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Semi-Contraction: Axioms and Construction

In memory of Carlos E. Alchourrón

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Abstract Semi-contraction is a withdrawal operation defined by Fermé in "On the logic of theory change: Contraction without recovery." In this paper we propose: (1) an axiomatic characterization of semi-contraction; (2) an alternative construction for semi-contraction based on *semi-saturatable sets*, inspired by Levi's *saturatable sets*; (3) a special kind of semi-contraction that satisfies the *Lindström and Rabinowicz interpolation thesis*.

1 Introduction *Recovery* is the postulate of the AGM account of belief contraction that provokes most criticism (see Alchourrón and Makinson [3] and Alchourrón, Gärdenfors, and Makinson [1]). According to recovery, so much is retained after contraction that everything can be recovered by adding the contracted sentence again. This may therefore be interpreted as a *principle of minimal loss of information*. However, this simple principle provokes nonintuitive results and, consequently, several authors reject it. Contraction functions that satisfy the AGM basic contraction postulates except recovery have been dubbed *withdrawal functions* (See Makinson [17]). Levi ([14], pp. 80–81, p. 123) has argued that *measures of information* should be replaced by *measures of informational value*,[1] and proposed an alternative construction. Another important withdrawal function, severe withdrawal, was introduced by Rott in [21] and Rott and Pagnucco in [23]. Hansson [12] noted that severe withdrawal satisfies the implausible property of *expulsiveness* (if $\nvdash \alpha$ and $\nvdash \beta$, then either $\mathbf{K}-\beta \nvdash \alpha$ or $\mathbf{K}-\alpha \nvdash \beta$). Lindström and Rabinowicz [16] abstained from recommending either a particularly expulsive contraction (severe withdrawal) or a particularly retentive one (AGM contraction). They argued that these extremes should be taken as "upper" and "lower" bounds and that any "reasonable" contraction function should be situated between them. This condition was called the *Lindström and Rabinowicz interpolation thesis* [22].

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In [4], Fermé defined semi-contraction, a withdrawal function that allows satisfaction of both principles: *minimal loss of information* and *minimal loss of informational value*,[2]. In this paper we propose an axiomatic characterization of semi-contraction and a constructive approach based on *semi-saturatable sets*, inspired by Levi's construction; and we introduce a special kind of semi-contraction function that satisfies the *interpolation thesis*.

2 Preliminaries

2.1 Partial meet AGM and Levi contraction In the AGM [1] and Levi [14] accounts the beliefs of a rational agent are represented by a belief set $\mathbf{K}$, which is a set of sentences in a language $\mathcal{L}$ closed under logical consequence Cn, where Cn satisfies: $A \subseteq Cn(A)$, $Cn(Cn(A)) \subseteq Cn(A)$, and $Cn(A) \subseteq Cn(B)$ if $A \subseteq B$, as well as supraclassicality, deduction, and compactness. We use $\vdash \alpha$ as an alternative notation for $\alpha \in Cn(\varnothing)$, $\mathbf{H} \vdash \alpha$ for $\alpha \in Cn(\mathbf{H})$, $\alpha \vdash \beta$ for $\beta \in Cn(\{\alpha\})$. $\mathbf{K}_\perp$ denotes the inconsistent belief set. $\mathbf{K}+\alpha$ denotes the expansion of $\mathbf{K}$ by α and is defined by $\mathbf{K}+\alpha = Cn(\mathbf{K} \cup \{\alpha\})$.

The *partial meet AGM contraction function* (See Alchourrón and Makinson [2]; and also [1]) of $\mathbf{K}$ by a sentence α of $\mathcal{L}$ is defined by the following identity:

$$\mathbf{K}-\alpha = \cap\gamma(\mathbf{K}\perp\alpha) \tag{1}$$

where $\mathbf{K}\perp\alpha$ is the remainder set from $\mathbf{K}$ by α, that is, the set of all inclusion-maximal subsets of $\mathbf{K}$ that do not imply α, and γ is a selection function such that $\gamma(\mathbf{K}\perp\alpha)$ is a nonempty subset of $\mathbf{K}\perp\alpha$ unless the latter is empty, in which case $\gamma(\mathbf{K}\perp\alpha) = \mathbf{K}$. A selection function γ, and consequently the contraction operator are transitively relational if and only if γ is based on some transitive relation $\sqsubseteq$ in the sense that $\gamma(\mathbf{K}\perp\alpha) = \{\mathbf{H} \in \mathbf{K}\perp\alpha \mid \mathbf{H}' \sqsubseteq \mathbf{H} \text{ for all } \mathbf{H}' \in \mathbf{K}\perp\alpha\}$. The following lemmas will be useful in the following sections.

Lemma 2.1 ([2]) *Let $\mathbf{K}$ be a belief set. If $\mathbf{H} \subseteq \mathbf{K}$ and $\mathbf{H} \nvdash \alpha$, then there exists some $\mathbf{H}' \in \mathbf{K}\perp\alpha$ such that $\mathbf{H} \subseteq \mathbf{H}'$.*

Lemma 2.2 ([3]) *Let $\mathbf{K}$ be a belief set. If $\alpha \in \mathbf{K}$ and $\nvdash \alpha$, then for all $\mathbf{H}$ in $\mathbf{K}\perp\alpha$, $\mathbf{H}+\neg\alpha$ is a maximal consistent subset of the language.*

Partial meet AGM contraction can be characterized by the following set of postulates [1]:

$(\mathbf{K-1})$	$\mathbf{K}-\alpha$ is a belief set.	(closure)
$(\mathbf{K-2})$	$\mathbf{K}-\alpha \subseteq \mathbf{K}$	(inclusion)
$(\mathbf{K-3})$	if $\alpha \notin \mathbf{K}$, then $\mathbf{K}-\alpha = \mathbf{K}$	(vacuity)
$(\mathbf{K-4})$	if $\nvdash \alpha$, then $\alpha \notin \mathbf{K}-\alpha$	(success)
$(\mathbf{K-5})$	if $\vdash \alpha \leftrightarrow \beta$ then $\mathbf{K}-\alpha = \mathbf{K}-\beta$	(extensionality)
$(\mathbf{K-6})$	$\mathbf{K} \subseteq (\mathbf{K}-\alpha) + \alpha$	(recovery).

Furthermore, $-$ is a *transitively relational partial meet AGM contraction* if and only if it also satisfies:

$(\mathbf{K}-\mathbf{7})$	$\mathbf{K}-\alpha \cap \mathbf{K}-\beta \subseteq \mathbf{K}-(\alpha \wedge \beta)$	(conjunctive overlap)
$(\mathbf{K}-\mathbf{8})$	If $\alpha \notin \mathbf{K}-(\alpha \wedge \beta)$, then $\mathbf{K}-(\alpha \wedge \beta) \subseteq \mathbf{K}-\alpha$	(conjunctive inclusion).

