Notre Dame Journal of Formal Logic
Volume 41, Number 4, 2000

NEO-FREGEAN FOUNDATIONS FOR REAL ANALYSIS: SOME REFLECTIONS ON FREGE'S CONSTRAINT

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Abstract We now know of a number of ways of developing real analysis on a basis of abstraction principles and second-order logic. One, outlined by Shapiro in his contribution to this volume, mimics Dedekind in identifying the reals with cuts in the series of rationals under their natural order. The result is an essentially structuralist conception of the reals. An earlier approach, developed by Hale in his "Reals by Abstraction" program differs by placing additional emphasis upon what I here term *Frege's Constraint*, that a satisfactory foundation for any branch of mathematics should somehow so explain its basic concepts that their applications are immediate. This paper is concerned with the meaning of and motivation for this constraint. Structuralism has to represent the application of a mathematical theory as always posterior to the understanding of it, turning upon the appreciation of structural affinities between the structure it concerns and a domain to which it is to be applied. There is, therefore, a case that Frege's Constraint has bite whenever there is a standing body of informal mathematical knowledge grounded in direct reflection upon sample, or schematic, applications of the concepts of the theory in question. It is argued that this condition is satisfied by simple arithmetic and geometry, but that in view of the gap between its basic concepts (of continuity and of the nature of the distinctions among the individual reals) and their empirical applications, it is doubtful that Frege's Constraint should be imposed on a neo-Fregean construction of analysis.

1. Two Approaches

The basic formal prerequisite for a successful neo-Fregean—or as I shall sometimes say, *abstractionist*—foundation for a mathematical theory is to devise presumptively consistent abstraction principles strong enough to ensure the existence of a range of objects having the structure of the objects of the intended theory. In the case of

Received March 29,2001; printed October 28, 2002
2001 Mathematics Subject Classification: Primary, 00A30; Secondary, 03A05
Keywords: abstraction principles, analysis, Frege, structuralism

number theory, for instance, the task is to devise presumptively consistent abstraction principles sufficient to ensure the existence of a series of objects having the structure of the natural numbers: a series of objects that constitute an ω-sequence. As is now familiar, second-order logic, augmented by the single abstraction, Hume's Principle, accomplishes this formal prerequisite.[1] The outstanding question is therefore whether Hume's Principle, beyond being presumptively consistent, may be regarded as acceptable in a fuller, *philosophically interesting* sense. The neo-Fregean program inherits from Frege the anterior conviction that in mainstream classical mathematics, we deal in bodies of necessary truths of which we have a priori knowledge. So in order for Hume's Principle to serve the neo-Fregean purpose, the least that will have to be argued is that it too is necessary and knowable a priori (and that second-order logic can serve as a medium for the transmission of those characteristics). That raises an intriguing complex of metaphysical and epistemological issues—with which I will not here be primarily concerned.

In parallel, the basic formal prerequisite for a successful abstractionist foundation of real analysis must be to find presumptively consistent abstraction principles which, again in conjunction with a suitable—presumably second-order—logic, suffice for the existence of an array of objects that collectively comport themselves like the classical real numbers; that is, compose a complete, ordered field. Recently a number of ways have emerged for achieving this result. In this volume Shapiro [13] has described one which I'll call the *Dedekindian Way*. We start with Fregean arithmetic, that is, Hume's Principle plus second-order logic. Then we use the *Pairs* abstraction:

$$(\forall x)(\forall y)(\forall z)(\forall w)(\langle x, y\rangle = \langle z, w\rangle \leftrightarrow x = z \;\&\; y = w)$$

to arrive at the ordered pairs of the finite cardinals so provided.[2] Next we abstract over the *Differences* between such pairs,

$$\text{Diff}(\langle x, y\rangle) = \text{Diff}(\langle z, w\rangle) \leftrightarrow x + w = y + z,$$

and proceed to identify the *integers* with these differences. We proceed to define addition and multiplication on the integers so identified and then, where m, n, p, and q are any integers, form *Quotients* of pairs of integers in accordance with this abstraction:

$$\text{Q}\langle m, n\rangle = \text{Q}\langle p, q\rangle \leftrightarrow (n = 0 \;\&\; q = 0) \vee (n \neq 0 \;\&\; q \neq 0 \;\&\; m \times q = n \times p).$$

We now identify a *rational* with any quotient $\text{Q}\langle m, n\rangle$ whose second term n is non-zero. Then, defining addition and multiplication and the natural linear order on the rationals so generated, we can move on to the objects which are to compose the sought-for completely ordered field via the Dedekind-inspired *Cut Abstraction*:

$$(\forall P)(\forall Q)(\text{Cut}(P) = \text{Cut}(Q) \leftrightarrow (\forall r)(P \leq r \leftrightarrow Q \leq r))$$

where 'r' ranges over rationals and the relation '$\leq$' holds between a property P of rationals and a specific rational number r just in case any instance of P is less than or equal to r under the constructed linear order on the rationals. Cuts are the same, accordingly, just in case their associated properties have exactly the same rational upper bounds. Finally we identify the *real numbers* with the cuts of those properties P which are both bounded above and instantiated in the rationals.

