Notre Dame Journal of Formal Logic
Volume 40, Number 2, Spring 1999

An Open Formalism against Incompleteness

FRANCESC TOMÀS

Abstract An *open* formalism for arithmetic is presented based on first-order logic supplemented by a very strictly controlled constructive form of the omega-rule. This formalism (which contains Peano Arithmetic) is proved (nonconstructively, of course) to be complete. Besides this main formalism, two other complete open formalisms are presented, in which the only inference rule is modus ponens. Any closure of any theorem of the main formalism is a theorem of each of these other two. This fact is proved constructively for the stronger of them and nonconstructively for the weaker one. There is, though, an interesting counterpart: the consistency of the weaker formalism can be proved finitarily.

1 Introduction Vaguely stated, we understand that an *open* formalism differs from what is usually called a formal system or axiomatic theory in that it is not demanded that the collection of its axioms be decidable, or generated by a mechanical process or something of the sort, so that an open formalism for arithmetic will be invulnerable to Gödel's proof of incompleteness or other such proofs, such as the one that is based on the incompleteness of the halting problem (see Ebbinghaus [4], Chapter V, §6). As a matter of fact, Gödel's method becomes a procedure, among others, in the description of the collection of axioms of the formalism (or formalisms) to be defined. The central idea, in the definition of the formalism, is to describe the collection of its axioms by means of a constructive and very strictly controlled form of the ω-rule and of the notion of "segment" to be introduced. To this purpose transfinite ordinals or related devices are needed. About the ω-rule and its uses in the foundations of arithmetic see Feferman [6] and the more recent Ignjatović [10] and their references.

In the present introduction the open formalism $\mathbf{OA_1}$ is described and its completeness is proved nonconstructively. The constructive proof that $\mathbf{OA_1}$ contains Peano Arithmetic is carried on in Section 2 where a subformalism $\mathbf{OA}$ of $\mathbf{OA_1}$ is described in which advantage is taken of the perfect notation and clear constructiveness

Received September 11, 1995; revised February 20, 1997

of the ordinals $< \epsilon_0$. In Sections 3, 4, and 5, besides proving that **OA** contains Peano Arithmetic, it is illustrated how, as said before, Gödel's and related methods can be used to introduce in **OA** axioms that are not theorems of Peano Arithmetic.

There is an appendix where complete formalisms $\mathbf{OA_1'}$ and $\mathbf{OA_1''}$ are defined in which modus ponens is the only inference rule. The consistency of $\mathbf{OA_1''}$ is proved constructively. Its completeness makes it as strong as $\mathbf{OA_1}$ (and, consequently, stronger than Peano Arithmetic), in some nonconstructive sense.

We now proceed to define $\mathbf{OA_1}$. Previous to it we need to describe a collection or class C of constructive totally ordered countable sets without infinite descent.

Definition 1.1 Each member of C will be provided by a procedure that

1. defines a set σ and a decidable (by the same procedure) total order $<$ on it;
2. has the means to prove, for any given decreasing sequence $a_1 > a_2 > \cdots$ of elements of σ, that the sequence is finite;
3. its correctness (i.e., of the procedure) is proved constructively without having recourse to the excluded third for infinite sets.

The possibility of thinking of correct procedures whose correctness we do not know how to prove constructively will be illustrated in Section 4.3 and in the appendix (end of Section A.1).

We now have the following definition.

Definition 1.2 $\mathbf{OA_1}$ is a first-order theory with equality in which there is only one individual constant 0 and the functional symbols are $+$, $\cdot$, and $'$. Its collection of nonlogical axioms and the notion of *segment* are described as follows.

(a_1) The set B of *basic* axioms of $\mathbf{OA_1}$ consists of the following.

$$
\begin{aligned}
&x_1 = x_2 \Longrightarrow (x_1 = x_3 \Longrightarrow x_2 = x_3),\\
&x_1 = x_2 \Longrightarrow x_1' = x_2',\\
&\neg 0 = x_1',\\
&x_1' = x_2' \Longrightarrow x_1 = x_2,\\
&x_1 + 0 = x_1,\\
&x_1 + x_2' = (x_1 + x_2)',\\
&x_1 \cdot 0 = 0,\\
&x_1 \cdot (x_2') = (x_1 \cdot x_2) + x_1.
\end{aligned}
$$

(b_1) A *segment* $\mathbf{T}$ of $\mathbf{OA_1}$ consists of a member $\sigma_{\mathbf{T}}$ of the class C and a procedure that defines a function $j_{\mathbf{T}}$ from $\sigma_{\mathbf{T}}$ to the set of formulas of $\mathbf{OA_1}$ of the form $\forall x\, A(x)$ (for any variable x) and provides, for each $\delta \in \sigma_{\mathbf{T}}$, a proof of

$$B \cup \{ j_{\mathbf{T}}(\gamma) : \gamma \in \sigma_{\mathbf{T}},\ \gamma < \delta \} \vdash A_\delta^{\mathbf{T}}(\overline{n})$$

for each natural n, where it is understood that $\forall x\, A_\delta^{\mathbf{T}}(x) = j_{\mathbf{T}}(\delta)$ (and where $\overline{0}$ is 0, $\overline{1}$ is $0'$, and so on). The set $AX(\mathbf{T})$ of nonlogical axioms of $\mathbf{T}$ is then defined as

$$AX(\mathbf{T}) = B \cup \{ j_{\mathbf{T}}(\gamma) : \gamma \in \sigma_{\mathbf{T}},\ \gamma \in \sigma_{\mathbf{T}} \}$$

The procedure that consists of the definition of $j_{\mathbf{T}}$ and provides the demanded proofs must be correct, of course. It is demanded, moreover, that its correctness be proved constructively, as has already been demanded of the procedure that defined $\sigma_{\mathbf{T}}$ and satisfied (1), (2), and (3) of Definition 1.2. This demand of constructivity of the proof of the correctness of the whole procedure has the clear intention of staying in accordance with Hilbert's program.

(c_1) The collection $AX(\mathbf{OA_1})$ of nonlogical axioms of $\mathbf{OA_1}$ consists of all the axioms of all its segments $\mathbf{T}$:

$$AX(\mathbf{OA_1}) = \bigcup AX(\mathbf{T}),$$

where the union runs over all the segments $\mathbf{T}$ of $\mathbf{OA_1}$.

This describes the open formalism $\mathbf{OA_1}$. We shall write $\mathbf{T} \vdash A$ or $\mathbf{OA_1} \vdash A$ to indicate that A is derived from $AX(\mathbf{T})$ (and the logical axioms) or from $AX(\mathbf{OA_1})$ (and the logical axioms), respectively, and say in such cases that A is a theorem of $\mathbf{T}$ or of $\mathbf{OA_1}$.

The schema of induction, absent in B, will appear as a schema of axioms in some segment. This will be proved in Section 3, where it is shown that Peano Arithmetic is contained in a segment of $\mathbf{OA}$ and, consequently, of $\mathbf{OA_1}$ (see Section 2).

We now proceed to prove, nonconstructively, the completeness of $\mathbf{OA_1}$. We define the *sum* of a finite or infinite sequence of segments, $\mathbf{T_1}$, $\mathbf{T_2}$, ... , of $\mathbf{OA_1}$, denoted

$$\mathbf{T} = \mathbf{T_1} + \mathbf{T_2} + \cdots$$

by setting

$$\sigma_{\mathbf{T}} = (\sigma_{\mathbf{T_1}} \times \{1\}) \cup (\sigma_{\mathbf{T_2}} \times \{2\}) \cup \cdots ,$$

$$(\gamma, i) < (\delta, j) \ \text{ if } \ i < j, \quad \text{or if } \ i = j \ \text{ and } \ \gamma < \delta$$

$$j_{\mathbf{T}}(\gamma, i) = j_{\mathbf{T_i}}(\gamma)$$

Clearly, $\mathbf{T}$ is a segment of $\mathbf{OA_1}$, the collection of procedures for the $\mathbf{T_i}$ providing a procedure for their sum and the collection of constructive correctness proofs providing a constructive correctness proof for the sum.

By taking $\sigma_{\mathbf{T_i}} = \varnothing$ for $i > n$ the apparently infinite sum reduces to the finite sum $\mathbf{T_1} + \cdots + \mathbf{T_n}$. As each theorem of $\mathbf{OA_1}$ depends only of a finite number of axioms we see, by considering in each case the segments in which these axioms arise, that each theorem of $\mathbf{OA_1}$ is a theorem of one of its segments.

Observation 1.3 If $\mathbf{OA_1} \vdash A$ then $\mathbf{T} \vdash A$ for some segment $\mathbf{T}$ of $\mathbf{OA_1}$.