Lemma 2.2 tells us that in the principal case that $\alpha \in \mathbf{K}$ and $\nvdash \alpha$, the elements of $\mathbf{K}\perp\alpha$ are *saturatable*, that is, they become maximal consistent subsets of the language when $\neg\alpha$ is added. In [14], pp. 134, Levi argued that not only do the elements of $\mathbf{K}\perp\alpha$ guarantee minimal loss of informational value but all the saturatable sets do; and that by means of partial meets functions defined for saturatable sets it is possible to capture all possible admissible ways of contracting a belief set $\mathbf{K}$ by a sentence α. According to this argument, he presented an alternative contraction, *partial meet Levi contraction*, based on a selection among all the saturatable subsets of $\mathbf{K}$ with respect to α:

$$\mathbf{K} \sim_{\gamma} \alpha = \cap\gamma(S(\mathbf{K}, \alpha)) \tag{2}$$

where $S(\mathbf{K}, \alpha)$ is the set of all saturatable belief subsets of $\mathbf{K}$ with respect to α, that is, $\mathbf{H} \in S(\mathbf{K}, \alpha)$ if and only if $\mathbf{H} \subseteq \mathbf{K}$, $\mathbf{H} = Cn(\mathbf{H})$, and $\mathbf{H}+\neg\alpha$ is a maximal consistent subset of the language. γ is a selection function defined in the same way as in the AGM account. Hansson and Olsson [13] proved that an operator $-$ on $\mathbf{K}$ is a *partial meet Levi contraction* if and only if it satisfies *closure, inclusion, vacuity, success, extensionality*, and *failure* (if $\vdash \alpha$, then $\mathbf{K}-\alpha = \mathbf{K}$).

2.2 Epistemic entrenchment and severe withdrawal The notion of epistemic entrenchment for theories was introduced in [8][3] by Gärdenfors to define the properties that an order between sentences of the language should satisfy. Gärdenfors proposed the following set of axioms:

(**EE1**)	if $\alpha \leq_{\mathbf{K}} \beta$ and $\beta \leq_{\mathbf{K}} \delta$, then $\alpha \leq_{\mathbf{K}} \delta$	(transitivity)
(**EE2**)	if $\alpha \vdash \beta$, then $\alpha \leq_{\mathbf{K}} \beta$	(dominance)
(**EE3**)	$\alpha \leq_{\mathbf{K}} (\alpha \wedge \beta)$ or $\beta \leq_{\mathbf{K}} (\alpha \wedge \beta)$	(conjunctiveness)
(**EE4**)	if $\mathbf{K} \neq \mathbf{K}_{\perp}$, then $\alpha \notin \mathbf{K}$ iff $\alpha \leq_{\mathbf{K}} \beta$ for all β	(minimality)
(**EE5**)	if $\beta \leq_{\mathbf{K}} \alpha$ for all β, then $\vdash \alpha$	(maximality)

A relation satisfying (**EE1**)–(**EE5**) is a *standard epistemic entrenchment ordering*. Gärdenfors investigated the connections between orders of epistemic entrenchment and contraction functions. The two are connected by the following equivalences, where we write $\alpha <_{\mathbf{K}} \beta$ when $\alpha \leq_{\mathbf{K}} \beta$ and $\beta \nleq_{\mathbf{K}} \alpha$:

$(C \leq)$	$\alpha \leq_{\mathbf{K}} \beta$ if and only if $\alpha \notin \mathbf{K}-(\alpha \wedge \beta)$ or $\vdash (\alpha \wedge \beta)$;
$(-_{\mathrm{G}})$	$\beta \in \mathbf{K}-\alpha$ if and only if $\beta \in \mathbf{K}$ and either $\vdash \alpha$ or $\alpha <_{\mathbf{K}} (\alpha \vee \beta)$.

Gärdenfors and Makinson [9] presented representation theorems linking the AGM postulates and $(-_G)$. Later Rott [21] related $(-_G)$ with *transitively relational partial meet contraction* and pointed out that the comparison $\alpha <_{\mathbf{K}} (\alpha \vee \beta)$ is not intuitive. He proposed an alternative definition, later called *severe withdrawal* [23]:

$(-_{\mathrm{R}})$ $\beta \in \mathbf{K}-\alpha$ if and only if $\beta \in \mathbf{K}$ and either $\vdash \alpha$ or $\alpha <_{\mathbf{K}} \beta$.

Rott proved that severe withdrawal satisfies all the AGM postulates except recovery. This construction was later axiomatized in Fermé [6], Pugnucco [19], Rott [20], and Rott and Pagnucco [23]. Rott [21] proved that for all α, $\mathbf{K} -_{\mathrm{R}} \alpha \subseteq \mathbf{K} -_{\mathrm{G}} \alpha$. Lindström and Rabinowicz [16], pp. 115 suggested that $\mathbf{K} -_{\mathrm{R}} \alpha$ and $\mathbf{K} -_{\mathrm{G}} \alpha$ may be taken as lower and upper limits for a reasonable contraction function. This suggestion was called the *Lindström and Rabinowicz interpolation thesis* [22].

2.3 Construction of semi-contraction Let us consider the following example from [11], deliberately modified to eliminate psychological aspects.

Example 2.3 I previously entertained the two beliefs, 'x is divisible by 2' (α) and 'x is divisible by 6' (β). When I received new information that induced me to give up the first of these beliefs (α), the second (β) had to go as well (since α would otherwise follow from β).

I then received new information that made me accept the belief 'x is divisible by 8' (ϵ). Since α follows from ϵ, $(\mathbf{K}-\alpha) + \alpha$ is a subset of $(\mathbf{K}-\alpha) + \epsilon$, so by recovery I obtain that 'x is divisible by 24' (δ), contrary to intuition.

In the above example we show that retaining the sentence $\mu = \alpha \to \beta$ in the contraction of $\mathbf{K}$ by α provokes unintuitive results. Therefore μ must be removed in the process of contraction by α. Due to recovery, AGM contraction does not eliminate μ.

However, not all the '$\alpha \to \beta$' sentences are undesirable. Makinson ([19], p. 478) noted that "as soon as contraction makes use of the notion *y is believed only because of x*, we run into counterexamples to recovery . . . but when a theory is taken as *naked*, that is, as a bare set $A = Cn(A)$ of statements closed under consequence, then recovery appears to be free of intuitive counterexamples." He also noted that "a theory may be *clothed* with additional structure without damaging recovery, if that structure is read as expressing something different from grounding or justification."

In our model, to determine which '$\alpha \to \beta$' sentences must be discarded, we need to "clothe" the theory with a justificatory structure that allows us to determine the justificational dependence among the sentences of the belief set. Semi-contraction does just this, through the combined use of a unique AGM contraction and a selection function Sel.

Definition 2.4 Let $\mathbf{A}$ be a set of sentences. A semi-selection function for $\mathbf{A}$ is a function Sel such that

(1) if $\mathbf{A}$ is nonempty, then $Sel(\mathbf{A}) \in \mathbf{A}$,
(2) If $\mathbf{A}$ is empty, then $Sel(\mathbf{A}) = \top$.