On the Dedekindian Way then, successive abstractions take us from one-to-one correspondence on concepts to cardinals, from cardinals to pairs of cardinals, from pairs of finite cardinals to integers, from pairs of integers to rationals, and finally from concepts of rationals to (what are then identified as) reals. Although the path is quite complex in detail and the proof that it indeed succeeds in the construction of a completely ordered field is at least as untrivial as Frege's Theorem, it does make for a near-perfect abstractionist capture of the Dedekindian conception of a real number as the cut of an upper-bounded nonempty set of rationals. True, the series of abstractions used do not, of course, collectively provide for the transformability of any statement about the reals, so introduced, back into the vocabulary of pure second-order logic with which we started. But that—pure logicist—desideratum was already compromised at the very first stage, in the construction of number theory on the basis of Hume's Principle. Something weaker but still interesting remains in prospect. Suppose we are persuaded that each of the successive abstractions serves to fix the meaning of contexts of the type schematized on its left-hand side just provided one already understands the corresponding right-hand side: then we allow that there is a route of successive concept formations that starts in second-order logic and winds up with an understanding of the Cuts and a canonical mathematical theory of them. If the abstraction principles involved can be regarded as epistemologically definitionlike—as a kind of implicit definition of the type of contexts they serve to introduce on their left-hand sides—then the effect of the Dedekindian Way is to provide a foundation for analysis in second-order logic and (implicit) definitions. Dedekind did not have the notion of an abstraction principle. But it seems likely that his logicist sympathies would have applauded this construction and its philosophical potential.[3]

The Dedekindian Way contrasts significantly, however, with the route followed by Hale [8] in his important recent study.[4] In claiming to supply a foundation for analysis—in particular, in claiming that the series of abstractions involved effectively leads to the real numbers—the Dedekindian Way may be viewed as resting on an essentially *structural* conception of what a real number is: in effect, the idea of a real number as a location in a certain kind of—completely—ordered series. For one following the Dedekindian Way, success just consists in the construction of a field of objects—the Cuts, as defined—having the structure of the classical continuum. Against that, contrast what is accomplished by Hume's Principle in providing neo-Fregean foundations for number theory. The corresponding formal result is that Hume's Principle plus second-order logic suffices for the construction of an ω-sequence. That is certainly of mathematical interest. But it doesn't distinguish the situation from what can be accomplished in a system consisting, say, of second-order logic and Boolos's axiom New V.[5] What gives Frege's Theorem its distinctive *philosophical* interest is that Hume's Principle also purports to encapsulate an account of *what cardinal numbers are*. The philosophical payload turns not on the mathematical reduction as such but on the specific character of the abstraction by which the reduction is effected. Hume's Principle effectively incorporates a variety of philosophical claims about the nature of number for which Frege prepares the ground philosophically in the sections of *Grundlagen* preceding its first appearance—for example, the claims

(i) that number is a second-level property—a property of concepts; concepts are the things that *have* numbers,

which is incorporated by the feature that the cardinality operator is introduced as taking concepts for its arguments; and

(ii) that the numbers themselves are objects;

which is incorporated by the feature that terms formed using the cardinality operator are singular terms. And in addition, of course, Hume's Principle purports to explain

(iii) what sort of things numbers are.

It does so by framing an account of their criterion of identity in terms of when the things that have them have the same one: numbers, according to Hume's Principle, are the sort of things that concepts share when one-to-one correspondent.

Now you could, it is true, read a corresponding set of claims about real number off the Cut Abstraction principle featured in the Dedekindian Way. You would then conclude, correspondingly, that real numbers are objects, that the things which have real numbers are properties of rationals, and that real numbers are the sort of things that properties of rationals share just when their instances have the same rational upper bounds. One could draw these conclusions. But—apart from the first—they are strange-seeming conclusions to draw. There is no philosophical case that real number is a property of properties of rationals which stands comparison with Frege's case that cardinal number is a property of sortal concepts. On the contrary, the intuitive case is that real number belongs to things like *lengths, masses, temperatures, angles*, and *periods of time*. We could conclude that the Dedekindian Way incorporates poor answers to questions whose analogues about the natural numbers Hume's Principle answers relatively well. But a better conclusion is that the Dedekindian Way was not designed to take *those* questions on.

The fact is that Hume's Principle accomplishes two quite separate tasks. There is, a priori, no particular reason why a principle intended to incorporate an account of the nature of a particular kind of mathematical entity should also provide a sufficient axiomatic basis for the standard mathematical theory of that kind of entity. It's one thing to characterize what kind of entity we are concerned with, another thing to show that and why there are all the entities of that kind that we standardly take there to be, and that they compose a structure of the kind we intuitively understand them to do. Of course we can expect the two projects to interact. But the striking feature of the neo-Fregean foundations for number theory is that the one core principle, Hume's Principle, discharges *both* roles. This is not a feature which we should expect to be replicated in general when it comes to providing abstractionist foundations for other classical mathematical theories. And what the reflections of a moment ago suggest is that the Dedekindian Way, for its part, is best conceived as addressing only the second project.

It is the distinction between these two projects—the *metaphysical* project of explaining the nature of the objects in a given field of mathematical enquiry and the *epistemological* project of providing a foundation for our standard mathematical theory of those objects—that, as I read his discussion, drives the approach taken by Hale and—so far as one can judge from the incomplete discussion in *Grundgesetze*—by Frege himself. If we start with the metaphysical questions: what kind of thing are real numbers, what is real number a property of—what are the things that have real numbers—and what is the criterion of identity for reals, we are taken straight to the territory to which Hale devotes the initial part of his discussion. Real numbers, as remarked, are things possessed by lengths, masses, weights, velocities, and so on—things which allow of some kind of magnitude or, in Hale's preferred term, *quantity*. To stress, though, quantities, or magnitudes, are not themselves the reals, but the things which the reals *measure*. As Frege says,

> the same relation that holds between lines also holds between periods of time, masses, intensities of light, etc. The real number thereby comes off these specific kinds of quantities and somehow floats above them. (*Grundgesetze*, §185)[6]

