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Kripke Completeness of Infinitary Predicate Multimodal Logics

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Abstract Kripke completeness of some infinitary predicate modal logics is presented. More precisely, we prove that if a normal modal logic **L** above **K** is $\mathcal{D}$-persistent and universal, the infinitary and predicate extension of **L** with BF_{ω_1} and BF is Kripke complete, where BF_{ω_1} and BF denote the formulas $\bigwedge_{i\in\omega}\Box p_i \supset \Box\bigwedge_{i\in\omega} p_i$ and $\forall x\Box\varphi \supset \Box\forall x\varphi$, respectively. The results include the completeness of extensions of standard modal logics such as **K**, and its extensions by the schemata $\mathrm{T}, \mathrm{B}, 4, 5, \mathrm{D}$, and their combinations. The proof is obtained by extending the correspondence between the representation of modal algebras and the completeness of propositional modal logic to infinite.

1 Introduction The study of logics with infinitary connectives based on classical logic started in the1950s at the latest. There are two main motives to introduce infinitary connectives into the language: one comes from model theory. There exist some concepts in mathematics which cannot be described by a theory of finitary logics, and infinitary connectives are introduced to strengthen the expressive power of theories (see Barwise and Feferman [1]). Others come from proof theory. Infinitary connectives are used as an instrument to give a proof of consistency of finitary formal systems ([16], [14]). The completeness theorem for the classical infinitary predicate logic is given in [12] by using the properties of Boolean algebras and then [13] by the Henkin methods (cf. [6]).

Now we discuss modal logics. Let $\mathbf{K}_{\omega_1}$ be an infinitary extension of propositional **K** and BF_{ω_1} be the formula $\bigwedge_{i\in\omega}\Box p_i \supset \Box\bigwedge_{i\in\omega} p_i$ of infinitary propositional modal logic which corresponds to the Barcan formula BF, that is, the formula $\forall x\Box\varphi \supset \Box\forall x\varphi$ of predicate modal logic. The completeness theorem for infinitary propositional modal logic $\mathbf{K}_{\omega_1} \oplus \mathrm{BF}_{\omega_1}$ with respect to the class of Kripke frames is given, for example, in [17], [5], and [21]. In [17], the interpolation theorem is also proved. In [5], the completeness of the infinitary extension of graded modal logic is proved which includes the completeness of $\mathbf{K}_{\omega_1} \oplus \mathrm{BF}_{\omega_1}$ as a special case. In [21],

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the completeness of infinitary multimodal logic and some of its extensions is proved. However, most completeness studies of infinitary modal logic have not directed attention to predicate logic.

In this article, we present Kripke completeness of some infinitary predicate multimodal logics above **K**. More precisely, we prove that if a (finitary) propositional modal logic **L** above **K** is $\mathcal{D}$-persistent and universal then the infinitary and predicate extension of **L** with BF_{ω_1} and BF is Kripke complete, by an algebraic method. It is known that the representation theorem of modal algebras corresponds to the completeness theorem of propositional modal logic ([2], [9]). Similarly infinitary representation theorem, that is, a representation theorem which preserves countable infinite meets and joins, corresponds to the completeness theorem of infinitary predicate modal logics, as we will see in Section 4. The results include the completeness of extensions of standard modal logics such as **K** and its extensions by the schemata T, B, 4, 5, D, and their combinations.

2 *Infinitary representation of modal algebras*

A multimodal algebra is a modal algebra with countably many modal operators. Here we assume that there is no interaction between modal operators. We introduce a representation theorem for multimodal algebras which preserves countably many infinite meets and joins (cf. [24], [23]).

Definition 2.1 An algebra $(A, \wedge, \vee, -, \Box_i\ (i \in \omega), 0, 1)$ is called a *multimodal algebra* if for each $i \in \omega$, $(A, \wedge, \vee, -, \ell, 0, 1)$ is a Boolean algebra and

1. $\Box_i 1 = 1$;
2. $\Box_i(x \wedge y) = \Box_i x \wedge \Box_i y$ for any x and y in A.

For any x and y in A, we sometimes write $x \to y$ for $-x \vee y$. Also, we write $\mathcal{F}_{\mathrm{p}}(A)$ for the set of all prime filters of A.

Definition 2.2 Let A and B be multimodal algebras. A function $f : A \to B$ is called a *homomorphism of multimodal algebras* if f is a homomorphism of Boolean algebras and satisfies $f(\Box_i x) = \Box_i f(x)$ for all $i \in \omega$.

Proposition 2.3 *Let A be any set and $\{R_i\}_{i\in\omega}$ be any set of binary relations on A. Then $(\mathcal{P}(A), \cap, \cup, -, \Box_i\ (i \in \omega), \varnothing, A)$ is a multimodal algebra, where*

$$-X := A \setminus X,\ \Box_i X := \{x \in A : \forall y (x <_{R_i} y \implies y \in X)\},$$

for any $i \in \omega$. Especially, for any multimodal algebra A, the binary relations $R_i (i \in \omega)$ on $\mathcal{F}_{\mathrm{p}}(A)$ given by $F <_{R_i} G \iff \Box_i^{-1} F \subset G$ define a multimodal algebra on $\mathcal{P}(\mathcal{F}_{\mathrm{p}}(A))$.

Proof: Since $(\mathcal{P}(A), \cap, \cup, -, \varnothing, A)$ is a Boolean algebra, it is enough to show that the operator $\Box_i$ is well defined for any $i \in \omega$, but this is straightforward. □

Definition 2.4 ([18]) Let A be a Boolean algebra and Q be a pair $(\{X_n\}_{n\in\omega}, \{Y_n\}_{n\in\omega})$ of subsets of $\mathcal{P}(A)$ such that $\bigwedge X_n \in A$ and $\bigvee Y_n \in A$ for any $n \in \omega$. A prime filter F is called a *Q-filter* if the following conditions are satisfied:

1. $\forall n \in \omega (X_n \subset F \implies \bigwedge X_n \in F)$;

2. $\forall n \in \omega(\bigvee Y_n \in F \implies Y_n \cap F \neq \varnothing)$.

Obviously, the two conditions in Definition 2.4 are infinitary extensions of the conditions for prime filters:

1. $x,\ y \in F \implies x \wedge y \in F$;
2. $x \vee y \in F \implies x \in F$ or $y \in F$.

We write $\mathcal{F}_Q(A)$ for the set of all Q-filters in A. It is easy to see that the binary relation on $\mathcal{F}_Q(A)$ defined in Proposition 2.3 yields a multimodal algebra $\mathcal{P}(\mathcal{F}_Q(A))$.

The following proposition, an infinitary extension of the prime filter theorem for Boolean algebras, is sometimes called the Rasiowa-Sikorski Lemma ([18], [19]).