Observation 1.4 If $\mathbf{T_i} \vdash A_i$ for $i = 1, 2, \ldots$, then

$$\mathbf{T_1} + \mathbf{T_2} + \cdots \vdash A_i \ \text{ for } \ i = 1, 2, \ldots ,$$

where the sequence of values of i may be finite or infinite.

As a simple consequence we see that $\mathbf{OA_1}$ is consistent if and only if each one of its segments is consistent. This last assertion can be proved nonconstructively very easily by showing that $(\mathbb{N}, +, \cdot, ')$ is a model of each segment (with the natural interpretations of the operations). The proof of this fact depends of the noninfinite-descent condition 2 of Definition 1.1.

We now have (nonconstructively, of course) the following theorem.

Theorem 1.5 $\mathbf{OA_1}$ *is complete.*

Proof: Let us consider the closed formulas of $\mathbf{OA_1}$ and say that A is *simpler* than B if the number of presences of symbols $\vee$, $\wedge$, $\Longrightarrow$, $\exists$, or $\forall$ (not counting the presences of $\neg$) is lesser in A than in B. Suppose now, on the contrary, that $\mathbf{OA_1}$ is incomplete. This means there are undecidable closed formulas. Suppose A is one of the simplest of them. As any two formulas C and $\neg C$ are decidable or undecidable at the same time we may suppose that A is not of the form $\neg B$. We also see easily that A, being simplest undecidable, is not of the forms $B \vee C$, $B \wedge C$, or $B \Longrightarrow C$. Finally, A is not of the form $a = b$ for constant terms a and b because all such formulas are decidable, so that A must be of the form $\exists x B(x)$ or $\forall x C(x)$. But, as $\exists x B(x)$ and $\forall x \neg B(x)$ are undecidable or decidable at the same time, we may always suppose that A is of the form $\forall x C(x)$. We now observe that $C(\overline{n})$ is decidable for each natural n because each $C(\overline{n})$ is simpler than $\forall x C(x)$. Moreover, we must have $\mathbf{OA_1} \vdash C(\overline{n})$ for each n because otherwise we would have $\mathbf{OA_1} \vdash \neg C(\overline{n})$ for some n, and then $\mathbf{OA_1} \vdash \neg \forall x C(x)$, against the undecidability of $\forall x C(x)$. Due to the previous observations, if the segments $\mathbf{S_n}$ are such that $\mathbf{S_n} \vdash C(\overline{n})$ for each n, then we consider the segment

$$\mathbf{S} = \mathbf{S_0} + \mathbf{S_1} + \cdots$$

and will have

$$\mathbf{S} \vdash C(\overline{n}) \text{ for all natural } n.$$

We now define the object (not a segment) $\mathbf{P}$ consisting of $\sigma_{\mathbf{P}} = \{\varnothing\}$ (a monic set) and $j_{\mathbf{P}}(\varnothing) = \forall x C(x)$. Then we consider the object $\mathbf{S} + \mathbf{P}$, for which, as in a previous definition,

$$\sigma_{\mathbf{S+P}} = (\sigma_{\mathbf{S}} \times \{1\}) \cup (\sigma_{\mathbf{P}} \times \{2\}),$$

$(\gamma, 1) < (\varnothing, 2)$ for every $\gamma \in \sigma_{\mathbf{S}}$, $(\gamma, 1) < (\delta, 1)$ whenever $\gamma < \delta$, $j_{\mathbf{S+P}}((\gamma, 1)) = j_{\mathbf{S}}(\gamma)$ and $j_{\mathbf{S+P}}((\varnothing, 2)) = j_{\mathbf{P}}(\varnothing)$. We assert that the complex $\mathbf{S} + \mathbf{P}$ is a segment of $\mathbf{OA_1}$. To be convinced of it we need only to see that there is a procedure to prove $\mathbf{S} \vdash C(\overline{n})$ for each n and that the correctness of this procedure is proved constructively. But such a procedure is nothing else than the collection of the proofs of the $\mathbf{S_n} \vdash C(\overline{n})$ and, as the correctness of each of these proofs is constructively proved, the correctness of the collection of proofs is also constructively proved. Thus, $\mathbf{S} + \mathbf{P}$ being a segment of $\mathbf{OA_1}$, $\forall x C(x)$ is an axiom of it, against the hypothesis. This proves that $\mathbf{OA_1}$ is not incomplete. $\square$

***2 The open arithmetic* OA** For the description of the notion of *segment* of **OA** the ordinals less than ϵ_0 are required. As is well known, Gentzen made use of these ordinals in his proof of the consistency of arithmetic. But they have also been useful in the description of formalisms in which, as in the present one, an alleviation of incompleteness theorems was aimed at. In this line Turing [19] and Feferman [5] may be mentioned. We shall open this section with an exposé on ordinals $< \epsilon_0$, following Gentzen [8], Section 4.4. About ordinals in general Suppes [17] and Takeuti [18] may be consulted.

We introduce recursively, simultaneously, the definition of the set S of ordinals $< \epsilon_0$ and of the relation $<$ on it. The equality will be the identity.

1. $S_0 = \{0\}$; not $0 < 0$.
2. Let us suppose the set S_r and the relation $<$ on it have been defined for a natural r. We then define S_{r+1} as the set consisting on 0 and all the expressions

$$\omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_n}$$

in which the α_i are members of S_r such that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. We define, in S_{r+1},

$$\omega^{\alpha_1} + \cdots + \omega^{\alpha_n} < \omega^{\beta_1} + \cdots + \omega^{\beta_m}$$

if we have, for some $j \leq n, m$, that $\alpha_i = \beta_i$ for all $i < j$ and $\alpha_j < \beta_j$; or if $m > n$ and we have $\alpha_i = \beta_i$ for all $i \leq n$. We also define $0 < \gamma$ for every $\gamma \in S_{r+1}$ different from 0 itself.

It is clear, by induction on r, that $S_r \subseteq S_{r+1}$, that the definition of $<$ in S_{r+1} coincides in S_r with the one given for it, that $<$ is irreflexive transitive and that we can decide for every two different elements of S_{r+1}, whether $\alpha < \beta$ or $\beta < \alpha$.
3. Finally, we define $S = \cup S_i$

S is, as said before, the set of ordinals $< \epsilon_0$. In the previous definition the signs 0, ω, and $+$ are devoid of any significance, the sign $+$ between expressions is only a concatenation and there is no question of associativity. We shall use the notations or identifications $1 \equiv \omega^0$, $2 \equiv \omega^0 + \omega^0$, and so on; $\omega \equiv \omega^1$, $2\omega \equiv \omega^1 + \omega^1$, and so on. The ordinal 1 and all ordinals of the form $\cdots + 1$ will be called the *successors*. All the other elements of S will be the *limit* ordinals. According to this 0 is a limit ordinal. We may agree that 0+1 denotes 1, in order to be able to describe the successor ordinals as all those of the form $\cdots + 1$. We have $n < \omega$ for every finite ordinal n; and ω is less than any other transfinite ordinal. We observe that for any successor $\beta + 1$ there is a unique limit ordinal γ such that $\gamma < \beta + 1 < \gamma + \omega$.

In fact we shall make use of the (proper?) subset S' of the elements $\gamma \in S$ for which the set of successor ordinals $\leq \gamma$ belongs to C (see Definition 1.1). This is a technical trick. In the first place it will help to make clear that **OA** is a subformalism of $\mathbf{OA_1}$; and, on the other hand, all the concrete members of S of which we shall make use will obviously belong to S'. In this way we avoid the problem of having to know if S' is, or is not, a proper subset of S.

Before entering into the definition of **OA** we still need something else: we shall recall some constructive proof of the countability of each S_n. If, in the definition of the S_n, we take the empty set $\varnothing$ instead of 0 we see, for each $n \geq 0$, that the elements of S_{n+1} may be viewed as the combinations with repetition, or multisets, of S_n. Indeed, in the first place the element $\varnothing$ of S_{n+1} is the empty multiset of S_n and, given $\omega^{\alpha_1} + \cdots + \omega^{\alpha_r} \in S_{n+1}$, we identify it with the multiset consisting of the elements $\alpha_1, \ldots, \alpha_r$, respecting their multiplicities. Conversely, given any nonempty multiset, consisting of $\beta_1, \ldots, \beta_r$, we order these elements so that $\beta_{\varphi(1)} \geq \cdots \geq \beta_{\varphi(t)}$ for some permutation φ and then identify it with the ordinal $\omega^{\beta_{\varphi(1)}} + \cdots + \omega^{\beta_{\varphi(t)}} \in S_{n+1}$. We can then accept inductively that for each positive n we can go on constructing a list $\alpha_0^n, \alpha_1^n, \ldots$ of all the elements of S_n. For S_0 the list reduces to $\varnothing$.

Definition 2.1 **OA** is a first-order theory with equality with the same language of $\mathbf{OA_1}$. Its collection of nonlogical axioms and the notion of *segment* are defined as follows.