Definition 2.5 ([4]) Let $\mathcal{L}$ be the set of all the sentences of the language and $\mathcal{K}$ the set of all theories in $\mathcal{L}$. Let Sel be a semi-selection function as defined in Definition 2.4. A function $\bar{-}_{s} : \mathcal{K} \times \mathcal{L} \to \mathcal{L}$ is a semi-contraction function if and only if there is a *partial meet AGM contraction function* such that for all $\mathbf{K} \in \mathcal{K}$ and $\alpha \in \mathcal{L}$

$$\mathbf{K} \bar{-}_{s} \alpha = (\mathbf{K}-\alpha) \cap (\mathbf{K}-(\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}-\alpha))). \tag{3}$$

Sel selects an element of $(\mathbf{K} \setminus \mathbf{K}-\alpha)$; this is equivalent to selecting some finite subset of $(\mathbf{K} \setminus \mathbf{K}-\alpha)$, as we see in the following property.

Property 2.6 If $\beta_1 \in \mathbf{K} \setminus \mathbf{K}-\alpha$ and $\beta_2 \in \mathbf{K} \setminus \mathbf{K}-\alpha$, then $\beta_1 \wedge \beta_2 \in \mathbf{K} \setminus \mathbf{K}-\alpha$.

Sel is a selection function that depends on the original belief set **K** and the sentence α (in the sense that it is used over the set $\mathbf{K} \setminus \mathbf{K}-\alpha$). This function provides the theory **K** with an additional apparatus to determine the dependencies among the sentences of the belief set. In our example $\alpha \to \beta$ is believed "just because" β is believed, consequently, Sel must select β to discard $\alpha \to \beta$ in the contraction.

One interesting point is the relationship between the semi-contraction and recovery.

Definition 2.7 ([5]) Let **K** be a belief set, $-$ a contraction function for **K** and α a sentence. $-$ satisfies *α-recovery* if and only if $\mathbf{K} \subseteq (\mathbf{K}-\alpha)+\alpha$.

Observation 2.8 ([4], [5]) Every *semi-contraction function* satisfies *α-recovery* if and only if $\vdash \alpha \to Sel(\mathbf{K} \setminus \mathbf{K}-\alpha)$.

As we have seen in the last observation, semi-contraction allows us to define a contraction function that

1. does not satisfy recovery for all the sentences of the language or
2. always satisfies recovery or
3. satisfies recovery only for specific sentences of the language.

3 Axioms for a sensible withdrawal function Example 2.3 showed that in AGM contractions the recovery postulate can give rise to unintuitive results. Our purpose is to define axioms for a sensible withdrawal function that preserves the principle of minimal loss of information but removes the sentences that provoke these nonintuitive results. In this context closure, inclusion, vacuity, success, and extensionality must hold.

However, finding counterexamples of recovery does not mean that recovery must be eliminated completely. There are many cases where recovery is a desired property. We must find a new postulate that preserves recovery in certain cases but allows us to eliminate the '$\alpha \to \beta$' sentences that provoke unintuitive results. In the last case, we also want to retain the possibility of recovering the original belief set.

If when contracting by α we eliminate sentences of the form $\alpha \to \beta$, we cannot recover the original set of sentences by simply adding. To re-obtain the whole original set of beliefs we must reintroduce not only α but also all the $\alpha \to \beta$ sentences lost in the contraction, that is, this should happen when adding: $\alpha \wedge (\alpha \to \beta_1) \wedge \cdots (\alpha \to \beta_n)$, which is equivalent to: $\alpha \wedge \beta_1 \wedge \cdots \beta_n$. Consequently, we delegate the task of recovering the whole set to a sentence $\beta = \alpha \wedge \beta_1 \wedge \cdots \beta_n$. We formalize this idea in the following postulate.

Proxy Recovery If $\mathbf{K} \neq \mathbf{K}-\alpha$ then there exists some $\beta \in \mathbf{K}$ such that $\mathbf{K}-\alpha \nvdash \beta$ and $\mathbf{K} \subseteq (\mathbf{K}-\alpha)+\beta$.

Proxy recovery is a weaker version of recovery. When recovery is satisfied, proxy recovery holds taking $\beta = \alpha$. As has been pointed out to us by the referees, the converse of the last formula of this postulate follows from inclusion.

In the limiting case in which the sentence to be removed is a tautology (which is impossible to remove) recovery and inclusion guarantee that the result of this contraction is the original belief set $\mathbf{K}$. If we reject recovery we must explicitly add this intuitive condition.

Failure [7] If $\vdash \alpha$, then $\mathbf{K}-\alpha = \mathbf{K}$.

Definition 3.1 Let $\mathbf{K}$ be a belief. An operator $-$ on $\mathbf{K}$ is a *sensible withdrawal* function if and only if it satisfies closure, inclusion, vacuity, success, extensionality, failure, and proxy recovery.

Note that when the language is finite, every withdrawal function satisfies proxy recovery, and then all Levi contractions are semi-contractions and conversely (just let $\beta : Cn(\beta) = \mathbf{K}$). The intuitions that guide the axioms for sensible withdrawals are the same as those that inspire semi-contraction, as we can see in the following lemma, part of which was already proven in [4].

Lemma 3.2 *Every semi-contraction function defined as in Definition 2.5 satisfies closure, inclusion, vacuity, success, extensionality, failure, and proxy recovery.*

This lemma and the axiomatic characterization of Levi contraction [13] imply that semi-contraction is a special case of withdrawal; more general than AGM contraction but less general than Levi contraction. It can be stated formally as follows.

Observation 3.3

1. Every *semi-contraction function* defined as in Definition 2.5 is a *partial meet Levi contraction function*.
2. Every *partial meet AGM contraction function* is a *semi-contraction function* defined as in Definition 2.5.

4 Semi-saturatable contraction We have shown that semi-contraction functions are situated between Levi and AGM contractions. In this section our purpose is to find an alternative construction in terms of the remainder sets and Levi's saturatable sets. Since semi-contraction is equivalent to the intersection of the same AGM contraction applied to α and $\alpha \to \beta$, respectively, an obvious approach is

$$\mathbf{K} \overline{\,_s} \alpha = \cap\gamma(\mathbf{K}\bot\alpha) \bigcap \cap\gamma(\mathbf{K}\bot(\alpha \to \beta)). \tag{4}$$

Since in semi-contraction $\beta \in \mathbf{K} \setminus \mathbf{K}-\alpha$, we also need to add the constraint that $\exists \mathbf{H} \in \cap\gamma(\mathbf{K}\bot\alpha)$: $\beta \notin \mathbf{H}$. This constraint and the use of two different remainder sets encourage us to find a simple selection function over a unique set.

Since the semi-contractions are withdrawals, $S(\mathbf{K}, \alpha)$ appears as a candidate, but again, the selection function must be constrained to select at least one $\mathbf{H}$ such that $\beta \notin \mathbf{H}$. This condition is given by the set $S(\mathbf{K}, (\alpha \vee \beta))$. However, there remains the constraint that we want to recover the whole set $\mathbf{H}$ by adding $\alpha \wedge \beta$. Consequently we add this constraint and define the *semi-saturatable* sets for α and β as subsets of $S(\mathbf{K}, (\alpha \vee \beta))$ as follows.

