (see Section 5), is a "hidden contradiction" innocuous in both senses of the term? In this section I shall argue that hidden contradictions are not always innocuous, from which it follows that RF is untenable.

Wittgenstein's principal point about syntactical contradictions, hidden or otherwise, seems to be that there is no need to worry about a syntactically defined contradiction in a purely formal calculus, because a "contradiction" such as "$a \neq a$" is just another theorem of the calculus: "If I call an arbitrary configuration a contradiction, then this has no essential significance, at least not for the game qua game" ([14], p. 119). In a purely formal calculus, 'p' and '$\neg p$' are just two different well-formed formulas, just as "$a \neq a$" is merely a well-formed formula. One can say that '$\neg p$' is the "negation" of 'p,' and that "$a \neq a$" and "$p \& \neg p$" are (syntactical) "contradictions," but in what way is this problematic if we consider that calculus as only a *formal* calculus? (Cf. [4], p. 194). The fact that the rules of the calculus allow one to derive any proposition from a 'contradiction' makes no difference, because qua uninterpreted calculus, the fact that we can derive any well-formed formula and its negation is not in and of itself problematic. If the game (i.e., calculus) is just a game, then the "discovery of a contradiction [cannot] mean the destruction of the calculus . . . " ([14], p. 196), for "if I arrange the rules in such a way that this configuration of pieces cannot come about, I have merely made up another game" ([14], p. 119).

But here lies the rub. On Wittgenstein's account, the matter is very different when we are dealing with a *descriptive theory* that is the extrasystemic application of a purely formal calculus, because the descriptive theory *can* have statements that contradict one another and contradictory propositions cannot both be true. The problem here, as Wittgenstein himself points out on p. 126 of [14], is that "whether a theory *can* describe anything depends on whether the logical product of its axioms is a contradiction." Given that "[a] calculus can be applied in such a way that true and false propositions correspond to the configurations of the calculus," it follows that there is an isomorphism between true and false propositions of the calculus and *true* and *false propositions* of the descriptive theory (e.g., between "$3+4=7$" and "3 plums $+$ 4 plums $=$ 7 plums"). Thus, *if* the axioms of the descriptive theory must be consistent for the theory to describe a real world domain *and* the aforementioned isomorphism must obtain, *then* the axioms of the calculus must also be (syntactically) consistent.

For the most part,[29] Wittgenstein tries to deny this by arguing that "if a contradiction were now actually found in arithmetic—that would only prove that an arithmetic with *such* a contradiction in it could render very good service" ([19], 7:35). Once we realize that we do not need a consistency proof to "rely on the calculus" ([19], 3:84), the idea that a "contradiction destroys the calculus" can "with a little imagination . . . certainly be shaken" ([19], 7:15). The point is that we are still "really doing mathematics" in the absence of a consistency proof. "If I *see* a contradiction," Wittgenstein says, "then will be the time do to something about it" ([19], 3:81; cf. [14], p. 120).

These, I believe, are *good* points, but the question is: How far do they get us? All that Wittgenstein has shown is that a calculus that contains an unseen contradiction *may not* lead to any trouble in an extrasystemic application. What he has not shown is that such a contradiction *cannot* cause a disaster in an extrasystemic application, which is precisely Turing's point when he says, "What I object to is the bridge falling down" ([15], p. 218). Turing's point is that we may not notice a hidden contradiction, and that if we do not we may construct a bridge using an inconsistent calculus, which may collapse because of the unseen contradiction.

To see this point, suppose that we construct a "descriptive theory" by applying such a calculus to (a system of) contingent propositions. Suppose further that, in building a bridge, a single engineer uses this theory to do a large number of calculations. In one part of the descriptive theory, s/he performs derivations or calculations in accordance with certain systemic rules and arrives at 'p', while in another part of the theory s/he performs derivations or calculations in accordance with different systemic rules and thereby arrives at '$\neg p$'. Because s/he has done so many calculations, s/he does not notice that in constructing part A of the bridge proposition 'p' was used, while in constructing part B of the bridge proposition '$\neg p$' was used. This very real possibility is magnified if we have many different engineers working, somewhat independently, with the same descriptive theory in which they all have confidence. If they have had success applying the descriptive theory (and derivatively, the formal calculus) in the past, and they do not know it contains a hidden contradiction, they will have no reason to suspect that their different calculations have led to a contradiction. The fact that such a calculus has hitherto rendered "very good service" may mean that the contradiction has thus far not been used in two different calculations in the descriptive theory. Or, it may mean that it has been used, but the bridge did not collapse because the derived propositions were only contraries, which agreed, say, to the hundredth decimal place;[30] because they so closely approximate one another, the fact that they are contraries may not be physically significant. That is, perhaps only a difference at the tenth decimal place would make a physically significant difference (i.e., the bridge would collapse). This means that if the isomorphism between formal calculus and descriptive theory holds, and the calculus contains a "hidden contradiction" in the sense of proved contraries or contradictories which have not been seen, that contradiction will have an analogue contradiction in the descriptive theory, which *can* create a problem in an extrasystemic application. That Wittgenstein is right in saying that it will not *necessarily* create a problem is beside the point.