(a) The set of *basic* axioms of **OA** is the same B listed in (a_1) of Definition 1.2 of $\mathbf{OA_1}$.

(b) A *segment* **S** of **OA** consists of an element of S', denoted $\lambda(\mathbf{S})$, the *length* of **S**, and a procedure that defines a function $j_{\mathbf{S}}$ from the set of successor ordinals $\leq \lambda(\mathbf{S})$ to the set of formulas of **OA** of the form $\forall x A(x)$ and provides, for each $\beta + 1 \leq \lambda(\mathbf{S})$, a proof of

$$B \cup \{j_S(\gamma+1) : \gamma + 1 \leq \beta\} \vdash A^{\mathbf{S}}_{\beta+1}(\overline{n})$$

for each natural n, where we understand that $\forall x A^{\mathbf{S}}_{\beta+1}(x) = j_{\mathbf{S}}(\beta+1)$. The set of axioms of **S** is then

$$AX(\mathbf{S}) = B \cup \{j_{\mathbf{S}}(\delta+1) : \delta + 1 \leq \lambda(\mathbf{S})\}.$$

(c) The collection $AX(\mathbf{OA})$ of axioms of **OA** consists of $\bigcup AX(\mathbf{S})$, for **S** running over all the segments of **OA**.

The same demand of constructivity in the correctness proofs made before for $\mathbf{OA_1}$ is made now for **OA**. The same notations, $\mathbf{S} \vdash A$, and so forth, used for $\mathbf{OA_1}$ will be used for **OA**.

Proposition 2.2 **OA** *is a subformalism of* $\mathbf{OA_1}$.

In fact, each segment **S** of **OA** is trivially interpreted as the segment **T** of $\mathbf{OA_1}$ defined by

$$\sigma_{\mathbf{T}} = \{\beta + 1 : \beta + 1 \leq \lambda(\mathbf{S})\},$$

with the order $<$ inherited from S' and with $j_{\mathbf{T}} = j_{\mathbf{S}}$. That $\sigma_{\mathbf{T}}$ belongs to the collection C of Section 1 is a consequence of the observations made at the beginning of the present section.

Proposition 2.3 *Each finite collection of axioms (theorems) of* **OA** *is a subset of the set of axioms (theorems) of one of its segments.*

For the (constructive) proof, and in order to avoid the sums of segments used in Section 1, the following device may be used. Given two segments of **OA**, **S**, and **T**, we define their *fusion*, denoted **ST**, as follows, supposing $\lambda(\mathbf{S}) \leq \lambda(\mathbf{T})$ (and the symmetrical case can be treated in a similar way).

We first define the length $\lambda(\mathbf{ST})$ of **ST**: $\lambda(\mathbf{ST}) = \lambda(\mathbf{T})$ if $\lambda(\mathbf{T})$ is a limit ordinal, or if $\lambda(\mathbf{T}) = \delta + n$ and $\lambda(\mathbf{S}) \leq \delta$, where δ is limit and n finite. $\lambda(\mathbf{ST}) = \delta + n + m$ if $\lambda(\mathbf{T}) = \delta + n$ and $\lambda(\mathbf{S}) = \delta + m$, where δ is limit and n and m are finite. Now, for each limit ordinal $\alpha \leq \lambda(\mathbf{T})$ we consider the lists of values of $j_{\mathbf{S}}$ and $j_{\mathbf{T}}$ in the ordinals $\alpha + s$ for s finite:

$$j_{\mathbf{S}}(\alpha+1),\ j_{\mathbf{S}}(\alpha+2), \ldots$$

$$j_{\mathbf{T}}(\alpha+1),\ j_{\mathbf{T}}(\alpha+2), \ldots$$

in which it may happen that both are infinite or that the second is infinite but not the first, or that both are finite and the first not longer than the second. We then take as values of $j_{\mathbf{ST}}(\alpha + s)$ for $s = 1, 2, \ldots$ alternately the members of the first and the second lists, continuing only with the second if and when the first is exhausted. Once this has been done it is clear how the procedures of 2.1(b) for **S** and for **T** furnish a procedure for **ST**. It is also clear that $AX(\mathbf{ST}) = AX(\mathbf{S}) \cup AX(\mathbf{T})$.

This ends the definition of the fusion of two segments. Given three segments we can define the fusion **STU** as $(\mathbf{ST})\mathbf{U}$, and so forth, for any finite collection of segments. So that every finite collection of axioms of **OA** is clearly a subset of the set of axioms of some segment. Consequently, for every finite collection of theorems of **OA** we can find a segment of which all of them are theorems.

3 A segment* P *of* OA*: Peano Arithmetic In this section and the following one we shall rely on Mendelson [13] whenever we need to refer to a formal presentation of Peano Arithmetic. For any $A(x)$ let us consider the formula

$$\forall x\,[A(0) \wedge \forall x\,(A(x) \Longrightarrow A(x')) \Longrightarrow A(x)].$$

For each n we shall have

$$B \vdash A(0) \wedge \forall x\,(A(x) \Longrightarrow A(x')) \Longrightarrow A(\overline{n}). \tag{3.1}$$

The way to prove it is obviously the following: let us suppose

$$A(0) \wedge \forall x\,(A(x) \Longrightarrow A(x')). \tag{3.2}$$

For $n = 0$ we have $A(0)$. Suppose now $A(\overline{s})$ proved with the hypothesis (3.2). We have, from (3.2), $A(\overline{s}) \Longrightarrow A(\overline{s+1})$, from which we obtain $A(\overline{s+1})$. We have in this way, for each n, a proof of $A(\overline{n})$ from (3.2). As the inference rule generalization has not been used we have a proof of (3.1) for each n.

If we enumerate the formulas $A(x)$,

$$A_1(x),\ A_2(x), \ldots$$

we define a segment **P** of length ω by putting

$$j_{\mathbf{P}}(m) = \forall x\,[A_m(0) \wedge \forall x\,(A_m(x) \Longrightarrow A_m(x')) \Longrightarrow A_m(x)] \tag{3.3}$$

and take as a proof of $A_m(0) \wedge \forall x\,(A_m(x) \Longrightarrow A_m(x')) \Longrightarrow A_m(\overline{n})$ from $B \cup \{j_{\mathbf{P}}(k) : k < m\}$ for each n the one we have just described, which makes use only of B. This segment **P** is, clearly, first-order Peano Arithmetic since the second member of (3.3) is equivalent to the axiom of induction for A_m in the usual form (see [13]).

4 *Gödel's incompleteness method as a way of introducing axioms in* OA

4.1 Let us consider the segment **P** described before. From it a new segment $\mathbf{P}'$ of length $\omega + 1$ will be defined. The results we shall now recall are taken from [13], pp. 143 and following. We have a primitive recursive binary relation $\mathbf{W_1}(u, y)$ in $\mathbb{N}$ and a formula $W_1(x_1, x_2)$ with the following properties:

(a) if $\mathbf{W_1}(k_1, k_2)$ then $\mathbf{P} \vdash W_1(\overline{k_1}, \overline{k_2})$;
(b) if not $\mathbf{W_1}(k_1, k_2)$ then $\mathbf{P} \vdash \neg W_1(\overline{k_1}, \overline{k_2})$;
(c) $\mathbf{W_1}(u, y)$ if and only if u is the Gödel number of a formula $A(x_1)$ that contains the free variable x_1 and y is the Gödel number of a proof of $A(\bar{u})$ in **P**.

We now consider the formula

$$\forall x_2 \neg W_1(x_1, x_2). \tag{4.1.1}$$

Let m be the Gödel number of this formula and let us consider

$$\forall x_2 \neg W_1(\overline{m}, x_2). \tag{4.1.2}$$

A procedure will be provided to prove

$$\mathbf{P} \vdash \neg W_1(\overline{m}, \overline{b}) \text{ for each natural } b. \tag{4.1.3}$$

As the characteristic function of $\mathbf{W_1}(u, y)$ is primitive recursive we can decide, for each b, whether $\mathbf{W_1}(m, b)$ or not. We have the two following cases.

Case 1: If not $\mathbf{W_1}(m, b)$ then $\mathbf{P} \vdash \neg W_1(\overline{m}, \overline{b})$, according to (4.1.b); and (4.1.3) is proved for this b.

Case 2: If $\mathbf{W_1}(m, b)$ then, by (4.1.c) and because m is the Gödel number of (4.1.1), b is the Gödel number of a proof of $\forall x_2 \neg W_1(\overline{m}, x_2)$ in **P**. We can reproduce that proof and, by specializing x_2 to $\overline{b}$ we obtain a proof of (4.1.3) for this b.