If we want to formulate an abstraction principle incorporating an answer to the metaphysical question, what kind of thing are the reals, after the fashion in which Hume's Principle incorporates an answer to the metaphysical question, what kind of thing are the cardinal numbers, then quantities will feature not as the domain of reference of the new singular terms which that abstraction will introduce but rather as the *abstractive domain*: as the terms of the abstractive relation on the right-hand side. On the other hand, it's clear that individual quantities don't have their real numbers after the fashion in which a particular concept, say *speaker at the Notre Dame 2001 Logicism Reappraisal conference*, has its cardinal number. We are familiar with different systems of measurement, such as the imperial and metric systems for lengths, volumes and weights, or the Fahrenheit and Celsius systems for temperature, but there is no conceptual space for correspondingly different systems of counting. Of course, there can be different systems of counting *notation*: we can count in a decimal or binary system, for instance, or in roman or arabic numerals. But if they are used correctly, they won't differ in the cardinal number they deliver to any specified concept, but only in the way they name that number. By contrast, the imperial and metric systems do precisely differ in the real numbers they assign to the length of a specified object. One inch is 2.54 centimeters. The real number properly assigned to a length depends on a previously fixed unit of comparison. So real numbers are *relations* of quantities, just as Frege says.

Quick as they are, these reflections seem to enforce a view about what a principle would have broadly to be like whose metaphysical accomplishment for the real numbers matches that of Hume's Principle for the cardinals. Where Hume's Principle introduces a monadic operator on concepts, our abstraction for real numbers will feature a dyadic operator taking, in each use, as its arguments, a pair of terms standing for quantities of the same type; more specifically, it will be a *first-order* abstraction:

Real Abstraction $\quad R\langle a, b\rangle = R\langle c, d\rangle \leftrightarrow E(\langle a, b\rangle\langle c, d\rangle)$

where a and b are quantities of the same type, c and d are quantities of the same type (but not necessarily of the type of a and b), and E is an equivalence relation on pairs of quantities whose holding ensures that a is proportionately to b as c is to d. In effect, the analogy is between the abstraction of cardinal numbers from one-one correspondence on concepts, and abstraction of real numbers from equi-proportionality on pairs of suitable quantities.

With this preliminary analogy in place, it's clear that the neo-Fregean now has his work cut into three large subtasks:

1. A philosophical account is owing of what in the first place a *quantity* is—what the ingredient terms of the abstractive relation on the right-hand side of the Real Abstraction principle are.

2. If the aspiration is to give a *logicist* treatment in the sense in which Hume's Principle provides a logicist treatment of number theory, it must be shown that, parallel to the definability of one-one correspondence using just the resources of second-order logic, both the notion of *quantity* and the relevant

equivalence relation E allow of (ancestral)[7] characterization in (second-order) logical terms. (Should it prove impossible to do this, that would not necessarily deprive the abstractionist project of interest. But the point would have to be faced that an abstractionist treatment of analysis would apparently have to originate in a special *nonlogical* subject matter, with significant possible impact on the epistemological pay-off of the project.)

3. A result needs to be established analogous to Frege's Theorem: specifically, it needs to be shown that there are sufficiently many appropriately independent truths of the type depicted by the right-hand side of Real Abstraction to ground the existence of a full continuum of real numbers. And while, as stressed, Hume's Principle itself suffices for the corresponding derivation for the natural numbers, here it is clear that additional input is going to be required to augment the Real Abstraction principle.

Although not explicitly structured by a separation of these three issues, it is an achievement of Hale's discussion that it contains points of response to each of them. It is informed by taking to heart Frege's injunction that to take the question What is a quantity? head on is to put

> the wrong question. There are many different kinds of quantities: lengths, angles, periods of time, masses, temperatures, etc., and it will hardly be possible to specify in virtue of what the members of these various kinds of quantities are distinct from objects that do not belong to any kind of quantity. And nothing would be gained thereby anyway; for we would still lack the means to recognise which of these quantities belonged to the same realm of quantities.
>
> Instead of asking: which properties must an object have in order to be a quantity? one must ask: what must a concept be like in order for its extension to be a realm of quantities? For brevity's sake, let us now use 'class' instead of 'extension of a concept'. Then we can put the question as follows: which properties must a class have in order to be a realm of quantities? Something is not a quantity all by itself, rather it is a quantity only insofar as it belongs, with other objects, to a class which is a realm of quantities. (Frege [7], §161)

The leading idea in [8] is to distinguish a number of different *kinds* of quantitative domain—"realms of quantities"—with more complex kinds obtainable from simpler ones by successive abstractions on the latter, culminating in a quantitative domain of a kind to which the Real Abstraction principle can be applied so as to generate the full continuum of real numbers. I shall not here attempt to do justice to the detail, but the basic moves are not dissimilar to those followed in the Dedekindian Way. The route goes once again via the natural numbers, as provided by Hume's Principle, and then via a ratio abstraction principle to what Hale calls a *full* quantitative domain in which the ingredients exhibit a structure corresponding to that of the positive rationals. Since such a domain is countable, and since the Real Abstraction principle is first order and so delivers uncountably many reals only if applied to an uncountable domain of quantities on its right-hand side, an intermediate step is now required in Hale's construction to take us from a full quantitative domain to what he calls a *complete* quantitative domain in which, in addition, every class of elements which is bounded above has a least upper bound. Hale's proposal to turn this trick is an abstraction principle he too calls Cut. We consider a full quantitative domain—like the rationals—and restrict our attention to a special kind of property—what Hale calls *cut properties*—of its elements. Cut properties of the elements of such domains are nonempty, have no greatest instance, and are such that anything in the domain smaller than any instance of them is likewise an instance. With F and G restricted to such properties, and the range of the objectual variable on the right-hand side restricted to elements in the domain in question, the relevant principle—*Hale-Cut abstraction*—then (ironically enough) turns out to be a syntactic *doppelganger* of Basic Law V:

$$(\forall P)(\forall Q)(\mathrm{Cut}(P) = \mathrm{Cut}(Q) \leftrightarrow (\forall x)(Px \leftrightarrow Qx)).$$

