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Failure of Interpolation in Combined Modal Logics

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Abstract We investigate transfer of interpolation in such combinations of modal logic which lead to interaction of the modalities. Combining logics by taking products often blocks transfer of interpolation. The same holds for combinations by taking unions, a generalization of Humberstone's inaccessibility logic. Viewing first-order logic as a product of modal logics, we derive a strong counterexample for failure of interpolation in the finite variable fragments of first-order logic. We provide a simple condition stated only in terms of frames and bisimulations which implies failure of interpolation. Its use is exemplified in a wide range of cases.

1 Introduction In 1957, Craig proved the interpolation theorem for first-order logic [3]. Comer [2] showed that the property fails for all finite variable fragments except the one-variable fragment. The n-variable fragment of first-order logic—for short L_n—contains all first-order formulas using just n variables and containing only predicate symbols of arity not higher that n (we assume the language has only variables as terms). Here we will show that the axiom which makes the quantifiers commute can be seen as the reason for this failure.

Since Craig's paper, interpolation has become one of the standard properties that one investigates when designing a logic, though it hasn't received the status of a completeness or a decidability theorem. One of the main reasons why a logic should have interpolation is because of "modular theory building." As we will see below interpolation in modal logic is equivalent to the following property (which is the semantical version of Robinson's consistency lemma).

> If two theories T_1, T_2 both have a model, and they don't contradict each other on the common language (i.e., there is no formula θ built up from atoms occurring both in T_1 and in T_2 such that $T_1 \models \theta$ and $T_2 \models \neg\theta$), then $T_1 \cup T_2$ has a model.

The property is not only intuitively valid for scientific reasoning, it also has practical (and computational) consequences. In practice it shows up in the incremental design,

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specification and development of software and has received quite some attention in that community (cf. Maibaum and Sadler [13], Renardel [18]). Below we will give a more technical reason why interpolation is desirable: it can help in showing that irreflexivity style rules in an unorthodox axiom system are conservative over the orthodox part. In this paper we look at interpolation in combined modal logics (and we will see that first-order logic is just an instance of such a combination).

Combined modal logics (Gabbay [6]) are systems that are built up from simpler and familiar systems in very diverse ways. They are polymodal logics with some "additional structure" or requirements set over their classes of frames. One of the most interesting questions in the field of combining logics is that of *transfer theorems*: under which conditions does a metalogical property—such as finite axiomatizability, decidability, or interpolation—transfer to the combined system? We will show that interpolation usually does not transfer in products of modal logics (Gabbay and Shehtman [7]). (Compare this with combining through fibering, where we often have transfer of interpolation (see Marx [15].) We obtain our mentioned result for first-order logic by considering L_n as a product of modal logics. We will also show failure in Humberstone's logic of inaccessibility (a combination of a modal logic with its complement modality [12]) and several generalizations of this logic. Often, combined modal logics are proposed in an effort to capture some class of frames that the familiar modal systems cannot represent. Our article shows that the gain in expressive power has a price: in many cases the interpolation property is lost.

The article is organized as follows. In the next section we show failure of interpolation in first-order logic with finitely many variables. Section 3 presents different Interpolation Properties that can be found in the literature and explores their interconnections. We will also present a general proof-method for disproving interpolation which allows us to work solely with models, and truth preserving constructions like zigzag-morphisms. We then apply this method in the following sections to combinations of modal logics and see how certain types of combinations block transfer of interpolation.

Modal logic A *modal similarity type* S is a pair $\langle O, \rho\rangle$ with O a set of logical connectives and $\rho : O \mapsto \omega$ a function assigning to each symbol in O a finite rank or arity. We call $\mathcal{ML}(\mathsf{K}_S)$ a *modal logic* for type $S = \langle O, \rho\rangle$, if $\mathcal{ML}(\mathsf{K}_S)$ is a tuple $\langle \mathcal{L}_S, \mathsf{K}_S, \Vdash_S\rangle$ in which,

1. $\mathcal{L}_S$ is the smallest set containing countably many propositional variables, and which is closed under the Boolean connectives and the connectives in O.
2. K_S is a class of frames of the form $\langle W, R^{\Diamond}\rangle_{\Diamond\in O}$, in which W is a nonempty set and each $R^{\Diamond}$ is a subset of $W^{\rho(\Diamond)+1}$. We use calligraphic capitals $\mathcal{F}$ to denote frames and their corresponding Roman F for their domains.
3. $\Vdash_S$ is the usual truth-relation from modal logic between models over frames in K, worlds and formulas. For the modal connectives it is defined as

$$\begin{aligned}\mathfrak{M}, x \Vdash \Diamond(\varphi_1, \ldots, \varphi_{\rho(\Diamond)}) \quad \text{iff} \quad & (\exists x_1, \ldots, x_{\rho(\Diamond)}) : R^{\Diamond} x x_1, \ldots, x_{\rho(\Diamond)} \ \& \\ & \mathfrak{M}, x_1 \Vdash \varphi_1 \ \& \ \cdots \ \& \ \mathfrak{M}, x_{\rho(\Diamond)} \Vdash \varphi_{\rho(\Diamond)}.\end{aligned}$$

If the similarity type S is clear from the context, we usually omit it. A formula φ is *true in a model* $\mathfrak{M}$ (notation: $\mathfrak{M} \models \varphi$) if it holds in every world in $\mathfrak{M}$. A formula φ

is said to be *valid in* $\mathcal{ML}(\mathsf{K})$ (notation: $\models_{\mathsf{K}} \varphi$) if it is true in every model over every frame in K. We will often equate $\mathcal{ML}(\mathsf{K})$ with its set of valid formulas.

2 First order logic We will show that interpolation fails very badly in first-order logic with two variables. For every finite n, we create L_2 formulas φ, ψ such that validity of $\varphi \models \psi$ can be proved using only a minimum of resources from the derivation system, and there is no interpolant for $\varphi \models \psi$ in L_n. This strengthens a similar result of Andréka (unpublished) who used the complete derivation system of L_2. Our result shows that the axiom making the quantifiers commute causes failure of interpolation in the finite variable fragments.

We define a (highly incomplete) derivation system for L_n as follows. Let $\vdash_2$ denote the derivation system consisting of these axioms schemas and rules.

$Ax1$	Every propositional tautology is an axiom scheme.	
$Ax2_i$	$\forall v_i(\varphi \to \psi) \to (\forall v_i\varphi \to \forall v_i\psi)$,	for $i \in \{0, 1\}$.
$Ax3$	$\forall v_1\forall v_0\varphi \to \forall v_0\forall v_1\varphi$.	
MP	From φ and $\varphi \to \psi$ infer ψ.	
UG_i	From φ infer $\forall v_i\varphi$,	for $i \in \{0, 1\}$.

Clearly $\vdash_2$ is sound for first-order logic but hopelessly incomplete. Trivial validities like $\forall v_0(v_0 = v_0)$ and $\exists v_0\exists v_0\varphi \longleftrightarrow \exists v_0\varphi$ are not theorems of $\vdash_2$.

Theorem 2.1 *For every n, there exists L_2 formulas φ, ψ such that*

1. *$\varphi \vdash_2 \psi$, and*
2. *for every L_n formula θ in the common language of φ and ψ, either $\varphi \not\models \theta$ or $\theta \not\models \psi$.*

These formulas can be algorithmically obtained and have size polynomial in n. Either φ and ψ are in disjoint languages but both contain the equality symbol, or they are equality-free but the common language contains one binary predicate.

Proof: Fix n. Let $\forall^k v_i$ abbreviate k many $\forall v_i$. Since all our atomic formulas will be of the form $R(v_0, v_1)$, we might as well forget about the variables and write atomic formulas as lowercase variables p, q, and so on. We propose the following formulas.

$A1$	$(d \longleftrightarrow \bigvee\{p_i \mid 0 \le i \le n\})$.	
$A2$	$(p_i \to \neg p_j)$	$0 \le i, j \le n, i \ne j$.
$A3$	$(p_i \to \bigwedge\{\forall^k v_0(d \to p_i) \mid k \le n\})$	$0 \le i \le n$.
$A4$	$(p_i \to \bigwedge\{\forall^k v_1(d \to p_i) \mid k \le n\})$	$0 \le i \le n$.
$A5$	$\exists v_1\exists v_0(p_0 \wedge \exists v_1\exists v_0(p_1 \wedge \exists v_1\exists v_0(p_2 \ldots \exists v_1\exists v_0 p_n)) \ldots)$.	
$C1$	$\bigwedge_{k\le n+1} \forall^k v_1\forall^k v_0(d \longleftrightarrow \bigvee\{q_i \mid 0 \le i < n\})$.	
$C2$	$\bigvee_{k\le n+1} \exists^k v_1\exists^k v_0(\bigvee_{i<n}[q_i \wedge \bigvee\{\exists^k v_1(\neg d \wedge \exists^k v_0 q_i) \mid 1 \le i \le n\}])$.	

Clearly these formulas can be algorithmically obtained from n and their size is linear in n. The predicate d can stand for the equality statement $v_0 = v_1$, or alternatively it can be seen as an arbitrary formula $D(v_0, v_1)$.