So that, by defining $\lambda(\mathbf{P}') = \omega + 1$, $j_{\mathbf{P}'}(\beta + 1) = j_{\mathbf{P}}(\beta + 1)$ for every $\beta + 1 < \omega$ and $j_{\mathbf{P}'}(\omega + 1) = \forall x_2 \neg W_1(\overline{m}, x_2)$, we have the segment $\mathbf{P}'$ in which (4.1.2) is an axiom, while neither (4.1.2) nor its negation are theorems of **P** (Proposition 3.31 of [13]). It is in this sense that Gödel's method becomes a method to introduce axioms of **OA**.

The procedure just described may be repeated starting from $\mathbf{P}'$ instead of **P**, if $W_1(x_1, x_2)$ is substituted by the corresponding formula, and so on. A systematization of the process would give a new segment $\mathbf{P}''$ of length 2ω. From this last segment we can begin again, and so forth. Of course, the choice of different gödelizations would give different segments.

4.2 To describe another kind of segment let us begin, for some gödelization g, with

$$\exists y Pf(y, \overline{g(A)}) \Longrightarrow A, \tag{4.2.1}$$

where $Pf(y, z)$ corresponds to a primitive recursive relation $\mathbf{Pf}(u, v)$ such that $\mathbf{Pf}(u, v)$ if and only if u is the Gödel number of a proof in **P** of the formula of Gödel number v. We may suppose that y does not appear in A and then (4.2.1) is equivalent to

$$\forall y\,(\neg Pf(y, \overline{g(A)}) \vee A) \tag{4.2.2}$$

and we can prove easily that

$$\mathbf{P} \vdash \neg Pf(\overline{b}, \overline{g(A)}) \vee A \text{ for every natural } b.$$

(The way to prove it is: if not $\mathbf{Pf}(b, g(A))$, then $\mathbf{P} \vdash \neg Pf(\overline{b}, \overline{g(A)})$; if $\mathbf{Pf}(b, g(A))$ then we reproduce the proof of Gödel number b, which is a proof of A, and we have $\mathbf{P} \vdash A$.)

We can then construct, from $\mathbf{P}$, a segment $\mathbf{U}$ of length $\omega + 1$ in which (4.2.2) is an axiom. We can also iterate the process, like in Section 4.1. Formula 4.2.1 is the *reflection principle* for A. It depends, of course, on the particular gödelization. About the reflection principles see Feferman [7]. The effects of interpreting (4.2.1) as the box of modal logic are investigated in Boolos [1], Chapter 3. Löb proved that for all sentences A if (4.2.1) is a theorem of $\mathbf{P}$ then A is also a theorem of $\mathbf{P}$ (see the same [1] or Smorinski [15]).

4.3 Another example is the following. Let us consider the set

$$K = \{x : x \text{ is the Gödel number of a proof of } \neg 0 = 0 \text{ in } \mathbf{P}\}.$$

K is diophantine (recursively enumerable), so that we can find a polynomial $P \in \mathbb{Z}[x, y_1, \ldots, y_t]$ such that

$$x \in K \text{ iff there exist naturals } y_i \text{ such that } P(x, y_i) = 0$$

(see Jones [11] or Davis [3], about diophantine sets). As the coefficients of P are integers there are, clearly, terms $U(z_i)$ and $V(z_i)$ of the language of $\mathbf{P}$, in which no other variables than $z_0, \ldots, z_t$ appear, such that if $P(b_i)$ is zero (respectively, not zero) then $U(\overline{b_i}) = V(\overline{b_i})$ (respectively, $\neg U(\overline{b_i}) = V(\overline{b_i})$) is a theorem of $\mathbf{P}$. To simplify, let us denote the relation $U(z_i) = V(z_i)$ by $R(z_i)$, so that we have

$$\text{if } P(b_i) = 0, \text{ then } \mathbf{P} \vdash R(\overline{b_i}); \tag{4.3.1}$$

$$\text{if not } P(b_i) = 0, \text{ then } \mathbf{P} \vdash \neg R(\overline{b_i}). \tag{4.3.2}$$

There is a procedure to prove

$$\mathbf{P} \vdash \neg R(\overline{b_0}, \ldots, \overline{b_t}) \text{ for each } (b_i) \in \mathbb{N}^{t+1}. \tag{$*$}$$

We shall discuss that simple procedure later, in order to insist in the demand that its correctness be proved finitarily. Now, supposing ($*$) proved, we shall see how to construct a segment $\mathbf{T}$ of length $(t+1)\omega + 1$ having $\forall z_t \ldots \forall z_0 \neg R(z_0, \ldots, z_t)$ as an axiom. We begin by defining $j_{\mathbf{T}}(\delta + 1) = j_{\mathbf{P}}(\delta + 1)$ for $\delta + 1 < \omega$. We then enumerate the t-tuples of naturals:

$$(b^1_{1,1}, \ldots, b^1_{1,t}), (b^2_{1,1}, \ldots, b^2_{1,t}), \ldots.$$

For the first one we have, from $(*)$, that for each b_0,

$$B \cup \{ j_{\mathbf{T}}(\delta+1) : \delta_1 < \omega+1\} \vdash \neg R(\overline{b_0}, \overline{b^1_{1,1}}, \dots, \overline{b^1_{1,t}}).$$

We then define $j_{\mathbf{T}}(\omega+1) = \forall z_o \neg R(z_0, \overline{b^1_{1,1}}, \dots, \overline{b^1_{1,t}})$. In a similar way we make

$$j_{\mathbf{T}}(\omega+2) = \forall z_0 \neg R(z_0, \overline{b^2_{1,1}}, \dots, \overline{b^2_{1,t}}),$$

and so on, and shall have that $j_{\mathbf{T}}(\delta+1)$ is defined for every $\delta+1 < 2\omega$ and that $\forall z_0 \neg(z_0, \overline{b_1}, \dots, \overline{b_t})$ is an axiom of the segment defined until now (of length 2ω) for every t-tuple of naturals $(b_1, \dots, b_t)$.

We now enumerate the $t-1$-tuples of naturals,

$$(b^1_{2,2}, \dots, b^1_{2,t}), (b^2_{2,2}, \dots, b^2_{2,t}), \dots$$

and define

$$j_{\mathbf{T}}(2\omega+i) = \forall z_1 \forall z_0 \neg R(z_0, z_1, \overline{b^i_{2,1}}, \dots, \overline{b^i_{2,t}})$$

for $i = 1, 2, \dots$ and obtain a segment of length 3ω in which $\forall z_1 \forall z_0 \neg R(z_0, z_1, \overline{b_2}, \dots, \overline{b_t})$ is an axiom for each $t-1$-tuple of naturals $(b_2, \dots, b_t)$.

Continuing in this way we arrive to a segment of length $(t+1)\omega$ in which

$$\forall z_{t-1}, \dots, \forall z_0 \neg R(z_0, \dots, z_{t-1}, \overline{b_t})$$

is an axiom for each b_t. We then define

$$j_{\mathbf{T}}((t+1)\omega+1) = \forall z_t, \dots, \forall z_0 \neg R(z_0, \dots, z_t)$$

and the construction will be finished.

Let us now return to the proof of $(*)$. It would be inadmissible to rely on some proof of the consistency of $\mathbf{P}$ and say: as $\mathbf{P}$ is consistent there is no proof of $\neg 0 = 0$ in it, so that for every (b_i) we have not $P(b_i) = 0$ and, according to (4.3.2), $\mathbf{P} \vdash \neg R(\overline{b_i})$. This proof of the correctness would not be finitary (unless the consistency proof were finitary). The finitary proof of $(*)$ would be as follows.

For each (b_i) we decide whether $P(b_i) = 0$ or not in $\mathbb{N}$. If not, then we have $\mathbf{P} \vdash \neg R(\overline{b_i})$, by (4.3.2). If the answer is yes, then there is a proof of $\mathbf{P} \vdash \neg 0 = 0$ of Gödel number b_0 which we can reproduce. Then from this and $\mathbf{P} \vdash 0 = 0$ we can prove anything, in particular $(*)$.

5 *About the Paris-Harrington theorem* Paris and others obtained some true arithmetical statements which are interesting in themselves and not provable in Peano Arithmetic. One of them is the theorem of Paris and Harrington (see Paris and Harrington [14], Smorinski [16], Graham, Rothschild, and Spencer [9] and their references). It is asserted in [14] that this theorem, originally combinatorial (expressed as $\forall e, r, k \exists M (M \overset{*}{\to} (k)^e_r))$ is also expressible in the language of first-order Peano Arithmetic. But it turns out that it is not provable there (see [14]). If $Q(e)$ is the arithmetical formulation of $\forall r, k \exists M (M \overset{*}{\to} (k)^e_r)$ it is also asserted that

$$\mathbf{P} \vdash Q(\bar{e}) \text{ for each natural } e \tag{5.1}$$

(whereas, as remarked before, $\forall x\, Q(x)$ is not provable in **P**). Let us consider the procedure that proves (5.1) for each e. If its correctness is provable constructively or finitarily, then we have $\mathbf{OA} \vdash \forall x\, Q(x)$. The author has no reason to doubt that it is so but, not being familiar with the subject, is not able at this moment to assure that such correctness is indeed constructively provable.

