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EXISTENCE RESULTS FOR EVOLUTION EQUATIONS WITH SUPERLINEAR GROWTH

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Dedicated to the memory of Professor Ioan I. Vrabie

Abstract. By combining an approximation technique with the Leray–Schauder continuation principle, we prove global existence results for semilinear differential equations involving a dissipative linear operator, generating an extendable compact C_0-semigroup of contractions, and a Carathéodory nonlinearity $f\colon [0,T] \times E \to F$, with E and F two real Banach spaces such that $E \subseteq F$, besides imposing other conditions. The case $E \neq F$ allows to treat, as an application, parabolic equations with continuous superlinear nonlinearities which satisfy a sign condition.

1. Introduction

Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ be two real Banach spaces such that $E \subseteq F$. This work deals with the study of mild solutions for semilinear differential equations

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of the form

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), & \text{for a.e. } t \in [0, T], \\ u(0) = u_0 \in E, \end{cases} \tag{1.1}$$

where $A\colon D(A) \subset E \to E$ is a linear operator generating a compact C_0-semigroup of contractions and $f\colon [0,T] \times E \to F$ is a Carathéodory map such that for every bounded subset $\Omega \subset E$ there exists a function $\nu_\Omega \in L^1([0,T], \mathbb{R}_+)$ such that

$$\|f(t, v)\|_F \leq \nu_\Omega(t), \tag{1.2}$$

for almost every $t \in [0,T]$ and all $v \in \Omega$.

Differential problems of the type (1.1) with $E = F$ are often studied by means of topological methods. In particular, assuming sublinear growth conditions on the nonlinearity allows to apply directly classical fixed point theorems. This is no longer the case for nonlinear terms for which only a local bounded condition such as (1.2) is required. However, as showed in [1] where condition (1.2) is assumed, it is again possible to use some topological techniques by considering the invariance of a suitable topological degree by an homotopic field. We recall that, in order that such an invariance is satisfied, there must be no fixed points on the homotopic field domain's boundary. This is usually known as the transversality condition (or property). Sufficient conditions for getting the transversality property are introduced and discussed in [7] and [10], for finite dimensional systems. These techniques were then extended to infinite dimensional Banach spaces in [1], when $A\colon E \to E$ is a linear and bounded operator. Very recently, they have been generalized to generators of C_0 semigroups in [3]. In [2], it is showed that, in the particular case of a Banach space X with a strictly convex dual X^*, the transversality property is given by the existence of two constants $r_0 > 0$ and $R_0 > \max\{r_0, \|u_0\|_X\}$ such that

$$\langle J(v), f(t, v)\rangle_X \leq 0,$$

for almost every $t \in [0,T]$ and all $v \in X \cap \{r_0 < \|v\|_X < R_0\}$, where $J\colon X \to X^*$ is the duality map (see (2.1) and Proposition 2.1); notice that since X^* is strictly convex, J is a single valued map. In the present paper, we use a similar sign condition but taking into account that there are two different Banach spaces (see condition (f4)).

The most common application of the mentioned results are partial differential equations considering as Banach space E the space $L^p(\Omega; \mathbb{R})$, with $p \geq 1$, and Ω a domain in $\mathbb{R}^k$, $k \geq 1$. However, due to the Vainberg Theorem, the Nemytskiĭ operator associated to a Carathéodory function $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ maps continuously the space $L^p(\Omega)$ into itself if and only if g is sublinear, as stated in the following theorem (see Theorem 19.1 in [12]).

Theorem 1.1. *Let B be a measurable set in a s-dimensional euclidean space and $g\colon B \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, i.e. continuous with respect to $u \in \mathbb{R}$ for almost every $x \in B$ and measurable with respect to $x \in B$ for every $u \in \mathbb{R}$. Then the Nemytskiĭ operator associated to g, $h(u)(x) = g(x, u(x))$, is a continuous and bounded operator from $L^p(B;\mathbb{R})$ into $L^q(B;\mathbb{R})$, $p, q \in [1, +\infty)$, if and only if there exist a function $a \in L^q(B;\mathbb{R})$ and a constant $b \geq 0$ such that, for every $v \in \mathbb{R}$,*

$$|g(x, v)| \leq a(x) + b|v|^{p/q}.$$

By considering two different Banach spaces, we overcome this difficulty handling nonlinearities with superlinear growth. More precisely, we assume that the semigroup generated by A on E can be extended to the space F (see hypothesis (A2)) and, by means of an approximation technique, we prove an existence result for mild solutions of (1.1) applying the Leray–Schauder continuation principle below, see e.g. [6] or the original paper [9].

