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Book Review

Stewart Shapiro. *Philosophy of Mathematics: Structure and Ontology.* Oxford University Press, Oxford, 1997. xii + 279 pages

1 Structuralism heretofore has not had quite the same standing in the field of philosophy of mathematics as logicism, formalism, constructivism, nominalism, and so on: it has not been an 'ism any writer surveying field with a pretension to comprehensiveness would feel absolutely obliged to examine at length. That may change with Stewart Shapiro's book. The book is an extended exposition and defense of a distinctive version of structuralism in which the author considers the familiar questions in philosophy of mathematics, and also—this being one of the book's more significant if less conspicuous contributions—raises several less familiar questions, and in every case articulates a structuralist response. Shapiro is the first philosopher to devote a whole book to defending structuralism (though Michael Resnik's recent book and Geoffrey Hellman's older one are also much concerned with structuralism) and the book can be expected to remain required reading for some time to come. In examining the issues it treats here, the most efficient strategy will be the least imaginative one: to follow the author, chapter by chapter, through the questions, summarizing his response to each, and interspersing any critical commentary of my own as we go along.

2 Realism Passing over an Introduction which may be more effective as a summary to be read after reading the book, Shapiro begins, in his first chapter, by considering what the relationship between philosophical principle and mathematical practice has been and should be. He decides for the space of this book to confine his attention to philosophical positions that aim only to *interpret* rather than to *change* mathematical practice. What does this exclude from consideration? Mainly certain among the so-called *antirealist* positions.

So-called antirealists are all troubled by one or another aspect of *orthopraxis*, or currently accepted practice in mathematics, and specifically by the practice of making certain kinds of assertions when doing mathematics. Antirealists may be classified one way by the kind of assertion that troubles them. Thus the *constructivists* are troubled by assertions like (1) below and *nominalists* by assertions like (2) below (or more precisely by the implication of such assertions that there are such things as numbers):

1. Either there is a number greater that is greater than a googolplex and is a twin prime, or there is not.
2. There is a number that is greater than a googolplex and is a prime.

Antirealists may be classified another way by what they propose should be done about what troubles them. Thus *revisionists* propose that the troubling assertions should no longer be made while doing mathematics. By contrast, *fictionalists* maintain that mathematics is a useful mythology or fiction, and allow that on account of its being useful the troublesome assertions may be made while doing mathematics, only insisting that on account of its being mythology they must be taken back when doing philosophy. Also by contrast, *figurativists* maintain that the assertions only have troublesome implications when construed straightforwardly or nonfiguratively, and allow that they may be made even while doing philosophy, provided they are explained away through some nonstraightforward reconstrual.

Shapiro's restriction excludes from consideration revisionism, but probably not fictionalism, and certainly not figurativism. Theoretically the two modes of classification are independent, though actually constructivists tend to be revisionists, and nominalists to be fictionalists or figurativists. So the restriction does not exclude nominalism (which is in fact extensively discussed in later chapters), but does exclude constructivism (though there is one extended digression on the latter in a later chapter).

In his second chapter, Shapiro contrasts *working realism*, which is more or less just orthopraxis, plus perhaps some degree of self-consciousness about it, with *philosophical realism*, and struggles to define what more is involved in the latter than in the former. One element, *truth-value realism*, involves a willingness to go beyond (1) to (3) below, while another element, *ontological realism*, involves a willingness to go beyond (2) to (4) below:

3. Either "there exists a number that is greater than a googolplex and is a twin prime" is true, or its negation is true.
4. There does exist a number, and moreover one that satisfies "it is greater than a googolplex and is prime."

Now, given orthopraxis and hence (1) and (2), truth-value and ontological realism as in (3) and (4) will be forthcoming if one accepts the kind of biconditionals that figure prominently in the work of model theorists in general and Alfred Tarski in particular:

5. "___" is true if and only if ___;
6. "___ or ---' is true if and only if "___" is true or " —— " is true;
7. "there is something such that it ___s" is true if and only if there is something such that it satisfies "it ___s."