Put differently, though it is true that according to SF a mathematical calculus need not have an extrasystemic application, this does not dissolve the problem, for Wittgenstein's SF allows that a mathematical calculus *can* be given such an application. Thus, the problem for Wittgenstein lies in the *possibility* of application. So long as a formal calculus may be applied to the real world, an unseen syntactical contradiction is a very real danger. From this it follows that RF is untenable in the sense that a calculus must be syntactically consistent if correct calculations and derivations within the calculus can never lead, by means of an extrasystemic application, to unseen contradictions within a descriptive theory. This presupposition is particularly important for the later Wittgenstein, for, as we shall now see, in [19] Wittgenstein makes extrasystemic application a necessary condition of a mathematical calculus.

5 Extrasystemic application as a necessary condition of meaningfulness in [19]

In [19], Wittgenstein still maintains that the operations within a mathematical calculus are purely formal, syntactical operations governed by rules of syntax (i.e., the solid core of formalism).

> It is of course clear that the mathematician, in so far as he really is 'playing a game' . . . [is] acting in accordance with certain rules. ([19], 5:1)

> To say mathematics is a game is supposed to mean: in proving, we need never appeal to the meaning of the signs, that is to their extramathematical application. ([19], 5:4)

> For example, it is the property of '5' to be the subject of the rule '$3 + 2 = 5$'. For only as the subject of the rule is this number the result of the addition of the other numbers. ([19], 1:83)

Just as Wittgenstein speaks of "arithmetic [as] a kind of geometry" in ([17], 109, 111), in ([19], 3:38) Wittgenstein speaks of a "geometrical application" according to which the "transformation of signs" in accordance with "transformation rules" shows that "when mathematics is divested of all content, it would remain that certain signs can be constructed from others according to certain rules."[31] From these passages it is clear that Wittgenstein still conceives of proofs and refutations within a calculus in a formalist vein. In this sense, he still maintains that "mathematics is always a machine, a calculus" ([14], p. 106). The difference in [19], the shift that Frascolla and others fail to appreciate,[32] is that Wittgenstein now requires that a sign-game must have a real world application to constitute a mathematical "language-game." "It is essential to mathematics that its signs are also employed in *mufti*," Wittgenstein states, for "it is the use outside mathematics, and so the meaning of the signs, that makes the sign game into mathematics" (i.e., a mathematical "language-game") ([19], 5:2).[33] As Wittgenstein says, "concepts which occur in 'necessary' propositions *must also* occur and have a meaning in nonnecessary ones" ([19], 5:42) [italics mine].

That Wittgenstein shifts to WF in [19], by demanding extrasystemic application for meaningfulness, is also made clear when he discusses the question of whether two different proofs can prove the same proposition.

> It all depends what settles the sense of a proposition, what we chose to say settles its sense. The use of the signs must settle it; but what do we count as the use?— That these proofs prove the same proposition means, e.g.: both demonstrate it as a suitable instrument for the same purpose. And the purpose is an allusion to something outside mathematics. ([19], 7:10)

This "allusion to something outside mathematics" is an allusion to extrasystemic application. Two different proofs of a mathematical proposition only count as proofs of the same proposition if "they demonstrate it as a suitable instrument for the same purpose" (i.e., for the same extrasystemic application).

In demanding an extrasystemic, real world application for a meaningful mathematical calculus (i.e., that the concepts of mathematics must also occur in contingent propositions), Wittgenstein returns to the weak formalism of the *Tractatus*.