Theorem 1.2. *Let Q be a closed subset of a Banach space $\mathcal{B}$ and let $\Sigma\colon Q \times [0, 1] \to \mathcal{B}$ be a continuous map sending bounded subsets of $Q \times [0, 1]$ into relatively compact subsets of $\mathcal{B}$. Assume that*

(a) $\Sigma(x, 0) = x_0 \in \operatorname{int}(Q)$, *for all $x \in Q$;*
(b) *The fixed point set*

$$F = \{x \in Q, x = \Sigma(x, \lambda),\ \text{for some } \lambda \in [0, 1]\}$$

is bounded and does not meet the boundary ∂Q of Q.

Then the map $x \mapsto \Sigma(x, 1)$ has a fixed point in Q.

As a consequence, we obtain the localization of this solution in the ball of radius R_0 and center 0.

Setting $E = L^p(\Omega;\mathbb{R})$ and $F = L^q(\Omega;\mathbb{R})$ with $2 \leq q \leq p < \infty$, where $\Omega \subset \mathbb{R}^k$, $2 \leq k \leq 2pq/(p - q)$, with $2 \leq q < p < \infty$, ($k \geq 2$, in the case $p = q$), is a bounded domain with C^2-boundary, and exploiting the Vainberg Theorem, we obtain an existence result for mild solutions of the following class of parabolic differential equations

$$\begin{aligned}
&u_t = \Delta u + g(t, x, u(t, x)) && \text{for } (t, x) \in \,]0, T[\, \times \Omega, \\
&u(t, x) = 0 && \text{for } (t, x) \in \,]0, T[\, \times \partial\Omega, \\
&u(0, x) = u_0(x) && \text{for } x \in \Omega,
\end{aligned} \tag{1.3}$$

allowing $g\colon [0, T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ to have superlinear growth.

The paper is organized as follows: in Section 2, we recall some basic results on semi-inner products, on the duality map, on the generator of semigroups, and recall different notions of solutions, including their relations (which turn out useful to prove the main result); the statement of the problem and the main result are contained in Section 3; in Section 4, the approximation technique is described; the proof of the main result of the paper is contained in Section 5; finally, in Section 6 the existence of mild solution of (1.3) is proved.

2. Preliminaries

Let $(X, \|\cdot\|_X)$ be a real Banach space and X^* its dual, let $x, y \in X$ and $h \in \mathbb{R} \setminus \{0\}$, define

$$[x,y]_h := \frac{1}{h}\left(\|x+hy\| - \|x\|\right).$$

The limits $[x,y]_+ = \lim\limits_{h\downarrow 0}[x,y]_h$ and $[x,y]_- = \lim\limits_{h\uparrow 0}[x,y]_h$ exist and are finite. In addition, the function $[\,\cdot\,,\,\cdot\,]_+$ is upper semicontinuous from $X\times X$ into $\mathbb{R}$, while $[\,\cdot\,,\,\cdot\,]_-$ is lower semicontinuous from $X\times X$ into $\mathbb{R}$; the function $[\,\cdot\,,\,\cdot\,]_+$ is called the normalized upper semi-inner product on X and $[\,\cdot\,,\,\cdot\,]_-$ is called the normalized lower semi-inner product on X (see Lemma 1.4.1 and Definition 1.4.2 of [13]). Denoting by $\langle\,\cdot\,,\,\cdot\,\rangle$ the duality product between X^* and X and by $J\colon X \multimap X^*$ the duality map, i.e.

$$J(x) = \left\{x^* \in X^*\colon \|x^*\|_{X^*} = \|x\|_X \text{ and } \langle x^*, x\rangle = \|x\|_X^2\right\} \tag{2.1}$$

for every $x, y \in X$, $x \neq 0$, we have

$$[x,y]_+ = \frac{1}{\|x\|_X}\sup\{\langle x^*, y\rangle\colon x^* \in J(x)\},$$
$$[x,y]_- = \frac{1}{\|x\|_X}\inf\{\langle x^*, y\rangle\colon x^* \in J(x)\}.$$

Moreover, we have

$$[x,y]_+ = -[x,-y]_- = -[-x,y]_-,$$

see Lemmas 1.4.2 and 1.4.3 in [13]. The next proposition contains some useful properties of the duality map, see Proposition 12.3 in [4] and Theorem 1 in [5].