Applied to the full domain provided by a neo-Fregean construction of the rationals, Hale-Cut abstraction will generate a completely ordered field in just the way in which the Dedekindian Way's Cut abstraction principle did so. (Indeed, Hale could just as well have used the latter at this stage of his construction.) But whereas, on the Dedekindian Way, the game ends once acceptable abstraction principles have been provided which lead to such a domain, all that construction serves to achieve, on the Hale route, are the needed raw materials for the right-hand side of the Real Abstraction principle itself. It remains, via that principle, to advance to the real numbers themselves, and to prove that they correspondingly compose a completely ordered field, thus bringing the mathematical construction into mesh with the overarching metaphysical account of what a real number is.

2. Frege's Constraint

Now we can get to our main issue. In the foregoing comparison I have deliberately encouraged the impression that the Dedekindian Way is best viewed as passing over certain legitimate general metaphysical questions: What is the nature of real number? and What is real number characteristic of?—What are the things which have real numbers?—to which the Frege/Hale approach rightly gives a central place. But *are* those questions rightly given a central place? Two well-known lines of thought converge on the contention that they are.

First there is the tendency, exemplified for instance by Heck, to think that there is a good distinction to be drawn between (the neo-Fregean delivery of) a theory *which allows of interpretation* as, say, number theory, or analysis, or geometry and (the delivery of) number theory, or analysis, or geometry *itself*. Heck writes,

> What is required if logicism is to be vindicated is not just that there is *some conceptual truth or other* from which what *look like* axioms for arithmetic follow, given certain definitions: That would not show that the truths of arithmetic, *as we ordinarily understand them*, are analytic, but only that arithmetic can be interpreted in some analytically true theory. To put the point differently, if we are so much as to evaluate logicism, we must first uncover the 'basic laws of arithmetic', laws which are not just sufficient to allow us to prove translations of arithmetical truths, but laws from which arithmetical truths themselves can be *proven*. (The distinction is not a mathematical one, but a philosophical one.) (Heck [11], p. 596–97)

The distinction seems plausibly made in at least some cases. There is no reason, for instance, why a derivation within ZFC of a theory which allows a *geometrical* interpretation, say, should do anything to illuminate the status of geometry—it all depends on the status of the principles from which the derivation proceeds after they receive whatever may be the corresponding interpretation. But if we restrict attention to second-order axiomatizations, then a theory will allow of interpretation as number theory in particular—or so we may take it for the present purpose—if and only if it is categorical: if all its (standard) models have domains comprising ω-sequences. So one who follows the tendency exemplified by Heck's remarks is urging a type of distinction illustrated by that between a second-order theory which is so categorical and one which somehow, beyond that, *genuinely concerns the finite cardinals themselves*. Such a distinction can make no sense unless the finite cardinals have a *nature* which goes beyond their collective composition of an ω-sequence. One who presses Heck's distinction is accordingly committed to taking seriously the general questions about the nature of numbers (of different kinds) which the Dedekindian Way, I have suggested, should be seen as passing by.

Compare Frege's own thought in *Grundgesetze* [7], §159. There he writes,

> The path that is to be pursued here thus lies between the old way of founding the theory of irrational numbers, the one H. Hankel used to prefer,

in which geometrical quantities were predominant,

> and the paths followed more recently [Cantor and Dedekind]. We retain the former's conception of real number as a relation of quantities . . . , but dissociate it from geometrical or any other specific kinds of quantities and thereby approach more recent efforts. At the same time, on the other hand, we avoid the drawback showing up in the latter approaches, namely that any relation to measurement is either completely ignored or patched on solely from the outside without any internal connection grounded in the nature of the number itself . . . our hope is thus neither to lose our grip on the applicability of [analysis] in specific areas of knowledge nor to contaminate it with the objects, concepts and relations taken from those areas and so to threaten its peculiar nature and independence. The display of such possibilities of application is something one should have the right to expect from [analysis] notwithstanding that that application is not itself its subject matter.
>
> Whether our plan can be carried out is something the attempt must show.

This is one of the clearest passages in which Frege gives expression to something that I propose we call *Frege's Constraint*: that a satisfactory foundation for a mathematical theory must somehow build its applications, actual and potential, into its core—into the content it ascribes to the statements of the theory—rather than merely "patch them on from the outside." The constraint is repeatedly emphasized, with approval, by Dummett in [5]. A typical passage is as follows:

> A correct definition of the *natural numbers* must, on [Frege's] view, show how such a number can be used to say how many matches there are in a box or books on a shelf. Yet number theory has nothing to do with matches or with books: its business in this regard is only to display what, in general, is involved in stating the cardinality of the objects, of whatever source, that fall under some concept, and how the natural numbers can be used for their purpose. In the same way, analysis has nothing to do with electric charge or mechanical work, with length or temporal duration; but it must display the general principle underlying the use of the real numbers to characterise the magnitude of quantities of these and other kinds. A real number does not directly represent the magnitude of a quantity, but only the ratio of one quantity to another of the same type; and this is in common to all the various types. It is because one mass can bear to another the very same ratio that one length bears to another that the principle

> governing the use of real numbers to state the magnitude of a quantity, relatively to a unit, can be displayed without the need to refer to any particular type of quantity. It is what is in common to all such uses, and only that, which must be incorporated into the characterisation of the real numbers as mathematical objects: that is how statements about them can be allotted a sense which explains their applications, without violating the generality of arithmetic by allusion to any specific type of empirical application. ([5], pp. 272–73)

What is it to observe Frege's Constraint? To insist that the general principle governing the application of a type of number be built into their characterization from the start is in effect just to insist such numbers be characterized by reference to a principle which explains what kind of entities they apply to—are *of*—and what it is for such entities to be associated with the same or different such numbers. And of course that is exactly what a suitable abstraction principle will do. It is a feature shared by both Hume's Principle and the Real Abstraction principle. To view such principles as philosophically and mathematically foundational is accordingly to view the applications of the sorts of mathematical objects they concern as belonging to the essence of objects of those sorts.