> Indeed in real life a mathematical proposition is never what we want. Rather, we make use of mathematical propositions only in inferences from propositions that do not belong to mathematics to others that likewise do not belong to mathematics. ([20], 6.211)

But given that Wittgenstein abandons WF (i.e., extrasystemic application) during the middle period, the question arises: Why does Wittgenstein reintroduce WF in [19]?

5.1 A tension dissolved We can see the answer to this question, I believe, if we recall that the intermediate Wittgenstein has considerable contempt for TST, saying at 173 of [17] that "mathematics is ridden through and through with the pernicious idioms of set theory," that set theory is "nonsense" because "it builds on a fictitious symbolism [which cannot possibly exist]" ([17], 174), that he has "abolish[ed] every set theory" ([17], 159), and later, tacitly claiming that set theory cannot be applied to the real world ([16], 468).[34] At [16], 464, Wittgenstein pulls no punches as he ridicules Dedekind's definition of an "infinite class" by saying that the idea of a one to one correspondence between a class and one of its subclasses is "laughable"; that the very attempt is "nonsense." We should be "ashamed," says Wittgenstein, "of this paradoxical form [$m = 2n$ correlates a class with one of its proper subclasses] as something ridiculous" ([16], 465–66).

Similarly, Wittgenstein rails against the Multiplicative Axiom (Axiom of Choice).

> What gives the multiplicative axiom its plausibility? Surely that in the case of a finite class of classes we can in fact make a selection [choice]. But what about the case of infinitely many subclasses? It's obvious that in such a case I can only know the law for making a selection. Now I can make something like a random selection from a finite class of classes. But is that conceivable in the case of an infinite class of classes? It seems to me to be nonsense. ([17], 146)

Wittgenstein's intermediate attack on TST gives rise to a serious tension, mentioned in Section 3.2.3, between Wittgenstein's intermediate SF view of mathematical calculi ("For me one calculus is as good as another" ([16], 334); TST is not "something incorrect (it would be at worst uninteresting)" ([16], 469–70)) and his extreme criticism of the meaning and meaningfulness of TST. On Wittgenstein's own intermediate terms, TST *should* qualify as a meaningful mathematical calculus, but it does not.

At [19], 5:5 Wittgenstein meets this problem head on, by considering the idea that TST, qua formal calculus, must be a mathematical calculus.

> If the intended application of mathematics is essential, how about parts of mathematics whose application—or at least what mathematicians take for their application—is quite fantastic? So that, as in set theory, one is doing a branch of mathematics of whose application one forms an entirely false idea. Now, isn't one doing mathematics none the less?

Here Wittgenstein poses a hypothetical problem for his (new) idea that an extrasystemic application is essential to a meaningful mathematical calculus. A defender of the meaningfulness of TST, Wittgenstein suggests, might ask, "Isn't it evident that there are concepts here—even if we are not clear about their application?" Wittgenstein's answer is unequivocal: "How is it possible to have a concept and not be clear about its application?" ([19], 5:7; cf. [19], 5:42, above).

In [19], with application as a necessary condition of a meaningful mathematical calculus, Wittgenstein no longer needs to vacillate between admitting TST as a mathematical calculus and questioning its meaningfulness.

> If it were said: "Consideration of the diagonal procedure shews you that the concept 'real number' has much less analogy with the concept 'cardinal number' than we, being misled by certain analogies, are inclined to believe," that would have a good and honest sense. But just the opposite happens: one pretends to compare the 'set' of real numbers in magnitude with that of cardinal numbers. The difference in kind between the two conceptions is represented,

> by a skew form of expression, as difference of extension. I believe, and hope, that a future generation will laugh at this hocus pocus. ([19], 2:22)
>
> The sickness of a time is cured by an alteration in the mode of life of human beings . . . . ([19], 2:23)

Thus, Cantor's diagonal proof does not show what it purports to show;[35] it is a piece of *legerdemain*, which Wittgenstein hopes future generations, having undergone an "alteration in [their] mode of life," will see as such and accordingly "laugh at." In a similar vein, Wittgenstein takes issue with the claim that a proposition of transfinite cardinal arithmetic has the same meaning, or an analogous meaning, to a similarly constructed proposition of (finite) cardinal arithmetic.

