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Book Review

Raymond M. Smullyan and Melvin Fitting. *Set Theory and the Continuum Problem*. Clarendon Press, Oxford, 1996, xiii + 288 pages.

1 This rewarding, exasperating book is largely drawn from lecture notes used by the authors in their set theory classes. Although we receive the standard assurance that it is a suitable text for "advanced undergraduates and graduates in mathematics and philosophy," this reviewer doubts that many undergraduates will be sufficiently advanced to benefit from it. Though flawed, the book does have many virtues. Its ups and downs might best be conveyed by some commentary on each of its twenty-two chapters.

2 Part I: Axiomatic Set Theory

2.1 Chapter 1: General background The opening chapter provides a happy-go-lucky introduction to size comparisons between infinite sets. The use of the first person singular helps to warm an atmosphere already sweetened with cozy good-humor (perplexing though the 'I' might be in a work with two authors). If this chapter is meant to entice and entertain readers rather than to instruct them, then it succeeds admirably.

2.2 Chapter 2: Some basics of class-set theory According to the first line of this chapter, the authors " . . . presume familiarity with the notions of a *class* or *collection* of objects and the notion of an object *x* being a *member* or an *element* of a class *A*." Since subsequent pages characterize these very notions in detail, one wonders what sort of familiarity is presumed. This chapter offers a theory of finite sets in the style of von Neumann and Bernays. The presentation is clear and spritely. Particularly noteworthy and successful is the introduction of model theoretic considerations from the very start. Nonetheless, the reviewer allows himself one quibble, two pet peeves, and an emendation. (1) Without any warning, the authors use 'contains' to express the converse of membership. (2) In note 1 of §4, the authors claim to use extensionality when they actually employ Leibniz's law. (3) According to the initial remark

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of §7, "The term 'Cartesian product' is named in honor of René Descartes." What name of the term 'Cartesian product' do our authors have in mind? (One might have guessed that the Cartesian products *themselves* were named in honor of Descartes.) (4) Exercise 5.6(d) was probably supposed to read '$B - (A - B) = B$'.

2.3 Chapter 3: The natural numbers Herein we meet the axiom of infinity and explore a construction of the natural numbers. The authors take special pride in their extensive use of Smullyan's double induction principle. "It is high time," we are told, "that [this principle] appear in a textbook." As in the prior chapter, the exercises seem well chosen. (One is puzzled, though, by the placement of Exercise 8.1. Since 8.1 is an immediate consequence of Exercises 6.1 and 7.1, one wonders what role the intervening discussion of recursion is supposed to play.) Dutiful readers of Chapter 2 will have made an effort to remember that 'P_1' is our authors' name for the extensionality axiom. It is not so helpful for this to reappear in Chapter 3 as a name for the first Peano postulate. 'P_2' is made to suffer from a similar ambiguity.

2.4 Chapter 4: Superinduction, well ordering, and choice Choice is shown to be equivalent to several other celebrated propositions. The results are standard, but the proofs are unusually clear and economical. The chapter concludes with a derivation of Cowen's theorem (on the existence of minimally superinductive classes). Errors abound, four of which could cause students special grief. (1) Exercise 1.2 can be done only if proper lower sections are allowed to be empty (as the authors do allow on p. 71 of Chapter 6). This change having been made, we can drop condition (a) from Exercise 1.3. (2) On line 2 of Exercise 1.6, replace 'proper' with 'nonempty'. (3) If we assume the axiom of choice, Exercise 4.3 is trivial. If, as the authors apparently intend, we do not assume it, then 4.3 is impossible. (The authors seem unaware that 4.3 is, in fact, equivalent to choice.) (4) Definition 7.1 is a disaster. The authors write

$$(z \in y \wedge z \subseteq x) \supset g(z) \in y$$

when, evidently, they mean

$$(z \in y \wedge g(z) \subseteq x) \supset g(z) \in y.$$

This might not have been so harmful if the authors had not offered the alternative formulation

$$z \in (y \cap \mathcal{P}(x)) \supset g(z) \in y$$

taking the garbled, rather than the intended, version as their guide. Readers who accept the definition as written will be especially irritated by the "proof" of Lemma 7.4. It consists of the single word "Obvious"—whereas 7.4 is (not so obviously) false under the definition readers have innocently adopted.