Definition 4.1 Let $\mathbf{K}$ be a belief set and α, β sentences. Then the *semi-saturatable* set $SS(\mathbf{K}, \alpha, \beta)$ is the set such that $\mathbf{H} \in SS(\mathbf{K}, \alpha, \beta)$ if and only if

$$\begin{cases} \mathbf{H} \subseteq \mathbf{K}; \\ \mathbf{H} = Cn(\mathbf{H}); \\ \mathbf{H}+(\neg\alpha \wedge \neg\beta) \text{ is a maximal consistent subset of the language;} \\ \mathbf{K} \subseteq \mathbf{H}+(\alpha \wedge \beta). \end{cases}$$

The following observations formalize the relationship between the elements of $SS(\mathbf{K}, \alpha, \beta)$ and $S(\mathbf{K}, \alpha \vee \beta)$ and also relate them to $\mathbf{K} \perp (\alpha \vee \beta)$.

Observation 4.2 If $\alpha \vee \beta \in \mathbf{K}$, then $\mathbf{K}\perp(\alpha \vee \beta) \subseteq SS(\mathbf{K}, \alpha, \beta)$.

Observation 4.3 $SS(\mathbf{K}, \alpha, \beta) \subseteq S(\mathbf{K}, \alpha \vee \beta)$.

Similarly to the construction of partial meet AGM and Levi contraction, we now build contraction functions by means of a selection function over the semi-saturatable set $SS(\mathbf{K}, \alpha, \beta)$.

Definition 4.4 Let $\mathbf{K}$ be a belief set. A *selection function* for $\mathbf{K}$ is a function γ such that for all sentences α

1. if $SS(\mathbf{K}, \alpha, \beta)$ is nonempty, then $\gamma(SS(\mathbf{K}, \alpha, \beta))$ is a nonempty subset of $SS(\mathbf{K}, \alpha, \beta)$;
2. if $SS(\mathbf{K}, \alpha, \beta)$ is empty, then $\gamma(SS(\mathbf{K}, \alpha, \beta)) = \mathbf{K}$.

Definition 4.5 Let $\mathbf{K}$ be a belief set. An operation $\overline{}_{\overline{s}}$ on $\mathbf{K}$ is a *semi-saturatable contraction* if and only if there is a selection function γ for $\mathbf{K}$ defined as in Definition 4.4, such that for all sentences α: $\mathbf{K}\overline{}_{\overline{s}}\alpha = \cap\gamma(SS(\mathbf{K}, \alpha, \beta))$, where $\beta = f(\mathbf{K}, \alpha)$ for a function $f : \mathbf{K} \times \mathcal{L} \to \mathcal{L}$.

Clearly, the role of f is the same as the role of Sel in semi-contraction, that is, $Sel(\mathbf{K} \setminus \mathbf{K}-\alpha) = f(\mathbf{K}, \alpha)$. The next lemma shows the relationship between *semi-saturatable contraction* and semi-contraction.

Lemma 4.6 *Let* $\mathbf{K}$ *be a belief set and* $\sim$ *a semi-saturatable contraction function for* $\mathbf{K}$. *Then* $\sim$ *is a semi-contraction function, that is, there exists a partial meet AGM contraction function* $-$ *such that* $\mathbf{K}{\sim}\alpha = \mathbf{K}-\alpha \cap \mathbf{K}-(\alpha \to \beta)$, $\beta \in \mathbf{K} \setminus \mathbf{K}-\alpha$.

Finally, we relate the axioms for a *sensible withdrawal* with the construction by means of semi-saturatable sets.

Lemma 4.7 *Let* $\mathbf{K}$ *be a belief set and* $\sim$ *a sensible withdrawal for* $\mathbf{K}$. *Then there is a selection function* γ *on* $\mathbf{K}$ *such that* $\mathbf{K}{\sim}\alpha = \cap\gamma(SS(\mathbf{K}, \alpha, \beta))$, *where* $\beta = f(\mathbf{K}, \alpha)$ *for a function* $f : \mathbf{K} \times \mathcal{L} \to \mathcal{L}$.

5 Characterizations of semi-contraction Based on Lemmas 3.2, 4.6, and 4.7 we can characterize semi-contraction functions as follows.

Theorem 5.1 *Let* $\mathbf{K}-$ *be a belief set and operator on* $\mathbf{K}-$. *Then the following conditions are equivalent:*

1. $\overline{}_{\overline{s}}$ *is a semi-contraction function as defined in Definition 2.5, that is, there is a partial meet AGM contraction function* $-$ *and a semi-selection function* Sel *such that for all* α ,$\mathbf{K}\overline{}_{\overline{s}}\alpha = \mathbf{K}-\alpha \cap \mathbf{K}-(\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}-\alpha))$.

2. *$\overset{-}{_S}$ is a sensible withdrawal as defined in Definition 3.1, that is, it satisfies closure, inclusion, vacuity, success, extensionality, failure, and proxy recovery.*
3. *$\overset{-}{_S}$ is a semi-saturatable contraction function as defined in Definition 4.1, that is, there is a selection function γ on $\mathbf{K}$ such that $\mathbf{K}\overset{-}{_S}\alpha = \cap\gamma(SS(\mathbf{K}, \alpha, \beta))$, where $\beta = f(\mathbf{K}, \alpha)$ for a function $f : \mathbf{K} \times \mathcal{L} \to \mathcal{L}$.*

6 Epistemic entrenchment for semi-contraction In Subsection 2.2 we recalled the relations between *transitively relational partial meet AGM contraction* function and epistemic entrenchment. Since semi-contraction is defined using a unique *partial meet AGM contraction*, if the latter is transitively relational then it is easy to construct a semi-contraction function based on an epistemic entrenchment relation and $(C \leq)$.

For the first contraction, $\mathbf{K}-\alpha$, the condition is the same as $(-_G)$. For the second contraction, $\mathbf{K}-(\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}-\alpha))$, we use $(-_G)$ again, using $(\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}-\alpha))$ instead of α; that is, $\beta \in \mathbf{K}-(\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}-\alpha))$ if and only if $\beta \in \mathbf{K}$ and, either $\vdash (\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}-\alpha))$ or $(\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}-\alpha)) <_{\mathbf{K}} ((\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}-\alpha)) \vee \beta)$. The next step is to define $(\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}-\alpha)$ in terms of an entrenchment ordering: $\mathbf{K} \setminus \mathbf{K}-\alpha = \{\epsilon \mid \epsilon \in \mathbf{K}$ and $\nvdash \alpha$ and $(\alpha \vee \epsilon) \leq_{\mathbf{K}} \alpha\}$.

We combine all the above conditions and obtain the following definition.