Proposition 2.5 ([18]) *Let A be a Boolean algebra and $Q = (\{X_n\}_{n\in\omega}, \{Y_n\}_{n\in\omega})$ be a pair of countable subsets of $\mathcal{P}(A)$ such that $\bigwedge X_n \in A$ and $\bigvee Y_n \in A$ for any $n \in \omega$. Then for any a and b in A with $a \not\leq b$, there exists a Q-filter F such that $a \in F$ and $b \notin F$.*

Proof: We define two sequences $\{\alpha_n : n \in \omega\}$ and $\{\beta_n : n \in \omega\}$ of elements of A which satisfy the conditions

1. $\alpha_0 = a,\ \beta_0 = b$;
2. $\forall n \in \omega(\alpha_{n+1} \leq \alpha_n,\ \beta_n \leq \beta_{n+1},\ \alpha_n \not\leq \beta_n)$;
3. $\forall n \in \omega(\alpha_{2n+1} \leq \bigwedge X_n$ or $\exists x \in X_n(x \leq \beta_{2n+1}))$;
4. $\forall n \in \omega(\exists y \in Y_n(\alpha_{2n+2} \leq y)$ or $\bigvee Y_n \leq \beta_{2n+2})$.

Suppose α_{2k} and β_{2k} are constructed. We may assume that $\alpha_{2k} \not\leq \beta_{2k} \vee \bigwedge X_k$ or $\alpha_{2k} \wedge \bigwedge X_k \not\leq \beta_{2k}$, since otherwise,

$$\alpha_{2k} \leq (\alpha_{2k} \vee \beta_{2k}) \wedge (\beta_{2k} \vee \bigwedge X_k) = \beta_{2k} \vee (\alpha_{2k} \wedge \bigwedge X_k) \leq \beta_{2k}.$$

Case 1: $\alpha_{2k} \not\leq \beta_{2k} \vee \bigwedge X_k$. There exists $x \in X_k$ such that $\alpha_{2k} \not\leq \beta_{2k} \vee x$, for if not,

$$\alpha_{2k} \leq \bigwedge_{x\in X_k} (\beta_{2k} \vee x) = \beta_{2k} \vee \bigwedge X_k.$$

Take one such x and define $\alpha_{2k+1} := \alpha_{2k}$ and $\beta_{2k+1} := \beta_{2k} \vee x$.

Case 2: $\alpha_{2k} \wedge \bigwedge X_k \not\leq \beta_{2k}$. Define $\alpha_{2k+1} := \alpha_{2k} \wedge \bigwedge X_k$ and $\beta_{2k+1} := \beta_{2k}$.

We construct α_{2k+2} and β_{2k+2} similarly. It is easy to see that α_n and β_n are well defined. Let G be the filter generated by the set $\{\alpha_n : n \in \omega\}$ and H be the ideal generated by the set $\{\beta_n : n \in \omega\}$. It is obvious from (2) that H and G are disjoint. Hence, there exist a prime ideal I and a prime filter F such that $G \subset F$, $H \subset I$, and $I \cap F = \varnothing$ by the prime filter theorem. Now it is straightforward by (3) and (4) that F is a Q-filter. □

Let A be a Boolean algebra and F be a filter of A. Then the binary relation $\sim_F$ on A defined by

$$x \sim_F y \iff \exists a \in F(x \wedge a = y \wedge a)$$

is a congruence relation. We write A/F for $A/\sim_F$, $|z|$ for the equivalence class of an element $z \in A$, and $|Z|$ for the set $\{|z| : z \in Z\}$ for any $Z \subset A$. However, $\sim_F$ does not preserve infinite meets and joins in A, in general. Hence, we need the following lemma ([24], [23]).

Lemma 2.6 *Let A be a multimodal algebra, $Q = (\{X_n\}_{n\in\omega}, \{Y_n\}_{n\in\omega})$ be a pair of countable subsets of $\mathcal{P}(A)$. Let F be a filter of A such that $X_n \subset F$ implies $\bigwedge X_n \in F$ for any $n \in \omega$. Suppose the following conditions are satisfied:*

1. $\forall n \in \omega(\bigwedge X_n \in A,\ \bigvee Y_n \in A)$;
2. $\forall i \in \omega \forall n \in \omega(\bigwedge \Box_i X_n \in A,\ \bigwedge \Box_i X_n = \Box_i \bigwedge X_n)$;
3. $\forall i \in \omega \forall z \in A \forall n \in \omega \exists m \in \omega(\{\Box_i(z \to x) : x \in X_n\} = X_m)$;
4. $\forall i \in \omega \forall z \in A \forall n \in \omega \exists m \in \omega(\{\Box_i(y \to z) : y \in Y_n\} = X_m)$.

Then, for any $i \in \omega$, $A/(\Box_i^{-1}F)$ is a Boolean algebra which satisfies the following conditions:

1. $\forall n \in \omega(\bigwedge |X_n| \in A/(\Box_i^{-1}F),\ \bigwedge |X_n| = |\bigwedge X_n|)$;
2. $\forall n \in \omega(\bigvee |Y_n| \in A/(\Box_i^{-1}F),\ \bigvee |Y_n| = |\bigvee Y_n|)$.

Proof: We only show the second one. Take any i and n in ω and let $G = \Box_i^{-1}F$. Then A/G is a Boolean algebra. For any $y \in Y_n$, it is obvious that $|y| \le |\bigvee Y_n|$. Suppose z is an upper bound of the set $|Y_n|$. Then

$$
\begin{aligned}
\forall y \in Y_n(|y| \le |z|) &\iff \forall y \in Y_n(y \to z \in G) \\
&\iff \forall y \in Y_n(\Box_i(y \to z) \in F) \\
&\iff \bigwedge_{y\in Y_n} \Box_i(y \to z) \in F \\
&\iff \Box_i \bigwedge_{y\in Y_n} (y \to z) \in F \\
&\iff \Box_i(\bigvee Y_n \to z) \in F \\
&\iff \bigvee Y_n \to z \in G \\
&\iff |\bigvee Y_n| \le |z|.
\end{aligned}
$$

Hence, $|Y_n|$ has the least upper bound $|\bigvee Y_n|$ in A/G. □

Now we show the main lemma for the completeness theorem of infinitary and predicate modal logic and the infinitary representation theorem ([24], [23]).

Lemma 2.7 *Let A be a multimodal algebra and $Q = (\{X_n\}_{n\in\omega}, \{Y_n\}_{n\in\omega})$ be a pair of countable subsets of $\mathcal{P}(A)$. Suppose Q satisfies the conditions in Lemma 2.6. Then for any $F \in \mathcal{F}_Q(A)$ and $\Box_i a \notin F$, there exists $G \in \mathcal{F}_Q(A)$ such that $\Box_i^{-1}F \subset G$ and $a \notin G$.*

Proof: Let $H = \Box_i^{-1}F$. By Lemma 2.6, A/H is a Boolean algebra which satisfies

1. $\forall n \in \omega(\bigwedge |X_n| \in A/H,\ \bigwedge |X_n| = |\bigwedge X_n|)$;
2. $\forall n \in \omega(\bigvee |Y_n| \in A/H,\ \bigvee |Y_n| = |\bigvee Y_n|)$.