2.5 Chapter 5: Ordinal numbers The authors spare reviewers the task of crafting laudatory prose to describe this chapter. It offers, they say, "a particularly smooth and intuitive development of the ordinals." Indeed, we are guaranteed "a beautifully natural and elegant treatment." The authors cannot help but remark, "It is high time this neat approach should be known!" Any praise this reviewer might offer would be superfluous.

2.6 Chapter 6: Order isomorphism and transfinite recursion None of the foregoing required the axiom of replacement, which is used only now to prove that every well-ordered set is isomorphic to an ordinal. Work from previous chapters yields a swift justification of various forms of transfinite recursion. The authors assure us, "A semester course in symbolic logic is more than enough background [for their book]." Most novices at set theory, though, will have no idea what counts as a proof of, say, Exercise 5.1.

2.7 Chapter 7: Rank The usual hierarchy of $R(\alpha)$'s is introduced. An admirable feature of this chapter is the swiftness with which the authors move to model theoretic applications. This reviewer detected five misprints, none likely to impede readers for long. However, if the authors are going to declare a proof "obvious," they really ought to state the theorem correctly. (In Proposition 1.2, '$\subseteq$' should be '$\in$'.) Lemma 4.4 presents a more serious problem. The authors seem to claim that

$$\mathrm{VNB} \vdash \forall M (Z^M \longrightarrow R(\omega \cdot 2) \subseteq M)$$

where Z^M is the Zermelian part of VNB with all quantifiers relativized to M. Their proof, however, seems to establish no more than that

$$\mathrm{VNB} \vdash \forall \alpha (Z^{R(\alpha)} \longrightarrow R(\omega \cdot 2) \subseteq R(\alpha)).$$

It is not so clear what more they *could* prove, since Zermelo set theory does not imply the existence of $R(\omega)$.

2.8 Chapter 8: Foundation, $\in$-induction, and rank The authors discuss the axiom of foundation, but refrain from adopting it. (Perhaps by way of justification, they mention recent applications of non-well-founded sets.) Theorem 3.6 is puzzling. It would seem to imply $\forall M \subset WF(\mathrm{Z}^M \longrightarrow \exists \alpha M = R(\alpha))$ where $WF = \cup\{R(\alpha) : \alpha \in ON\}$. Yet $H(\beth_\omega) \cap WF$, the set of well-founded sets hereditarily of cardinality less than $\beth_\omega$, would then be a counterexample. The authors' intentions are not clarified by their "proof."

2.9 Chapter 9: Cardinals This chapter offers a cornucopia of interesting and fundamental results about cardinals, culminating in a proof that GCH implies AC. The exposition is lucid and well paced. (Even the misprints are easily detected and corrected!) One senses with regret that the preceding chapters were just a revision away from this high standard.

3 Part II: Consistency of the Continuum Hypothesis

3.1 Chapter 10: Mostowski-Shepherdson mappings We seem (temporarily) to have entered a new world, less amiable on the surface, but crafted with great care and concern for the reader. (The poltergeist haunting the typesetter has not, however, been exorcised. It strikes at least eight times in the first twelve pages.) This chapter offers a smooth presentation of the Mostowski collapsing theorem. It also improves on some results from Part I.

3.2 Chapter 11: Reflection principles Sections 1 and 2 share the clarity of the preceding chapter, though some of the exercises are unrealistic. Seven energetic pages lead painlessly to a powerful and highly abstract reflection theorem. So far, so good. Unfortunately, most students will find the final three sections deeply mysterious. Some will be tempted to read no further. There is also a substantial problem that students might overlook: identity is not counted as a logical notion, but neither is it ever defined. So readers are left to guess what '$=$' might mean. Suppose we define identity "from above" as membership in the same sets. Then '$A = B$' can be true in a class K even when A and B are different sets. (We just have to make sure that A and B are members of the same members of K.) Indeed, K can even be transitive: note, for example, that $\{\{\varnothing\}\}$ and $\{\varnothing, \{\varnothing\}\}$ are distinct, but belong to the same members of $R(3)$ (namely, none). Since this would torpedo two of the proofs on p. 141, it ought not to be what the authors intend. One might guess that they mean to define identity "from below" as extensional equivalence. We would then want to make sure that our models are extensional. (That is, we would want to pick K such that, for any $x, y \in K$, if $x \cap K$ and $y \cap K$ are the same set, then x and y are the same set.) This would help to make sense of some details in Chapter 12. But, then, one hardly knows what to say about footnote 8 of Chapter 13 in which the authors insist that they really mean to define identity from above. (By the way, they also insist that '$(\forall z)(z \in x \equiv z \in y)$' says that x and y are members of the same sets!!!!!!!!!!)