Let A abbreviate $A1 \wedge \cdots \wedge A5$. We claim that

$$A \vdash_2 (C1 \to C2). \tag{1}$$

$$\text{There is no interpolant for } A \models (C1 \to C2) \text{ in } L_n. \tag{2}$$

Before we look at the proof, let us see the intuition behind the formula and its validity in first-order logic. In classical first-order logic, $\forall^k v_i \varphi$ is equivalent to just $\forall v_i \varphi$. Whenever A is true on a model, the predicate d is partitioned in the $n+1$ p_i-predicates. $A5$ tells us that all the p_i occur, so the domain of any model satisfying A should contain at least $n+1$ elements. The intended interpretation of d is $v_0 = v_1$. Then $A3$ and $A4$ are trivially true. With that interpretation of d, the formula $C1 \to C2$ says that if d is partitioned in n q_i-atoms $(C1)$, then "there must be two different elements having the same q_i-value" (C2).

We have to use the more lengthy formulations of our formulas because we want to use as little from the first-order proof system as possible. We first prove (1). Instead of a derivation using the axioms, we give a semantic proof using the fact that $\vdash_2$ completely axiomatizes a *normal modal logic*. If we read $\forall v_i$ as a modal box operator $[i]$, then $\vdash_2$ axiomatizes the bimodal logic over the class of frames $\langle W, R_0, R_1\rangle$ where the following law holds:

$$\forall xyz((xR_0y \wedge yR_1z) \to \exists y(xR_1y \wedge yR_0z)), \tag{3}$$

this by virtue of (the Sahlqvist) axiom $Ax3$. We will show that in this semantics the validity of (1) must hold. Now our way of writing binary predicates $P(v_0, v_1)$ as (propositional) variables p comes in handy because the formulas involved are in the appropriate modal language. Suppose to the contrary that (1) fails. So we find a model $\mathfrak{M} = \langle W, R_0, R_1, v\rangle$ and a world $w \in W$ such that $\mathfrak{M} \models A$ and $\mathfrak{M}, w \Vdash C1 \wedge \neg C2$. By $A5$ there are $n+1$ worlds w_i such that for some x, $wR_1xR_0w_0$ and for all i there exists an x, $w_iR_1xR_0w_{i+1}$, and $\mathfrak{M}, w_i \Vdash p_i$. By $A1$, $A2$, they are all d-worlds and all different. Moreover, since (3) holds in this model, we have

$$\text{for all } i \neq j \text{ such that } i+k=j, \text{ there is an } x \text{ such that } w_iR_1^kxR_0^kw_j, \tag{4}$$

where xR^ky stands for a k-long R-path. We claim that all the intermediate x in (4) make $\neg d$ true. Suppose to the contrary that for one such x, $\mathfrak{M}, x \Vdash d$. Then by $A4$, also $\mathfrak{M}, x \Vdash p_i$, and by $A3$, also $\mathfrak{M}, w_j \Vdash p_i$. But $\mathfrak{M}, w_j \Vdash p_j$ and $i \neq j$. So this is impossible by $A2$.

Because $\mathfrak{M}, w \Vdash C1$, and there is one q-variable less than there are $p_i \wedge d$-worlds, there must be two w_i making the same q-atom true. But then, by (4), we can go from a q_i-world to a q_i-world, via a $\neg d$ world. This is just what $C2$ says and that is false at w: our desired contradiction. This proves (1).

Let $\mathfrak{M}_A = \langle\{0, \ldots, n\}, I\rangle$ be the first-order model where $I(d) = \{(x, y) \mid x = y\}$ and $I(p_i) = \{(i, i)\}$, and $\mathfrak{M}_C = \langle\{0, \ldots, n-1\}, I\rangle$ where d is also interpreted as the equality, and $I(q_i) = \{(i, i)\}$. It is easy to see that $\mathfrak{M}_A \models A$ and $\mathfrak{M}_C \models C1 \wedge \neg C2$. Let θ be any L_n-sentence constructed from the atom d which is true in $\mathfrak{M}_A$. Because in $\mathfrak{M}_A$, d is interpreted as the equality, θ is equivalent to a pure (i.e., containing only $=$ as atomic symbols) L_n-sentence. But then also $\mathfrak{M}_C \models \theta$ because pure L_n formulas cannot distinguish between models of size at least n, and also in $\mathfrak{M}_C$, d is interpreted as the equality. But then θ cannot be an interpolant. This proves (2). $\square$

The last theorem shows that by just looking at the number of variables in $\varphi \to \psi$, we cannot predict how many variables are needed for the interpolant. Our counterexample showed two variable formulas of length polynomial in n, which didn't have an interpolant in n variables. Is there some way of predicting the number of variables needed for an interpolant as a function of some combination of the parameters, number of variables in $\varphi \to \psi$, number of relation symbols in $\varphi \to \psi$, and the length of $\varphi \to \psi$?

3 *Kinds of interpolation* For first-order logic we find the following definitions of interpolation in the literature. Let $I\!P(\varphi)$ be the set of atomic symbols occurring in φ (propositional variables in modal logic, relation symbols in first-order logic.)

AIP A logic L has the *Arrow Interpolation Property* (AIP) if, whenever $\models_L \varphi \to \psi$, there exists a formula θ such that $\models_L \varphi \to \theta$, $\models_L \theta \to \psi$, and $I\!P(\theta) \subseteq I\!P(\varphi) \cap I\!P(\psi)$.

TIP A logic L has the *Turnstile Interpolation Property* (TIP) if, whenever $\varphi \models_L \psi$, there exists a formula θ such that $\varphi \models_L \theta$, $\theta \models_L \psi$, and $I\!P(\theta) \subseteq I\!P(\varphi) \cap I\!P(\psi)$.

SIP A logic L has the *Splitting Interpolation Property* (SIP) if, whenever $\varphi_0 \wedge \varphi_1 \models_L \psi$, there exists a formula θ such that $\varphi_0 \models_L \theta$, $\varphi_1 \wedge \theta \models_L \psi$, and $I\!P(\theta) \subseteq I\!P(\varphi_0) \cap (I\!P(\varphi_1) \cup I\!P(\psi))$.

For first-order logic they are all equivalent but in general this is not the case (as we see below this depends on both compactness and the availability of a deduction theorem, cf. also Czelakowski [4]). The meaning of TIP and SIP in modal logic depends on the way we define the consequence relation $\varphi \models \psi$. There are two options: a local and a global one (cf. van Benthem [20] or Marx and Venema [16] for a discussion of their relative merits). Let K be a class of frames and Γ, ψ (set of) $\mathcal{ML}(\mathsf{K})$-formulas.

$\models^{loc}$ The *local consequence relation* $\Gamma \models^{loc} \psi$ holds iff for every $\mathcal{F} \in \mathsf{K}$, for every valuation v, for every world w in F, $\langle \mathcal{F}, v\rangle, w \Vdash \Gamma$ implies $\langle \mathcal{F}, v\rangle, w \Vdash \psi$.

$\models^{glo}$ The *global consequence relation* $\Gamma \models^{glo} \psi$ holds iff for every $\mathcal{F} \in \mathsf{K}$, for every valuation v, $\langle \mathcal{F}, v\rangle \models \Gamma$ implies $\langle \mathcal{F}, v\rangle \models \psi$.

The global relation is the one familiar from first-order logic, but it is usually defined for Γ a set of *sentences* (if they are formulas, the universal closure is considered). If we view the world w as an assignment, then for sentences as premises, the two notions are equivalent. Indeed, when Γ is a set of formulas—and they are treated as formulas—the local definition becomes the more interesting (cf. the definition just before Proposition 2.3.6 in Chang and Keisler [1]). In modal logic, the different interpolation properties are related as follows.

Proposition 3.1

(i) With the local consequence relation, AIP, TIP, and SIP are all equivalent.

(ii) If $\models^{loc}$ is compact and the class K of frames of the logic is closed under taking point-generated subframes, then AIP implies TIP, and TIP and SIP are equivalent.

For this reason, we will only use TIP and SIP defined using the global consequence relation. As compactness is a common notion in modal logic (e.g., every modal logic of an elementary class of frames is compact), AIP and TIP are often referred to as the strong and weak interpolation property, respectively. We note that the splitting interpolation version is the one used in connection with modularization of programs [13, 18]. In the rest of the article $\models$ always refers to the global consequence relation.

Proof: For (i), use the fact that with the local relation the deduction theorem $\varphi \models^{loc} \psi$ if and only if $\models \varphi \rightarrow \psi$ holds. We prove (ii) for the unimodal case only. The proof extends easily to any modal similarity type. For (ii) we use that we can switch from the global to the local perspective by $\varphi \models^{glo} \psi$ iff $\{\Box^n \varphi \mid n < \omega\} \models^{loc} \psi$ ([20], Lemma 2.33). Here we use the assumption of being closed under point-generated subframes.

AIP implies TIP. Assume $\varphi \models^{glo} \psi$. This holds if and only if $\{\Box^n \varphi \mid n < \omega\} \models^{loc} \psi$, if and only if (by compactness) $\Box^{m*}\varphi \models^{loc} \psi$ for some m, where $\Box^{m*}\varphi = \varphi \wedge \Box\varphi \wedge \Box\Box\varphi \wedge \cdots \wedge \Box^m\varphi$. If and only if, by the deduction theorem, $\models \Box^{m*}\varphi \rightarrow \psi$. But then, by AIP, there is an interpolant θ such that $\models \Box^{m*}\varphi \rightarrow \theta$ and $\models \theta \rightarrow \psi$. Whence $\varphi \models^{glo} \theta$ and $\theta \models^{glo} \psi$.