Let us take stock. Frege's Constraint and the insistence on a contrast between establishing a mathematical theory and merely establishing a theory which allows of interpretation as that theory have in common the thought that the objects of, for example, the classical theories of the natural and real numbers, or of classical geometry, have an essence which transcends whatever is shared by the respective types of models of even categorical (second-order) formulations of those theories. Frege's Constraint explicitly incorporates the additional thought that this essence is to be located in the applications; and so much was tacitly built into my characterizations above of the basic metaphysical questions which a satisfactory foundation for a particular pure mathematical theory should address, in particular in the central role accorded to the question what kinds of thing the numbers in question are numbers *of*? Heck's distinction—between deriving the axioms of number theory or analysis and merely deriving a body of statements which allow of interpretation as those axioms—might in principle, I suppose, be grounded in some other kind of conception of what makes for the essence of natural or real number. But no candidate is on the table besides that incorporated in Frege's Constraint. And it is hard to see what alternative there could be. For the pure mathematical theories of those entities make no distinction between them and any other isomorphic structure—so what could distinguish them except something to do with application?

Well, it should now stand out quite clearly what is arguably tendentious about Frege's, Hale's, Heck's, and Dummett's position. It is, in effect, the presupposition that there has to be more to the natural, or real, numbers than any broadly *structuralist* view of them can accommodate. For structuralism, there *is* no essence shared by the natural numbers beyond their composition of an ω-sequence; and there *is* no essence shared by the real numbers beyond their composition of a complete, ordered field. We may, for certain purposes, reify the "elements" in these respective types of structure as though they were entities in their own right. But for structuralism, the real "objects" of pure mathematical enquiry are the structures themselves; and the applications of the relevant pure mathematical theories derive from the appreciation of structural affinities between (segments of) the pure structures and certain structured collections of entities taken from the domain of application. From this perspective, the Dedekindian Way is not to be seen as *neglecting* a range of bona fide metaphysical questions which the Frege/Hale approach rightly takes seriously, but rather as *discounting* them—or better, as answering them in their only legitimate form, by providing for the derivation of a theory which appropriately characterizes the collective structure which is the true subject matter of analysis. No doubt there is a good philosophical issue about what provides for the applicability of a pure mathematical theory—what enables it to bestow on us knowledge of certain characteristics of the domains to which we do apply it. But structuralism may insist that it does not neglect this question; to the contrary, it provides a general rubric for a response to it—again, the applications of pure mathematical theories are grounded in our recognizing certain structural affinities between (segments of) the pure structures they concern and situations they are applied to. (To assimilate, for example, applications of arithmetic, conceived as the pure science of ω-sequences, to the purpose of simple counting of ordinary collections of objects, we conceive the latter as suitably serially ordered in some way and ask which initial segment of the naturals (excepting 0) is isomorphic to that ordering.) Structuralism does not—in intention anyway—neglect the issue of application. Its contention is rather—in flat contradiction to Frege, Dummett, Heck, and Hale—that it is a philosophical *mistake* to think of natural or real numbers as having an objectual essence at all, whether or not grounded in their applications, which a satisfactory metaphysical account of them must build in from the start, rather than "patch on as an afterthought."[8]

3. When Frege's Constraint Applies

At this point it appears that a decision between the Dedekindian Way and something akin to the Hale construction must ultimately depend on a verdict about the adequacy of a broadly structuralist conception of the classical continuum. More generally, it appears that whether the abstractionist should respect Frege's Constraint in recovering a given region of mathematics depends on whether we should think of that region structurally or not. If we should—if a full understanding of the mathematical theory in question invokes no specific conception of the kind of entities it is concerned with save as occupants of particular nodes in the structure—then there is no need for the abstractionist to observe Frege's Constraint, whatever aesthetic or other merits may be possessed by accounts which do so. But if understanding the theory requires grasping that its characteristic objects have a kind of distinguishing feature going beyond their occupancy of places in a structure—if, in particular, it requires grasping that they are *of* certain kinds of item, in the way that, for Frege, natural numbers belong to concepts, directions belong to lines and geometrical shapes belong to figures—then an abstractionist account which ignores Frege's Constraint will not succeed in recovering the whole content of the targeted statements, and its claim to provide a foundation will thereby be compromised.

It is implicit in the foregoing that the exigency of Frege's Constraint may vary as a function of field. But how should we decide whether we should "think of a region of mathematics structurally?" Let me close on one type of consideration that might move us *not* to do so—the crucial question will be how wide a range of cases it covers.