Shapiro defines the essence of philosophical realism to be "model-theoretic or Tarskian semantics."

I confess that I dislike this formulation, since I don't think use of the word "true" is a monopoly of philosophical theorists as opposed to practicing mathematicians.

This is partly acknowledged by Shapiro, but I think it should be acknowledged that it is equally part of orthopraxis to accept arguments of either of the two forms below:

8. If the power of the continuum is the smallest uncountable cardinal, then ___; and if the power of the continuum is not the smallest uncountable cardinal, then ___: therefore ___.
9. If the Continuum Hypothesis is true, then ___; and if the negation of the Continuum Hypothesis is true, then ___; therefore ___.

To me, what seems essential is not making such assertions when doing mathematics, but refusing when doing philosophy to take them back or explain them away.

Given his preferred formulation, Shapiro immediately confronts a number of questions. First, Tarski's decision to use as a label for his theory of *models* a term—namely, the term "semantics"—that was already in use as a label for the theory of *meaning* introduced a serious ambiguity. So there is a question of what is meant by "semantics" and of what is the status of the biconditionals above: should they be taken in the *Tarskian* way, as parts of a definition of how the term "true" is to be understood, an understanding of mathematical terms being presupposed; or should they be taken in the diametrically opposed *Davidsonian* way, as part of an account of what understanding of mathematical notions consists in, an understanding of the truth notion being presupposed?

Second, there is the old question about the nature of "truth": is it an *anemic* notion, used for undoing quotations and other intralinguistic purposes, or is it a *robust* notion, indicating a metaphysical correspondence between language and extralinguistic reality?

Third, there is for *anyone* writing about "realism" a question about the distinction(s) advocated by philosophers like Rudolf Carnap and Arthur Fine between some sort of "internal" or "natural" lowercase realist attitude and some sort of "external" or "metaphysical" Uppercase Realist Attitude: is the former enough, or is the latter also required?

Shapiro's discussion of such issues is very hard going, and after reading through the chapter thrice, I am still unclear as to what the conceptions of "semantics" and "truth" put forward are and are not supposed to involve, and as to whether I myself would count as a "philosophical realist." Fortunately, whatever the writer intended, readers can understand most of the rest of the book if they just replace "realism" by "anti-anti-realism", where *anti-anti-realism* is simply rejection of revisionism and fictionalism and figurativism. And once the barrier of the second chapter is crossed, the going becomes easier.

3 Structuralism as a variety of realism Shapiro at last arrives, in his third chapter, at his main topic, structuralism. Now structuralism is the view that mathematics is the study of structures; but the term *structure* in mathematics has several senses, and therefore the term *structuralism* in philosophy of mathematics must correspondingly have several senses. Much of the chapter is devoted to sorting some of these out. I will attempt such a sorting out myself, before indicating what is distinctive about the version of structuralism that Shapiro advocates.

In a first sense of the term, a "structure" is just a *system*, which is to say a set (whose elements are by abuse of language also called elements of the system) together with a sequence of distinguished relations or operations on them. Thus an *ordered set* is a system consisting of a set and a single distinguished two-place relation, while a *field* is a system consisting of a set and a pair of two-place operations. (Bells and whistles would have to be added to take in vector spaces, where there are two sorts of elements, namely, scalars and vectors, or topological spaces, where what is relevant is a distinguished property of subsets of the set of elements, namely, openness, rather than of the elements themselves.)

One system is a *subsystem* of another if the set of elements of the former is a subset of the set of elements of the latter, and the distinguished relations or operations of the former are just the restrictions of the distinguished relations or operations of the latter to that subset. Two systems are *isomorphic* if there is a one-to-one correspondence between their elements, such that whenever elements of the one stand in one of its distinguished relations, the corresponding elements of the other stand in its corresponding distinguished relation, and analogously for operations. One system is *embeddable* in another if the former is isomorphic to a subsystem of the latter.