SIP is equivalent to TIP. The direction from SIP to TIP is trivial. For the other direction, assume $\varphi_0 \wedge \varphi_1 \models^{glo} \psi$. As above we obtain, $\Box^{m*}\varphi_0 \wedge \Box^{k*}\varphi_1 \models^{loc} \psi$. Then by the deduction theorem, $\Box^{m*}\varphi_0 \models^{loc} \Box^{k*}\varphi_1 \rightarrow \psi$. Whence, $\Box^{m*}\varphi_0 \models^{glo} \Box^{k*}\varphi_1 \rightarrow \psi$. By TIP, we find an interpolant θ such that $\Box^{m*}\varphi_0 \models^{glo} \theta$ and $\theta \models \Box^{k*}\varphi_1 \rightarrow \psi$. Whence, $\varphi_0 \models^{glo} \theta$ and $\varphi_1 \wedge \theta \models^{glo} \psi$. $\Box$

In the absence of the notion of a sentence, Robinson's consistency property is rather hard to formulate globally. The local version follows.

RCP A logic L has the Robinson consistency property if whenever Γ_1, Γ_2 are both satisfiable and there is no θ such that $I\!P(\theta) \subseteq I\!P(\Gamma_1) \cap I\!P(\Gamma_2)$ and $\Gamma_1 \models^{loc} \theta$ and $\Gamma_2 \models^{loc} \neg\theta$, then also $\Gamma_1 \cup \Gamma_2$ is satisfiable.

It is a standard proof to show the following proposition.

Proposition 3.2 *Assuming that the local consequence relation is compact, AIP and RCP are equivalent.*

Relevance property The version of the interpolation property where there are no common variables in the given formulas is sometimes called the *relevance property*. Again we have three versions of this property corresponding to AIP, TIP, and SIP. If in a modal logic $\Diamond\top \longleftrightarrow \top$ and $\Diamond\bot \longleftrightarrow \bot$ are valid for all modalities, then the AIP relevance property is equivalent to the *disjunction property* for formulas φ, ψ without common variables: if $\models \varphi \vee \psi$, then $\models \varphi$ or $\models \psi$. The standard term for this property is *Halldén-completeness* (see e.g., van Benthem and Humberstone [21]). The relevance property—insignificant as it may look at first sight—is a strong weapon for axiomatizing "difficult logics." We mean logics for which it is not easy to find a finite (Sahlqvist) axiomatization, but there is a finite axiomatization using irreflexivity-style rules. The relevance property can help to decide whether such rules are really needed, viz., Proposition 2.9.2 in Venema [22]. The result states that for a logic axiomatized using unorthodox rules, these rules are conservative (i.e., not needed) if the axiom system without these rules has the AIP relevance property and the two axiom systems derive precisely the same formulas built up from constants only.

We will now provide some simple semantical conditions on frames that imply the failure of SIP. The proof is given for unary monomodal logics (the similarity type

$S = \{\Diamond\}$ is assumed fixed throughout the proof) for notational convenience, but the result can be easily extended to n-ary polymodal logics. First we recall the notion of bisimulation and zigzag-morphism.

Bisimulation Let $\mathcal{G}$ and $\mathcal{H}$ be two frames of type S. Let $B \subseteq G \times H$, B nonempty.

1. We say that B is a *bisimulation* between $\mathcal{G}$ and $\mathcal{H}$ if for any operator $\langle i \rangle \in S$ the following clauses (called forth and back) hold:

$$\text{if } Bxx' \ \& \ R_{\mathcal{G}}^{\langle i \rangle} xy, \text{ then } (\exists y')(Byy' \ \& \ R_{\mathcal{H}}^{\langle i \rangle} x'y')$$

   and similarly in the other direction,

$$\text{if } Bxx' \ \& \ R_{\mathcal{H}}^{\langle i \rangle} x'y', \text{ then } (\exists y)(Byy' \ \& \ R_{\mathcal{G}}^{\langle i \rangle} xy).$$

   If Bxx' holds we will call x and x' *bisimilar*.
2. If B is a total function f, then it is called a *zigzag morphism*. If f is also surjective we use notation $\mathcal{G} \overset{f}{\twoheadrightarrow} \mathcal{H}$ and call $\mathcal{H}$ the *zigzag morphic image* of $\mathcal{G}$ by f.
   Note that in this case, it is equivalent to say that f is a homomorphism that furthermore satisfies the (zag) condition

$$\text{if } R_{\mathcal{H}}^{\langle i \rangle} f(x)y', \text{ then } (\exists y)(f(y) = y' \ \& \ R_{\mathcal{G}}^{\langle i \rangle} xy).$$

3. The notions of bisimulation and zigzag morphism can also be defined for models $\mathfrak{M}_{\mathcal{G}} = \langle \mathcal{G}, v_{\mathcal{G}} \rangle$ and $\mathfrak{M}_{\mathcal{H}} = \langle \mathcal{H}, v_{\mathcal{H}} \rangle$, relative to a given set of propositional variables V by adding the following condition:

$$\text{if } Bxx' \text{ then for all } p_i \in V, \mathfrak{M}_{\mathcal{G}}, x \Vdash p_i \text{ iff } \mathfrak{M}_{\mathcal{H}}, x' \Vdash p_i.$$

   We will say in this case that B is a *V-bisimulation* or a *V-zigzag morphism*.

Lemma 3.3 *Let K be a class of frames.*

1. *SIP fails in the modal theory of K if there are finite frames $\mathcal{G}, \mathcal{H} \in \mathsf{K}$, a frame $\mathcal{F}$ and surjective zigzag morphisms m, n such that $\mathcal{G} \overset{m}{\twoheadrightarrow} \mathcal{F} \overset{n}{\twoheadleftarrow} \mathcal{H}$, $\mathcal{F}$ is generated by one point w, every m-pre-image of w in G generates $\mathcal{G}$, and similarly for $\mathcal{H}$, and there is no frame $\mathcal{J} \in \mathsf{K}$ with commuting surjective zigzag morphisms g and h from $\mathcal{J}$ onto $\mathcal{G}$ and $\mathcal{H}$ (i.e., $\mathcal{G} \overset{g}{\twoheadleftarrow} \mathcal{J} \overset{h}{\twoheadrightarrow} \mathcal{H}$ and $m \circ g = n \circ h$).*
   Moreover, an explicit counterexample for SIP can be algorithmically constructed from the frames and functions $\mathcal{G} \overset{m}{\twoheadrightarrow} \mathcal{F} \overset{n}{\twoheadleftarrow} \mathcal{H}$.
2. *If in addition, K is elementary and closed under point-generated subframes, then also AIP and TIP fail.*

The proof relies on the fact that for any finite frame $\mathcal{F}$ generated by a point there is an algorithmically constructible formula $\Sigma_{\mathcal{F}}$ that characterizes the frame up to bisimulation. The formulas that describe frames $\mathcal{G}$ and $\mathcal{H}$ together with a description of the zigzag morphisms m and n, will play the role of formulas φ_0 and φ_1 in the premise of SIP, whereas ψ is simply a negated propositional symbol that will be "standing" in a world in $\mathcal{F}$. From $\mathcal{G} \overset{m}{\twoheadrightarrow} \mathcal{F} \overset{n}{\twoheadleftarrow} \mathcal{H}$ we will be able to prove that there is no splitting interpolant for $\varphi_0 \wedge \varphi_1, \psi$, whereas the inexistence of a frame $\mathcal{J}$ implies $\varphi_0 \wedge \varphi_1 \models \psi$.

We start by proving that we are able to syntactically characterize finite frames, up to bisimulation. The following lemma is a generalization of Lemma 1 in Fine [5].

Lemma 3.4 *Let $\mathcal{F} = \langle F, R\rangle$ be a finite frame generated by w_1 and let $|F| = n$. Let $\mathfrak{M} = \langle \mathcal{F}, v\rangle$ be a model such that $v(p_i) = \{w_i\}$ for $p_1, \ldots, p_n$. Define $\Sigma_{\mathcal{F}}$ as the conjunction of the following formulas.*

A_1: $\bigvee p_i$,
A_2: $p_i \to \bigwedge\{\neg p_j \mid i \neq j\}$,
A_3: $p_i \to \bigwedge\{\langle i\rangle p_j \mid Rw_iw_j\}$,
A_4: $p_i \to \bigwedge\{\neg\langle i\rangle p_j \mid \neg Rw_iw_j\}$.

Let $\mathfrak{M}' = \langle \mathcal{F}', v'\rangle$ be any model such that

1. $\mathfrak{M}' \models \Sigma_{\mathcal{F}}$, *and*
2. $\mathfrak{M}', w' \Vdash p_1$ *for some* w'.

Then the relation $B \subseteq F' \times F$ defined as

$$Bw'w \text{ iff } w' \text{ and } w \text{ agree in the truth value assigned to } \{p_1, \ldots, p_n\}$$

is a surjective $\{p_1, \ldots, p_n\}$-zigzag morphism from $\mathfrak{M}'$ onto $\mathfrak{M}$.