Let $|Q| = (\{|X_n|\}_{n\in\omega}, \{|Y_n|\}_{n\in\omega})$. Since $a \notin H$, $|a| \neq |1|$. Then, by Proposition 2.5, there exists a $|Q|$-filter $\tilde{G}$ of A/H such that $|a| \notin \tilde{G}$. Define a set $G \subset A$ by $\{x \in A : |x| \in \tilde{G}\}$. We claim that G is a Q-filter of$A$. It is easy to see that G is a prime filter. Take any $n \in \omega$. Since $\tilde{G}$ is a$|Q|$-filter, $\bigwedge |X_n| \in \tilde{G}$ if and only if $|X_n| \subset \tilde{G}$. Hence,

$$X_n \subset G \iff |X_n| \subset \tilde{G}$$

$$\begin{aligned} &\iff \bigwedge |X_n| \in \tilde{G} \\ &\iff |\bigwedge X_n| \in \tilde{G} \\ &\iff \bigwedge X_n \in G. \end{aligned}$$

Similarly, $\bigvee Y_n \in G$ if and only if $Y_n \cap G \neq \varnothing$. It is trivial that $a \notin G$ and $H \subset G$. □

Then we have the infinitary representation theorem for multimodal algebras ([24], [23]).

Theorem 2.8 *Let A be any multimodal algebra and $Q = (\{X_n\}_{n\in\omega}, \{Y_n\}_{n\in\omega})$ be a pair of countable subsets of $\mathcal{P}(A)$. Suppose Q satisfies the conditions in Lemma 2.6. Then the function $\eta : A \to \mathcal{P}(\mathcal{F}_Q(A))$ defined by*

$$\eta : x \mapsto \{F \in \mathcal{F}_Q(A) : x \in F\}$$

is a monomorphism of multimodal algebras such that $\eta(\bigwedge X_n) = \bigcap \eta[X_n]$ and $\eta(\bigvee Y_n) = \bigcup \eta[Y_n]$ for all $n \in \omega$.

Proof: It is easy to see that η is a morphism of Boolean algebras. Moreover, η is an injection by Proposition 2.5. We first show that $\eta(\Box_i x) = \Box_i \eta(x)$ for any $x \in A$ and $i \in \omega$. Take any $i \in \omega$. Suppose $F \in \eta(\Box_i x)$. Then

$$\begin{aligned} \Box_i x \in F &\implies \forall G \in \mathcal{F}_Q(A)(\Box_i^{-1} F \subset G \implies x \in G) \\ &\iff \forall G \in \mathcal{F}_Q(A)(F <_{R_i} G \implies G \in \eta(x)) \\ &\iff F \in \Box_i \eta(x). \end{aligned}$$

Hence, $\eta(\Box_i x) \subset \Box_i \eta(x)$. Conversely, suppose $F \notin \eta(\Box_i x)$. Then, by Lemma 2.7, there exists a Q-filter G such that $\Box_i^{-1} F \subset G$ and $x \notin G$. Hence, $F \notin \Box_i \eta(x)$. Therefore, η is a monomorphism of multimodal algebras. We next show that η preserves infinite meets and joins in Q. Take any $n \in \omega$. Then

$$\begin{aligned} F \in \eta(\bigwedge X_n) &\iff \bigwedge X_n \in F \\ &\iff X_n \subset F \\ &\iff \forall x \in X_n (F \in \eta(x)) \\ &\iff F \in \bigcap \eta[X_n]. \end{aligned}$$

Hence, $\eta(\bigwedge X_n) = \bigcap \eta(X_n)$, and similarly, $\eta(\bigvee X_n) = \bigcup \eta[X_n]$. □

Remark 2.9 It is known that the same infinitary representation theorem holds for Heyting algebras ([8], [15], [20], and [23]). In this case, the equality $\bigwedge_{x\in X}(x \vee y) = \bigwedge_{x \in X} X \vee y$ is essential. Moreover, the intuitionistic counterpart of Section 5 holds. Let $\mathbf{L}$ be an intermediate propositional logic and $\mathbf{L}_{\omega_1\omega}$ be the infinitary and predicate extension of $\mathbf{L}$. Suppose D_{ω_1} and D denote the formulas $\bigwedge_{i\in\omega}(p_i \vee q) \supset \bigwedge_{i\in\omega} p_i \vee q$ and $\forall x(\varphi(x) \vee q) \supset \forall x \varphi(x) \vee q$, respectively. Then, if $\mathbf{L}$ is $\mathcal{D}$-persistent and universal, $\mathbf{L}_{\omega_1\omega} + \mathrm{D}_{\omega_1} + \mathrm{D}$ is Kripke complete (for details see [22], [23]).

3 Infinitary predicate multimodal logics In this section, we discuss an infinitary and predicate extension of multimodal **K**.

The language $\mathcal{L}$ of infinitary predicate multimodal logic consists of the following symbols:

1. logical connectives: $\bigwedge, \bigvee, \neg, \Box_i (i \in \omega)$;
2. quantifiers: $\forall, \exists$;
3. the set of variables of cardinality $\aleph_1$;
4. countably many constant symbols: $c, d, e, \ldots$;
5. countably many predicate symbols: $P, Q, R, \ldots$.

It should be remarked that $\mathcal{L}$ includes uncountably many variables. This makes it possible to show the proof theoretic equivalence of renaming bound variables in a standard manner. Indeed, in infinitary predicate logics, renaming bound variables is a delicate problem and there are several ways to avert this difficulty:

1. The set of variables is countable and is divided into two disjoint sets FV and BV: for each formula φ, every free variable of φ belongs to FV and every bound variable of φ belongs to BV; there is no inference rule for renaming bound variables (e.g., [6], [10], and [11]);
2. The set of variables is uncountable and assume a special inference rule for renaming bound variables (e.g., [13]);
3. The set of variables is uncountable and there is no inference rule for renaming bound variables (e.g., [12], [3], and [4]).

Note that $\mathcal{L}$ does not have any function symbols. So, a *term* in $\mathcal{L}$ is a variable or a constant symbol, and a *closed term* in $\mathcal{L}$ is a constant symbol. We write T for the set of all terms. The set of formulas of the language $\mathcal{L}$ is the smallest set which satisfies the following:

1. if P is a predicate symbol of arity n and $t_1, \ldots, t_n$ are terms, then $P(t_1, \ldots, t_n)$ is a formula;
2. if Γ is a countable set of formulas then $(\bigwedge \Gamma)$ and $(\bigvee \Gamma)$ are formulas;
3. if φ is a formula then $(\neg\varphi)$ and $(\Box_i\varphi)$ are formulas $(i \in \omega)$;
4. if φ is a formula and x is a variable of $\mathcal{L}$ then $(\forall x\varphi)$ and $(\exists x\varphi)$ are formulas.

A *Kripke frame* is a pair $(W, \{R_i\}_{i\in\omega})$, where W is a set and R_i is a binary relation on W for each $i \in \omega$. Let D be a set. A *Kripke model* $\mathcal{M}$ with constant domain D is a triple (F, D, I), where F is a Kripke frame $(W, \{R_i\}_{i\in\omega})$ and I is a mapping from W called an *interpretation* which satisfies the following conditions:

1. for any $w \in W$, I_w assigns an element $I_w(c) \in D$ to a constant symbol c, and for any constant symbol c and $w,\ w' \in W$, $I_w(c) = I_{w'}(c)$;
2. for any $w \in W$ and predicate symbol P of arity n, $I_w(P) \subset D^n$.