> These considerations may lead us to say that $2^{\aleph_0} > \aleph_0$.
>
> That is to say: we can *make* the considerations lead us to that.
>
> Or: we can say *this* and give *this* as our reason.
>
> But if we do say it—what are we to do next? In what practice is this proposition anchored? It is for the time being a piece of mathematical architecture which hangs in the air, and looks as if it were, let us say, an architrave, but not supported by anything and supporting nothing. ([19], 2:35)

What Wittgenstein means by this, I believe, is that '$2^{\aleph_0} > \aleph_0$' is presently not anchored in any real world language-game (i.e., it has not been given a real world application), and hence, it is not a mathematical proposition. It is worth noting, however, that contrary to the intimation of [19], 2:22 that future generations will laugh at TST, presumably because it cannot be given an application (see [16], 468, above), in the present passage, when Wittgenstein says "for the time being," he seems to allow that TST might one day be successfully given an application, and thereby become a meaningful mathematical calculus.[36] It follows that when Wittgenstein says that TST "is for the time being a piece of mathematical architecture which hangs in the air," ([19], 2:35) he means that a purely formal calculus, such as TST, which has yet to be given a real world application, "looks as if it were . . . an architrave" (i.e., supported by a column, and supporting a frieze and cornice), but in reality it is a piece of (meaningless mathematical) architecture hanging in midair. A piece of mathematical architecture (i.e., a "mathematical sign-game") becomes a mathematical calculus (i.e., a "mathematical language-game") only when it is given a real world application (cf. [19], 5:2, above). If I am right that Wittgenstein's [19] application criterion requires syntactical consistency, then in allowing that TST may one day be given an application, Wittgenstein must demand that TST be syntactically consistent if it is "a piece of mathematical architecture." Given, however, the impossibility of an absolute consistency proof for some mathematical calculi, "a good angel will always be necessary" ([19], 7:16) if we are to successfully apply some formal calculi to real world domains.

6 Conclusion From 1929 through 1944, Wittgenstein maintains that mathematics is invented, not discovered. A crucial component of this view is that mathematical meaningfulness is entirely an intrasystemic and epistemological affair. A mathematical calculus is essentially a calculating machine, and whether or not a given expression is a meaningful proposition of a given calculus is exclusively determined by its syntax and our knowledge of an applicable decision procedure. If we "prove" a meaningless expression nonalgorithmically, we necessarily construct a new calculus, for after the construction we now know how to use the new machine part (i.e., proved proposition) in a new calculating machine.

The most significant departure from the middle period in [19] is that Wittgenstein returns to the weak formalism of the *Tractatus* by introducing extrasystemic application as a necessary condition of a mathematical calculus. In doing so, however, he does not abandon the core idea of formalism, namely, that the meaningfulness of 'propositions' relative to a given calculus is essentially a syntactical and intrasystemic affair.[37] Though a mathematical calculus must now contain 'concepts' which occur also in contingent propositions, as regards meaningful propositions and mathematical activity, the calculus itself is no different from a purely formal sign-game. Whether an expression is a 'proposition' of a formal calculus is exclusively a function of syntax and knowledge of the calculus (e.g., its decision procedures). Such a 'proposition' is only a mathematical proposition if its calculus has been given an extrasystemic application. Thus, the principal change, and indeed gain, of the new application criterion of [19], is that there is no longer any conflict between Wittgenstein's rejection of TST as a mathematical calculus and his intermediate SF. For the later Wittgenstein, a formal calculus is only a mathematical calculus if it has successfully been given a real world application; TST is not a mathematical calculus because it has no such application. If, however, the argument of Section 4 is sound, though Wittgenstein's application criterion resolves a glaring intermediate tension, he can no longer maintain, as he does, the innocuousness of hidden contradictions.

NOTES

1. See also [19], 3:31, and especially [15], p. 22: "I shall try again and again to show that what is called a mathematical discovery had much better be called a mathematical invention."

2. Instead, Frascolla calls Wittgenstein a "quasi-formalist," just as he prefers to call Wittgenstein a "quasi-revisionist." I shall argue in Section 3.2.3 that Wittgenstein is a revisionist, not a quasi-revisionist.

3. One exception to this occurs in Hilbert's Dec. 29, 1899 letter to Frege, in [5], pp. 39–40: "If the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist. This is for me the criterion of truth and existence." The problem with construing this as SF is that it is hard to see how it can be reconciled with Hilbert's Kantianism.