Proof: Trivially, bisimilar worlds agree on the variables $p_1, \ldots, p_n$. The back and forth clauses hold precisely because of A_3 and A_4. So B is a $\{p_1, \ldots, p_n\}$-bisimulation. B is functional by A_2 and it is always defined by A_1. Finally B is surjective because $\mathcal{F}$ was generated by the p_1-world w_1, there exists a p_1-world in $\mathfrak{M}'$, and B is a zigzag morphism. □

Now we are ready for the proof of Lemma 3.3.

Proof of Lemma 3.3: Let $\mathcal{G} \overset{m}{\twoheadrightarrow} \mathcal{F} \overset{n}{\twoheadleftarrow} \mathcal{H}$ be given as in the Lemma, and suppose $\mathcal{F}$ is generated by w_1. We use three disjoint sets of propositional variables:

$f_1, \ldots, f_{\lvert F\rvert}$	one for each point in $\mathcal{F}$,
$g_1, \ldots, g_{\lvert G\rvert}$	one for each point in $\mathcal{G}$,
$h_1, \ldots, h_{\lvert H\rvert}$	one for each point in $\mathcal{H}$.

We create three models by making each variable true at precisely one point in the respective model, and by making the f_i true in $\mathcal{G}$ and $\mathcal{H}$ at precisely those worlds which are mapped to an f_i-world in $\mathcal{F}$ by m and n, respectively. Formally we define models $\mathfrak{M}_{\mathcal{F}} = \langle \mathcal{F}, v_{\mathcal{F}}\rangle$, $\mathfrak{M}_{\mathcal{G}} = \langle \mathcal{G}, v_{\mathcal{G}}\rangle$ and $\mathfrak{M}_{\mathcal{H}} = \langle \mathcal{H}, v_{\mathcal{H}}\rangle$, by setting

$$\begin{aligned} v_{\mathcal{F}}(f_i) &= \{w_i\} \\ v_{\mathcal{G}}(g_i) &= \{w_i\}, \quad v_{\mathcal{G}}(f_i) = \{w \in G \mid m(w) = w_i\} \\ v_{\mathcal{H}}(h_i) &= \{w_i\}, \quad v_{\mathcal{H}}(f_i) = \{w \in H \mid n(w) = w_i\}. \end{aligned}$$

(Any value can be assigned to the other propositional letters.) We define two formulas describing $\overset{m}{\twoheadrightarrow}$ and $\overset{n}{\twoheadrightarrow}$:

$$\Gamma_m = \bigwedge_{1\leq i\leq |F|}(f_i \longleftrightarrow \bigvee\{g_j \mid m(w_j) = w_i\})$$
$$\Gamma_n = \bigwedge_{1\leq i\leq |F|}(f_i \longleftrightarrow \bigvee\{h_j \mid n(w_j) = w_i\}).$$

Let $\Sigma_{\mathcal{G}}$ and $\Sigma_{\mathcal{H}}$ be the descriptions of $\mathfrak{M}_{\mathcal{G}}$ and $\mathfrak{M}_{\mathcal{H}}$ in the variables $g_1, \ldots, g_{|G|}$ and $h_1 \ldots, h_{|H|}$, respectively, just as in Lemma 3.4. By the valuations it is immediate that

$$m, n \text{ are surjective } \{f_1, \ldots, f_{|F|}\} - \text{zigzag morphisms from } \mathfrak{M}_{\mathcal{G}} \text{ and } \mathfrak{M}_{\mathcal{H}} \text{ onto } \mathfrak{M}_{\mathcal{F}}, \tag{5}$$

$$\mathfrak{M}_{\mathcal{G}} \models \Sigma_G \wedge \Gamma_m \text{ and } \mathfrak{M}_{\mathcal{H}} \models \Sigma_H \wedge \Gamma_n. \tag{6}$$

Note that Σ_G, Σ_H, Γ_m, and Γ_n can be algorithmically obtained from $\mathcal{G} \overset{m}{\twoheadrightarrow} \mathcal{F} \overset{n}{\twoheadleftarrow} \mathcal{H}$. These formulas will provide the counterexample to SIP.

Claim 3.5

$$(\Sigma_{\mathcal{G}} \wedge \Gamma_m) \wedge (\Sigma_{\mathcal{H}} \wedge \Gamma_n) \models \neg f_1, \tag{7}$$

$$\textit{there is no splitting interpolant for (7).} \tag{8}$$

Proof: We start with the easy part, (8). Suppose to the contrary that there is an interpolant θ for (7). Then we have $\Sigma_{\mathcal{G}} \wedge \Gamma_m \models \theta$ and $(\Sigma_{\mathcal{H}} \wedge \Gamma_n) \wedge \theta \models \neg f_1$ and θ is constructed from the variables $\{f_1, \ldots, f_{|F|}\}$.

We will derive a contradiction. By (6), $\mathfrak{M}_{\mathcal{G}} \models \Sigma_{\mathcal{G}} \wedge \Gamma_m$. So by hypothesis, also $\mathfrak{M}_{\mathcal{G}} \models \theta$. But then by (5) and the fact that θ is in the common $\{f_i, \ldots, f_{|F|}\}$-language, also $\mathfrak{M}_{\mathcal{F}} \models \theta$. Then again by (5) but for n, also $\mathfrak{M}_{\mathcal{H}} \models \theta$. By (6) now, $\mathfrak{M}_{\mathcal{H}} \models (\Sigma_{\mathcal{H}} \wedge \Gamma_n) \wedge \theta$. So by hypothesis, $\mathfrak{M}_{\mathcal{H}} \models \neg f_1$. But $\mathfrak{M}_{\mathcal{F}}$ contains an f_1-point and n is surjective, so $\mathfrak{M}_{\mathcal{H}}$ must contain an f_1-point as well: the desired contradiction. This proves Claim 3.5(8).

Now we show (7). Suppose (7) is not true. Then, there is a frame $\mathcal{J} \in \mathsf{K}$ and a valuation $v_{\mathcal{J}}$ such that

$$\langle \mathcal{J}, v_{\mathcal{J}}\rangle \models (\Sigma_{\mathcal{G}} \wedge \Gamma_m) \wedge (\Sigma_{\mathcal{H}} \wedge \Gamma_n)$$

and there is

$$w \in J \text{ such that } \langle \mathcal{J}, v_{\mathcal{J}}\rangle, w \Vdash f_1.$$

Define two relations B_G and B_H as follows.

$$B_G = \{\langle x, y\rangle \in J \times G \mid x \text{ and } y \text{ agree on the } g_i\},$$
$$B_H = \{\langle x, y\rangle \in J \times H \mid x \text{ and } y \text{ agree on the } h_i\}.$$

Let $x \in G$ and $y \in H$ be points such that $B_G wx$ and $B_H wy$ hold (they exist because $\mathfrak{M}_{\mathcal{J}} \models \Sigma_{\mathcal{G}} \wedge \Sigma_{\mathcal{H}}$). As $\mathfrak{M}_{\mathcal{J}} \models \Gamma_m \wedge \Gamma_n$, also $\mathfrak{M}_{\mathcal{G}}, x \Vdash f_1$ and $\mathfrak{M}_{\mathcal{H}}, y \Vdash f_1$. Whence, $m(x) = n(y) = w_1$, the generating point of $\mathcal{F}$. Since we assumed that any $x \in G$ such that $m(x) = w_1$ generates $\mathcal{G}$, and similarly for $\mathcal{H}$, $\mathcal{G}$ and $\mathcal{H}$ are generated from x and y, respectively. Thus the frames satisfy all the conditions in Lemma 3.4 and we can derive the following.

$$B_G \text{ is a surjective } \{g_1, \ldots, g_{|G|}\}\text{-zigzag morphism from } \mathfrak{M}_{\mathcal{J}} \text{ onto } \mathfrak{M}_{\mathcal{G}}. \tag{9}$$

$$B_H \text{ is a surjective } \{h_1, \ldots, h_{|H|}\}\text{-zigzag morphism from } \mathfrak{M}_{\mathcal{J}} \text{ onto } \mathfrak{M}_{\mathcal{H}}. \tag{10}$$

Because $\mathfrak{M}_{\mathcal{J}} \models \Gamma_m \wedge \Gamma_n$, B_G and B_H are also $\{f_1, \ldots, f_{|F|}\}$-zigzag morphisms. But then the diagram must commute, since every world in $\mathfrak{M}_{\mathcal{F}}$ satisfies precisely one f_i. So we found a frame in K with commuting zigzag morphisms onto $\mathcal{G}$ and $\mathcal{H}$, contrary to our assumption. This proves Claim 3.5(7). □

Part (1) of the lemma follows immediately from this claim. If K is also elementary, then the local consequence relation of Modal-Th(K) is compact (by compactness of first-order logic, using the standard translation), so by Proposition 3.1 also AIP and TIP fail. □

If we slightly strengthen the conditions imposed on $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ in Lemma 3.3 we obtain a method for disproving the relevance version of SIP.