$(-_S)$ $\beta \in \mathbf{K}\overset{-}{_S}\alpha$ if and only if $\beta \in \mathbf{K}$ and either $\vdash \alpha$ or $\alpha <_{\mathbf{K}} (\alpha \vee \beta)$ and either $\vdash (\alpha \to Sel(\mathbf{H}))$ or $(\alpha \to Sel(\mathbf{H})) <_{\mathbf{K}} ((\alpha \to Sel(\mathbf{H})) \vee \beta)$, where $\mathbf{H} = \{\epsilon \mid \epsilon \in \mathbf{K}$ and $\nvdash \alpha$ and $(\alpha \vee \epsilon) \leq_{\mathbf{K}} \alpha\}$.

Due to the construction of $(-_S)$, we can relate this to semi-contraction.

Observation 6.1 Let $\leq_{\mathbf{K}}$ be a standard epistemic entrenchment ordering on a consistent belief set $\mathbf{K}$. Furthermore, let $\overset{-}{_S}$ be an entrenchment-contraction on $\mathbf{K}$ based on $\leq_{\mathbf{K}}$ defined via condition $(-_S)$. Then $\overset{-}{_S}$ is a semi-contraction function and $(C \leq)$ also holds.

Observation 6.2 Let $\sim$ be a semi-contraction function on the consistent belief set $\mathbf{K}$ and $-$ its associate *partial meet AGM contraction* such that $-$ is also transitively relational. Furthermore, let $\leq_{\mathbf{K}}$ be the relation that is derived from $-$ through $(C \leq)$. Then $\leq_{\mathbf{K}}$ satisfies the standard entrenchment postulates and $(-_S)$ also holds.

7 Construction of interpolated semi-contraction We saw in Section 2.2 that according to the *Lindström and Rabinowicz interpolation thesis*, a reasonable contraction function must be situated between *partial meet AGM contraction* and *severe withdrawal*. We show in this section what additional restrictions on $\overset{-}{_S}$ are needed to obtain an interpolated semi-contraction function; that is, such that for all α, $\mathbf{K} -_R \alpha \subseteq \mathbf{K}\overset{-}{_S}\alpha \subseteq \mathbf{K} -_G \alpha$.

We will introduce the basic ideas informally. We will assume an epistemic entrenchment ordering $\leq_{\mathbf{K}}$ for $\mathbf{K}$ and the partial meet AGM contraction and severe withdrawal $-_G$ and $-_R$ based on $\leq_{\mathbf{K}}$. $\overset{-}{_S}$ is the semi-contraction based on $-_G$, and Sel its associated selection function.

It is trivial that $\mathbf{K}\overset{-}{_S}\alpha \subseteq \mathbf{K} -_G \alpha$. For the other condition, $\mathbf{K} -_R \alpha \subseteq \mathbf{K}\overset{-}{_S}\alpha$, we must show $\mathbf{K} -_R \alpha \subseteq \mathbf{K} -_G \alpha \cap \mathbf{K} -_G (\alpha \to \beta)$ for $\beta = Sel(\mathbf{K} \setminus \mathbf{K} -_G \alpha)$. $\mathbf{K} -_R \alpha \subseteq$

$\mathbf{K} -_{\mathrm{G}} \alpha$ so we only have to prove that $\mathbf{K} -_{\mathrm{R}} \alpha \subseteq \mathbf{K} -_{\mathrm{G}} (\alpha \rightarrow \beta)$. This condition holds if $\vdash \alpha \rightarrow \beta$ or $\alpha \rightarrow \beta \notin \mathbf{K} -_{\mathrm{R}} \alpha$. When $\nvdash \alpha \rightarrow \beta$, then $\alpha \rightarrow \beta \notin \mathbf{K} -_{\mathrm{R}} \alpha$ if and only if $\alpha \rightarrow \beta \leq_{\mathbf{K}} \alpha$. By means of $(C \leq)$, we write it as follows: $\alpha \rightarrow \beta \notin \mathbf{K} -_{\mathrm{G}} ((\alpha \rightarrow \beta) \wedge \alpha)$, or equivalently $\alpha \rightarrow \beta \notin \mathbf{K} -_{\mathrm{G}} (\alpha \wedge \beta)$.

We can formalize the above explanation in the following theorem.

Theorem 7.1 *Let $\mathbf{K}$ be a belief set, $\leq_{\mathbf{K}}$ an epistemic entrenchment ordering for $\mathbf{K}$, $-_{\mathrm{R}}$ the severe withdrawal, and $-_{\mathrm{G}}$ the partial meet AGM contraction function associated with the epistemic entrenchment ordering $\leq_{\mathbf{K}}$. Let $\overline{-}_{S}$ be the associated semi-contraction of $-_{\mathrm{G}}$, and Sel its selection function. If $\beta = Sel(\mathbf{K} \setminus \mathbf{K} -_{\mathrm{G}} \alpha)$ satisfies $\alpha \rightarrow \beta \notin \mathbf{K} -_{\mathrm{G}} (\alpha \wedge \beta)$, then $\mathbf{K} -_{\mathrm{R}} \alpha \subseteq \mathbf{K} \overline{-}_{S} \alpha \subseteq \mathbf{K} -_{\mathrm{G}} \alpha$ for all α.*

The converse of this theorem is not true, since there are contraction functions that satisfy the *interpolation thesis* but they are not semi-contractions. An example can be found in the Appendix.

Appendix Proofs

A Lemmas

The following lemmas will be used in the demonstrations.

Lemma A.1 *Let $\mathbf{K}$ be a belief set and $-$ an operator on $\mathbf{K}$ that satisfies success, vacuity and failure. Then $-$ satisfies proxy recovery if and only if it satisfies*

Weak Recovery *If $\mathbf{K} \neq \mathbf{K}-\alpha$ then there exists some β such that $\mathbf{K} \vdash \beta$, $\mathbf{K}-\alpha \nvdash (\alpha \vee \beta)$ but $\mathbf{K} \subseteq (\mathbf{K}-\alpha) + (\alpha \wedge \beta)$.*

Proof of Lemma A.1: Weak recovery to proxy recovery is trivial. For the converse, let δ be a sentence that satisfies the proxy recovery conditions and let $\beta = \alpha \wedge \delta$. It is trivial to prove that β satisfies weak recovery. □

Lemma A.2 *Let $\mathbf{K}$ be a belief set. Let $\alpha \in \mathbf{K}$, and $\nvdash \alpha$. Then $\mathbf{K}\perp(\alpha \vee \beta) \subseteq \mathbf{K}\perp\alpha$.*

Proof of Lemma A.2: If $\vdash \beta$, then the proof is trivial. For the principal case, let $\nvdash \beta$ and $\mathbf{H} \in \mathbf{K}\perp(\alpha \vee \beta)$. Then $\mathbf{H} = Cn(\mathbf{H})$ and $\mathbf{H} \nvdash \alpha$. We must prove that $\mathbf{H}$ is a maximal subset of $\mathbf{K}$ that does not imply α.