Proposition 2.1. *The following statements are true*:

(a) *For each $x \in X$, the set $J(x)$ is convex and nonempty in X^**;

(b) *J is monotone in the following sense*

$$\langle x^* - y^*, x - y\rangle \geq 0 \quad \textit{for every } x, y \in X \textit{ and } x^* \in J(x), y^* \in J(y);$$

(c) *for every $\lambda \in \mathbb{R}$ and $x \in X$, it holds $J(\lambda x) = \lambda J(x)$.*

In particular, if X^* is strictly convex then J is a single valued map, and if X is reflexive then J is demicontinuous, i.e. if $x_n \to x$ in X, $J(x_n) \rightharpoonup J(x)$ in X^*, see e.g. [5]. Thus, in the case of X^* strictly convex, we get

$$[x,y]_+ = \frac{1}{\|x\|}\langle J(x), y\rangle = [x,y]_-. \tag{2.2}$$

Consider the linear problem

$$\begin{cases} u'(t) = Au(t) + f(t), & \text{for a.e. } t \in [0,T], \\ u(0) = u_0 \in X, \end{cases} \tag{2.3}$$

where $A\colon D(A) \subset X \to X$ is the infinitesimal generator of a semigroup $\{S(t)\}_{t\geq 0}$ and $f\colon [0,T] \to X$ is a given map. There are several definitions of solutions of (2.3). For instance, if $f \in L^1([0,T];X)$ we have the following definitions (see Definition 3.5.1 in [8] and Definition 1.7.4 in [13]).

Definition 2.2. A function $u\colon [0,T] \to X$ is called an integral solution of (2.3) on $[0,T]$ if $u \in C([0,T];X)$ satisfies $u(0) = u_0$ and

$$\|u(t) - x\|_X \leq \|u(s) - x\|_X + \int_s^t [u(\tau) - x, f(\tau) - Ax]_+ \, d\tau$$

for each $x \in D(A)$ and $0 \leq s \leq t \leq T$.

To give the concept of ε-approximate solution we need the definition of Λ_ε-discretization.

Definition 2.3. Let $\varepsilon > 0$ be given. Then $\Lambda(\varepsilon; t_0, \ldots, t_n; f_1, \ldots, f_n)$ is called a Λ_ε-discretization of (2.3) on $[0,T]$ if

(D1) $0 = t_0 < t_1 < \ldots < t_n = T$, $f_1, \ldots, f_n \in X$;
(D2) $t_i - t_{i-1} \leq \varepsilon$, for $i = 1, \ldots, n$;
(D3) $\displaystyle\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|f(s) - f_i\|_X \, ds \leq \varepsilon$.

For the next two definitions see [8, pp. 96–97] and [13, pp. 33–34].

Definition 2.4. Let $\varepsilon > 0$ be given and $\Lambda(\varepsilon; t_0, \ldots, t_n; f_1, \ldots, f_n)$ be a Λ_ε-discretization of (2.3) on $[0,T]$. A function $v\colon [0,T] \to X$ is called a ε-approximate solution of $\Lambda(\varepsilon; t_0, t_1, \ldots, t_n; f_1, \ldots, f_n)$ on $[0,T]$ if there exists $v_0, \ldots, v_n \in D(A)$ such that

(S1) $\dfrac{v_i - v_{i-1}}{t_i - t_{i-1}} + Av_i = f_i$ for $i = 1, \ldots, n$;
(S2) $v(t_0) = v_0, v(t) = v_i$ for $t \in \,]t_{i-1}, t_i]$ and $i = 1, \ldots, n$.