Lemma 3.6 *Assume the conditions of Lemma 3.3. If, in addition, $\mathcal{F}$ consists of one world and $\mathcal{G}$ and $\mathcal{H}$ are both* simple *(i.e., every generated subframe is the frame itself), then there are formulas φ and ψ without common variables such that*

$$\varphi \wedge \psi \models \bot, \tag{11}$$

$$\textit{there is no splitting interpolant for 11.} \tag{12}$$

Proof: A copy of the proof of Lemma 3.3 will do. We have to prove that in Claim 3.5 we can delete Γ_m, Γ_n, and f_1 from the given formula. We used the Γ's to show that the functions commute. But now that is always the case since $\mathcal{F}$ consists of just one point. We used f_1 to guarantee that the functions B_G and B_H are surjective. But since $\mathcal{G}$ and $\mathcal{H}$ are simple, the defined B_G and B_H are always surjective. Γ_m, Γ_n, and f_1 were not used any further in the proof of Lemma 3.3. □

4 Transfer of interpolation in combined modal logics In [16] the following tool is presented to prove interpolation in canonical modal logics. Let $\mathcal{G}$ and $\mathcal{F}$ be two modal frames. A frame $\mathcal{H}$ is called a *zigzag product* of $\mathcal{G}$ and $\mathcal{F}$ if $\mathcal{H}$ is a substructure of the direct product $\mathcal{G} \times \mathcal{F}$ in the standard model-theoretic sense, where in addition the projections are surjective zigzag morphisms (also called bounded or p-morphisms).

The notion of zigzag product is a generalization of the (a_1, a_2)-products in Maksimova [14]. ((a_1, a_2)-products are defined for unimodal Kripke frames with a reflexive and transitive accessibility relation.) The following Lemma generalizes Theorem 2 in [14] to arbitrary classes of frames.

Lemma 4.1 ([16]: Theorem B.4.5) *If the modal logic of a class K of frames is canonical and K is closed under zigzag products, then the logic enjoys (Arrow) interpolation.*

An immediate consequence of being closed under zigzag products (because universal Horn sentences are preserved under substructures of direct products) is the following theorem.

Theorem 4.2 *Every Sahlqvist axiomatizable modal logic whose axioms correspond to universal Horn formulas enjoys the (Arrow) interpolation property.*

Our examples show how existential, or universal but disjunctive, frame conditions can indeed lead to failure of interpolation.

For the relevance property/Halldén-completeness there exists a similar criterion provided in [21]. We present the short proof for completeness.

Lemma 4.3 *Let* K *be a class of frames closed under taking finite direct products in which the condition* $\forall x \exists y R_i xy$ *holds for all relations* R_i*. Then the relevance property holds in the modal logic of* K*.*

Proof: On such frame classes there are up to logical equivalence only two formulas built up from constants, $\top$ and $\bot$. So we might as well prove the disjunction property. Suppose $\not\models \varphi$ and $\not\models \psi$. Then we have K models $\mathfrak{M}$ and $\mathfrak{M}'$, satisfying $\neg\varphi$ and $\neg\psi$, respectively. Because in K every relation is serial, the two models have the one-element reflexive frame as a $\varnothing$-zigzag morphic image. From this it follows quickly that the product of the two frames underlying $\mathfrak{M}$ and $\mathfrak{M}'$ is a zigzag product. The obvious valuation now turns this product into a model satisfying $\neg\varphi \wedge \neg\psi$. □

In Figure 1 we have listed a few well-known conditions on frames, together with the axioms that characterize them. Note that these axioms give rise to canonical modal logics, so by Theorem 4.2 every modal logic defined by these axioms enjoys interpolation.

We will see that interpolation does not transfer for any of these logics by taking products or by forming unions in the sense defined below.

T.	$p \to \Diamond p$	$\forall x Rxx$	*reflexivity*
4.	$\Diamond\Diamond p \to \Diamond p$	$\forall xyz((Rxy \wedge Ryz) \to Rxz)$	*transitivity*
B.	$p \wedge \Diamond q \to \Diamond(q \wedge \Diamond p)$	$\forall xy(Rxy \to Ryx)$	*symmetry*
5.	$\Diamond p \wedge \Diamond q \to \Diamond(q \wedge \Diamond p)$	$\forall xyz((Rxy \wedge Rxz) \to Ryz)$	*euclidicity*

Figure 1: Conditions on frames.

4.1 Products of modal logics In [7], *bi-dimensional products logics* are defined as follows. The product $\mathcal{F} \times \mathcal{G}$ of two standard modal frames $\mathcal{F} = \langle F, R_{\mathcal{F}} \rangle$ and $\mathcal{G} = \langle G, R_{\mathcal{G}} \rangle$ is the modal frame $\langle F \times G, H, V \rangle$, where H and V are defined as

$$\begin{aligned} (x, y)H(x', y') &\quad \text{iff} \quad R_{\mathcal{F}}xx' \text{ and } y = y' \\ (x, y)V(x', y') &\quad \text{iff} \quad R_{\mathcal{G}}yy' \text{ and } x = x'. \end{aligned}$$

The product of two unimodal frames leads to a bimodal frame. We will use φ-diamond $\Diamond\!\!\!|$ and $\ominus$-diamond for the modalities defined over the V-relation and the H-relation (V and H are for vertical and horizontal), respectively. Their meaning is defined in the standard way, for example, $\mathfrak{M}, w \Vdash \Diamond\!\!\!|\, \varphi$ if and only if there exists a w' such that wVw' and $\mathfrak{M}, w' \Vdash \varphi$.

For classes of modal frames K and K' the product $\mathsf{K} \times \mathsf{K}'$ is the class of frames $\{\mathcal{F} \times \mathcal{G} \mid \mathcal{F} \in \mathsf{K} \text{ and } \mathcal{G} \in \mathsf{K}'\}$. If $\mathsf{K} = \mathsf{K}'$ we also use the notation K^2 to denote $\mathsf{K} \times \mathsf{K}$. For familiar modal logics such as **K**, **S4**, **S5**, and so forth, we will use **K**×**K** and so on, to denote the product of the largest classes of frames for which these logics are complete.

The notion of product logic can very easily be extended to n-dimensional product logics by just taking the product of n unimodal systems, but for simplicity we will restrict ourselves to bi-dimensional logics.

Completeness theorems are known for several cases, cf. [7]. We only mention the complete inference systems for $\mathbf{K}^2$ and $\mathbf{S5}^2$. The class $\mathbf{K}^2$ of all product frames can be axiomatized by adding the axioms of commutativity $\diamondvert\diamondminus p \longleftrightarrow \diamondminus\diamondvert p$ and confluence $\diamondvert\boxminus p \rightarrow \boxbar\diamondminus p$ to the standard axiomatization for a bi-modal system. The class $\mathbf{S5}^2$ of all product frames where V is the universal relation on the columns and H the universal relation on the rows, can be axiomatized by adding to the above system the axioms that make both $\diamondvert$ and $\diamondminus$ **S5**-modalities.

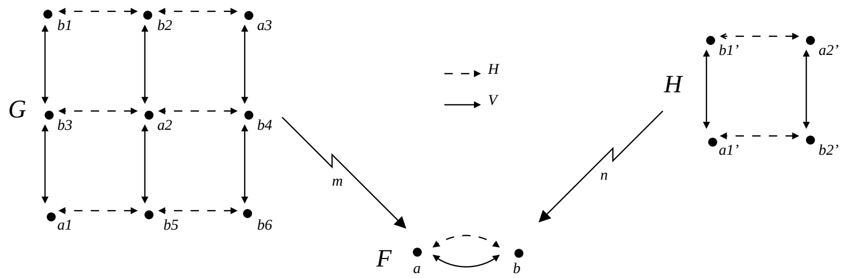
Products of modal logics have applications in computer science through their connection with labeled transition systems and are closely related to (finite variable fragments of) first-order logic as follows. Let (D, I) be a first-order model. Create the modal frame $({}^nD, \equiv_i)_{i<n}$ where $s \equiv_i t$ if and only if $s(j) = t(j)$ for all j except possibly i. Then that frame is just a product of n frames $(D, D \times D)$. Its set of "worlds" is composed by all the n-tuples over D, which we can view as all assignments of the first n variables on D. Every relation $\equiv_i$ corresponds to a diamond $\langle i \rangle$, which in turn is the modal counterpart of the first-order quantifier $\exists v_i$, as is easy to see by writing out the truth-definition. For more on this way of modalizing first-order logic, we refer to [16].

Taking products of modal logics is a method of combining logics [6] which immediately leads to interaction between the modalities (viz. the commutativity and the confluence axioms above). [15] shows that the method of dovetailing (a special case of the fibering logic approach) usually lets interpolation transfer to the combined system. The difference between dovetailing and taking products is that in the dovetailed system there is no interaction between the combined modalities. We will show that the existential nature of the interaction obtained by taking products often prohibits transfer of interpolation. Both the commutativity and the confluence axiom are Sahlqvist formulas which correspond to $\forall xy(\exists z(xVzHy) \longleftrightarrow \exists z(xHzVy))$ and $\forall xyz(xVy \wedge xHz \rightarrow \exists w(yHw \wedge zVw))$, respectively.