Let $\mathbf{H}'$ be such that $\mathbf{H} \subset \mathbf{H}' \subseteq \mathbf{K}$. Then there exists some $\delta \in \mathbf{H}'$ such that $\delta \notin \mathbf{H}$. Since $\mathbf{H} \in \mathbf{K}\perp(\alpha \vee \beta)$, $\delta \rightarrow (\alpha \vee \beta) \in \mathbf{H}$ and $(\alpha \vee \beta) \rightarrow \alpha \in \mathbf{H}$. Thus $\delta \rightarrow \alpha \in \mathbf{H}$, so that $\mathbf{H}' \vdash \alpha$. Hence $\mathbf{H} \in \mathbf{K}\perp\alpha$. □

Lemma A.3 *Let $\mathbf{B}$ be a belief set. If $\mathbf{B} \in SS(\mathbf{K}, \alpha, \beta)$, then there is exactly one belief set $\mathbf{H}$ such that $\mathbf{B} = \mathbf{H} \cap \Delta^{\sim} \cap \Pi^{\sim}$ where:*

$$\mathbf{H} \in \mathbf{K} \perp (\alpha \vee \beta)$$

$$\Delta = \{\mathbf{I} \in \mathbf{K} \perp (\alpha \vee \neg\beta) \mid \mathbf{B} \subseteq \mathbf{I}\}$$

$$\Pi = \{\mathbf{J} \in \mathbf{K} \perp (\neg\alpha \vee \beta) \mid \mathbf{B} \subseteq \mathbf{J}\}$$

$$\Delta^{\sim} = \begin{cases} \cap\Delta \ \textit{if}\ \Delta \neq \varnothing \\ \mathbf{B} \text{ otherwise} \end{cases}$$

$$\Pi^{\sim} = \begin{cases} \cap\Pi \ \textit{if}\ \Pi \neq \varnothing \\ \mathbf{B} \text{ otherwise} \end{cases}$$

Proof of Lemma A.3: We must prove (1) that $\mathbf{H}$ exists, (2) that $\mathbf{H}$ is unique, and finally (3) that $\mathbf{B} = \mathbf{H} \cap \Delta^{\sim} \cap \Pi^{\sim}$.

Case 1: By definition of SS $\mathbf{B} \subseteq \mathbf{K}$ and $(\alpha \vee \beta) \notin \mathbf{B}$. Then by Lemma 2.1 there is some $\mathbf{H}$ such that $\mathbf{H} \in \mathbf{K}\bot(\alpha \vee \beta)$.

Case 2: To prove that $\mathbf{H}$ is unique suppose for *reductio ad absurdum* that there is $\mathbf{H}'$ such that $\mathbf{H}' \neq \mathbf{H}$, $\mathbf{H}' \in \mathbf{K}\bot(\alpha \vee \beta)$. Since $\mathbf{H}' \neq \mathbf{H}$ and both are maximal subsets of $\mathbf{K}$ failing to imply $(\alpha \vee \beta)$, then there is some $\delta \in \mathbf{H}'$ such that $\delta \notin \mathbf{H}$. We have two subcases.

Subcase 1: $\mathbf{H}+(\neg\alpha \wedge \neg\beta) = \mathbf{H}'+(\neg\alpha \wedge \neg\beta)$, then by the *Cn* deduction theorem, $(\neg\alpha \wedge \neg\beta) \to \delta \in \mathbf{H}$ that is, $(\alpha \vee \beta \vee \delta) \in \mathbf{H}$ and since $(\neg\delta \vee \alpha \vee \beta) \in \mathbf{H}$ then $(\alpha \vee \beta) \in \mathbf{H}$. Contradiction.

Subcase 2: $\mathbf{H}+(\neg\alpha \wedge \neg\beta) \neq \mathbf{H}'+(\neg\alpha \wedge \neg\beta)$ then $\mathbf{B}+(\neg\alpha \wedge \neg\beta) \subseteq \mathbf{H}+(\neg\alpha \wedge \neg\beta) \cap \mathbf{H}'+(\neg\alpha \wedge \neg\beta)$, hence $\mathbf{B}+(\neg\alpha \wedge \neg\beta)$ is not a maximal subset, contrary to $\mathbf{B} \in SS(\mathbf{K}, \alpha, \beta)$. Contradiction.

Case 3: It is trivial that $\mathbf{B} \subseteq \mathbf{H} \cap \Delta^{\sim} \cap \Pi^{\sim}$. For the other inclusion suppose that $\mathbf{H} \cap \Delta^{\sim} \cap \Pi^{\sim} \nsubseteq \mathbf{B}$. Then there exists some $\delta \in \mathbf{H} \cap \Delta^{\sim} \cap \Pi^{\sim}$ such that $\delta \notin \mathbf{B}$. Since $\delta \notin \mathbf{B}$, then (by Lemma 2.1) there is some $\mathbf{H}'$ such that $\mathbf{H}' \in \mathbf{K}\bot(\alpha \vee \beta \vee \delta)$, and $\mathbf{B} \subseteq \mathbf{H}'$. By Lemma A.2 $\mathbf{H}' \in \mathbf{K}\bot(\alpha \vee \beta)$, then by part **(b)** $\mathbf{H} = \mathbf{H}'$, which is absurd since $\delta \in \mathbf{H}$ and $\delta \notin \mathbf{H}'$. □

A.1 Proofs

Proof of Lemma 3.2: Closure, inclusion, vacuity, success, extensionality, and failure are proved in [4].

In order to prove proxy recovery, let $\mathbf{K}$ be a belief set, $\overline{-}_{s}$ a semi-contraction function for $\mathbf{K}$; $-$ its associated *partial meet AGM contraction function* and β such that $\mathbf{K}\overline{-}_{s}\alpha = \mathbf{K}-\alpha \cap \mathbf{K}-(\alpha \to \beta)$, $\beta \in Sel(\mathbf{K} \setminus \mathbf{K}-\alpha)$. Let $\mathbf{K} \neq \mathbf{K}\overline{-}_{s}\alpha$ and $\delta = \alpha \wedge \beta$. Since $\mathbf{K} \neq \mathbf{K}\overline{-}_{s}\alpha$ it follows that $\alpha \in \mathbf{K}$ and $\beta \in \mathbf{K}$, from which it follows that $\delta \in \mathbf{K}$. We need to show (a) that $\delta \notin \mathbf{K}-\alpha$ and (b) that $\mathbf{K} \subseteq (\mathbf{K}\overline{-}_{s}\alpha) + \delta$.

(a) It follows by the definition of semi-contraction that $\mathbf{K} \neq \mathbf{K}-\alpha$ and that $\mathbf{K} \setminus \mathbf{K}-\alpha \neq \varnothing$; then $\beta \in \mathbf{K} \setminus \mathbf{K}-\alpha$, hence $\delta \in \mathbf{K} \setminus \mathbf{K}-\alpha$.