3.3 Chapter 12: Constructible sets Elements of a model are allowed to appear in formulas as names of themselves. This may strike students as insane, but it yields a gloriously seductive presentation of constructibility and absoluteness. There is, however, a problem with the exposition in §2 of this chapter. Whatever the authors' intentions, their presentation could easily lead uninitiated readers to believe the following falsehood: if ψ is Δ_0 and VNB $\vdash \forall x_1, \ldots, x_n(\varphi(x_1, \ldots, x_n) \longleftrightarrow \psi(x_1, \ldots, x_n))$, then φ is absolute over all transitive classes. Students should be taught to check how much of VNB a model must satisfy if it is to verify φ's equivalence to a Δ_0 formula. (Even if one only uses equivalences that are theorems of logic or definitional truths, this itself is worth mentioning.) There is also a surprising turnabout in this chapter: in spite of the qualms expressed in Chapter 8, we are informed (parenthetically) that the universe is well founded after all!

3.4 Chapter 13: L ***is a well-founded first-order universe*** In Part I, our official (but unformalized) formal theory was VNB. What a surprise, then, to learn that the rest of the book is devoted to models of ZF! (This chapter shows that L is such a model.) It would have made more sense to offer an unashamedly *informal* account in Part I. One could then have described the universe V quasi-categorically using plural locutions of English in place of the class vocabulary of VNB. (This would, by the way, have made Cowen's theorem unnecessary.) As noted before, the authors now define identity as membership in the same sets. But, then, '$x = y$' is not Δ_0 and, indeed, is not absolute over all transitive classes. ('$\{1\} = 2$' is true in $R(3)$.) So two steps in the proof of Theorem 2.1 are fallacious (but easily enough repaired) and the proof of Lemma 3.2 is incomplete. On p. 211 of Chapter 17, identity is once again defined from above. But on p. 227 of Chapter 18, the authors assure us (parenthetically) that they have all along intended to define identity as extensional equivalence. Then, on

p. 252, this very definition is treated as an innovation specially tailored to meet the needs of Chapter 20.

3.5 Chapter 14: Constructibility is absolute over L In this chapter, we learn that the axiom of constructibility is true in L, that L is the smallest transitive model of ZF that contains all ordinals, and that L can be well ordered. As a preliminary to proving $(V = L)^L$, we face the dreary task of showing that the definable power set operation is absolute. The authors try to relieve the tedium by omitting some details.

3.6 Chapter 15: Constructibility and the continuum hypothesis Of course, GCH turns out to be true in L as well. So ZF + GCH is consistent. So GCH is not refutable in ZF. At least, that is what one expects the authors to conclude. Instead, they allow only that ZF + GCH has a model if the universe V exists. That is probably a good qualification to introduce. But, up to now, the existence of V has been taken for granted. So readers might like some reason for this unexpected surge of agnosticism.

4 Part III: Forcing and Independence Results

4.1 Chapter 16: Forcing, the very idea This chapter introduces a technique for translating each sentence φ in the usual language of set theory into a sentence $[\varphi]$ in a modal language. A set theoretic sentence φ is logically true if and only if $[\varphi]$ is true in every possible world of every **S4** model. So the goal is to construct an **S4** model that makes our translation of every ZF axiom true in every possible world, but makes CH false in at least one possible world. This approach will be welcomed by scholars who have struggled with forcing, but are comfortable with modal logic. Many students, however, may lack both the motivation and the background to make sense of it.

4.2 Chapter 17: The construction of* S4 *models for ZF Given any transitive model M of ZFC and any preordered structure $\langle W, \leq\rangle \in M$, we learn how to construct an **S4** model $\langle W, \leq, M^W, \Vdash\rangle$ where W is the set of possible worlds, $\leq$ is the accessibility relation, $M^W \subseteq M$ is the universe, and $\Vdash$ is the valuation function (so that $p \Vdash \varphi$ means that φ is true at possible world p). The authors then demonstrate that if φ is an axiom of ZFC, then $p \Vdash [\varphi]$ whenever $p \in W$. The proof that $\langle W, \leq, M^W, \Vdash\rangle$ has this property is not exactly gripping, but is wonderfully clear.