We first provide a general result. Afterward we derive some corollaries. Let n^2, for n a natural number viewed as the set $\{0, 1, \ldots, n-1\}$, denote the product frame with domain $n \times n$ where V and H are universal relations on columns and rows, respectively.

Theorem 4.4 *Let K be a class of bi-dimensional product frames containing the frames 2^2 and 3^2. Then SIP fails in the modal logic of K.*

Proof: The proof is a purely semantical recast of the proof of Theorem 2.1, now using Lemma 3.3. Take the frames 2^2 and 3^2 and define a frame $\mathcal{F}$ consisting of two states with H and V universal accessibility relations. The functions mapping all states on the diagonal to one state in $\mathcal{F}$, and all states not in the diagonal to the other are surjective zigzag morphisms. Cf. the picture below (the relations in $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{F}$ are actually the reflexive transitive closure of the relations shown).



Since all three frames are simple, in the sense that every world generates the whole frame, the conditions on generation of Lemma 3.3 are met. We will prove now that there is no frame $\mathcal{J}$ such that

$$\mathcal{J} \models \diamondminus \diamondvert p \to \diamondvert \diamondminus p, \text{ or equivalently,}$$

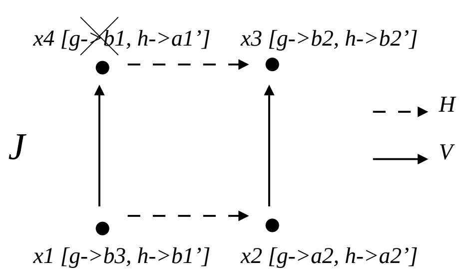
$$\mathcal{J} \models \forall xyz((xHy \wedge yVz) \to \exists w(xVw \wedge wHz)), \tag{13}$$

$$\text{there are surjective zigzag morphisms } g \text{ and } h \text{ from } J \text{ onto } G \text{ and } H, \tag{14}$$

$$\text{the morphisms commute, } m \circ g = n \circ h. \tag{15}$$

By (15) we have for every element x in J, either $m \circ g(x) = n \circ h(x) = a$ or $m \circ g(x) = n \circ h(x) = b$.

We will now try to construct $\mathcal{J}$ and show we will fail. Since g must be surjective, there should be an element $x_1 \in J$ such that $g(x_1) = b_3$. By (15) then either $h(x_1) = b'_1$ or $h(x_1) = b'_2$. Say $h(x_1) = b'_1$ (by the symmetrical nature of $\mathcal{H}$, the proof also goes through if we start from $h(x_1) = b'_2$). Because b_3Ha_2, by (14) we must have an $x_2 \in J$ such that x_1Hx_2 and $g(x_2) = a_2$. The homomorphism condition on h makes $h(x_2)$ either a'_2 or b'_1, but using restriction 15 the former should hold. In the same way we obtain, by the zigzag condition of g, x_2Vx_3 and $g(x_3) = b_2$, and by homomorphism of h and (15), $h(x_3) = b'_2$. Now by (13), from $x_1Hx_2Vx_3$ we can infer the existence of an x_4 such that $x_1Vx_4Hx_3$. But then to keep the projections homomorphisms we have to make $g(x_4) = b_1$ and $h(x_4) = a'_1$ and this is excluded by (15). Hence we cannot find a $\mathcal{J}$ as asked for in Lemma 3.3 and SIP fails. The picture below shows where the contradiction is found.



□

Corollary 4.5 *Let $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ be modal logics both weaker than $\boldsymbol{S5}$. Then SIP fails in the product of $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$.*

Section 7 of [7] shows that the product of two elementary frame classes is itself elementary. So in these cases, the local consequence relation is compact and failure of SIP implies failure of all three types of interpolation. Now we can infer many non-transference results, for example:

Corollary 4.6 *Let K_1 and K_2 be two classes of frames defined by some subset of the list of axioms in Figure 1. Both the logics of K_1 and K_2 enjoy all types of interpolation but all of them fail in the logic of the product $\mathsf{K}_1 \times \mathsf{K}_2$.*

In general we can conclude that interpolation does not transfer when taking products. (A noticeable exception is the product of two classes where the accessibility relation is a (partial) function. Interpolation for this class can easily be shown using Lemma 4.1.)

Sain [19], Theorem 2 implies that the Beth definability property fails for the class **S5** $\times$ **S5** but that the AIP relevance property holds. We conjecture that the Beth property also fails in the product of two tense logics (where we assume nothing about the accessibility relations). The proof would be a combination of Sain's counterexample and the proof of Theorem 1.1.

We have some positive news concerning the relevance property though.

Theorem 4.7 *Let K_1, K_2 be two classes of frames, both closed under finite direct products in the model-theoretic sense. If the relations in K_1 and K_2 are serial, then the logic of the bi-dimensional product $\mathsf{K}_1 \times \mathsf{K}_2$ has the AIP relevance property.*

Proof: By Lemma 4.3 it is sufficient to show that $\mathsf{K}_1 \times \mathsf{K}_2$ is closed under finite direct products. Let $\times$ denote the bi-dimensional product, and $\otimes$ the direct product of structures. We claim that for all frames $\mathcal{A}, \mathcal{B}, C, \mathcal{D}$,

$$(\mathcal{A} \times \mathcal{B}) \otimes (C \times \mathcal{D}) \cong (\mathcal{A} \otimes C) \times (\mathcal{B} \otimes \mathcal{D}). \tag{16}$$

(16) is simple to prove using the obvious isomorphism which sends $\langle\langle a, b\rangle, \langle c, d\rangle\rangle$ to $\langle\langle a, c\rangle, \langle b, d\rangle\rangle$. Because K_1 and K_2 are closed under finite direct products, (16) implies that $\mathsf{K}_1 \times \mathsf{K}_2$ is closed under them as well. □

We will now turn our attention to other combined modal logics: Humberstone's logic of inaccessibility and its generalization to unions of modal logics.

4.2 Humberstone's inaccessibility logic

In [12], Humberstone presented the logic of inaccessibility HIL, an extension of the classical modal systems through the introduction of a new modality $\langle - \rangle$ that has as associated relation the complement of the accessibility relation of $\langle\ \rangle$ (which in this case we will denote by $\langle + \rangle$). Humberstone proved that the inaccessibility operator $\langle - \rangle$ greatly increases the expressive power of the logic. New properties of frames such as *irreflexivity*, *asymmetry*, and *intransitivity* can now be captured by the system. For this logic, the questions about finite axiomatization and finite model property were already solved (Gabbay [8]; Gargov, Passy, and Tinchev [10]) but interpolation was still open. We will show that interpolation fails.

A *frame* for HIL is a structure $\mathcal{F} = \langle F, R_+, R_- \rangle$ where F is a nonempty set and R_+, R_- are binary relations on F that satisfy the condition $(R_-)^c = R_+$ (R^c stands for the complement of R). Truth is defined as usual.

$$\text{For } j \in \{+,-\}, \langle \mathcal{F}, v\rangle, w \Vdash \langle j\rangle\varphi \quad \text{iff} \quad \exists w' \in F, R_j ww' \,\&\, \langle \mathcal{F}, v\rangle, w' \Vdash \varphi.$$

[10] contains an axiom system for HIL. They show that the class of HIL-frames and the class of frames $\langle W, R_1, R_2\rangle$ where $R_1 \cup R_2$ is an equivalence relation have the same modal $\{\langle+\rangle, \langle-\rangle\}$-theory. (The notion of "conditions over unions" will be generalized in Section 4.3). But then, by a simple Sahlqvist argument, the basic bi-modal axiom system enriched with axioms which make the defined modality $\langle *\rangle\varphi \triangleq (\langle-\rangle\varphi \vee \langle+\rangle\varphi)$ an **S5**-modality is sound and complete for HIL.

Theorem 4.8 *All three types of interpolation fail for Humberstone's Inaccessibility Logic, even in the strong sense of the relevance property.*

Proof: Let K be the class of all frames $\langle W, R_1, R_2\rangle$ where $R_1 \cup R_2$ is an equivalence relation. We will show that SIP fails for the bi-modal logic of K. In Section 4.3 we will show that each of the conditions reflexivity, symmetry and transitivity of the union alone leads to failure of interpolation (Corollary 4.11). Since K is elementary and obviously closed under point-generated subframes this implies that all interpolation properties fail, and because the intended HIL-frames and K have the same modal theory, this implies the theorem. □

This is the "lazy" proof using Lemma 3.3. We will now provide an explicit counterexample, which also works for the expansion of HIL with the "past" or inverse operators.

Goranko [11] extended the expressive power of HIL by defining the system HIL_i which includes not only the complement operator $\langle-\rangle$, but also the inverse operators $\langle+\rangle_i$ and $\langle-\rangle_i$ that have as associated relations the converse of R_+ and R_-, respectively. This system is so powerful that it can give a categorical characterization of the natural order $\langle\mathbb{N}, <\rangle$ (which cannot be achieved in, for example, first-order logic).

A *frame* for HIL_i is a structure $\mathcal{F} = \langle F, R_+, R_-, R^i_+, R^i_-\rangle$ where F is a nonempty set and R_+, R_-, R^i_+, R^i_- are relations on $F \times F$ that satisfy the conditions $(R_-)^c = R_+$, $R^i_+ = R^{-1}_+$, $R^i_- = R^{-1}_-$. Notice that given a HIL_i frame we can obtain a HIL frame just "forgetting" the inverse relations.