According to structuralism, the appreciation of any pure mathematical truth is the appreciation of a statement of it as holding good of any particular instance of a targeted kind of structure; applications of pure mathematics will then depend upon an *additional* appreciation of structural affinities between any such instance and the intended realm of application. Because additional, this appreciation may be lacking in one who understands the statement. So, the structuralist should claim, a grasp of the content of a pure mathematical statement need never per se involve knowledge of its applications. But this claim promised to be difficult to sustain in full generality. It seems clear that one kind of access to, for example, simple truths of arithmetic precisely proceeds *through* their applications. Someone can—and our children surely typically do—first learn the concepts of elementary arithmetic by a grounding in their simple empirical applications and then, on the basis of the understanding thereby acquired, advance to an a priori recognition of simple arithmetical truths. I say "a priori" because I see no reason to deny that a child who reasons on her fingers, or with a diagram, say—

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 3 & \\ 4_1 & & & & 5_4 \\ & 6_2 & & 7_3 & \end{array}$$

Figure 1

—that $4 + 3 = 7$ has indeed acquired a piece of knowledge a priori in much the same way that a general geometrical intuition can be facilitated by means of a construction with paper and pencil. But if that is right, then there is a kind of a priori arithmetical knowledge which flows from an antecedent understanding of the way that arithmetical concepts are applied. It is not that pure knowledge comes first, as the apprehension of an a priori truth about structures, with the applicability of the knowledge so acquired only dawning on one after one has grasped how certain empirical situations can be viewed as, in effect, modeling aspects of that structure. Rather the content of the a priori knowledge in question already configures concepts drawn directly from the applications.

The last is the important point. The objection to the structuralist account in such a case is not that it misrepresents the actual typical order and nature of the acquisition of at least some basic arithmetical knowledge—coming from a neo-Fregean, that would be pretty rich, for no one actually gets their arithmetical knowledge by second-order reasoning from Hume's Principle either! Rather, the significant consideration is that simple arithmetical knowledge, so acquired, has to have a content in which the potential for application is absolutely *on the surface*, since the knowledge is induced precisely by reflection upon sample, or schematic, applications. By contrast, the structuralist reconstruction of this knowledge will involve a representation of its content from which an appreciation of potential application will be an additional step, depending upon an awareness of certain structural affinities. So the structuralist will be open to the charge of changing the subject: whatever the detail of her epistemological story about the simplest truths of arithmetic, the content of the knowledge thereby explained will not be that of the knowledge we actually have—for, again, *that* can be grounded in reflection upon sample, or schematic applications.

The point will also bear plausible illustration by simple geometrical knowledge. It is no part of a grasp of analytic geometry, as structurally conceived, to think of it as concerned with spatial figures at all. So the kind of account of knowledge of geometrical features of space that such a structuralist theory can provide will have to go via the recognition (a priori?) that space puts up a model of the pure theory. Again, the point is not that one *could* not arrive at geometrical knowledge that way, though it is manifest that in general we do not. Rather, it is that there is a route that goes through reflection—by all means, diagram-assisted reflection—on geometrical concepts as given by ordinary rough and ready empirical illustrations and leads to—apparently—a priori knowledge of simple geometrical truths on the basis of the concepts thereby understood. Think of how you first persuade yourself that “A straight line divides a circle at exactly two points if at any.” Again, the crucial consideration is what this shows about the *content* of the knowledge thereby achieved.

This suggests a distinction which, wherever it can be upheld, will mandate something close to Frege’s Constraint. It is one thing to explain how (a priori) knowledge could be acquired of a system which, taken in conjunction with certain supplementary reflections, can then be applied in the same ways as an entrenched mathematical theory. But that will not suffice to provide a correct (if idealized) reconstruction of the content of *what we actually know* in knowing that theory if at least some of that knowledge can be achieved just by the reflective exercise of concepts acquired and applied in the course of ordinary counting and calculation, measurement, and the kind of geometrical routines employed in joinery. For in that case the (simple) pure mathematical statements thereby known take on a content which makes those applications *immediate*. It is accordingly not knowledge of *those* contents of which we have given an account—even an idealized account—if the statements to which a given theoretical reconstruction leads are ones which, even if knowable a priori, can be wholly grasped without any inkling of their applications at all.

Perhaps it is by a development of this thought—and perhaps only thereby—that Frege’s Constraint can be made to prevail against what, I have suggested, are the essentially structuralist roots of resistance to it. But if that is right, then—to emphasize again—there is no reason to think it should prevail right across the board—and a doubt in particular about whether it should do so in the case we are now concentrating on: real analysis.

The immediate obstacle is, briefly, that it is simply not the case that the distinctive concepts of real analysis can be grounded in their applications after the fashion in which, at least in principle, arithmetical concepts and simple geometrical concepts can. For instance, while the cardinal number of a group can be empirically determined, and the application of at least small cardinals schematized in thought, as in Figure 1 above, *no* real number can ever be given as the measure of any particular empirically given quantity. There is simply no such thing as determining a *real* value of a quantity by measurement or indeed by any other empirical procedure—any set of measurements we take will be finite, and even in the best case there will be no empirical distinction between their convergence upon a particular real value as opposed to uncountably many others sufficiently close to but distinct from it. How then can analogues grip the reals of the kind of thought-experimental or imaginative routines that can engage the objects of arithmetic and geometry and which form the basis of the simplest kinds of reflective knowledge of them? And if no such analogues are possible, what reason is there to suppose that any of our knowledge of analysis is of propositions whose applications are immediate?

That, at any rate, is the issue. Frege’s Constraint is justified, it seems to me, when—and I am tempted to say, only when—we are concerned to reconstruct a branch of mathematics at least some—if only a very basic core—of whose distinctive concepts can be communicated just by explaining their empirical applications. However, the fact is that both our concepts of the identity of particular real numbers and, more importantly, the entire overarching conception of continuity, as classically conceived—the density and completeness of the range of possible values within a parameter determined by measurement—are simply not manifest in empirical applications at all. Rather, so one would think, the flow of concept-formation goes *in the other direction*: the classical mathematics of continuity is made to inform a *nonempirical* reconceptualization of the parameters of potential variation in the empirical domains to which it is applied.