Other ways of proving the incompleteness of Peano Arithmetic are discussed in Kotlarski [12].

Appendix A constructively consistent open formalism for arithmetic

A.1 The intermediate formalism $\mathbf{OA_1'}$ Let us remember that we are relying on [13] for the general presentation of arithmetic, so that $\vee$, $\wedge$, and $\exists$ are defined symbols and modus ponens and generalization are the only inference rules of $\mathbf{OA_1}$. The "intermediate" open formalism $\mathbf{OA_1'}$ to be described now differs from the formalism $\mathbf{OA_1}$ described in Section 1.2 in the following ways (1, 2, and 3).

1. All the formulas susceptible of being axioms or theorems of $\mathbf{OA_1'}$ will be closed. Following rigorously the terminology of [13] we could then say that the relevant formulas of $\mathbf{OA_1'}$ are the closed well-formed formulas of first-order arithmetic, shortened as cwf. If A is a well-formed formula of $\mathbf{OA_1}$ in which all the variables that have a free presence belong to the set $\{x_1, \dots, x_n\}$ we say that $\forall x_1, \dots, \forall x_n\, A$ is a *closure* of A. We say that the closure is *strict* if all the variables of the set do have a free presence in A. For instance, $\forall x \forall y\; x+y=y+x$, $\forall x \forall y \forall z\; x+y=y+x$, and $\forall y \forall x\; x+y=y+x$ are closures of $x+y=y+x$, and the first and third are strict ones. If all the variables appearing free in A are $x_1, \dots, x_s$ and have their first free appearances in A in precisely this order, then we say that $\forall x_1, \dots, \forall x_s\; A$ is the natural closure of A, and denote it $\overline{A}$. If A is closed we understand that $\overline{A} = A$. The first of the three closures in the previous example is the natural one.
2. The *logical* (*basic*) axioms of $\mathbf{OA_1'}$ are the natural closures of all the logical (basic) axioms of $\mathbf{OA_1}$. The set of basic axioms of $\mathbf{OA_1'}$ will be denoted B'.
3. The generalization rule is not an inference rule of $\mathbf{OA_1'}$ (it does not even make much sense in it, according to 1). Modus ponens is then its only inference rule. Its exact form in $\mathbf{OA_1'}$ is: whenever A, B, and C are cwf we infer B from $A \Longrightarrow B$ and A.

Our aim is to prove that $\mathbf{OA_1'}$ has essentially the same deductive power of $\mathbf{OA_1}$. To this end we need to discuss a device already used in Section 4.3. It is, in general, in the context of $\mathbf{OA_1'}$, the following.

A.1.1 Device If, in a certain segment **T** of $\mathbf{OA_1'}$ we have, for a certain well-formed $A(x_1, \dots, x_r)$, that $A(\overline{n}_1, \dots, \overline{n}_r)$ is a theorem of **T** for every R-tuple of naturals (n_i), then we take

$$\tau = \{\beta+1 : \beta+1 \in S',\ \beta+1 \leq (r-1)\omega+1\},$$

with the order inherited from S', and

$$\zeta = (\sigma_{\mathbf{T}} \times \{0\}) \cup (\tau \times \{1\})),$$

with $(x, i) < (y, j)$ if $i < j$, or if $i = j$ and $x < y$.

We then consider any permutation i of $\{1, \ldots, r\}$ and a list

$$(n_1^1, \ldots, n_r^1), (n_1^2, \ldots, n_r^2), \ldots \tag{A.1.1}$$

of all the r-tuples of naturals and define a new segment $\mathbf{U}$ in the following way. We first put

$$\sigma_{\mathbf{U}} = \zeta, \;\; j_{\mathbf{U}}((\alpha, 0)) = j_{\mathbf{T}}(\alpha) \text{ for all } \alpha \in \sigma_{\mathbf{T}}.$$

We then define the values $j_{\mathbf{T}}((\beta, 1))$ for $\beta \in \tau$ as follows. We first substitute $n_{i(1)}^j$ by $x_{i(1)}$ in all the members of the list (A.1.1), then suppress the members that repeat previous ones, and then renumerate the upper indices, obtaining a new list

$$(n_1^1, \ldots, x_{i(1)}, \ldots, n_r^1), (n_1^2, \ldots, x_{i(1)}, \ldots, n_r^2), \ldots \tag{A.1.2}$$

and define

$$j_{\mathbf{U}}((1, 1)) = \forall x_{i(1)}\, A(\overline{n}_1^1, \ldots, x_{i(1)}, \ldots, \overline{n}_r^1)$$
$$j_{\mathbf{U}}((2, 1)) = \forall x_{i(1)}\, A(\overline{n}_1^2, \ldots, x_{i(1)}, \ldots, \overline{n}_r^2)$$
$$\ldots$$

In all the members of the new list (A.1.2) we now substitute $n_{i(2)}^j$ by $x_{i(2)}$, suppress all the members that repeat previous ones and then renumerate it, thus obtaining (supposing, to simplify the notation, that $i(1) \leq i(2)$) a new list

$$(n_1^1, \ldots, x_{i(1)}, \ldots, x_{i(2)}, \ldots, n_r^1), \;(n_1^2, \ldots, x_{i(1)}, \ldots, x_{i(2)}, \ldots, n_r^2), \ldots \tag{A.1.3}$$

and define

$$j_{\mathbf{U}}((\omega + 1, 1)) = \forall x_{i(2)} \forall x_{i(1)}\, A(\overline{n}_1^1, \ldots, x_{i(1)}, \ldots, x_{i(2)}, \ldots, \overline{n}_r^1)$$
$$j_{\mathbf{U}}((\omega + 1, 2)) = \forall x_{i(2)} \forall x_{i(1)}\, A(\overline{n}_1^2, \ldots, x_{i(1)}, \ldots, x_{i(2)}, \ldots, \overline{n}_r^2)$$
$$\ldots$$

By iterating the process we arrive finally to

$$j_{\mathbf{U}}(((r-1)\omega + 1, 1)) = \forall x_{i(r)}, \ldots, \forall x_{i(1)}\, A(x_1, \ldots, x_r)$$

It is clear that $\mathbf{U}$ is a segment of $\mathbf{OA}_1'$. This describes the device.

Proposition A.1 *The natural closure of every theorem of* $\mathbf{OA}_1$ *is a theorem of* $\mathbf{OA}_1'$.

Proof: The proof will be constructive. As every well-formed formula is interdeducible in $\mathbf{OA}_1$ with any of its closures, then it is clear that the proposition will enable us to think of $\mathbf{OA}_1'$ as a presentation of $\mathbf{OA}_1$ having modus ponens as sole inference rule.

The proof will consist in describing the way, given any segment $\mathbf{S}$ of $\mathbf{OA_1}$, of constructing a segment $\mathbf{S}'$ of $\mathbf{OA}'_\mathbf{1}$ having among its theorems the natural closures of all the theorems of $\mathbf{S}$.

We begin, given $\mathbf{S}$, by considering any $\beta \in \sigma_\mathbf{S}$ and the segment $\mathbf{S}_{<\beta}$ defined by

$$\sigma_{\mathbf{S}_{<\beta}} = \{\gamma : \gamma \in \sigma_\mathbf{S},\ \gamma < \beta\},$$

$j_{\mathbf{S}_{<\beta}}$ being the restriction of $j_\mathbf{S}$ to $\sigma_{\mathbf{S}_{<\beta}}$. We write a list of all the logical and basic axioms of $\mathbf{S}_{<\beta}$ (the same as for $\mathbf{S}$). As $\sigma_{\mathbf{S}_{<\beta}}$ belongs to C (remember Definition 1.1) we can make a list of all its elements and then a list of all the axioms $j_\mathbf{S}(\gamma)$ for $\gamma \in \sigma_{\mathbf{S}_{<\beta}}$. From these two lists we can make a list of all the axioms of $\mathbf{S}_{<\beta}$ and then a list

$$A_1^\beta,\ A_2^\beta, \ldots \tag{A.1.4}$$

such that

(i) it contains all the theorems of $\mathbf{S}_{<\beta}$,
(ii) each member of the list is an axiom (logical or nonlogical) of $\mathbf{S}_{<\beta}$ or is obtained from previous members by generalization or by modus ponens.

We then consider the list

$$\overline{A_1^\beta},\ \overline{A_2^\beta},\ \ldots \tag{A.1.5}$$

of the natural closures of the previous one.