Truth is defined using the standard clauses.

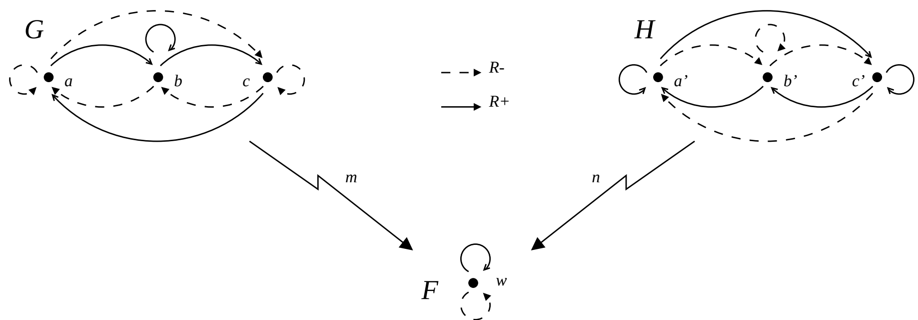
$$\begin{array}{l}
\text{For } j \in \{+,-\}, \langle \mathcal{F}, v\rangle, w \Vdash \langle j\rangle\varphi \quad \text{iff} \quad \exists w' \in F, R_j ww' \,\&\, \langle \mathcal{F}, v\rangle, w' \Vdash \varphi. \\
\text{For } j \in \{+,-\}, \langle \mathcal{F}, v\rangle, w \Vdash \langle j\rangle_i\varphi \quad \text{iff} \quad \exists w' \in F, R^i_j ww' \,\&\, \langle \mathcal{F}, v\rangle, w' \Vdash \varphi.
\end{array}$$

A complete Sahlqvist axiomatization for this system is given in [10]. Again defining $\langle *\rangle\varphi \triangleq (\langle-\rangle\varphi \vee \langle+\rangle\varphi)$ the axiomatic system for HIL_i is built from the distributive axioms, the basic temporal axioms and an **S5** system for $\langle *\rangle$. Thus, just as in the case of HIL, the intended HIL_i frames and the class HIL^*_i of frames $\langle F, R_1, R_2, R_3, R_4\rangle$ where $R_1 = R_3^{-1}$, $R_2 = R_4^{-1}$ and $R_1 \cup R_2$ is an equivalence relation have the same modal $\{\langle+\rangle, \langle-\rangle, \langle+\rangle_i, \langle-\rangle_i\}$-theory.

We will prove that the addition of the inverse modalities to HIL is not enough to regain interpolation.

Theorem 4.9 *All three types of interpolation fail for Humberstone's Inaccessibility Logic with Inverse Operators. Furthermore all three relevance properties fail.*

Proof: Applying Lemma 3.6 to the following frames proves the theorem. Instead of using that lemma we extract two formulas which describe the crucial properties of these frames.



(Here we don't show the converse relations.)

We propose the following HIL-formulas (the subscripted g and h are propositional variables):

$$\varphi: \quad \langle - \rangle (g_a \wedge [+]\neg g_a) \wedge \langle + \rangle (g_c \wedge [+]\neg g_c)$$
$$\psi: \quad \langle - \rangle ((h_{a'} \to \langle - \rangle h_{a'}) \wedge (\neg h_{a'} \to \langle - \rangle \neg h_{a'})) \wedge \langle + \rangle ((h_{b'} \to \langle - \rangle h_{b'}) \wedge (\neg h_{b'} \to \langle - \rangle \neg h_{b'})).$$

We claim that $\varphi \to \psi$ is HIL_i-valid, whence also HIL-valid.

Take any HIL_i frame $\mathcal{F}$, any valuation $v_{\mathcal{F}}$ and any world $w \in F$ and assume that $\langle \mathcal{F}, v_{\mathcal{F}} \rangle, w \Vdash \varphi$. By the semantic definitions of $\langle - \rangle$ and $\langle + \rangle$, there exist worlds $w', w'' \in F$ such that wR_-w' and wR_+w'' and $\langle \mathcal{F}, v_{\mathcal{F}} \rangle, w' \Vdash g_a \wedge [+]\neg g_a$, $\langle \mathcal{F}, v_{\mathcal{F}} \rangle, w'' \Vdash g_c \wedge [+]\neg g_c$. As $\mathcal{F}$ is a HIL_i-frame this implies $w'R_-w'$ and $w''R_-w''$ ($w' \Vdash g_a \wedge [+]\neg g_a$ and $w'' \Vdash g_b \wedge [+]\neg g_b$ forbid $w'R_+w'$ and $w''R_+w''$, respectively.) But this directly implies $\langle \mathcal{F}, v_{\mathcal{F}} \rangle, w \Vdash \psi$. Thus $\varphi \to \psi$ is valid in HIL_i.

To prove that there is no interpolant we use the frames proposed above. Transform frames $\mathcal{G}$ and $\mathcal{H}$ into models by providing the valuations $v_{\mathcal{G}}$ and $v_{\mathcal{H}}$ such that $v_{\mathcal{G}}(g_w) = \{w\}$ for $w \in \{a, b, c\}$ and $v_{\mathcal{H}}(h_{a'}) = \{a'\}$, $v_{\mathcal{H}}(h_{b'}) = \{b', c'\}$ and $v_{\mathcal{H}}(h_{c'}) = \{c'\}$.

From this, $\langle \mathcal{G}, v_{\mathcal{G}} \rangle, b \Vdash \varphi$. Let θ be any HIL_i-formula in the common language of φ and ψ, that is, it is constructed from constants, such that $\varphi \to \psi$ is HIL_i valid. Then also $\langle \mathcal{G}, v_{\mathcal{G}} \rangle, b \Vdash \theta$. Now, using the fact that the function m sending all elements in G to w in F is a zigzag morphism (for HIL_i), $\langle \mathcal{F}, v_{\mathcal{F}} \rangle, w \Vdash \theta$. As all elements in H are mapped to w and n (defined just as m over H) is also a zigzag morphism, all force θ. But, as is easy to check, $\langle \mathcal{H}, v_{\mathcal{H}} \rangle \models \neg\psi$. Thus $\varphi \to \psi$ has no interpolant, whence the relevance version of AIP fails in HIL_i. Because $\varphi \to \psi$ is already HIL-valid, this is also a counterexample for AIP in HIL.

We provided a counterexample to AIP and not to TIP. But the fact that in HIL (and then also in HIL_i) the universal diamond $\langle * \rangle$ can be defined, makes it easy to transform it in such a proof ($\langle * \rangle\varphi \equiv \langle - \rangle\varphi \vee \langle + \rangle\varphi$ means "φ somewhere in the frame"). Because $\varphi \to \psi$ is HIL_i-valid, $\langle * \rangle\varphi \models \langle * \rangle\psi$. If we assume there is a (turnstile) interpolant θ we can again use models $\langle \mathcal{F}, v_{\mathcal{F}} \rangle$, $\langle \mathcal{G}, v_{\mathcal{G}} \rangle$ and $\langle \mathcal{H}, v_{\mathcal{H}} \rangle$ to derive a contradiction. Clearly $\langle \mathcal{G}, v_{\mathcal{G}} \rangle \models \langle * \rangle\varphi$ which implies $\langle \mathcal{G}, v_{\mathcal{G}} \rangle \models \theta$. As above, this makes

$\langle \mathcal{F}, v_{\mathcal{F}} \rangle \models \theta$ and hence $\langle \mathcal{H}, v_{\mathcal{H}} \rangle \models \theta$. As θ is an interpolant we obtain $\langle \mathcal{H}, v_{\mathcal{H}} \rangle \models \langle * \rangle \psi$. But as we said before, nowhere in $\langle \mathcal{H}, v_{\mathcal{H}} \rangle$ does ψ hold. $\square$

We will now provide a more fine-grained analysis of the failure of interpolation in HIL by generalizing it to unions of modal logics. This will show that there are several reasons for this failure and indicate the possible ways in which interpolation can be regained by expanding the language.

4.3 Unions of modal logics As we saw, the HIL-system is equivalent to the bimodal logic of the class of frames $\langle W, R_1, R_2 \rangle$ where the *union* of R_1 and R_2 is an equivalence relation. We can generalize this to a way of combining logics which we call *union logics*. Let I be some index set. An I-union logic is a polymodal logic (containing modalities $\{\langle i \rangle \mid i \in I\}$) of a class of I-frames $\langle W, R_i \rangle_{i \in I}$ where the union of all relations R_i satisfies some condition.

Union logics are a natural class in a polymodal framework. In many cases, a requirement over a relation in the model is too strong. Suppose for example that relations represent actions. Then we might want to require that in each state, there is some action that does not change the state. But perhaps this action is not always the same! If the temperature is increasing, for example, a stable state is obtained by cooling, whereas we need heating if the temperature is going down. In these cases a reflexive condition over the union of the relations representing the possible actions is what we need.