(b) $(\mathbf{K}-\alpha \cap \mathbf{K}-(\alpha \to \beta)) + (\alpha \wedge \beta)$
$= (\mathbf{K}-\alpha) + (\alpha \wedge \beta) \cap (\mathbf{K}-(\alpha \to \beta)) + (\alpha \wedge \beta)$
$= (\mathbf{K}-\alpha) + (\alpha \wedge \beta) \cap (\mathbf{K}-(\alpha \to \beta)) + ((\alpha \to \beta) \wedge \alpha)$
$= ((\mathbf{K}-\alpha) + \alpha) + \beta \cap ((\mathbf{K}-(\alpha \to \beta)) + (\alpha \to \beta)) + \alpha$
$=$ (by recovery and *inclusion*) $\mathbf{K}+\beta \cap \mathbf{K}+\alpha$
$= \mathbf{K}$ (since α and β are in $\mathbf{K}$). □

Proof of Observation 4.2: Let $(\alpha \vee \beta) \in \mathbf{K}$. If $\vdash \alpha \vee \beta$, then $\mathbf{K}\bot(\alpha \vee \beta) = \varnothing$ and we are finished. For $\nvdash \alpha \vee \beta$, let $\mathbf{H} \in \mathbf{K}\bot(\alpha \vee \beta)$. To prove that $\mathbf{H}$ is in $SS(\mathbf{K}, \alpha, \beta)$ we need to prove

(a) that $\mathbf{H} \subseteq \mathbf{K}$ and $\mathbf{H} = Cn(\mathbf{H})$: this follows trivially from the definition of $\mathbf{K}\bot\alpha \vee \beta$;

(b) that $\mathbf{H}+(\neg\alpha \wedge \neg\beta)$ is a maximal consistent subset of the language: this follows from Lemma 2.2, since $(\alpha \vee \beta) \in \mathbf{K}$ and $(\alpha \vee \beta) \notin \mathbf{H}$; and finally

(c) that $\mathbf{K} \subseteq \mathbf{H}+(\alpha \wedge \beta)$ which follows from $\mathbf{K} \subseteq \mathbf{H}+(\alpha \vee \beta)^4$ and $\mathbf{H}+(\alpha \vee \beta) \subseteq \mathbf{H}+(\alpha \wedge \beta)$. □

Proof of Observation 4.3: The demonstration is trivial, since the conditions for $S(\mathbf{K}, (\alpha \vee \beta))$ are the first three conditions for $SS(\mathbf{K}, \alpha, \beta)$. □

Proof of Lemma 4.6: In Lemma A.3, we show that for all $\mathbf{B}_i \in \gamma(SS(\mathbf{K}, \alpha, \beta))$, $\mathbf{B}_i$ can be expressed as $\mathbf{B}_i = \mathbf{H}_i \cap \Delta_i^{\sim} \cap \Pi_i^{\sim}$; then $\bigcap \gamma(SS(\mathbf{K}, \alpha, \beta)) = \bigcap_i \mathbf{H}_i \cap \Delta_i^{\sim} \cap \Pi_i^{\sim}$; where each $\mathbf{H}_i \cap \Delta_i^{\sim} \subseteq \mathbf{K}\bot\alpha$ and each $\Pi_i^{\sim} \subseteq \mathbf{K}\bot\neg\alpha \vee \beta$.

We can construct a *partial meet AGM contraction function* using a selection function that takes the elements of $\mathbf{H}_i \cap \Delta_i^{\sim}$ to construct $\mathbf{K}-\alpha$ and $\Pi_i^{\sim}$ to construct $\mathbf{K}-(\neg\alpha \vee \beta)$. Let γ_2 be an arbitrary selection function and γ_1 a selection function such that

$$\gamma_1(\mathbf{W}) = \begin{cases} \{\mathbf{M} \mid \mathbf{M} = \mathbf{H} \text{ or } \mathbf{M} \in \Delta_i^{\sim}\} & \text{if } \mathbf{W} = \mathbf{K}\bot\alpha \\ \{\mathbf{M} \mid \mathbf{M} \in \Pi_i^{\sim}\} & \text{if } \mathbf{W} = \mathbf{K}\bot(\neg\alpha \vee \beta) \\ \gamma_2(\mathbf{W}) & \text{otherwise.} \end{cases}$$

Clearly $\cap\gamma_1$ is a *partial meet AGM contraction* and it follows that $\bigcap \gamma(SS(\mathbf{K}, \alpha, \beta)) = \bigcap \gamma_1(\mathbf{K}\bot\alpha) \cap \bigcap \gamma_1(\mathbf{K}\bot(\neg\alpha \vee \beta))$. That concludes the proof. □

Proof of Lemma 4.7: If $\vdash \alpha$ or $\alpha \notin \mathbf{K}$, then it is trivial. Let $\nvdash \alpha$ and $\alpha \in \mathbf{K}$. Due to proxy recovery and Lemma A.1 there exists some β such that $\beta \in \mathbf{K}$ and $(\alpha \vee \beta) \notin \mathbf{K}{\sim}\alpha$. By inclusion, $\mathbf{K} = \mathbf{K}{\sim}\alpha + (\alpha \wedge \beta)$.

$$\begin{aligned} \text{Let } \Upsilon &= \{\mathbf{U} \in \mathbf{K}\bot(\alpha \vee \beta) \mid \mathbf{K}{\sim}\alpha \subseteq \mathbf{U}\} \\ \text{Let } \Delta &= \{\mathbf{I} \in \mathbf{K}\bot(\alpha \vee \neg\beta) \mid \mathbf{K}{\sim}\alpha \subseteq \mathbf{I}\} \\ \text{Let } \Pi &= \{\mathbf{J} \in \mathbf{K}\bot(\neg\alpha \vee \beta) \mid \mathbf{K}{\sim}\alpha \subseteq \mathbf{J}\} \\ \text{Let } \Delta^{\sim} &= \begin{cases} \cap\Delta \text{ if } \Delta \neq \varnothing \\ \mathbf{K}{\sim}\alpha \text{ otherwise} \end{cases} \\ \text{Let } \Pi^{\sim} &= \begin{cases} \cap\Pi \text{ if } \Pi \neq \varnothing \\ \mathbf{K}{\sim}\alpha \text{ otherwise} \end{cases} \end{aligned}$$

We must prove (a) that $\Upsilon \neq \varnothing$ and (b) that $\mathbf{K}{\sim}\alpha = \cap\mathbf{M}$, where $\mathbf{M} = \{\mathbf{M_i} : \mathbf{M}_i \in SS(\mathbf{K}, \alpha, \beta)\}$.

(a) $(\alpha \vee \beta) \notin \mathbf{K}{\sim}\alpha$ and by inclusion $\mathbf{K}{\sim}\alpha \subseteq \mathbf{K}$, then by Lemma 2.1 there exists some $\mathbf{U}$ such that $\mathbf{K}{\sim}\alpha \subseteq \mathbf{U}$ and $\mathbf{U} \in \mathbf{K}\bot(\alpha \vee \beta)$.