Now isomorphism is an equivalence relation, like the relation of parallelism between lines. Whenever we have an equivalence relation among items of a given kind, the *invariant* relations among items of that kind are those that, whenever they hold, continue to hold if the items are replaced by equivalent ones. For example, embeddability is an invariant relation between systems, as perpendicularity is an invariant relation between lines. Whenever we have an equivalence relation among items of a given kind, we may consider the associated *equivalence types*, where their equivalence type is what two items that are equivalent thereby have in common, as lines that are parallel have in common their direction. Associated with any invariant relation among the original items will be a relation among their equivalence types, often called by the same name. The equivalence types for the equivalence relation of isomorphism are called simply *isomorphism types*, with “structure” being a synonym, in a second sense of that term.

The *first-order language* appropriate to a system has variables for elements of the system and relation- and operation-symbols for its distinguished relations and operations, plus the usual apparatus of first-order logic. The properties of being an ordered set, or a field, or say an infinite discrete ordered set, or an orderable field, are expressible in such a language. Other properties, such as being an infinite discrete *well*-ordered set or a *complete* orderable field require a *second-order* language, or perhaps some even more extended language. Any property expressible in such a language is invariant.

Often when mathematicians study a particular system, they are interested only in its invariant properties, and in such a case they are often said to “ignore the nonstructural properties of its elements,” which distinguish them from the corresponding elements of isomorphic systems, and “attend only to their structural relations with each other.” In the nineteenth century, many commentators wrote as if they believed that the mind of the mathematician has the power not only of *ignoring* nonstructural properties, but actually of *annihilating* them, thus producing a new system isomorphic to the old but of a very special kind: a system whose elements, sometimes called *places*, simply *have no* nonstructural properties, but have only their structural relations with each other. Of course, this very fact distinguishes them from the elements of any isomorphic system.

Minus the notion that they are products of some kind of mental activity, the notion that there are such special systems has persisted into or been revived in the twentieth century. (Sometimes they are confused with isomorphism types, but this is a mistake: An isomorphism type is no more a special kind of system than a direction is a special kind of line.) They are often called before-the-things as opposed to in-the-things structures, or using the Latin singular to express generality in place of the English plural, *ante rem* as opposed to *in re* structures. But they are often just called "structures", this being yet another use of that term.

Now we may contrast three ways of interpreting a subject like number theory or real analysis. On the one hand, an *eliminative* structuralist interpretation would take the former to be the study of all infinite discrete well-ordered sets, it being known that all such ordered sets are isomorphic; and it would take the latter to be the study of all complete orderable fields, it again being known that all such fields are isomorphic. On the other hand, an *ante rem* structuralist interpretation would take each to be the study of a single specific system, *the* natural numbers in the case of the former, and *the* real numbers in the case of the latter, but would insist that the systems in question are *ante rem* structures, and that natural and real numbers have no *non*structural properties, but only structural relations to other natural or real numbers.

By contrast, a *non*structuralist interpretation would take the natural and real numbers to have nonstructural properties or individual natures: the natural numbers might be identified with the finite *cardinals*, which is to say the equivalence types of finite sets under the equivalence relation of equinumerosity, with various relations and operations on natural numbers then being associated with invariant relations and operations on sets; while the real numbers might be identified with *ratios* of pairs of geometric magnitudes, which is to say the equivalence classes of pairs of geometric magnitudes under the equivalence relation of proportionality, with various relations and operations on real numbers being associated with various invariant relations and operations on pairs of real numbers.

In a usage deriving from Michael Dummett, eliminativist structuralism has also been called *hard-headed* structuralism, while *ante rem* structuralism has also been called *mystical* structuralism—especially by philosophers who find the notion of an *ante rem* structure mysterious. Shapiro himself is not one of these. On the contrary, he is an advocate of *ante rem* structuralism, which he characterizes as a variety of realism in his sense of the term. In the chapter under discussion, besides broaching several topics to be treated more fully in later chapters, he works to make the notion of *ante rem* structure seem less mystifying. His most important contribution toward demystifying the notion consists in making explicit just what assumptions about the existence of *ante rem* structures the *ante rem* structuralist needs to make.