In preparation for the construction of the associated segment $\mathbf{S}'$ let us advance that for each $\beta \in \sigma_\mathbf{S}$, and related to the list (A.1.5), a certain sequence $\alpha_{\beta,1},\ \alpha_{\beta,2}, \ldots$ of ordinals that are 0 or of the form $t\omega + 1$, for natural t, will be defined, as well as the corresponding

$$\tau_{\beta,i} = \{\gamma + 1 : \gamma + 1 \in S,\ \gamma + 1 \leq \alpha_{\beta,i}\}.$$

We shall also define an $\alpha_{\beta,\omega}$ of the form $t\omega + 1$, and

$$\tau_{\beta,\omega} = \{\gamma + 1 : \gamma + 1 \in S,\ \gamma + 1 \leq \alpha_{\beta,\omega}\}.$$

The order in these sets is the restriction of the orders in S. We then will define

$$\beta_\beta = ((\tau_{\beta,1} \times \{1\}) \cup (\tau_{\beta,2} \times \{2\}) \cup \cdots \cup (\tau_{\beta,\omega} \times \{\omega\}) \times \{\beta\}$$

with the order defined as $((\delta, i), \beta) < ((\eta, j), \beta)$ if $i < j$, or if $i = j$ and $\delta < \eta$ (where i and j represent naturals or ω).

For the future segment $\mathbf{S}'$ of $\mathbf{OA}'_\mathbf{1}$ we shall have

$$\sigma_{\mathbf{S}'} = \bigcup_{\beta \in \mathbf{S}} \beta_\beta,$$

where, for $\delta_1 \in \beta_\beta$ and $\delta_2 \in \gamma_\gamma$, we have $\delta_1 < \delta_2$ if $\beta < \gamma$ in $\sigma_\mathbf{S}$, or if $\beta = \gamma$ and $\delta_1 < \delta_2$ in β_β. It is already clear that $\sigma_{\mathbf{S}'}$ will belong to C. We define, to begin with the description of $\mathbf{S}'$,

$$j_{\mathbf{S}'}(((\alpha_{\beta,\omega}, \omega), \beta)) = \overline{j_\mathbf{S}(\beta)} \tag{A.1.6}$$

for each β, where $\alpha_{\beta,\omega}$ is, of course, the last or greatest element of $\tau_{\beta,\omega}$.

More concretely, what shall be done is, for each $\beta \in \sigma_{\mathbf{S}}$, the following.

$\boldsymbol{\beta}$.I To define the $\alpha_{\beta,i}$ and $\tau_{\beta,i}$ and the not yet defined $j_{\mathbf{S}'}(\gamma)$ for $\gamma \in \beta_\beta$, in accordance, of course, with (b_1) of Definition 1.2; and to prove that every member of the list (A.1.5) is a theorem of the segment $\mathbf{S}'_{<\beta'}$, where the notation $\beta' = ((\alpha_{\beta,\omega}, \omega), \beta)$ is used, defined by

$$\sigma_{\mathbf{S}'_{<\beta'}} = \{\gamma : \gamma \in \sigma_{\mathbf{S}'}, \gamma < \beta'\},$$

$j_{\mathbf{S}'_{<\beta'}}$ being the restriction of $j_{\mathbf{S}'}$ to $\sigma_{\mathbf{S}'_{<\beta'}}$.

$\boldsymbol{\beta}$.II If $j_{\mathbf{S}'}(((\alpha_{\beta,\omega}, \omega), \beta)) = \overline{j_{\mathbf{S}}(\beta)} = \forall x\ B(x)$, to prove that $B(\overline{n})$ is a theorem of $\mathbf{S}'_{<\beta'}$ for each natural n.

Once this is done it will have been proved that $\mathbf{S}'$ is a segment of $\mathbf{OA}'$ in which the natural closure of every theorem of $\mathbf{S}$ is a theorem. In this process it is very important to note the following

Note A.2 The accomplishments of $\boldsymbol{\beta}$.I and $\boldsymbol{\beta}$.II for each $\beta \in \sigma_{\mathbf{S}}$ will be completely independent of the same actions for every other such β; they will depend only on $\mathbf{S}$. In fact, in each of these accomplishments only the basic and logical axioms and those of the form $\overline{j_{\mathbf{S}}(\gamma)}$ for $\gamma \leq \beta$ will be used, apart from those of the form $j_{\mathbf{S}'}(\delta)$ for $\delta \in \beta_\beta$ that may appear in the same process. This will avoid completely any recourse to transfinite induction and maintain the proof completely constructive.

To accomplish $\boldsymbol{\beta}$.I we run through the lists (A.1.4) and (A.1.5) performing the steps described as follows.

For positive i, the ith step takes $\tau_{\beta,j}$ and the values of $j'_{\mathbf{S}}(\gamma)$ for

$$\gamma \in \bigcup_{j<i} (\tau_{\beta,j} \times \{j\}) \times \{\beta\}$$

as already given and supposes that it has been proved, by means of all the logical and basic axioms of $\mathbf{S}'$ (the natural closures of those of $\mathbf{S}$), the axioms $j_{\mathbf{S}'}(\gamma)$ for γ as before and the axioms $j_{\mathbf{S}'}(((\alpha_{\delta,\omega}, \omega)\delta))$ for $\delta < \beta$ (see (A.1.6)), that all the $\overline{A_j^\beta}$ for $j < i$ are theorems of $\mathbf{S}'$. We then want to define $\tau_{\beta,i}$ and the corresponding values of $j_{\mathbf{S}'}$, in order to make appear A_i^β as a theorem. We have three cases to consider.

Case 1: A_i^β is an axiom of $\mathbf{S}_{<\beta}$. In this case we take $\alpha_{\beta,i} = 0$, so that $\tau_{\beta,i} = \varnothing$. And $\overline{A_i^\beta}$ is a theorem of $\mathbf{S}'_{<\beta'}$ because then it is either a logical or basic axiom of $\mathbf{OA}'_{\mathbf{1}}$ or, according to (A.1.6), it is an axiom of it of the form

$$j_{\mathbf{S}'}(((\alpha_{\gamma,\omega}, \omega), \gamma))$$

for some $\gamma < \beta$.

Case 2: A_i^β is obtained by modus ponens from previous members A_j^β and $A_k^\beta = A_j^\beta \Longrightarrow A_i^\beta$. By induction hypotheses $\overline{A_j^\beta}$ and $\overline{A_j^\beta \Longrightarrow A_i^\beta}$ are theorems proved by means of the said axioms. Suppose $x_1, \dots, x_r$ are the variables appearing free in A_k^β. Then, with the obvious notation,

$$A_j^\beta(\overline{n}_1, \dots, \overline{n}_r) \Longrightarrow A_i^\beta(\overline{n}_1, \dots, \overline{n}_r) \text{ and } A_j^\beta(\overline{n}_1, \dots, \overline{n}_r)$$

are theorems (proved with the same means) and, by the device A.1.1, we may define $\alpha_{\beta,\omega}$ (in fact equal to $(r-1)\omega+1$) and the values of $j_{\mathbf{S}'}(((\delta, i)\beta))$ for $\delta \in \tau_{\beta,i}$ in order to have $\overline{A_i^\beta}$ as the axiom $j_{\mathbf{S}'}(((\alpha_{\beta,i}, i), \beta))$.

Case 3: A_i^β is obtained by generalization from a previous member of the list. This case is easier to deal with than the previous one.

We must now accomplish $\boldsymbol{\beta}$.II. Let us suppose

$$j_{\mathbf{S}}(\beta) = \forall y\, C(x_1, \dots, x_r) \quad \text{and} \quad \overline{j_{\mathbf{S}}(\beta)} = \forall x\, B(x),$$

where the x_i are all the variables that have free presences in $C(x_i)$ and the variables x, y may, or not, coincide with one another or with some x_i. To concretize, we shall suppose x is x_1 and y is x_2. The proof corresponding to any other possibility will be a more or less trivial variant of the one that shall be given now.

According to the first identity, now written as

$$j_{\mathbf{S}}(\beta) = \forall x_2\, C(x_i),$$

the formula $C(x_1, \overline{n}_2, \dots, x_r)$ is, for each natural n_2, a theorem that must appear in (A.1.4), so that every $C(\overline{n}_i)$ must also appear in the list, for all (n_i). And, once accomplished $\boldsymbol{\beta}$.I, they must also be theorems of $\mathbf{S}'_{<((1,\omega),\beta)}$. Now, by using the same device A.1.1, we can define $\tau_{\beta,\omega}$ and the values of $j_{\mathbf{S}'}$ in $\tau_{\beta,\omega} \times \{\omega\}) \times \{\beta\}$, so that we obtain

$$j_{\mathbf{S}'}(((\alpha_{\beta,\omega}, \omega), \beta)) = \overline{j_{\mathbf{S}}(\beta)},$$

as wanted.