An *assignment* $\mathcal{A}$ is a function from the set of all variables to D. For each $w \in W$ and assignment $\mathcal{A}$, define the function $v_{I_w,\mathcal{A}}$ from T to D by

$$v_{I_w,\mathcal{A}}(t) = \begin{cases} \mathcal{A}(x) & \text{if } t \text{ is a variable } x \\ I_w(c) & \text{if } t \text{ is a constant symbol } c. \end{cases}$$

Then, the relation $\models_{\mathcal{A}}$ between $w \in W$ and a formula φ is defined by

1. $w \models_{\mathcal{A}} P(t_1, \ldots, t_n)$ if and only if $(v_{I_w,\mathcal{A}}(t_1), \ldots, v_{I_w,\mathcal{A}}(t_n)) \in I_w(P)$, for any predicate symbol P of arity n and terms $t_1, \ldots, t_n$;
2. $w \models_{\mathcal{A}} \bigwedge \Gamma$ if and only if $w \models_{\mathcal{A}} \gamma$ for any $\gamma \in \Gamma$;
3. $w \models_{\mathcal{A}} \bigvee \Gamma$ if and only if $w \models_{\mathcal{A}} \gamma$ for some $\gamma \in \Gamma$;
4. $w \models_{\mathcal{A}} \neg\varphi$ if and only if $w \not\models_{\mathcal{A}} \varphi$;
5. $w \models_{\mathcal{A}} \forall x\varphi$ if and only if $w \models_{\mathcal{A}'} \varphi$ for any $\mathcal{A}'$ such that $\mathcal{A}(y) = \mathcal{A}'(y)$ for any $y \neq x$;
6. $w \models_{\mathcal{A}} \exists x\varphi$ if and only if $w \models_{\mathcal{A}'} \varphi$ for some $\mathcal{A}'$ such that $\mathcal{A}(y) = \mathcal{A}'(y)$ for any $y \neq x$;
7. $w \models_{\mathcal{A}} \Box_i\varphi$ if and only if for any w' in W, $w <_{R_i} w'$ implies $w' \models_{\mathcal{A}} \varphi$. ($i \in \omega$).

Suppose $w \in W$ and φ is a closed formula. Then it is easy to see that $w \models_{\mathcal{A}} \varphi \iff w \models_{\mathcal{A}'} \varphi$ for any $\mathcal{A}$ and $\mathcal{A}'$. Therefore, for a closed formula φ, we write $w \models \varphi$ for $w \models_{\mathcal{A}} \varphi$. If $w \models \varphi$ for any $w \in W$, we write $\mathcal{M} \models \varphi$. If $\mathcal{M} \models \varphi$ for any $\mathcal{M}$, we write $\models \varphi$.

Now we discuss formal systems. First, we present a system $\mathrm{LK}_{\omega_1\omega}$ for classical infinitary logic, given in [6]. A sequent $\Gamma \to \Delta$ is a pair of finite sets Γ and Δ of formulas. We write Γ, Δ for $\Gamma \cup \Delta$ and Γ, φ for $\Gamma, \{\varphi\}$. The axiom schema of $\mathrm{LK}_{\omega_1\omega}$ is $p \to p$, and the derivation rules are the following:

set

$$\frac{\Gamma \to \Delta}{\Gamma' \to \Delta'}\ (\text{set}) \quad (\Gamma \subset \Gamma',\ \Delta \subset \Delta')$$

cut

$$\frac{\Gamma \to \Delta, \varphi \qquad \varphi, \Lambda \to \Xi}{\Gamma, \Lambda \to \Delta, \Xi}\ (\text{cut})$$

conjunction

$$\frac{\Gamma \to \Delta, \varphi\ (\forall\varphi \in \Theta)}{\Gamma \to \Delta, \bigwedge\Theta}\ (\to \wedge) \qquad \frac{\varphi, \Gamma \to \Delta\ (\exists\varphi \in \Theta)}{\bigwedge\Theta, \Gamma \to \Delta}\ (\wedge \to)$$

disjunction

$$\frac{\Gamma \to \Delta, \varphi\ (\exists\varphi \in \Theta)}{\Gamma \to \Delta, \bigvee\Theta}\ (\to \vee) \qquad \frac{\varphi, \Gamma \to \Delta\ (\forall\varphi \in \Theta)}{\bigvee\Theta, \Gamma \to \Delta}\ (\vee \to)$$

negation

$$\frac{\varphi, \Gamma \to \Delta}{\Gamma \to \Delta, \neg\varphi}\ (\to \neg) \qquad \frac{\Gamma \to \Delta, \varphi}{\neg\varphi, \Gamma \to \Delta}\ (\neg \to)$$

forall

$$\frac{\Gamma \to \Delta, \varphi[y/x]}{\Gamma \to \Delta, \forall x\varphi}\ (\to \forall) \qquad \frac{\varphi[t/x], \Gamma \to \Delta}{\forall x\varphi, \Gamma \to \Delta}\ (\forall \to)$$

exists

$$\frac{\Gamma \to \Delta, \varphi[t/x]}{\Gamma \to \Delta, \exists x\varphi}\ (\to \exists) \qquad \frac{\varphi[y/x], \Gamma \to \Delta}{\exists x\varphi, \Gamma \to \Delta}\ (\exists \to)$$

Here, t denotes any term which is free for x in φ and y denotes a variable which does not occur in any formulas in the lower sequent and free for x in φ.

We write LK_{ω_1} for the propositional fragment of $\mathrm{LK}_{\omega_1\omega}$. The system $\mathrm{LM}_{\omega_1\omega}$ (LM_{ω_1}) is defined by $\mathrm{LK}_{\omega_1\omega}$ (LK_{ω_1}) and the following inference rule:

$$\frac{\Gamma \to \varphi}{\Box_i\Gamma \to \Box_i\varphi}\ (\mathrm{nec}) \quad (\Box_i\Gamma := \{\Box_i\gamma : \gamma \in \Gamma\},\ i \in \omega).$$

In [6], Feferman proved the cut-elimination theorem for $\mathrm{LK}_{\omega_1\omega}$. In fact, by the methods in [6], the cut-elimination theorem for $\mathrm{LM}_{\omega_1\omega}$ is obtained immediately (see [23]).

Theorem 3.1 *If a sequent is derivable in* $\mathrm{LM}_{\omega_1\omega}$*, there exists a cut-free derivation of it.*

The logic $\mathbf{K}_{\omega_1\omega}$ ($\mathbf{K}_{\omega_1}$) is the set of all formulas which are derivable in $\mathrm{LM}_{\omega_1\omega}$ (LM_{ω_1}). We will see in Section 4 that $\mathbf{K}_{\omega_1\omega}$ and $\mathbf{K}_{\omega_1}$ are Kripke incomplete.