On the other hand, some mysteries do remain. These are least troubling in the cases to which Shapiro most often reverts. In the case of the natural numbers for instance, as Shapiro conceives them, though they are supposed to have no nonstructural properties, at least each has a structural property, expressible in the relevant first-order language, that distinguishes it from the others: only one of them comes first in the order on natural numbers, only one comes next-to-first, only one comes next-to-next-to-first, and so on—and that's all the natural numbers there are. Something similar, but more complicated, holds for real numbers as well.

The situation changes, however, when we come to the complex numbers. There we have two roots to the equation $z^2 + 1 = 0$, which are additive inverses of each other, so that if we call them i and j we have $j = -i$ and $i = -j$. But the two are not distinguished from each other by any algebraic properties, since there is a *symmetry* or *automorphism* of the field of complex numbers, which is to say an isomorphism with itself, which switches i and j. On Shapiro's view the two are distinct, though there seems to be *nothing* to distinguish them. The case is even worse with, say, the Euclidean plane, which is *homogeneous*: for any two points p and q there is an automorphism, and indeed there are many automorphisms, carrying the one to the other. Shapiro offers no extended discussion of the mystery of symmetry, and I consider this the most serious omission in the book. Even when he reverts to homely intuitive examples drawn from team sports, he fastens on the one major team sport, baseball, where there is no left-right symmetry in the game (because the runner is required to go around the bases in a counterclockwise, not a clockwise direction). If he had considered football, he would have had to confront the issue that, for instance, when a concrete team takes the field, the two tackles are distinguished by being one on the left and the other on the right, whereas in the abstract structure there seems to be nothing to distinguish these two positions.

4 Structuralism versus nominalism Shapiro next turns, in his fourth chapter, to consider the alleged epistemological difficulties that have been cited by nominalists as motivating their denial of the existence of numbers. The background is as follows. Compare the following two simple noun-verb, subject-predicate assertions:

10. Evelyn is prim.
11. Eleven is prime.

In the case of (10), we can find out that it is true by locating Evelyn spatiotemporally (perhaps she lives in New York) and interacting with her causally (perhaps by interviewing her). In the case of (11), we cannot find out that it is true in a parallel way, for sentences combining a mathematical subject like "Eleven" with a spatiotemporal or causal predicate like "lives in New York" or "grants an interview" have no meaning—which is to say, have no use—in our language.

Rather, one might work with some *instance* of eleven, say a collection of eleven poker chips, trying to arrange them in a rectangular array with equally many chips in each row and equally many chips in each column, and more than one row and more than one column, and finding that it can't be done. Or one might work with some *term* denoting eleven (such as the usual Arabic numeral '11') and write out various multiplications according to the usual rules with pencil and paper (such as '$2 \times 5 = 10$' and '$2 \times 6 = 12$' and '$3 \times 3 = 9$' and '$3 \times 4 = 12$' and so on), and finding that none gives the product '11'. In either case, by the standards of common sense and mathematical science, the result would justify the assertion of (11).

Now from any point of view, one question epistemology should address is how *in detail* mathematicians come to consider themselves justified in asserting (11) and more advanced mathematical results, and what *in detail* are the standards of justification to which they implicitly appeal. From the point of view of *naturalized* epistemology, which proposes that the epistemologist should become a citizen of the scientific community, conforming to its standards of justification, that is the *only* epistemological question about (11). But from the point of view of *alienated* epistemology, which proposes that the epistemologist should remain a foreigner to the scientific community, judging its standards by philosophical standards outside, above, and beyond science, there is a further question whether the assertion of (11), though justified by scientific standards, is *really* justified.