To explore that thought properly, one would have to take on a complex set of issues for which I have no space here, even if I were confident how to proceed. But if it is good, and if the only really compelling motivation for Frege's Constraint is the one I have reviewed, there will be no significant shortcoming, from the neo-Fregean point of view, in an abstractionist reconstruction of the reals that follows the Dedekindian Way.

Appendix: Abstractionism and Structuralism

I have suggested that an abstractionist reconstruction of a pure mathematical theory may be absolved from Frege's Constraint in any case where it is appropriate to take a structuralist view of the content of that theory. That may seem an unstable claim. After all, the whole *raison d'être* of abstractionism is the recovery of an account of what is preconceived as knowledge of certain specific kinds of mathematical objects. By contrast, structuralists characteristically do not view mathematics as, in the appropriate way, object-directed in the first place but see the mathematician's concern as being with the structural features that collections of objects—whose nature is otherwise irrelevant—may exemplify. So it may seem that Frege's Constraint must be in force at least in all cases where abstractionism has any point—where there is a range of specific mathematical objects, with a proper intrinsic nature, which a targeted theory concerns; and that in cases where the Constraint does not apply, according to my proposal, because a structuralist view is appropriate, there is anyway no point to the abstractionist project that it might have constrained. Let me briefly explain why I do not think this is so—explain why and how abstractionism and structuralism can cooperate.

There is a kind of structuralism whose whole purpose is *ontological frugality*. For this—eliminative—kind of structuralism, the point of the emphasis on pure mathematics' (alleged) structural concerns is by way of a counterweight to, and thereby to liberate one from, what is viewed as the problematic notion that it is really concerned with any *objects* at all—that there is any such thing as specifically mathematical existence. This structuralism is indeed at odds with neo-Fregeanism. But its spirit is quite different to that of a second kind of structuralism advocated by writers such as Resnik and Shapiro.[9] For these theorists, to emphasize the concern of number theory, or analysis, with certain distinctive kinds of structure, goes with the idea not that we should think of such theories as innocent of ontological commitments but rather that they are precisely about *articulated structures* and that it does not matter what, if any, objects we take to be configured within them so long as they collectively compose a structure of the appropriate kind. It is in that object—in the articulated structure itself—that the mathematical interest lies.

It is structuralism of this ontologically liberal—as Shapiro styles it, *ante rem*—kind which is, as it seems to me, potentially consonant with a program of neo-Fregean foundations. Ontologically frugal structuralism does not require there actually to be any examples of the various types of structure in which it represents the mathematician as interested. There might actually be no completely ordered fields; there may even be no ω-sequences; but for frugal structuralism, we can still investigate what such structures would be like if they existed. Mathematics, on this view, is the science of *hypothetical* structures. It describes how things *would be* if there were structured collections of entities of the various relevant kinds. By contrast, the *ante rem* structuralist takes a Platonic view of structures: they exist and are available for mathematical description as complex objects in their own right, whether or not exemplified by any independent collections of objects. The *ante rem* structuralist must therefore address the questions: what guarantee can be given that, so conceived, classical mathematical structures, like the continuum, do indeed exist? And how do we gain knowledge of them?

Shapiro's answer[10] is nuanced but, in the end, broadly Hilbertian. In the best case, he holds, it is by giving a (categorical) *characterization* of an intended structure that we grasp the structure—make it available as an object of intellection. And once so made available, it may be investigated by exploring the deductive and model-theoretic consequences of the characterization by which it was communicated. The mere intelligibility of an appropriate characterization is enough—enough not just to communicate a concept of the structure involved but to present the very object to the mind. For example, the second-order (categorical) Dedekind-Peano axioms themselves present the structure: ω-series. Thus in Shapiro's final view, all that the theorist needs to do in order to explain how a particular mathematical structure is accessible to us as an object of mathematical investigation is to call attention to the fact that we are capable of grasping a canonical axiomatic description of it. Mathematical access is achieved merely by mathematical understanding.

It seems to me that there are two ways that abstractionism may complement and assist this view. One is completely in keeping with the proposal and is in effect remarked on by Shapiro himself. I said above that Shapiro's position was broadly Hilbertian. But for Hilbert—at least so the legend goes—consistency was enough: the mere consistency of an axiom set sufficed to ensure the reality of a mathematical subject matter for those axioms to treat of. Shapiro's view is more qualified. He does not accept that just any old consistent description serves to communicate—make accessible to us as an object of intellection—an *ante rem* structure. Tighter constraints are wanted, and he flags them in his notion of *coherence*. He has much to say about how the relevant notion of coherence should be understood but I shall not attempt to evaluate the detail of his discussion here. Suffice it to say that, as he intends the notion, a characterization is coherent just in case it is satisfiable in the standard iterative hierarchy of sets. (That, anyway, is the intended *extension* of the notion of coherence: it would be a serious concern for Shapiro's account if the best that could be done to explicate coherence were simply to help oneself, in that way, to an assumed prior ontology and epistemology of sets, since one would be left with no (nontrivial) account of the coherence of the axioms of ZFC.)[11]

Now the first way in which abstractionism may marry with this form of structuralism is precisely by delivering an assurance of the coherence of a given axiomatic characterization. For however a notion of coherence, apt for Shapiro's purpose, should be elucidated in general, it ought manifestly to *suffice* for the coherence of an axiom set if we can reach for an independently given domain of objects which those axioms may then be recognized to characterize. There should thus be no question about the coherence, in a sense consistent with Shapiro's purpose, of axioms which we can model in a domain composed of objects independently furnished by suitable abstraction principles.