Now we define another formal system $\mathrm{LM}_{\omega_1\omega} \oplus \mathrm{BF}_{\omega_1\omega}$ by $\mathrm{LM}_{\omega_1\omega}$ and additional axiom schemata $\to \bigwedge_{n\in\omega} \Box_i p_n \supset \Box_i \bigwedge_{n\in\omega} p_n$ and $\to \forall x \Box_i \varphi \supset \Box_i \forall x \varphi$ for any $i \in \omega$. We use the symbol $\vdash_{\mathrm{BF}}$ for the existence of a derivation in $\mathrm{LM}_{\omega_1\omega} \oplus \mathrm{BF}_{\omega_1\omega}$. The logic $\mathbf{K}_{\omega_1\omega} \oplus \mathrm{BF}_{\omega_1\omega}$ is defined by

$$\mathbf{K}_{\omega_1\omega} \oplus \mathrm{BF}_{\omega_1\omega} := \{\varphi : \vdash_{\mathrm{BF}} \varphi\}.$$

Namely, the set of all formulas which are derivable in $\mathrm{LM}_{\omega_1\omega} \oplus \mathrm{BF}_{\omega_1\omega}$.

4 Completeness theorem In this section, we present the completeness theorem of $\mathbf{K}_{\omega_1\omega} \oplus \mathrm{BF}_{\omega_1\omega}$ with respect to the class of Kripke frames. Let C be a countable set of new constant symbols and $\mathcal{L}'$ be a new language consisting of symbols in $\mathcal{L}$ and C. A derivation $\mathcal{D}$ is said to be *of the language* $\mathcal{L}$ ($\mathcal{L}'$), if each sequent in $\mathcal{D}$ consists of formulas of the language $\mathcal{L}$ ($\mathcal{L}'$).

Lemma 4.1 *Let* $\Gamma \to \Delta$ *be a sequent of the language* $\mathcal{L}'$*. Suppose there exists a derivation* $\mathcal{D}'$ *of the language* $\mathcal{L}'$ *of* $\Gamma \to \Delta$*. Let* $(c_i)_{i\in\omega}$ *and* $(y_i)_{i\in\omega}$ *be any mutually distinct lists of constant symbols and variables, respectively. Suppose* $(x_i)_{i\in\omega}$ *is any mutually distinct list of variables such that none of them has any occurrences in the derivation* $\mathcal{D}'$*. Then*

1. *there exists a derivation* $\mathcal{D}$ *of the sequent* $\Gamma[x_i/c_i \mid i \in \omega] \to \Delta[x_i/c_i \mid i \in \omega]$ *such that any constant symbol of* $(c_i)_{i\in\omega}$ *does not occur in* $\mathcal{D}$*;*
2. *there exists a derivation* $\mathcal{D}$ *of the sequent* $\Gamma[x_i/y_i \mid i \in \omega] \to \Delta[x_i/y_i \mid i \in \omega]$.

Proof: With the aid of uncountably many variables, we can prove the lemma in a standard manner. □

Lemma 4.2 *Let* φ *be a closed formula of the language* $\mathcal{L}$*. If there exists a derivation* $\mathcal{D}'$ *of the language* $\mathcal{L}'$ *of* φ*, then there exists a derivation* $\mathcal{D}$ *of the language* $\mathcal{L}$ *of* φ.

Proof: Let $(c_i)_{i\in\omega}$ be an enumeration of C. It is obvious that $\mathcal{D}'$ includes at most countably many variables. Since there exist uncountably many variables in $\mathcal{L}$, there exists a mutually distinct list $(x_i)_{i\in\omega}$ of variables such that each of them does not occur in $\mathcal{D}'$. Hence, by Lemma 4.1, there exists a derivation $\mathcal{D}$ of the formula $\varphi[x_i/c_i \mid i \in \omega] = \varphi$ which does not include any constant symbols in C. □

The set $\mathrm{sub}(\varphi)$ of all subformulas of a formula φ is defined as usual. In particular, if $\varphi = \bigwedge \Gamma$ or $\bigvee \Gamma$, then $\mathrm{sub}(\varphi) = \{\varphi\} \cup \bigcup_{\gamma \in \Gamma} \mathrm{sub}(\gamma)$. By a simple induction, the cardinality of $\mathrm{sub}(\varphi)$ is at most countable for any formula φ. Also if φ contains only finite free variables, any formula in $\mathrm{sub}(\varphi)$ has only finite free variables. Let φ be any formula of the language $\mathcal{L}'$, $\mathrm{fv}(\varphi)$ be the set of all free variables in φ, and $\mathrm{subst}(\varphi)$ be the set of all instances of the substitutions of constant symbols of $\mathcal{L}'$ to some free variables in φ, that is,

$$\mathrm{subst}(\varphi) := \{\varphi[t_x/x \mid x \in X] : X \subset \mathrm{fv}(\varphi),\ \forall x \in X (t_x \in \mathcal{L}' \text{ is closed})\}.$$

Then the set $\mathrm{esub}(\varphi)$ of *extended subformulas* of φ is defined by

$$\mathrm{esub}(\varphi) := \mathrm{sub}(\varphi) \cup \bigcup_{\psi \in \mathrm{sub}(\varphi)} \mathrm{subst}(\psi).$$

It is easy to see that if a formula φ contains only finite free variables the cardinality of the set $\mathrm{esub}(\varphi)$ is also countable. A set Γ of formulas is said to be *closed under extended subformulas* if $\varphi \in \Gamma$ implies $\mathrm{esub}(\varphi) \subset \Gamma$. The closure $\mathcal{C}_{\mathrm{e}}(\Gamma)$ of extended subformulas of Γ is the smallest set of formulas which includes Γ and is closed under extended subformulas. A set Γ of formulas is said to be *closed under finitary connectives* if the following conditions are satisfied:

1. if φ and ψ are members of Γ, then $\varphi \wedge \psi$ and $\varphi \vee \psi$ are members of Γ;
2. if φ is a member of Γ, then $\neg\varphi$ and $\Box_i\varphi$ are members of Γ, for all $i \in \omega$;
3. if φ is a member of Γ, then $\forall x\varphi$ and $\exists x\varphi$ is a member of Γ for any variable x which has some free or bounded occurrences in some formulas in Γ;
4. $\bigwedge \varnothing \in \Gamma$ and $\bigvee \varnothing \in \Gamma$.

The closure $\mathcal{C}_{\mathrm{f}}(\Gamma)$ of finitary connectives of Γ is the smallest set of formulas which includes Γ and is closed under finitary connectives. A set Γ of formulas is said to be *closed* if $\mathcal{C}_{\mathrm{e}}(\Gamma) = \Gamma$ and $\mathcal{C}_{\mathrm{f}}(\Gamma) = \Gamma$. The closure $\mathcal{C}(\Gamma)$ of a set Γ is the smallest closed set which includes Γ. A set Ψ of closed formulas is said to be *saturated* if it is the set of all closed formulas of some closed set Γ of formulas. Then the following lemmas hold immediately.