4. Cf. [19], 1:83, quoted in Section 5.

5. Perhaps even more striking, Wittgenstein seems to think that Frege somehow misses the very formalist idea that a mathematical calculus is a game with signs played in accordance with rules of operation. This, however, is a mistake, for Frege is well aware that Heine, Thomae, and later, Hilbert viewed rules as essential to a mathematical calculus. As Frege says in describing Thomae's formalism, "[Formal] arithmetic is concerned only with the rules governing the manipulation of the arithmetic signs, not, however, with the meaning of the signs" ([4], pp. 164); "formal arithmetic knows nothing

but rules" ([4], p. 168). It is worth noting that, although Thomae introduces the analogy between formal arithmetic and chess, it is Frege who turns this analogy around and uses it against Thomae ([4], pp. 163–67, 169–70, 172, 182–86, 189).

6. In [19], Wittgenstein similarly speaks of the "geometrical cogency" of proofs ([19], 3:38, 43) and "the geometry of proofs" ([19], 1, 14, app. 3).

7. A formalist view of *axiomatic* calculi is prefigured in the *Tractatus* [20], 6.126, where Wittgenstein discusses the Propositional Calculus of *Principia Mathematica*.

8. One might object to this reading of [17], 109, however, on the grounds that Wittgenstein's antifoundationism is, perhaps partly, an objection to axiomatization ([19], 7:12), but what he is really rejecting at [19], 7:12 is the claim that mathematical calculi and mathematical activity are only "good" and "secure" if we have proved that they can never yield a contradiction (i.e., by providing absolute consistency proofs). Wittgenstein's point is that a mathematical calculus is not "bad mathematics" without an absolute consistency proof, and that even if a contradiction should arise, this will not "destroy" the calculus. See also [14], pp. 195–96, and [19], 7:12, 15, 81–82, 84.

9. Cf. Section 5, [19], 5:1 and [19], 5:4.

10. Brouwer and Hilbert were first brought to Wittgenstein's attention by F. P. Ramsey in 1923 (letter from Ramsey to Wittgenstein, dated Dec. 20, 1923, in [21], pp. 82–83), and then later in 1929 by Schlick, Waismann, and Carnap (in Vienna), and again by Ramsey (at Cambridge).

11. In a marginal note added to [17], 107, perhaps after the completion of [17], Wittgenstein similarly states: "Instead of a question of the definition of number, it's only a question of the grammar of numerals."

12. Cf. [9] where Maddy argues incorrectly (p. 286) that Wittgenstein regards (decidably) false propositions as meaningless. Though Maddy explicitly references [17], pp. 148–51 in her argument, she seems to overlook: "We come back to the question: In what sense can we assert a mathematical proposition? That is: what would mean nothing would be to say that I can only assert it if it's correct" ([17], 150). See also [17], 202, quoted in Section 2, above.

13. This passage continues: "Rather, it corresponds to a proof by induction." See also [17], 168. I shall have more to say about Wittgenstein's views on mathematical induction in Section 3.2.2.

14. The parenthetic addition "(or make a conjecture)" is explicated, in agreement with my construal, by the continuation of this passage.

15. Wittgenstein muddies matters somewhat by characterizing "unproved mathematical propositions" as "signposts for mathematical investigation, stimuli to mathematical constructions" ([16], 371). See also [17], 148; [17], 159; and [16], 379. What Wittgenstein means is that "I may let a formula stimulate me," where such a 'formula' "is a stimulus—but not a question" ([14], p. 144; cf. [16], 380). "What is here going [o]n is an unsystematic attempt at constructing a [new] calculus" ([14], pp. 174–75).

16. See also [16], 366 and Ambrose [1], pp. 199–200.

17. See also [16], 400, quoted in Section 3.2.2.

18. See, e.g., [19], 3:67 and [19], 6:45.

19. That this is Wittgenstein's aim is made clear in numerous passages, which are cited in note 27. Despite the fact that Wittgenstein wants only to describe mathematics and not legislate or interfere, in Section 3.2.3 I argue that Wittgenstein is at least a weak revisionist.

20. See also [17], 183; [16], 474; and [16], 484. The later Wittgenstein maintains this view, as is evidenced by [19], 5:19: "The concepts of infinite decimals in mathematical propositions are not concepts of series, but of the unlimited technique of expansion of series."