4.3 Chapter 18: The axiom of constructibility is independent Let $M = L$. Let W be the set of all pairs $\langle P, N\rangle$ where P and N are finite, disjoint subsets of ω. Say that $\langle P_1, N_1\rangle \leq \langle P_2, N_2\rangle$ just in case $P_1 \subseteq P_2$ and $N_1 \subseteq N_2$. Then $[\varphi]$ is true in every possible world of $\langle W, \leq, M^W, \Vdash\rangle$ whenever φ is an axiom of ZFC + GCH + $V \neq L$. If $p \Vdash [\neg\varphi]$, then $[\varphi]$ is not true at p. So $[V = L]$ is not true at any $p \in W$. But if ZFC + GCH $\vDash V = L$, then $[V = L]$ is true at every $p \in W$. So the axiom of constructibility is not a theorem of ZFC + GCH. The result, of course, is Cohen's—as are the essential features of the proof. But this reviewer found the modal-logical presentation to be blessedly, miraculously, triumphantly comprehensible. Here is a sketch of the proof that there is a Cohen generic real in the **S4** model. We are able to define a certain set f. People who live in a possible world $\langle P, N\rangle$ think $[\hat{z} \in f]$ is true whenever $z \in P$. ($\hat{z}$ is the canonical representative of z in M^W.) So if $z \notin (P \cup N)$, then the inhabitants of $\langle P, N\rangle$ have access to a world in which $[\hat{z} \in f]$ is true. (Just consider $\langle P \cup \{z\}, N\rangle$.) Suppose [$f$ is not constructible] is false somewhere. Then there is a constructible set c and a world $\langle P, N\rangle$ in which [f and $\hat{c}$ have the same members] is true. Suppose $c \subseteq P$ and $z \notin (P \cup N)$. Then $[\hat{z} \in f]$ is true in $\langle P \cup \{z\}, N\rangle$. But $\langle P \cup \{z\}, N\rangle$ is accessible from $\langle P, N\rangle$. So [f and $\hat{c}$ have the same members] is true in $\langle P \cup \{z\}, \mathrm{N}\rangle$ and, hence, so is $[\hat{z} \in \hat{c}]$. But, since $z \notin c$, $[\hat{z} \notin \hat{c}]$ is true everywhere. So $[\hat{z} \in \hat{c}]$ is false everywhere, contrary to our earlier conclusion. A contradiction also follows if c is not a subset of P. So [f is not constructible] is true everywhere. The idea is that if f were constructible, then the inhabitants of some world would know exactly what objects $\hat{z}$ are members of f. This knowledge would be inherited by the inhabitants of all accessible worlds. But, since some of those worlds disagree with one another on this very point, we obtain a contradiction.

4.4 Chapter 19: Independence of the continuum hypothesis Let M be a transitive model of ZFC + GCH. We learn how to define a set of worlds W that yields $\aleph_2$ non-constructible subsets of ω in $\langle W, \leq, M^W, \Vdash\rangle$. It then follows that the modal translation of $|\mathcal{P}(\omega)| = \aleph_2$ is everywhere true. Indeed, let k be any cardinal with uncountable cofinality. Then one can contrive a modal model that thinks $\mathcal{P}(\omega)$ is of size k. From this, the authors conclude that, "The seemingly simple power set operation turns out to be one of the least understood operations of set theory." But it is not so clear that this follows. Second order ZF characterizes power sets up to $\in$-isomorphism. So the power set operation would seem to be quite well understood (indeed, as well understood as a mathematician could hope). We do, of course, have a lot to learn about the relation between power sets and the sequence of alephs. (One wonders, does the first order indeterminacy of 2^λ mean that 2 is one of the least understood objects of set theory?)

4.5 Chapter 20: Independence of the axiom of choice A modal version of the Fraenkel-Cohen-Scott-Solovay proof is beautifully (though sometimes rather tersely) presented.

4.6 Chapter 21: Constructing classical models An extraordinarily wonderful chapter! This reviewer knows no better introduction to generic extensions. It is particularly helpful that the authors manage to present denseness and genericity as natural generalizations of Cohen's more immediately intuitive approach. The modal machinery of previous chapters is also put to excellent use.

4.7 Chapter 22: Forcing background The book concludes with a useful bit of intellectual history.

5 Overall assessment Parts I and II are one revision away from being an exemplary introduction to Gödel's inner model L. (What a shame they were published in their current form!) Part III offers a splendid presentation of forcing, well suited for highly motivated students with a substantial background in modal logic. Set theory students without such a background might want to go ahead and acquire it: for many, this could be the quickest and least painful way to make sense of Cohen's technique.

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