Lemma 4.3 *Let Γ be a countable set of formulas. Suppose each formula in Γ contains only finite free variables and constant symbols of C. Then $\mathcal{C}(\Gamma)$ is countable and any formula in $\mathcal{C}(\Gamma)$ has only finite free variables and constant symbols of C.*

Lemma 4.4 *Let Ψ be a saturated set of the language $\mathcal{L}'$. Let $\sim$ be a subset of $\Psi \times \Psi$ defined by $\psi \sim \varphi \iff \vdash_{\mathrm{BF}} (\psi \supset \varphi) \wedge (\varphi \supset \psi)$ where the derivation is of the language $\mathcal{L}'$. Then $\sim$ is an equivalence relation on Ψ and $\Psi/\sim$ is a multimodal algebra under the operations:*

1. $|\varphi| \wedge |\psi| = |\varphi \wedge \psi|$, *for any* $\varphi,\ \psi \in \Psi$;
2. $|\varphi| \vee |\psi| = |\varphi \vee \psi|$, *for any* $\varphi,\ \psi \in \Psi$;
3. $-|\varphi| = |\neg\varphi|$, *for any* $\varphi \in \Psi$;
4. $\Box_i|\varphi| = |\Box_i\varphi|$, *for any* $\varphi \in \Psi$ $(i \in \omega)$;
5. $0 = |\bigvee \varnothing|$;
6. $1 = |\bigwedge \varnothing|$.

To show the completeness theorem, we need the following lemma.

Lemma 4.5 *Let Ψ be a saturated set of the language $\mathcal{L}'$. Suppose each formula in Ψ contains only finite constant symbols of C. Then each of the right-hand side of the following equalities exists in the modal algebra $A = \Psi/\sim$ and each of the equalities holds in A:*

1. $|\bigwedge \Gamma| = \bigwedge |\Gamma|$ *and* $|\bigvee \Gamma| = \bigvee |\Gamma|$, *for all* $\bigwedge \Gamma \in \Psi$ *and* $\bigvee \Gamma \in \Psi$;
2. $\Box_i |\bigwedge \Gamma| = \bigwedge |\Box_i \Gamma|$, *for all* $\bigwedge \Gamma \in \Psi$ *and* $i \in \omega$;
3. $\Box_i(|\varphi| \to |\bigwedge \Gamma|) = \bigwedge_{\gamma \in \Gamma} |\Box_i(\varphi \supset \gamma)|$, *for all* φ, $\bigwedge \Gamma \in \Psi$, *and* $i \in \omega$;
4. $\Box_i(|\bigvee \Gamma| \to |\varphi|) = \bigwedge_{\gamma \in \Gamma} |\Box_i(\gamma \supset \varphi)|$, *for all* φ, $\bigvee \Gamma \in \Psi$, *and* $i \in \omega$;
5. $|\forall x \varphi| = \bigwedge \{|\varphi[t/x]| : t \text{ is closed}\}$ *and* $|\exists x \varphi| = \bigvee \{|\varphi[t/x]| : t \text{ is closed}\}$, *for all* $\forall x \varphi$ *and* $\exists x \varphi$ *in* Ψ;
6. $|\Box_i \forall x \varphi| = \bigwedge \Box_i \{|\varphi[t/x]| : t \text{ is closed}\}$ *for all* $\forall x \varphi \in \Psi$ *and* $i \in \omega$;
7. $|\forall x(\Box_i(\psi \supset \varphi))| = \bigwedge \{\Box_i(|\psi| \to |\varphi[t/x]|) : t \text{ is closed}\}$, *for all* $\forall x \varphi$ *and* ψ *in* Ψ, *and for all* $i \in \omega$;
8. $|\forall x(\Box_i(\varphi \supset \psi))| = \bigwedge \{\Box_i(|\varphi[t/x]| \to |\psi|) : t \text{ is closed}\}$, *for all* $\exists x \varphi$ *and* ψ *in* Ψ, *for all* $i \in \omega$.

Proof: (1) is straightforward. (2) follows from the axiom BF_{ω_1}. Now the equalities

$$|\varphi| \to \bigwedge |\Gamma| = \bigwedge_{\gamma \in \Gamma} (|\varphi| \to |\gamma|), \ \bigvee |\Gamma| \to |\varphi| = \bigwedge_{\gamma \in \Gamma} (|\gamma| \to |\varphi|)$$

always hold in any Boolean algebra. Hence, (3) and (4) are special cases of (2). As to the first part of (5), suppose $\forall x \varphi \in \Psi$. It is clear that $\{|\varphi[t/x]| : t \text{ is closed}\}$ is a well-defined subset of A and $|\forall x \varphi|$ is its lower bound. Suppose $|\psi|$ is another lower bound. Then, $\vdash_{\mathrm{BF}} \psi \to \varphi[t/x]$, for any t. Since ψ and $\forall x \varphi$ include only finite constant symbols of C, there exists c in C which does not occur in ψ and $\forall x \varphi$. Now there exists a variable y which does not occur in the derivation of $\psi \to \varphi[c/x]$. Then, $\vdash_{\mathrm{BF}} \psi \to \varphi[y/x]$ by Lemma 4.1. Hence, $\vdash_{\mathrm{BF}} \psi \to \forall x \varphi$ which means $|\psi| \leq |\forall x \varphi|$. Hence,

$$\bigwedge \{|\varphi[t/x]| : t \text{ is closed}\} = |\forall x \varphi| \in A.$$

The second part is similar. (6) follows from BF. Since Ψ is saturated, $\forall x(\Box_i(\psi \supset \varphi)) \in \Psi$ whenever $\forall x \varphi \in \Psi$, and $\forall x(\Box_i(\varphi \supset \psi)) \in \Psi$ whenever $\exists x \varphi \in A$. Hence, (7) and (8) are special cases of (6). □

Now we prove the completeness theorem of infinitary predicate multimodal logic.

Theorem 4.6 *A closed formula φ of the language $\mathcal{L}$ is a member of $\mathbf{K}_{\omega_1 \omega} \oplus \mathrm{BF}_{\omega_1 \omega}$ if and only if it is valid in every Kripke model with constant domain.*

Proof: An easy induction shows that if φ is derivable in $\mathrm{LM}_{\omega_1 \omega} \oplus \mathrm{BF}_{\omega_1 \omega}$ then it is valid in every Kripke model with constant domain. We show the converse. By Lemma 4.2, it is enough to show that if there exists no derivation of the language $\mathcal{L}'$ of φ, then there exists a Kripke model with constant domain which refutes φ. Let Ψ be the set of all closed formulas of the closure of the set $\{\varphi\}$. By Lemma 4.3, Ψ is countable and each formula in Ψ contains only finite constant symbols of C. Let A be the modal algebra $\Psi/\sim$ in Lemma 4.4. For each closed formula $\psi \in \Psi$ of the shape $\forall x\chi$ or $\exists x\chi$, let $[\psi]$ be the subset $\{|\chi[t/x]| : t \text{ is closed}\}$ of A. Define two subsets α_0 and β_0 of $\mathcal{P}(A)$ by

$$\alpha_0 = \beta_0 = \{[\psi] : \psi \in \Psi \text{ is of the shape } \forall x\chi \text{ or } \exists x\chi\}.$$

Then define

1. $\alpha_1 = \{|\Gamma| : \bigwedge\Gamma \in \Psi\}$ and $\beta_1 = \{|\Gamma| : \bigvee\Gamma \in \Psi\}$;
2. $\alpha_2 = \{\{\Box_i(y \to z) : y \in Y\} : i \in \omega, z \in A,\ Y \in \beta_1\}$;
3. $\alpha_{n+1} = \{\{\Box_i(z \to x) : x \in X\} : i \in \omega, z \in A,\ X \in \bigcup_{k \le n}\alpha_k\}$ $(n \ge 2)$.