21. See also [17], 181, 183, and 191 and [16], 373, 460, 461, and 473.

22. See also [14]: "A law is not another method of giving what a list gives. The list *cannot* give what the law gives" (p. 103). Cf. [19]: "The rational numbers cannot be *enumerated*, because they cannot be counted [i.e., to completion]" (5:15). This is clearly what Wittgenstein means, for he continues: "one cannot *set out* to enumerate the rational numbers, but one can perfectly well *set out* to assign numbers to them [italics mine]."

23. It is my contention that Wittgenstein is still a finitist in [19], as is indicated by [19], references in the present section and in Section 5.

24. Wittgenstein, of course, is not revisionistic regarding finitistic 'conjectures' since he admits that expressions such as "$23 \times 38 = 864$" are meaningful because we know of an applicable decision procedure.

25. Suggestions along these lines can be found in [14], pp. 36–37. See also [17], 181–86 and [16], 475–81.

26. See also [17], 159; [16], 295, 367, 469–70; [18], 109, 124, 126, 128; [19], 3:81; 5:52, and the baffling [19], 2:62.

27. Shanker admits that this "certainly looks like . . . 'mathematical interference' " ([13], p. 198).

28. Cf. Poincaré [10], p. 60.

29. It should be noted that, in conversation with Turing, Wittgenstein does say: "I don't say that a contradiction may not get you into trouble. Of course it may" ([15], p. 219). And in [19], 7:34, Wittgenstein allows that we may "draw conclusions" from a contradiction in a calculus, "accept these inferences," and a bridge may collapse, but curiously, he qualifies this admission by saying that "if a bridge collapses, . . . we find some other cause for it, or we call it an Act of God." His point here, as articulated in the very next sentence, is that our calculation was neither wrong, nor a noncalculation, but one wonders why he does not allow that we may look for and find the contradiction itself. It should also be noted that Wittgenstein says that "a language-game can lose its sense through a contradiction, can lose the character of a language-game" ([19], 3:80). Indeed, he goes one step further in claiming that we may be prevented "from sealing . . . off" a contradiction if "we do not know our way about in the calculus." And in considering a case in which a contradiction "turns up," Wittgenstein states: " 'Up to now a good angel has

preserved us from going this way.' Well, what more do you want? One might say, I believe: a good angel will always be necessary, whatever you do" ([19], 7:16). It is hard to know what to make of these four passages given that Wittgenstein argues to the end that unseen contradictions are innocuous. See, e.g., [17], 160; [14], p. 196; and [19], 7:11, 12, 15.

30. Wittgenstein says that "as long as we move within the calculus, we do not have any contradiction," only contaries, which "do not contradict each other as long as we do not add a rule that has the effect of making their logical product a contradiction" ([14], p. 127). The problem with this argument is that, although we can stipulate that "$s = 180^\circ$, $s = 181^\circ$ do not contradict each other," mathematical and logical calculi typically enable us to infer a contradiction from contraries. This is why mathematicians like Hilbert worried about "hidden contradictions" (Cf. [14], p. 132).

31. The fact that Wittgenstein concludes this passage by saying that "the sequence of signs in the proof does not necessarily carry with it any kind of acceptance" does not undermine the formalist import of the passage, for his point is, like in [19], 1:5, that "if we made a different inference," we do not necessarily "get into conflict with truth" (i.e., a fact).

32. In [3], Frascolla does mention Wittgenstein's [19] and its emphasis on 'application' (pp. 163–64), but he fails to recognize both the importance of this shift and the problems that it addresses and possibly solves.

33. Wittgenstein's [19] and its application criterion is prefigured in [15], pp. 140–41 and pp. 169–70.

34. "In set theory what is calculus must be separated off from what attempts to be (and of course cannot be) theory." See also [16], 461.

35. It is worth noting in passing that Poincaré, [10], pp. 61–62, argues, as Wittgenstein does in [19], that Cantor's proof does not prove what it purports to prove: "What did Cantor mean and what did he really prove? It is not possible to find, among the integers and the points in space which are definable in a finite number of words, a law of correspondence which satisfies the following conditions: . . . "

36. See also [19], 2:38. Cf. [19], 2:29–31 and [19], 5:15.

37. Cf., e.g., "The application of the calculation must take care of itself. And that is what is correct about 'formalism' " ([19], 3:4) and "The truth in formalism is that every syntax can be conceived of as a system of rules of a game" ([14], p. 103).

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