Union logics are also related to the field of Informations Systems (Pawlak [17]). Based on the notion of rough sets, these systems try to capture relationships of indiscernibility among objects. Clearly, these relations are equivalence relations, but usually a further condition of Local Agreement is needed to obtain an accurate model of the situation: for each frame $\langle F, R_i \rangle_{i \in I}$ there is a linear order $\leq$ on I such that $i \leq j$ implies $R_i \subseteq R_j$. This condition reflects the fact that the different relations are modeling different degrees of indiscernibility over the same objects. Garbov [9] proves that if R and S are equivalence relations then the Local Agreement condition over R and S is equivalent to transitivity of $R \cup S$. Because of this, Local Agreement Logics are a kind of union logic. It is at present open whether interpolation holds for this logic of local agreement.

For the axiomatization of union logics, we can also generalize the idea used for HIL. Let $\mathbf{S}$ be a modal system, defining $\langle * \rangle \varphi \triangleq \bigvee_{i \in I} \langle i \rangle \varphi$, the axiomatic system for $\bigcup_{i \in I} \mathbf{S}$ is built from the distribution axioms for each $\langle i \rangle$ plus an $\mathbf{S}$ system for $\langle * \rangle$. Areces (unpublished) proves that if $\mathbf{S}$ is an axiomatic system with axioms in Sahlqvist form, then the system given above for $\bigcup_{i \in I} \mathbf{S}$ is a correct and complete Sahlqvist axiomatization for its corresponding class of frames. The proof is simple and relies on the fact that changing a diamond modality $\langle\ \rangle$ in a Sahlqvist formula by a finite disjunction of modalities $\bigvee_{i \in I} \langle i \rangle$ gives again a Sahlqvist formula which characterizes the same property the former formula did but this time over the union of the accessibility relations.

We will now show that any of the conditions from Figure 1 leads to failure of interpolation when they are stipulated over a union of relations and when the class of frames contains a few, very small frames where the union is an equivalence relation. Recall that any monomodal logic defined by a subset of these conditions enjoys interpolation, so we obtain another example where transfer of interpolation fails. We first prove a general result, then mention some corollaries.

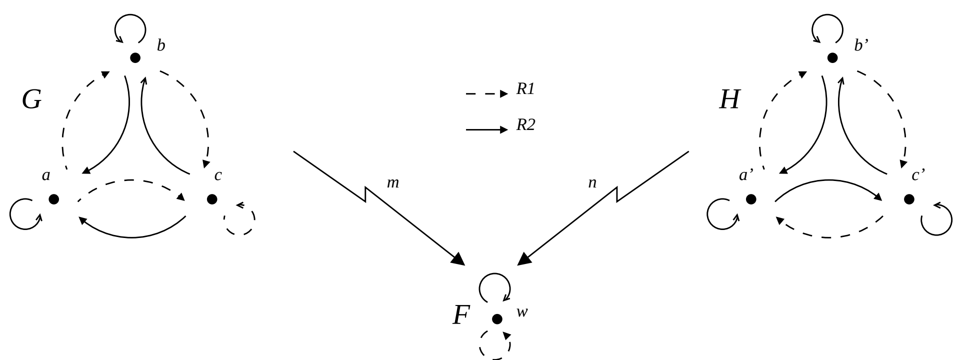
Theorem 4.10 *Let I be a set of indices of size larger than 1. Let K be a class of frames $\langle W, R_i\rangle_{i\in I}$ which satisfies*

min K *contains all finite frames $\langle W, R_i\rangle_{i\in I}$ where $\bigcup_{i\in I} R_i$ is an equivalence relation.*

max *at least one of the conditions from Figure 1, but now specified over $\bigcup_{i\in I} R_i$ is valid in K.*

Then the relevance version of SIP fails in the union logic $\mathcal{ML}(\mathsf{K})$.

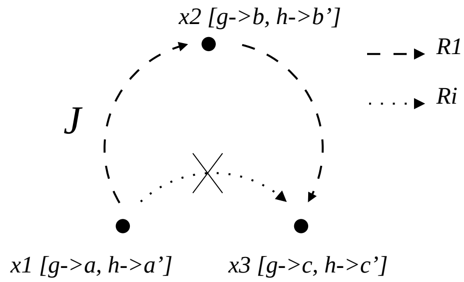
Proof: We will use Lemma 3.6. Fix an index set I. First we give three frames which lead to failure of SIP when the union is either reflexive or transitive.



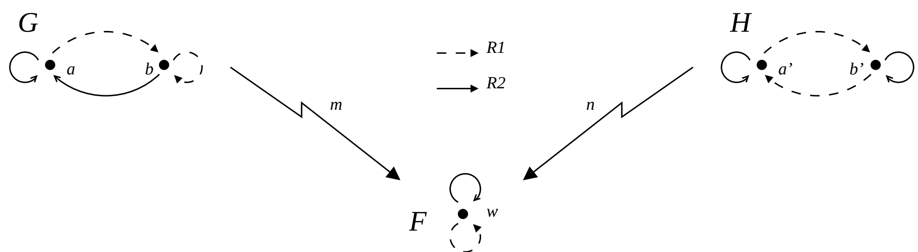
The relations R_i with $i \neq 1, 2$ are empty. Clearly m and n mapping all elements to w are zigzag morphisms and each element of $\mathcal{G}$ and $\mathcal{H}$ generates the full frame.

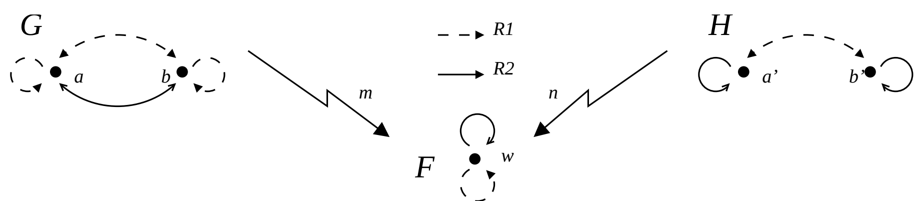
First assume $\bigcup_i R_i$ must be reflexive in K. Then no frame in K can exist with commuting zigzag morphisms onto $\mathcal{G}$ and $\mathcal{H}$, because $\mathcal{G}$ contains an R_1-reflexive point, while $\mathcal{H}$ does not.

Now assume $\bigcup_i R_i$ must be transitive. Again, we start the intended construction of $\mathcal{J}$ by an element $x_1 \in J$ which is mapped to a in G, $g(x_1) = a$ (it exists by surjectivity.) We analyze the case $h(x_1) = a'$ (for the other elements in H the argument is similar, note the symmetry of $\mathcal{H}$.) Using the relation R_1 in frame $\mathcal{G}$ and the conditions on g and h there exist elements x_2 and x_3 in J such that $R_1x_1x_2$ and $R_1x_2x_3$ and furthermore, $g(x_2) = b$, $h(x_2) = b'$, $g(x_3) = c$, and $h(x_3) = c'$. Then transitivity of the union would force $R_ix_1x_3$ for some i, but the homomorphism condition makes this impossible.



For the cases where the union of the relations is either symmetric or Euclidean, we only give the frames and leave it to the reader to check that the counterexample works. In both cases the relations R_i for $i \notin \{1, 2\}$ are empty and the zigzag morphisms m, n map all elements to w.





□

Corollary 4.11 *We just obtained four different reasons why SIP fails in the union logic of an equivalence relation, whence four reasons for failure of SIP in HIL.*

The next corollary is the counterpart of Corollary 4.6 (for products) which shows that transfer of interpolation fails for these unions of modal logics.

Corollary 4.12 *Consider the modal theory of the class of frames $\langle W, R_i \rangle_{i \in I}$ where $\bigcup_{i \in I} R_i$ satisfies some (nonempty) subset of the axioms from Figure 1. If $|I| = 1$, all types of interpolation hold. If $|I| > 1$ and finite, the relevance version of all types of interpolation fails.*

5 Conclusion and further directions We have seen that interaction of an existential or disjunctive kind between modalities often blocks transfer of interpolation in combinations of modal logics. If interpolation or the Robinson consistency property is important for the intended application of the combined modal logic, then further work in the logic-design phase is needed to fix the failure. Interpolation can show complex behavior when we consider reducts and expansions. For instance, monadic first-order logic with just one variable (i.e., modal logic **S5**) has interpolation, it fails in all other finite variable fragments, but it holds again in full first-order logic. If the counterexample is based on a "limited counting argument," then one often has to consider infinite similarity types to regain interpolation (e.g., interpolation fails for the difference operator, but is obtained when expanding the logic with all counting modalities). The four different reasons we provided for failure of interpolation in HIL each suggest an expansion of the language in which it might be recovered. For example, the symmetry-example leads one to consider modalities with the following truth-definition.

$$w \Vdash \langle i, j\rangle\varphi \quad \text{iff} \quad \exists w' : \; wR_iw' \;\&\; w'R_jw \;\&\; w' \Vdash \varphi.$$

Using them, we can eliminate the indeterminacy arising from the symmetry condition over the union. We think that the recipe provided by Lemma 3.3 is useful for a systematic search for expansions which lead to regaining interpolation. We finish with the following open problem concerning the logic of inaccessibility.

Problem 5.1 Find an expansion of HIL which enjoys interpolation and keeps the HIL-properties of decidability and finite (schema) axiomatizability. In the optimal case not even the complexity of the validity problem should go up.

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