The second point of complementarity is less friendly. The principal reservation abstractionism will have with Shapiro's approach concerns his idea that, merely by giving a coherent axiomatization, we can do more than *convey a concept*. Shapiro holds that we can, in addition, induce awareness of an articulate, archetypal *object*, at once representing the concept in question and embodying an illustration of it. But what ground is there for supposing so? Someone who writes a fiction, even the most coherent fiction, does not thereby create a range of entities whose properties and relations are exactly as the fiction depicts. Rather—it can be agreed on all hands—she merely creates a concept, a description of a possible scenario in which certain things, real or imagined, might be so qualified and related. It is implicit in Shapiro's view, by contrast, that there can be no such thing as a *fictionalized structure*. Try to write about merely imaginary structures, as about imaginary people, and the very description of your fiction, if coherent, will defeat your purpose. Only write coherently and, willy-nilly, a Platonic entity—a Shapironian structure—will step forward to fulfil your descriptive demand.

Shapiro is, naturally, fully self-conscious and deliberate about this aspect of his view. If he's right, mathematical fictionalism is simply an incoherent philosophy of mathematics from the start. With structures, the coherence of a description suffices for the existence of a realizer. Against that, abstractionism will set the orthodox idea, repeatedly stressed by Frege himself, that in mathematics, as elsewhere, there is a gap between concept and object, that it is one thing to give a however precise and coherent characterization and another to have reason for thinking that it is actually realized.

If one takes this view, the question becomes pressing: what, if not Shapiro's "fast track", could constitute a recognition of the existence of structures conceived as pure objects in their own right, in the fashion of *ante rem* structuralism? And the obvious suggestion, in the present context, is to attempt a view of structures as arrived at by abstraction, taking pure structures as in effect the order-types associated with given domains of objects and specific ordering relations.[12] Thus, for example, the structure, ω-series, is the order-type associated with the natural numbers under the less-than relation. And in general,

Structure Abstraction Structure(F, R) = Structure(G, S) iff the Fs under relation R are isomorphic to the Gs under relation S.

Cognoscenti will immediately protest that we cannot have exactly this form of abstraction, since in full generality, it will implicate a form of the Burali-Forti paradox.[13] It remains, however, that it does correctly encode the *ante rem* structuralist's implicit conception of the identity-conditions of pure structures, and that the resolution of the attendant paradox, as analogously with Basic Law V, must accordingly consist, as a first step, in the recognition that not every pair (F, R) determines a structure. Obviously there is much more to say; but it seems better to confront these questions squarely—even if it leaves the structuralist in difficulty in finding an overarching structure to accommodate the ordinals, for instance, or indeed the iterative hierarchy of sets—than to mask them with the, as the abstractionist will unkindly view it, mythology of coherence as sufficient for existence.

This range of issues about "large" structures to one side, it seems to me that the *ante rem* structuralist should welcome the situation whenever the abstractionist program goes well locally and it can be shown how it is possible to arrive at a specific collection of abstracts exemplifying a given interesting structure, and to recognize that they do indeed exemplify it; for that is all that should be needed to set up an abstraction to the structure itself. Very simply, there are all the—ontological and epistemological—reasons to attempt an abstractionist treatment of structures as of any other kind of mathematical object. The foreseeable difficulties in recovering structures comprehending all the ordinals, or sets, should be seen not as exposing undesirable limitations in the approach but as pointers to genuine difficulties in the ascription to ourselves of valid conceptions of the appropriately comprehensive structures.

Notes

1. This result is now commonly known as Frege's Theorem. It is prefigured by Frege in [6], §§ 82-83 and reconstructed in detail by Wright in [15], § xix. Other detailed accounts of the proof are given in Boolos [2], in an appendix to Boolos [3], and in Boolos and Heck [4].

2. Shapiro himself does not make direct use of the *Pairs* abstraction, moving directly to abstraction principles which "operate on objects taken two at a time." However, since the order in which the objects are taken matters for these principles, it seems better to signal the assumptions involved in an explicit principle and to treat their abstractive domains as composed by the appropriate ordered pairs delivered by it.

3. An illuminating brief discussion of Dedekind's "logicism" may be found in Shapiro [14], pp. 170–76.

4. Hale's was the first neo-Fregean treatment of the real numbers.

5. For discussion of which, see Hale [9], this volume pp. 379–98.

6. The translation of this passage, and others from *Grundgesetze* [7] given below, is by Sven Rosenkranz.

7. "Ancestral" characterization in the sense that a chain of effective implicit definitions, eventually grounding in concepts of second-order logic, may be reckoned good enough, even if it does not provide the resources for eliminative paraphrase of the definienda. This, of course, as noted, is the most that is achieved by the Dedekindian Way. But there is still a disanalogy: no issue arose, on that approach, concerning the logical character of the items in the abstractive domain for the abstraction that yields the reals. On the Dedekindian Way, those items were concepts (of ancestrally logical objects). On Hale's route they are (pairs of) *quantities*.

8. For discussion of the applications of mathematics from a structuralist point of view, see [14], Chapter 8.

9. Resnik [12] and Shapiro [14]. These two authors' similar ontological views are married to large differences however concerning mathematical epistemology. The remarks to follow are addressed to the overall structuralist position advocated by Shapiro. Resnik's epistemological views are more purely empiricist and Quinean.

10. Developed in Chapter 4 of [14].

11. If the observations at the conclusion of this Appendix are correct, this difficulty will come back to haunt the structuralist in any case.

12. This idea surfaces in Shapiro's own writing—see, for instance, [14], p. 123.

13. As Harold Hodes first noted. See Boolos [1], pp. 175–76.

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