Definition 2.5. A function $u\colon [0,T] \to X$ is called a limit solution of (2.3) on $[0,T]$ if $u \in C([0,T];X)$, $u(0) = u_0$ and for each $\varepsilon > 0$ there exists at least one Λ_ε-discretization $\Lambda(\varepsilon; t_0, \ldots, t_n; f_1, \ldots, f_n)$ of (2.3) on $[0,T]$ and a ε-approximate solution $v\colon [0,T] \to X$ of $\Lambda(\varepsilon; t_0, \ldots, t_n; f_1, \ldots, f_n)$ on $[0,T]$ such that

$$\|u(t) - v(t)\|_X \leq \varepsilon \quad \text{for each } t \in [0,T].$$

For the next definition see [11, Definition 2.3, Chapter 4].

Definition 2.6. A function $u\colon [0,T] \to X$ is called a mild solution of (2.3) on $[0,T]$ if satisfies

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)\,ds \quad \text{for each } t \in [0,T].$$

As a consequence of the next three theorems (see Theorems 1.7.3, 1.7.4, 1.8.2 in [13]), in the case of a m-dissipative operator A and a function $f \in L^1([0,T];X)$ all these type of solutions turn out to be equivalent.

Definition 2.7. An operator $A\colon D(A) \subset X \to X$ is called dissipative if

$$[x_1 - x_2, Ax_1 - Ax_2]_- \leq 0$$

for any $x_1, x_2 \in D(A)$ and m-dissipative if it is dissipative and for every $\lambda > 0$ the range of the operator $I - \lambda A$ is equal to X.

Theorem 2.8. *Let $A\colon D(A) \subset X \to X$ be a m-dissipative operator, let $f \in L^1([0,T];X)$. If $u\colon [0,T] \to X$ is a limit solution of* (2.3) *on* $[0,T]$ *satisfying* $u(0) = u_0$ *then u is the unique integral solution of* (2.3) *on* $[0,T]$ *satisfying* $u(0) = u_0$.

Theorem 2.9. *Let $A\colon D(A) \subset X \to X$ be a m-dissipative operator. Then, for each $u_0 \in X$ and $f \in L^1([0,T];X)$, there exists a unique limit solution u of* (2.3) *on* $[0,T]$.

Theorem 2.10. *Let $A\colon D(A) \subset X \to X$ be a linear, densely defined, m-dissipative operator and let $f \in L^1([0,T];X)$. A function $u\colon [0,T] \to X$ is a mild solution of* (2.3) *if and only if u is a limit solution of* (2.3) *on* $[0,T]$ *satisfying* $u(0) = u_0$.

Remark 2.11. Notice that if A is the infinitesimal generator of a C_0-semigroup of contraction then $D(A)$ is dense in X and A is a closed linear operator (see Corollary 2.5 in [11]). Moreover by the Lumer–Phillips Theorem (see Theorem 4.3, Chapter 1 in [11]) it is m-dissipative.

In what follows, we denote by $\|\cdot\|_p$ the norm in $L^p(\Omega;\mathbb{R})$, $1 \leq p \leq \infty$, where Ω is a domain in $\mathbb{R}^n$, $n \geq 1$ and with $\mathcal{L}(E)$ the space of linear and bounded operators in E.

3. Statement of the problem and main result

Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ be two real Banach spaces such that $E \subseteq F$ and assume that E is reflexive with dual E^* strictly convex and that there exists a constant $k > 0$ such that

$$\|v\|_F \leq k\|v\|_E, \quad \text{for every } v \in E. \tag{3.1}$$

Consider a semilinear differential equation of the form

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), & \text{for a.e. } t \in [0, T], \\ u(0) = u_0 \in E, \end{cases} \tag{3.2}$$

where $A\colon D(A) \subset E \to E$ and $f\colon [0,T] \times E \to F$. Assume the following hypotheses on the linear operator:

(A1) $A\colon D(A) \subset E \to E$ is a linear operator, with $0 \in D(A)$, generating a compact C_0-semigroup of contractions $\{S(t)\}_{t\geq 0}$ in E;

(A2) the semigroup $\{S(t)\}_{t\geq 0}$ can be extended to a semigroup in F, i.e.