(b) Let $\mathbf{M_i} = \mathbf{U_i} \cap \Delta^{\sim} \cap \Pi^{\sim}, \mathbf{U_i} \in \Upsilon$. It follows trivially that $\mathbf{M}_i = Cn(\mathbf{M_i})$, $\mathbf{M_i} \subseteq \mathbf{K}$ and $\mathbf{K} \subseteq \mathbf{M_i}+(\alpha \wedge \beta)$. $\mathbf{M_i}+(\neg\alpha \wedge \neg\beta) = \mathbf{U_i}+(\neg\alpha \wedge \neg\beta) \cap \Delta^{\sim}+(\neg\alpha \wedge \neg\beta) \cap \Pi^{\sim}+(\neg\alpha \wedge \neg\beta)$. Since $\Delta^{\sim}$ and $\Pi^{\sim}$ both satisfy recovery, $\Delta^{\sim}+(\neg\alpha \wedge \neg\beta) = \mathbf{K}_\bot$ and $\Pi^{\sim}+(\neg\alpha \wedge \neg\beta) = \mathbf{K}_\bot$, then $\mathbf{M_i}+(\neg\alpha \wedge \neg\beta) = \mathbf{U_i}+(\neg\alpha \wedge \neg\beta)$ that is a maximal consistent subset of the language. Hence $\mathbf{M}_i \in SS(\mathbf{K}, \alpha, \beta)$.

Finally, we must prove that $\mathbf{K}{\sim}\alpha = \cap\mathbf{M}$, where $\mathbf{M} = \{\mathbf{M_i} \mid \mathbf{M_i} = \mathbf{U_i} \cap \Delta^{\sim} \cap \Pi^{\sim}, \mathbf{U_i} \in \Upsilon\}$. It follows trivially that $\mathbf{K}{\sim}\alpha \subseteq \cap\mathbf{M}$. To prove that $\cap\mathbf{M} \subseteq \mathbf{K}{\sim}\alpha$ let $\delta \in \cap\mathbf{M}$, $\delta \notin$

$\mathbf{K}\sim\alpha$, then $\delta \in \mathbf{M_i}$, $\forall \mathbf{M_i} \in \Upsilon$. Since $\delta \notin \mathbf{K}\sim\alpha$, by Lemma 2.1 $\exists \mathbf{H} \in \mathbf{K}\bot(\alpha \vee \beta \vee \delta)$. By Lemma A.2, $\mathbf{H} \in \mathbf{K}\bot(\alpha \vee \beta)$, so that $\mathbf{K}\sim\alpha \subseteq \mathbf{H}$, and consequently $\mathbf{H} \in \Upsilon$, then $\delta \notin \cap\mathbf{M_i}$ and $\delta \notin \cap\mathbf{M}$. Absurd. □

Proof of Theorem 5.1:

(1) implies (2): this follows from Lemma 3.2.

(2) implies (3): this follows from Lemma 4.7.

(3) implies (1): this follows from Lemma 4.6. □

Proof of Example in Section 7: Let $\mathcal{L}$ be the closure under truth-functional operations of $\{\alpha, \beta\}$, and let $\mathbf{K} = Cn(\{\alpha \wedge \beta\})$. We will construct $\leq_{\mathbf{K}}$ explicitly. Due to (**EE2**) it is sufficient to order the sixteen formulas in the following ordering:

$$\left\{\begin{array}{c} \neg\alpha \wedge \beta \\ \alpha \wedge \neg\beta \\ \neg\alpha \wedge \neg\beta \\ \neg\alpha \\ \neg\beta \\ \alpha \not\longleftrightarrow \beta \\ \neg\alpha \vee \neg\beta \\ \bot \end{array}\right\} <_{\mathbf{K}} \left\{\begin{array}{c} \alpha \wedge \beta \\ \beta \\ \alpha \longleftrightarrow \beta \\ \neg\alpha \vee \beta \end{array}\right\} <_{\mathbf{K}} \left\{\begin{array}{c} \alpha \\ \alpha \vee \neg\beta \end{array}\right\} <_{\mathbf{K}} \left\{\ \alpha \vee \beta\ \right\} <_{\mathbf{K}} \{\top\}$$

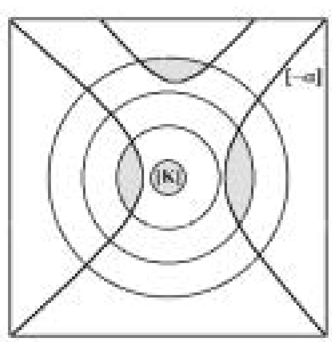
Let $-_{\mathrm{G}}$ and $-_{\mathrm{R}}$ be the AGM contraction and severe withdrawal based on $\leq_{\mathbf{K}}$ defined via $(-_{\mathrm{G}})$ and $(-_{\mathrm{R}})$ respectively. By definition of $-_{\mathrm{G}}$, we have

$$\begin{array}{lclclclcl} \mathbf{K}-_{\mathrm{G}}(\alpha \wedge \beta) & = & \mathbf{K}-_{\mathrm{G}}(\alpha \longleftrightarrow \beta) & = & \mathbf{K}-_{\mathrm{G}}(\beta) & = & \mathbf{K}-_{\mathrm{G}}(\neg\alpha \vee \beta) & = & Cn(\{\alpha\}) \\ \mathbf{K}-_{\mathrm{G}}(\alpha) & = & \mathbf{K}-_{\mathrm{G}}(\alpha \vee \neg\beta) & = & Cn(\{\beta\}) & & & & \\ \mathbf{K}-_{\mathrm{G}}(\alpha \vee \beta) & = & Cn(\{\alpha \longleftrightarrow \beta\}). & & & & & & \end{array}$$

Otherwise,

$$\mathbf{K}-_{\mathrm{G}}(x) \quad = \quad \mathbf{K}.$$

Trivially, $-_{\mathrm{R}}$ satisfies the interpolation thesis. For $\alpha \vee \beta$, $\mathbf{K}-_{\mathrm{R}}(\alpha \vee \beta) = Cn(\varnothing)$ and it is easy to show that there is no δ such that $Cn(\{\alpha \longleftrightarrow \beta\}) \cap \mathbf{K}-_{\mathrm{G}}((\alpha \vee \beta) \rightarrow \delta) = Cn(\varnothing)$. Hence $-_{\mathrm{R}}$ is not a semi-contraction. [5]



□

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NOTES

1. " . . . when seeking to answer a question, not all new information is relevant to the question being asked. This is, perhaps, the chief of several reasons why measures of informational value ought to be carefully distinguished from measures of information." ([14], p. 123)

2. According to Levi in [14] and in an unpublished manuscript, "Contraction and informational value" (1997), not all the information in the corpus of beliefs is of value to the inquiring agent; consequently, the agent tries to retain as much of the valuable information as possible, instead of as much of the information as possible.

3. Grove [10] introduced a different order between sentences (closely related to Lewis's ordering of comparative possibility [15]) that can be seen as a dual of epistemic entrenchment. ([8], p. 96)

4. Since each member of the remainder set satisfies recovery, see [2].

5. For readers with more background in belief revision, the following figure (Grove's sphere-system [10]) illustrates the example. If the contraction intersects more than two spheres (as in the figure), we cannot express it as a semi-contraction, since each AGM function can intersect only one sphere. This is the case of the example.

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