As it happens, those who adopt the alienated standpoint seem to tend sooner or later to adopt overtly or covertly a kind of principle of *grammatico-epistemological parallelism*, according to which sentences of parallel grammar, like (10) and (11), should have parallel conditions of justified assertability. And from there it is a short step to either revisionism, insisting that mathematicians should not after all assert (11), since it cannot be justified in the same way as (10); or to fictionalism, permitting the assertion of (11) in mathematics, but insisting that the assertion be taken back in philosophy, as having been merely useful fiction; or else to figurativism, insisting the assertion be explained away in philosophy through some nonstraightforward reconstrual showing that the grammar of (11) is not after all really parallel to that of (10).

Now what Shapiro tries to do is sketch a series of steps by which a naïve subject could come to make assertions like (11) about natural numbers, understood as he thinks they should be understood, namely, as assertions about places in an *ante rem* structure. *Any* such effort by a philosopher faces a dilemma. On the one hand, from a naturalized point of view, it must appear merely armchair philosophical speculation, about questions to which substantive answers can only be supplied by empirical psychological investigation. On the other hand, from an alienated point of view, it must appear merely a description of how mathematicians go more and more badly wrong, as they depart further and further from the principle of grammatico-epistemological parallelism. The *most* that such an effort could reasonably hope to achieve would be this. On the one hand, it could hope to produce some suggestive speculations in naturalized epistemology, ones that would be worthy of substantial empirical investigation. On the other hand, it could hope to show that the principle of grammatico-epistemological parallelism is violated at a stage in the sequence of development that comes so early and looks so innocent as to make the objections of alienated epistemology implicitly based on this principle seem unreasonable, and the alleged problems nominalism cites seem unreal. Shapiro does, in my opinion, succeed fairly well on both counts, though he himself seems to think he is doing something *more* than this, giving a substantive answer to a real problem.

Now the question how a sophisticated contemporary view like structuralism is arrived at may be considered either *ontogenetically*, as a question of developmental psychology, or *phylogenetically*, as a question of intellectual history. In moving from his fourth to his fifth chapter, Shapiro shifts from the one question to the other, and incidentally from armchair speculation about cognitive processes to empirical research citing historical documents.

The most important conclusion parallels the conclusion reached earlier, according to which the notion of *ante rem* structure is one that the individual student of mathematics develops only fairly late in the educational process, after experience working with less sophisticated kinds of abstracta. The conclusion is just this, that as a description of mathematicians' self-understanding, structuralism is *not* applicable to mathematics through the eighteenth century, but rather as something that came in at the earliest in the nineteenth century. Newton, for instance, clearly thought of the real numbers as ratios of geometric magnitudes. Dedekind, by contrast, isolated the assumptions that were actually being used about real numbers, the axioms for a complete orderable field, and so made a structuralist interpretation available. (If I have any critical comment at all to make on this chapter, it is that Shapiro sometimes seems too quick to take as manifestation of an interest in *structuralism* many cases of nineteenth-century work in algebraic axiomatics where there is no question of *categorical* axioms or a *single* isomorphism type of system.)

The sixth chapter is in the nature of an interlude or digression (a large part of it, which I will simply ignore here, it being concerned with constructivism, a topic apparently officially excluded from consideration in the opening chapter, though Shapiro wishes to bring it in by interpreting constructive mathematics nonrevisionistically, which is to say, not in the way it is conceived by the constructivists themselves, who regard it as the only legitimate form of mathematics, but as a kind of supplement to classical mathematics). I have written above as if it were characteristic of *nominalism* to insist on construing mathematical language nonstraightforwardly, but Shapiro here points out that a great deal of what mathematicians say must be taken nonliterally by *any* philosopher, because the mathematicians themselves indicate that it is intended nonliterally—where by mathematicians are meant professional mathematicians generally, speaking during working hours while doing their mathematical job, not just the minority among them who are also amateur philosophers, speaking after hours while pursuing their philosophical hobby. Such is the case with the very extensive use of "dynamic" language, in which functions, for instance, are spoken of as now doing this, then doing that.