   (i) there exists a semigroup $\{S^*(t)\}_{t\geq 0}$ on F generated by A such that for every $w \in E$, it holds $S^*(t)w = S(t)w$;

   (ii) for every $v \in F$ and $t > 0$, we have $S^*(t)v \in E$;

   (iii) there exists a function $c \in L^r([0,T]; \mathbb{R}_+)$, with $1 \leq r \leq \infty$ such that for any $v \in F$ it holds

$$\|S^*(t)v\|_E \leq c(t)\|v\|_F \quad \text{for every } t \in (0, T];$$

and the following hypotheses on the nonlinearity:

(f1) for every $v \in E$ the map $f(\,\cdot\,, v)\colon [0,T] \to F$ is measurable;

(f2) for almost every $t \in [0,T]$ the map $f(t, \cdot\,)\colon E \to F$ is continuous;

(f3) for every bounded subset $D \subset E$ there exists a function $\nu_D \in L^{r'}([0,T]; \mathbb{R}_+)$, with $1/r + 1/r' = 1$ and $r' = \infty$ if $r = 1$, such that

$$\|f(t,v)\|_F \leq \nu_D(t), \quad \text{for a.e. } t \in [0,T] \text{ and all } v \in D;$$

(f4) there exist constants $r_0 > 0$, $R_0 > \max\{r_0, \|u_0\|_E\}$ and $n_0 \in \mathbb{N}$ such that, for all $n > n_0$,

$$\left\langle J_E(v), S^*\left(\frac{1}{n}\right) f(t,v) \right\rangle \leq 0,$$

for almost every $t \in [0,T]$ and for all $v \in E$ such that $r_0 < \|v\|_E < R_0$, where J_E denotes the duality map on E.

Because of condition (A2), we can denote the C_0-semigroup generated by A, on the space E or on the space F, by the very same symbol $\{S(t)\}_{t\geq 0}$. Moreover, notice that, being $\{S(t)\}_{t\geq 0}$ a C_0-semigroup on the space F, there exists a constant $M > 0$ such that

$$\|S(t)\|_F \leq M, \quad \text{for every } t \in [0, T]. \tag{3.3}$$

Conditions (A1) and (A2) might seem quite restrictive, but a simple example of operator A that satisfies both these conditions is the Laplace operator subjected to the Dirichlet boundary conditions as the following example shows.

Example 3.1. Consider $p \geq 1$ fixed and a bounded domain $\Omega \subset \mathbb{R}^k$, $k \geq 2$, with a C^2-boundary. The Laplace operator $A_p\colon D(A_p) \subset L^p(\Omega;\mathbb{R}) \to L^p(\Omega;\mathbb{R})$ subjected to Dirichlet boundary conditions on $L^p(\Omega,\mathbb{R})$ and defined by

$$D(A_p) = W_0^{1,p}(\Omega,\mathbb{R}) \cap W^{2,p}(\Omega,\mathbb{R}), \qquad A_p w = \Delta w,$$

is the generator of a C_0-semigroup of contractions $\{S_p(t)\}_{t\geq 0}$ (see e.g. Theorem 4.1.3 and Remark 4.1.2 of [14]). Moreover, by Lemma 7.2.1 of [14], for each $p, q \in [1,+\infty]$, each $\xi \in C(\overline{\Omega};\mathbb{R})$ and each $t \geq 0$, we have $S_p(t)\xi = S_q(t)\xi$. Thus, we can denote the C_0-semigroup generated by the Laplace operator subjected to the Dirichlet boundary conditions on any of the spaces $L^p(\Omega;\mathbb{R})$ by the very same symbol $\{S(t)\}_{t\geq 0}$. By Theorem 7.2.5 of [14], $\{S(t)\}_{t\geq 0}$ is a compact semigroup. Finally, by Theorem 7.2.6 of [14], for each $1 \leq q \leq p \leq \infty$, each $\xi \in L^q(\Omega;\mathbb{R})$, and each $t > 0$, we have

$$\|S(t)\xi\|_p \leq (4\pi t)^{-k(1/q-1/p)/2}\|\xi\|_q.$$

Hence, A_p satisfies (A1) and (A2) with $c(t) = (4\pi t)^{-k(1/q-1/p)/2}$. Notice that $k(1/q-1/p)/2 < 1$, provided $2 \leq k < 2pq/(p-q)$, $2 \leq q < p < \infty$ and $k(1/q-1/p)/2 = 0$ for $p = q$, hence the function $c \in L^1([0,T],\mathbb{R}_+)$.

We look for *mild solutions* of (3.2), i.e. functions $u \in C([0,T];E)$ that satisfy

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s,u(s))ds, \quad \text{for all } t \in [0,T].$$

Theorem 3.2. *Let conditions* (A1), (A2) *and* (f1)–(f4) *hold, then the problem* (3.2) *admits at least one global mild solution* $u \in C([0,T],E)$*, satisfying* $\|u(t)\|_E < R_0$ *for every* $t \in [0,T]$.