Define $Q = (\{X_n\}_{n\in\omega}, \{Y_n\}_{n\in\omega})$ by $(\bigcup_{n\in\omega}\alpha_n, \beta_0 \cup \beta_1)$. By Lemma 4.5, Q satisfies the conditions in Lemma 2.6.

Now we define a Kripke model $(W, \{R_i\}_{i\in\omega}, D, I)$ with constant domain which refutes φ. Let $W = \mathcal{F}_Q(A)$, R_i be the binary relation in Proposition 2.3 for any $i \in \omega$; D be the set of all closed terms in $\mathcal{L}'$; and I be an interpretation defined by

1. $I_F(t) = t$, for any $F \in W$ and closed term t;
2. $(t_1, \ldots, t_n) \in I_F(P) \iff |P(t_1, \ldots, t_n)| \in F$, for any $F \in W$ and any predicate P of arity n.

Then, for any $\psi \in \Psi$ and $F \in \mathcal{F}_Q(A)$, ψ is valid in F if and only if $|\psi| \in F$, by an induction on ψ. We only show the case where $\psi = \Box_i\chi$. Suppose $|\Box_i\chi| \in F$. Then, $\Box_i^{-1}F \subset G$ implies $G \models \chi$, since $|\chi| \in G$. Hence, $F \models \Box_i\chi$. Suppose $|\Box_i\chi| \notin F$. Then, by Lemma 2.7, there exists $G \in \mathcal{F}_Q(A)$ such that $\Box_i^{-1}F \subset G$ and $|\chi| \notin G$. Hence, $F \not\models \Box_i\chi$. Now, since $|\varphi| \neq 1$ in A, there exists a Q-filter F such that $|\varphi| \notin F$, by Proposition 2.5. Hence, φ is not valid at F. □

It is known that the predicate extension of **K** plus BF is complete with respect to the class of Kripke frames with constant domain. On the other hand, since BF_{ω_1} is not derivable in $\text{LM}_{\omega_1\omega}$ by Theorem 3.1, we have the following corollary.

Corollary 4.7 *The logic $\mathbf{K}_{\omega_1\omega}$ plus* BF *is incomplete with respect to the class of Kripke frames with constant domain and the logic $\mathbf{K}_{\omega_1}$ is Kripke incomplete.*

5 Applications Let f_i be a function which replaces all occurrences of $\Box$ in a monomodal formula with $\Box_i$, for each $i \in \omega$. For any propositional monomodal logic **L**, we write $\mathbf{L}_{\omega_1\omega} \oplus \text{BF}_{\omega_1\omega}$ for the logic axiomatized by the system consists of $\text{LM}_{\omega_1\omega} \oplus \text{BF}_{\omega_1\omega}$ and additional axiom schemata

$$\{\to\ f_i(\varphi) : \varphi \in \mathbf{L},\ i \in \omega\}.$$

In this section, we give a sufficient condition on **L** for the completeness theorem of its extension $\mathbf{L}_{\omega_1\omega} \oplus \text{BF}_{\omega_1\omega}$.

A class C of (monomodal) Kripke frames is said to be *elementary* if there exists a set Ψ of first-order sentences in R and $=$ such that

$$C = \{F : F \text{ satisfies } \Psi \text{ as a first-order structure}\}.$$

An elementary class C of monomodal Kripke frames is said to be *universal* if any formula in Ψ is of the form $\forall x_1, \ldots, \forall x_n\psi$. Let **L** be a propositional modal logic. We write $C_{\mathbf{L}}$ for the class $\{F : F \models \mathbf{L}\}$ of Kripke frames. Then, $\mathbf{L}$ is said to be *elementary* (*universal*), if $C_{\mathbf{L}}$ is elementary (universal). For any class C of monomodal Kripke frames, we write C^* for its multimodal extension, namely,

$$C^* := \{(W, \{R_i\}_{i\in\omega}) : \forall i \in \omega((W, R_i) \in C)\}.$$

Let A be a modal algebra. An *assignment* v on A is a function from the set of all formulas of propositional modal logic to A which satisfies

1. $v(p) \in A$ for any propositional variable p;
2. $v(\varphi * \psi) = v(\varphi) * v(\psi)$ for any formulas φ and ψ, where $* \in \{\wedge, \vee\}$;
3. $v(\neg\varphi) = -v(\varphi)$ for any formula φ;
4. $v(\Box\varphi) = \Box v(\varphi)$ for any formula φ.

A formula φ of propositional modal logic is said to be *valid* in A if $v(\varphi) = 1$ for any assignment v on A, and a logic $\mathbf{L}$ is said to be *valid* in A if every $\varphi \in \mathbf{L}$ is valid in A.

The following is the generalized completeness theorem.

Theorem 5.1 *Let* $\mathbf{L}$ *be a propositional modal logic above* $\boldsymbol{K}$*. Suppose* C *is a universal class of Kripke frames such that for any modal algebra* A*, if* $\mathbf{L}$ *is valid in* A *then* $(\mathcal{F}_{\mathrm{p}}(A), R) \in C$*, where* $F <_R G \iff \Box^{-1}F \subset G$ *for any* F *and* G *in* $\mathcal{F}_{\mathrm{p}}(A)$*. Then,* $\mathbf{L}_{\omega_1\omega} \oplus \mathrm{BF}_{\omega_1\omega}$ *is complete with respect to the class* C^* *of Kripke frames.*

Proof: Suppose $\psi \notin \mathbf{L}_{\omega_1\omega} \oplus \mathrm{BF}_{\omega_1\omega}$. Consider the Lindenbaum algebra A and take Q as in the proof of Theorem 4.6. Since A validates $f_i(\varphi)$ for any $\varphi \in \mathbf{L}$, the frame $(\mathcal{F}_{\mathrm{p}}(A), R_i)$ is a member of C for any $i \in \omega$. Hence, the frame $(\mathcal{F}_Q(A), \{R_i\}_{i\in\omega})$ is a member of C^*, since any Q-filter is a prime filter and C is universal. However, ψ is not valid in the frame $(\mathcal{F}_Q(A), \{R_i\}_{i\in\omega})$. $\Box$

A triple $F = (W, R, P)$ is called a *general frame* if (W, R) is a Kripke frame and P is a subalgebra of $\mathcal{P}(W)$ in Proposition 2.3. P is called the *dual algebra* of F and written by F^+. Let A be a modal algebra. It is known that the function $\eta : A \to \mathcal{P}(\mathcal{F}_{\mathrm{p}}(A))$ defined by $\eta : x \mapsto \{F : x \in F\}$ is a monomorphism of modal algebras. Then the general frame $(\mathcal{F}_{\mathrm{p}}(A), R, \eta[A])$, where R is the binary relation in Proposition 2.3, is called the *dual frame* of A and written by A_+. A general frame F is said to be *descriptive* if it is isomorphic to $(F^+)_+$. A propositional formula φ is said to be valid in a general frame (W, R, P) if φ is valid in every Kripke model (W, R, v) such that $v(p) \in P$ for any propositional variable p. For any Kripke (general) frame F, we write $F \models \varphi$, if φ is valid in F. A logic $\mathbf{L}$ is said to be valid in a Kripke (general) frame F if $F \models \varphi$ for any $\varphi \in \mathbf{L}$ which is written as $F \models \mathbf{L}$. A logic $\mathbf{L}$ is said to be *$\mathcal{D}$-persistent* if $(W, R, P) \models \mathbf{L}$ implies $(W, R) \models \mathbf{L}$ for any descriptive frame (W, R, P).