(There is one strange slip in this chapter, which by the way is the only technical slip of any magnitude I have noted in the whole technically well-informed book, concerning the relationship between Euclid's geometry and Hilbert's, which though somewhat off the main topic perhaps should be set straight. Shapiro correctly notes, following Bernays and others, that Euclid's assumptions about what *can be constructed* are replaced by Hilbert with assertions about what *exists*. He also correctly notes that Hilbert, drawing on the work of nineteenth-century geometers, fills in some gaps in Euclid's proofs, which sometimes implicitly make constructibility assumptions beyond those explicitly listed as postulates. Now in fact there is a third difference. Greek geometers divided curves in the plane into three classes: the planar (lines and circles), the solid (conic sections), and the mechanical (including spirals). Euclid's *Elements of Geometry* is concerned only with curves of the first class, and does not need all the constructibility assumptions that would be needed for a treatment of the others; whereas Hilbert's *Foundations of Geometry* includes existence assumptions, in the form of a continuity or completeness axiom, sufficient for all three classes. This means that while it is a substantial question whether the constructibility of a regular pentagon or heptagon can be established on the basis of Euclid's explicit and implicit constructibility assumptions, it is a triviality that regular pentagons and heptagons exist given Hilbert's existence axioms. Shapiro seems to think that substantial questions *must* get lost in the transition from one style of formulation to the other, but it is not so. It is possible to produce a formulation like Hilbert's but with weaker existence assumptions, exactly corresponding to Euclid's constructibility assumptions, and this has been done by Tarski.)

The seventh chapter reverts to the topic of nominalism, and specifically considers the programs of Hartry Field, Charles Chihara, and especially Geoffrey Hellman, whose *modal structuralism* has already been alluded to in earlier chapters. The idea is, basically, to follow eliminative structuralism, but instead of taking number theory to be about all *actual* systems of abstract objects that are arranged in an infinite discrete well-order, to take it instead to be about all *possible* systems of concrete objects that would be so arranged. The hope is by such a nonstraightforward reconstrual to eliminate the assumption that there *are* any abstract entities.

Shapiro touches, however, on a serious obstacle to this approach. (He seems unaware of some other difficulties in the way of such a strategy, acknowledged in Hellman's later work.) The obstacle is that the very notion of a system involves the notions of *set* and *relation* and *sequence* (and the notion of set is again involved any time we need a *second*-order characterization of the particular systems of interest). Shapiro is not, of course, himself an advocate of modal structuralism, any more than of eliminative structuralism, but rather of *ante rem* structuralism. But actually, the point he brings out seems in fact an obstacle to extending *any* variety of structuralism to the *whole* of mathematics.

Shapiro, despite his preference for *ante rem* structuralism, thinks that the three varieties of structuralism are all in some sense equivalent, and works to make clearer what sense this is. In this connection, it is comparatively uncontroversial that there are no *mathematically* important differences among the three, and that the mathematician *qua* mathematician need not be concerned to choose among them. It, of course, does not follow immediately and without further argument from this that there are no *philosophically* important differences among the three, or that the philosopher *qua* philosopher need not be concerned to choose among them. It is true that *if* one accepts the modal notions of modal structuralism, *and* the variety of isomorphic systems of abstracta of eliminative structuralism, *and* the *ante rem* structures of *ante rem* structuralism, *then* each of the three can be interpreted in the other. But that is, philosophically speaking, a big *if*.

The eighth and last chapter perhaps calls for less comment, since it is more tentative in its conclusions. It undertakes preliminary exploration of the question of the applications of mathematics, and of the relevance of a structuralist approach even outside mathematics.

In sum, while I have reservations about the line Shapiro takes here or there or elsewhere on this or that or the other point, I think Shapiro has achieved his major goal: structuralism in general, and his version in particular, henceforth can still be criticized, but after this book they can hardly be ignored.

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