The proof of the Theorem 3.2 is based on an approximation technique and on a compactness result (see Section 5).

Given $\xi \in E$ and $g \in L^{r'}([0,T];F)$, where $1 \leq r' < \infty$ is such that $1/r + 1/r' = 1$, where $1 \leq r \leq \infty$ is defined in (A2) (iii), and $r' = \infty$ if $r = 1$, denote by $\mathcal{F}(\xi,g)\colon [0,T] \to E$ the mild solution of the linear problem

$$\begin{cases} u'(t) = Au(t) + g(t), & \text{for a.e. } t \in [0,T], \\ u(0) = \xi \in E, \end{cases} \tag{3.4}$$

that is

$$\mathcal{F}(\xi,g)(t) = S(t)\xi + \int_0^t S(t-s)g(s)\,ds, \quad \text{for all } t \in [0,T]. \tag{3.5}$$

Proposition 3.3. *If* $A\colon D(A) \subset E \to E$ *satisfies* (A1) *and* (A2)*, for every* $1 \leq r' < \infty$ *such that* $1/r + 1/r' = 1$*, where* $1 \leq r \leq \infty$ *is defined in* (A2) (iii)*, and* $r' = \infty$ *if* $r = 1$*, the operator* $\mathcal{F}\colon E \times L^{r'}([0,T];F) \to C([0,T];E)$*,* $\mathcal{F}(\xi,g)(t)$ *given in* (3.5)*, is well defined.*

PROOF. First of all we prove that, for every $\xi \in E$ and $g \in L^{r'}([0,T];F)$, the map $\mathcal{F}(\xi,g)(\,\cdot\,)$ has values in E. Let $\xi \in E$ and $g \in L^{r'}([0,T];F)$. By (A2), for every $v \in F$ and $t > 0$, we have $S(t)v \in E$, so $S(t-s)g(s) \in E$ for almost every $s \in [0,t)$; $\mathcal{F}(\xi,g)(0) = \xi \in E$, thus $\mathcal{F}(\xi,g)(t) \in E$ for every $t \in [0,T]$.

Now, we prove that for every $\xi \in E$ and $g \in L^{r'}([0,T];F)$ the map $\mathcal{F}(\xi,g)(\,\cdot\,)$ is continuous. Let $\xi \in E$ and $g \in L^{r'}([0,T];F)$. By the absolute continuity of the integral function of $\|g\|_F^{r'}$, for every $\varepsilon > 0$, there exists $\gamma(\varepsilon) > 0$ such that

$$\int_D \|g(s)\|_F^{r'}\,ds \le \frac{\varepsilon^{r'}}{\|c\|^{r'}_{L^r([0,T];\mathbb{R}_+)}} \tag{3.6}$$

for every measurable subset $D \subset [0,1]$ such that $\mu(D) \le \gamma(\varepsilon)$, where μ denotes the Lebesgue measure on $\mathbb{R}$. We want to prove that, for every $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ such that, for every $t_1, t_2 \in [0,T]$, with $|t_2 - t_1| \le \delta(\varepsilon)$ it holds

$$\|\mathcal{F}(\xi,g)(t_1) - \mathcal{F}(\xi,g)(t_2)\| \le \varepsilon.$$

First assume the case $t_1 = 0$. Let $\varepsilon > 0$ and $\gamma(\varepsilon) > 0$ satisfying (3.6). By the continuity of the semigroup $\{S(t)\}_{t\ge 0}$ we have that for the very same $\varepsilon > 0$, there exists $\delta(\varepsilon) \in (0,\gamma(\varepsilon)]$ such that, for every $0 < t_2 = t < \delta(\varepsilon)$, we have

$$\|S(t)\xi - \xi\|_E \le \varepsilon.$$

Thus we obtain

$$\begin{aligned}
\big\|\mathcal{F}(\xi,g)(t) - \mathcal{F}(\xi,g)(0)\big\|_E &\le \|S(t)\xi - \xi\|_E + \int_0^t \|S(t-s)g(s)\|_E\,ds \\
&\le \varepsilon + \int_0^t c(t-s)\|g(s)\|_F\,ds \le \varepsilon + \int_0^{\delta(\varepsilon)} c(t-s)\|g(s)\|_F\,ds \\
&\le \varepsilon + \|c\|_{L^r([0,T];\mathbb{R}_+)}\bigg(\int_0^{\delta(\varepsilon)} \|g(s)\|_F^{r'}\,ds\bigg)^{1/r'} \le 2\varepsilon.
\end{aligned}$$