Note A.3 (About the consistency of $\mathbf{OA}_1'$) Clearly, we may prove that $\mathbf{OA}_1'$ is consistent by showing that $(\mathbb{N}, +, \cdot, ')$ is a model of it. But let us look more closely at this consistency problem, considering that the only inference rule of $\mathbf{OA}'$ is modus ponens.

We admit in $\mathbf{OA}_1'$, as possible axioms or theorems, only cwf; and, following [13], we only make use of $\neg$, $\Longrightarrow$, and $\forall$ as undefined logical symbols. We say that a cwf of $\mathbf{OA}_1'$ is *irreducible* if it is not of the forms $\neg A$ or $A \Longrightarrow B$. The irreducible formulas of $\mathbf{OA}_1'$ are then the cwf that are atomic or of the form $\forall x\, A$. All the cwf of $\mathbf{OA}_1'$ are then obtained from the irreducible ones by means of repeated use of $\neg$ and $\Longrightarrow$. We now recall that the theorems of $\mathbf{OA}_1'$ are obtained from $AX(\mathbf{OA}_1')$ by iterated use of modus ponens and that every theorem requires for its proof only a finite set of those axioms. We may also remember that any such finite set of axioms is a subset of the set of axioms of some segment of $\mathbf{OA}_1'$. It is then clear that $\mathbf{OA}_1'$ is consistent if and only if there is a way, for any finite collection of its axioms, of assigning truth values T or F to every member of the finite set of irreducible formulas from which all the axioms of the set are built by means of $\neg$ and $\Longrightarrow$ in such a way that the logical valuation induced by these values in the finite set of axioms under consideration have value T in all its members. As we know that $\mathbf{OA_1'}$ is consistent, we know in consequence that this procedure does exist: we can find by trial an adequate set of values T or F for the irreducible cwf from which a finite set of axioms is built because there is only a finite number of possibilities in assigning these values. But, of course, the proof that this procedure is correct is not finitary: it depends on the original nonfinitary semantical proof of the consistency of $\mathbf{OA_1'}$. Had we not this proof at hand we could not be sure there is, in every case, a satisfactory assignment of values T, F to be found by trial.

The situation just described exemplifies anew the general phenomenon: there are correct procedures whose correctness is not proved finitarily. This is the kind of phenomenon that forced the demand of constructivity in the proofs of the correctness made in Definitions 1.2(b_1) and 2.1(b). □

A.2 The constructively consistent open formalism $\mathbf{OA}_1''$

A.2.1 The formalism $\mathbf{OA_1''}$ has, as $\mathbf{OA_1}$, $\mathbf{OA}$, or $\mathbf{OA_1'}$, the language of a first-order theory, with the constant 0 and the functional symbols $+$, $\cdot$, and $'$. Here, following [13], we take $\neg$, $\Longrightarrow$, and $\forall$ as the only primitive logical symbols.

A.2.2 The only inference rule, as in $\mathbf{OA_1'}$, will be modus ponens in the form 'B is inferred from A and $A \Longrightarrow B$, for any cwf A and B'.

A.2.3 The *logical* axioms will not be the same as for a first-order theory. They will be:

- (i) $A \Longrightarrow (B \Longrightarrow A)$, for any cwf A, B;
- (ii) $((A \Longrightarrow (B \Longrightarrow C)) \Longrightarrow ((A \Longrightarrow B) \Longrightarrow (A \Longrightarrow C))$, for any cwf A, B, C;
- (iii) $(\neg B \Longrightarrow \neg A) \Longrightarrow ((\neg B \Longrightarrow A) \Longrightarrow B)$, for any cwf A and B;
- (iv) $u = u$, for any constant term u;
- (v) $u = v \Longrightarrow (t_1 = t_2 \Longrightarrow t_1^* = t_2^*)$, for any constant terms u, v, t_1, t_2, whenever, for each i, t_i^* is obtained from t_i by substituting some presence of u by v, or inversely, or by leaving t_i unchanged.

A.2.4 The *arithmetical* axioms will be the following (vi) – (xi), for any natural n, n_1, n_2:

- (vi) $\neg 0 = \overline{n}'$;
- (vii) $\overline{n}_1' = \overline{n}_2' \Longrightarrow \overline{n}_1 = \overline{n}_2$;
- (viii) $\overline{n}_1 + 0 = \overline{n}_1$;
- (ix) $\overline{n}_1 + \overline{n}_2' = (\overline{n}_1 + \overline{n}_2)'$;
- (x) $\overline{n}_1 \cdot 0 = 0$;
- (xi) $\overline{n}_1 \cdot \overline{n}_2' = (\overline{n}_1 \cdot \overline{n}_2) + \overline{n}_1$.

If we denote the sets of logical and arithmetical axioms of $\mathbf{OA_1''}$ by B_L'' and by B_A'', then we define the set B'' of *initial* axioms of $\mathbf{OA_1''}$ as

$$B'' = B_L'' \cup B_A''.$$

A.2.5 We define the collection $AX(\mathbf{OA}_1'')$ of all axioms of $\mathbf{OA}_1''$ as follows, by means of the notion of segment, analogously to (b_1) of Definition 1.2. A *segment* $\mathbf{U}$ of $\mathbf{OA}_1''$ consists of a member $\sigma_{\mathbf{U}}$ of the class C (see Section 1) and a procedure that defines a function $j_{\mathbf{U}}$ from $\sigma_{\mathbf{U}}$ to the set of cwfs of $\mathbf{OA}_1''$ of the form $\forall x\, A(x)$ or $\neg\forall x\, B(x)$ and provides, for each $\delta \in \sigma_{\mathbf{U}}$, a proof of

$$B' \cup \{ j_{\mathbf{U}}(\gamma) : \gamma \in \sigma_{\mathbf{U}},\ \gamma < \delta\} \vdash A^{\mathbf{U}}_{\delta}(\overline{n})$$

for each natural n, if $\forall x\, A^{\mathbf{U}}_{\delta}(x) \ =\ j_{\mathbf{U}}(\delta)$; and a proof of

$$B'' \cup \{ j_{\mathbf{U}}(\gamma) : \gamma \in \sigma_{\mathbf{U}},\ \gamma < \delta\} \vdash \neg A^{\mathbf{U}}_{\delta}(\overline{n})$$

for some natural n, if $\neg\forall x\, A^{\mathbf{U}}_{\delta}(x) = j_{\mathbf{U}}(\delta)$. We then define

$$AX(\mathbf{U}) \ =\ B'' \cup \{ j_{\mathbf{U}}(\gamma) : \gamma \in \sigma_{\mathbf{U}}\}.$$

It is demanded, as always, that the correctness of the whole procedure be proved constructively. The collection of all the axioms of $\mathbf{OA}_1'$ is then

$$AX(\mathbf{OA}_1'') = \bigcup AX(\mathbf{U}),$$

where the union runs over all the segments $\mathbf{U}$ of $\mathbf{OA}_1''$. We include explicitly the logical axioms in $AX(\mathbf{OA}_1'')$ because, unlike for $\mathbf{OA}_1'$, they are not the natural closures of all the usual logical axioms of first-order logic. We now have the following proposition.

Proposition A.4 $\mathbf{OA}_1''$ *is consistent.*

Proof: The proof will be constructive. As we did in previous considerations about the consistency of $\mathbf{OA}_1'$, in (A.1.3), we say that the *irreducible* cwf are the atomic constant ones and those of the form $\forall x\, A$. All the other cwf are built from them by means of $\neg$ and $\Longrightarrow$. Given any finite collection of axioms we may consider the finite collection of all their irreducible "components" (from which the axioms of the collection are built using only $\neg$ and $\Longrightarrow$). If we can exhibit an assignation of truth values T, F, to these irreducible components in such a way that the logical valuations agreeing with it have value T in all the members of the given collection of axioms, then we will have proved that this collection is consistent. If we can do this operation for any such collection we will have proved the consistency of $\mathbf{OA}_1''$. This is what will be done, constructively.

We first observe: if a finite set D of axioms of $\mathbf{OA}_1''$ does not have as members two flagrantly contradictory formulas such as $\forall x\, A(x)$ and $\neg\forall x\, A(x)$ then we can assign the value T to each $\forall x\, B(x)$ in D and the value F to each $\forall z\, C(z)$ whose negation belongs to D; and there is no difficulty in assigning appropriate truth values to the components fo the members of D of the forms (iv) to (xi).