The following properties are well known (see, e.g., [2]).

Proposition 5.2 *Let* A *be a modal algebra. For any formula* φ *of propositional modal logic,* φ *is valid in* A *if and only if it is valid in the dual frame* A_+*.*

Proposition 5.3 *Let* $\mathbf{L}$ *be a* $\mathcal{D}$*-persistent propositional modal logic. If* $\mathbf{L}$ *is valid in a modal algebra* A*, then* $(\mathcal{F}_{\mathrm{p}}(A), R) \models \mathbf{L}$*.*

Proposition 5.4 *Let* $\mathbf{L}$ *be a* $\mathcal{D}$*-persistent propositional modal logic. Then* $\mathbf{L}$ *is complete with respect to the class* $C_{\mathbf{L}}$ *of Kripke frames.*

Then we have the following.

Theorem 5.5 *Let* **L** *be a* $\mathcal{D}$*-persistent and universal propositional modal logic. Then* $\mathbf{L}_{\omega_1\omega} \oplus \mathrm{BF}_{\omega_1\omega}$ *is complete with respect to the class* ${C_{\mathbf{L}}}^*$ *of Kripke frames.*

Proof: Let A be a modal algebra in which **L** is valid. Then **L** is valid in the Kripke frame $(\mathcal{F}_{\mathrm{p}}(A), R)$ by Proposition 5.3. Hence, $(\mathcal{F}_{\mathrm{p}}(A), R) \in C_{\mathbf{L}}$ and therefore $\mathbf{L}_{\omega_1\omega} \oplus \mathrm{BF}_{\omega_1\omega}$ is complete with respect to the class ${C_{\mathbf{L}}}^*$, by Theorem 5.1. □

A propositional modal logic **L** above **K** is called a *subframe logic* if it is characterized by a class of general frames which are closed under subframes (for more information see, e.g., [2], [26]). We also say that **L** has the *finite embedding property* if a Kripke frame F validates **L** whenever each finite subframe validates **L**. It is known that the following conditions are equivalent for each propositional subframe logic **L** above **K** (see [2] for details).

1. **L** is universal and Kripke complete;
2. **L** is $\mathcal{D}$-persistent;
3. **L** has the finite embedding property and is Kripke complete.

It is also known that the following conditions are equivalent for each finitary propositional subframe logic **L** above **K4** (see [2] for details).

1. **L** is universal;
2. **L** is $\mathcal{D}$-persistent;
3. **L** has the finite embedding property.

Then we have the following as corollaries of Theorem 5.5.

Corollary 5.6 *Suppose a propositional modal logic* **L** *above* ***K*** *is a subframe logic, has the finite embedding property, and is Kripke complete. Then* $\mathbf{L}_{\omega_1\omega} \oplus \mathrm{BF}_{\omega_1\omega}$ *is complete with respect to the class* ${C_{\mathbf{L}}}^*$ *of Kripke frames.*

Corollary 5.7 *Suppose a propositional modal logic* **L** *above* **K4** *is a subframe logic and has the finite embedding property. Then* $\mathbf{L}_{\omega_1} \oplus \mathrm{BF}_{\omega_1}$ *is complete with respect to the class* ${C_{\mathbf{L}}}^*$ *of Kripke frames.*

The following theorem is known as the Fine-van Benthem Theorem (see [25], [7], also [2]).

Theorem 5.8 *If a propositional modal logic* **L** *above* ***K*** *is characterized by an elementary class* C *of Kripke frames then* **L** *is* $\mathcal{D}$*-persistent.*

From the Fine-van Benthem Theorem and Theorem 5.5, we have the following immediately.

Theorem 5.9 *If a propositional modal logic* **L** *above* ***K*** *is characterized by a universal class* $C_{\mathbf{L}}$ *of Kripke frames, then* $\mathbf{L}_{\omega_1\omega} \oplus \mathrm{Bf}_{\omega_1\omega}$ *is complete with respect to the class* ${C_{\mathbf{L}}}^*$ *of Kripke frames.*

Conversely, Theorem 5.9 implies Theorem 5.5, as follows. Suppose **L** is a $\mathcal{D}$-persistent and universal propositional modal logic above **K**. By Proposition 5.4, **L** is complete with respect to the class $C_{\mathbf{L}}$ of Kripke frames. Hence, **L** is characterized by the universal class $C_{\mathbf{L}}$ of Kripke frames. Consequently, Theorem 5.5 and Theorem 5.9 are equivalent.

The foregoing completeness proofs rely on the universality of the class C of Kripke frames. On the other hand, for some propositional modal logic $\mathbf{L}$, we can immediately prove Kripke completeness of $\mathbf{L}_{\omega_1\omega} \oplus \mathrm{BF}_{\omega_1\omega}$ with respect to the class C^* without assuming that C is universal. Let D be the formula $\neg\Box\bot$. It is well known that $\mathbf{K} \oplus \mathrm{D}$ is complete with respect to the class C of serial frames. Now we show that $\mathbf{K}_{\omega_1\omega} \oplus \mathrm{BF}_{\omega_1\omega} \oplus \mathrm{D}$ is complete with respect to the class C^* of serial frames. Suppose $\psi \notin \mathbf{K}_{\omega_1} \oplus \mathrm{BF}_{\omega_1} \oplus \mathrm{D}$. Let A be the Lindenbaum algebra. Since A satisfies $-\Box 0 = 1$, we have $\Box 0 = 0$. Let F be any Q-filter of A. Since $\Box 0 = 0 \notin F$, there exists a Q-filter G such that $\Box^{-1} F \subset G$ and $0 \notin G$, by Lemma 2.7. Therefore, the frame $(\mathcal{F}_Q(A), R_i)$ belongs to C for any $i \in \omega$. Then from Theorem 5.1, we have the following.

Theorem 5.10 *The logic* $\mathbf{K}_{\omega_1\omega} \oplus \mathrm{BF}_{\omega_1\omega} \oplus \mathrm{D}$ *is complete with respect to the class of serial frames.*

Now the following corollary follows immediately.

Corollary 5.11 *Any infinitary predicate multimodal logic which is defined by* $\mathbf{K}_{\omega_1\omega} \oplus \mathrm{BF}_{\omega_1\omega}$ *plus additional axiom schemata* T, B, 4, 5, D, *and their combinations is complete with respect to the class of reflexive, symmetric, transitive, euclidean, serial, and their combined frames, respectively.*

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