It will be sufficient to show that no finite set D of axioms of $\mathbf{OA}_1''$ can contain two formulas $\forall(x)\, A(x)$ and $\neg\forall y\, A(y)$. To begin with, it is easy to see that any finite set of axioms of $\mathbf{OA}_1''$ is contained in the set $AX(\mathbf{U})$ for some segment $\mathbf{U}$. In fact, there is a least $\delta \in \sigma_{\mathbf{U}}$ such that $D \subseteq \{ j_{\mathbf{U}}(\gamma) : \gamma \leq \delta\}$. Now, if we had $\forall x\, A(x) \in D$ and $\neg\forall y\, A(y) \in D$ we would have a natural m and some $\delta' \in \sigma_{\mathbf{U}}$, $\delta' < \delta$, such that

$$B'' \cup \{ j_{\mathbf{U}}(\gamma) : \gamma \in \sigma_{\mathbf{U}}, \gamma \leq \delta'\} \vdash A(\overline{m})$$

and the same for $\neg A(\overline{m})$. If now D' were a finite set of axioms of **U** such that $D' \vdash A(\overline{m})$, $D' \vdash \neg A(\overline{m})$ and $D' \subseteq B' \cup \{ j_{\mathbf{U}}(\gamma) : \gamma \in \sigma_{\mathbf{U}}, \gamma \in \delta' \}$, then, by the previous observation, D' would have to contain two formulas such as $\forall v\, A'(v)$ and its negation. The iteration of this observation would then produce an impossible infinite descent in $\sigma_{\mathbf{U}}$. □

We also have the following proposition.

Proposition A.5 $\mathbf{OA_1''}$ *is complete.*

Proof: The proof is nonconstructive. The same proof of Theorem 1.5 applies here and it is now slightly simpler because we are taking $\neg$, $\Longrightarrow$, and $\forall$ as the only undefined logical symbols. □

Finally, we have this proposition.

Proposition A.6 $\mathbf{OA_1''}$ *contains (nonconstructively)* $\mathbf{OA_1'}$.

Proof: For the proof we need first to observe that any proof in $\mathbf{OA_1''}$ gives a proof in $\mathbf{OA_1'}$ because the first formalism is not stronger than the second (it is, at first sight, weaker). We now consider any cwf A and suppose $\mathbf{OA_1'} \vdash A$. As $\mathbf{OA_1''}$ is complete, either A or its negation A' is a theorem of $\mathbf{OA_1''}$ (where we understand that A is $\neg A'$ or A' is $\neg A$). If we had $\mathbf{OA_1''} \vdash A'$ we would have $\mathbf{OA_1'} \vdash A'$, by our first observation, against the consistency of $\mathbf{OA_1'}$. So that we must have $\mathbf{OA_1''} \vdash A$, as expected. □

Corollary A.7 *Every closure of every theorem of* $\mathbf{OA_1}$ *is a theorem of* $\mathbf{OA_1''}$.

Proof: This is a consequence of the previous proposition and Proposition A.6. □

Remark A.8 If, in the proof of Proposition A.6, we write thoroughly $\mathbf{OA_1}$ in the place of $\mathbf{OA_1'}$, we obtain a direct proof of the corollary, without needing to make use of the intermediate formalism $\mathbf{OA_1'}$.

A.2.6 $\mathbf{OA}_1''$ ***and Hilbert's program*** The proof that $\mathbf{OA_1''}$ contains $\mathbf{OA_1'}$ does not provide any effective way of repeating in $\mathbf{OA_1''}$, and by the only means of the finitary or constructive manipulations allowed in it, a proof that may have been given in $\mathbf{OA_1'}$; so that any conviction that $\mathbf{OA_1''}$ can develop arithmetic as satisfactorily as $\mathbf{OA_1'}$ must, perhaps, rely on empirical grounds. We may, for example, try to prove in $\mathbf{OA_1''}$ what has been proved in Sections 4.1, 4.2, and 4.3 for **OA**. In any case, $\mathbf{OA_1''}$—having been proved (constructively) to be consistent and (nonconstructively) to be complete—might perhaps be considered at least a partial fulfilment of Hilbert's program, if we admitted in that program a logical instrument other than the usual first-order logic. Let us remember that the admission of some form of the ω-rule was suggested, although vaguely, by Hilbert himself.

Acknowledgments The present paper dealt, in its original form, only with the formalism **OA**, defined here in Section 2 (see also Sections 3 and 4). On behalf of completeness, the referee suggested the use of constructive ordinals (see [2]) instead of just ordinals $< \epsilon_0$. The suggestion evolved into the consideration of the class C and the formalism $\mathbf{OA_1}$ (see Definition 1.1) and the completeness of the latter followed almost trivially. The Appendix, which dealt originally with a formalism related to **OA** was rewritten, with a first part dealing with the

"intermediate" formalism $\mathbf{OA}_1''$, related to $\mathbf{OA}_1$; and the completeness result made possible the second part, dealing with the already mentioned complete and constructively consistent $\mathbf{OA}_1''$.

REFERENCES

[1] Boolos, G., *The Logic of Provability*, Cambridge University Press, Cambridge, 1973. Zbl 0891.03004 MR 95c:03038 4.2, 4.2

[2] Church, A., "The constructive second number class," *Bulletin of the American Mathematical Society*, vol. 44 (1938), pp. 224–32. Zbl 0018.33803 A.2.6

[3] Davis, M., "Hilbert's tenth problem is unsolvable," *American Mathematical Monthly*, vol. 80 (1973), pp. 233–69. Zbl 0277.02008 MR 47:6465 4.3

[4] Ebbinghaus, H.-D., J. Flum, and W. Thomas, *Mathematical Logic*, Springer, New York, 1984. Zbl 0556.03001 MR 85a:03001 1

[5] Feferman, S., "Transfinite recursive projections of axiomatic theories," *The Journal of Symbolic Logic*, vol. 27 (1962), pp. 259–316. Zbl 0117.25402 MR 30:3011 2

[6] Feferman, S., "Introductory note to 1931c," *Kurt Gödel: Collected works*, vol. 1, edited by S. Feferman, Oxford University Press, London, 1986. 1

[7] Feferman, S. "Reflecting on incompleteness," *The Journal of Symbolic Logic*, vol. 56 (1991), pp. 1–49. Zbl 0746.03046 MR 93b:03097 4.2

[8] Gentzen, G., "New version of the consistency proof for elementary number theory," pp. 252–86 in *The Collected Papers of Gerhard Gentzen*, edited by M. E. Szabo, North-Holland, Amsterdam, 1969. (translated from the original: "Neue Fassung der Wiederspruchfreiheitsbeweises für die Reine Zahlentheorie," *Forschung zur Logic und zur Grundlegung der Exakten Wissenschaften*, new series vol. 4 (1938), pp. 19–44). 2

[9] Graham, R. L., B. L. Rothschild, and J. H. Spencer, *Ramsey Theory*, 2d edition, Wiley Interscience Series in Discrete Mathematics and Optimization, New York, 1990. Zbl 0705.05061 MR 90m:05003 5

[10] Ignjatović, A., "Hilbert's program and the ω-rule," *The Journal of Symbolic Logic*, vol. 59 (1994), pp. 322–43. Zbl 0820.03002 MR 95f:03094 1

[11] Jones, J. P., and Y. Matijasevič, "Proof of recursive unsolvability of Hilbert's tenth problem," *American Mathematical Monthly*, vol. 98 (1991), pp. 689–709. Zbl 0746.03006 MR 92i:03050 4.3

[12] Kotlarski, H., "On the incompleteness theorems," *The Journal of Symbolic Logic*, vol. 59 (1994), pp. 1414–19. Zbl 0816.03025 MR 96c:03111 5

[13] Mendelson, E., *Introduction to Mathematical Logic*, Van Nostrand, Princeton, 1964. Zbl 0192.01901 MR 29:2158 3, 3, 4.1, 4.1, A.1, 1, A.3, A.2.1

[14] Paris, J., and L. Harrington, "A mathematical incompleteness in Peano Arithmetic," pp. 1133–42 in *Handbook of Mathematical Logic*, edited by J. Barwise, North-Holland, Amsterdam, 1977. 5, 5, 5

[15] Smorinski, C., "The incompleteness theorems," pp. 821–65 in *Handbook of Mathematical Logic*, edited by J. Barwise, North-Holland, Amsterdam, 1977. 4.2

[16] Smorinski, C., "Some rapidly growing functions," *Mathematical Intelligencer*, vol. 2 (1980), pp. 149–154. 5

[17] Suppes, P., *Axiomatic Set Theory*, Dover, New York, 1972.
Zbl 0269.02028 MR 50:1883 2

[18] Takeuti, G. *Proof Theory*, 2d edition, North-Holland, Amsterdam, 1987.
Zbl 0609.03019 MR 89a:03115 2

[19] Turing, A., "Systems of logic based on ordinals," *Proceedings of the London Mathematical Society*, series 2, vol. 45, (1939) pp. 161–228. Zbl 0021.09704 2

Universitat Politècnica de Catalunya
Departament de Matemàtica Aplicada
Pau Gargallo 5
08028 Barcelona
SPAIN
email: ftomas@ma2.upc.es