Consider now the case $t_1 > 0$. Let $\varepsilon > 0$ and $\lambda = \lambda(\varepsilon) > 0$ such that $t_1 - 2\lambda > 0$ and $2\lambda \le \gamma(\varepsilon)$. Being compact, the semigroup $\{S(t)\}_{t\ge 0}$ is equicontinuous, i.e. the map $t \mapsto S(t)$ is continuous from $(0,+\infty)$ to $\mathcal{L}(E)$, endowed with the uniform operator norm $\|\cdot\|_{\mathcal{L}(E)}$, see Theorem 6.2.1 of [14]. Therefore, for the very same $\varepsilon > 0$, there exists $\delta(\varepsilon) \in (0,\lambda]$ such that for every $|h| < \delta(\varepsilon)$ and every $s \in [0, \min\{t_1 - \lambda, t_1 + h - \lambda\}]$, we have

$$\|S(t_1+h) - S(t_1)\|_{\mathcal{L}(E)} \le \min\bigg\{\frac{\varepsilon}{\|\xi\|_E}, \frac{\varepsilon}{\|g\|_{L^{r'}([0,T];F)}c(\lambda)T^{1/r}}\bigg\}$$

for $\xi \ne 0$,

$$\|S(t_1+h) - S(t_1)\|_{\mathcal{L}(E)} \le \frac{\varepsilon}{\|g\|_{L^{r'}([0,T];F)}c(\lambda)T^{1/r}}$$

for $\xi = 0$ and

$$\|S(t_1 + h - \lambda - s) - S(t_1 - \lambda - s)\|_{\mathcal{L}(E)} \leq \min\left\{\frac{\varepsilon}{\|\xi\|_E}, \frac{\varepsilon}{\|g\|_{L^{r'}([0,T];F)} c(\lambda) T^{1/r}}\right\}.$$

For each $h \in (-\delta(\varepsilon), \delta(\varepsilon))$, we have

$$\mathcal{F}(\xi, g)(t_1 + h) = S(\lambda + h)\mathcal{F}(\xi, g)(t_1 - \lambda) + \int_{t_1-\lambda}^{t_1+h} S(t_1 + h - s) g(s)\, ds.$$

Hence, denoting $t_1 = t$ and $t_2 = t + h$, we get

$$\begin{aligned}
\|\mathcal{F}(\xi, g)(t + h) - \mathcal{F}(\xi, g)(t)\|_E &\leq \|S(t + h) - S(t)\|_{\mathcal{L}(E)} \|\xi\|_E \\
&\quad + \int_0^{t-\lambda} \|(S(t + h - s) - S(t - s))g(s)\|_E\, ds \\
&\quad + \int_{t-\lambda}^{t+h} \|S(t + h - s)g(s)\|_E\, ds + \int_{t-\lambda}^{t} \|S(t - s)g(s)\|_E\, ds \\
&\leq \|S(t + h) - S(t)\|_{\mathcal{L}(E)} \|\xi\|_E \\
&\quad + \int_0^{t-2\lambda} \|(S(t + h - \lambda - s) - S(t - \lambda - s))S(\lambda)g(s)\|_E\, ds \\
&\quad + \int_{t-2\lambda}^{t-\lambda} \|(S(t + h - s) - S(t - s))g(s)\|_E\, ds \\
&\quad + \int_{t-\lambda}^{t+h} \|S(t + h - s)g(s)\|_E\, ds + \int_{t-\lambda}^{t} \|S(t - s)g(s)\|_E\, ds \\
&\leq \|S(t + h) - S(t)\|_{\mathcal{L}(E)} \|\xi\|_E \\
&\quad + \int_0^{t-2\lambda} \|(S(t + h - \lambda - s) - S(t - \lambda - s))\|_{\mathcal{L}(E)} c(\lambda) \|g(s)\|_F\, ds \\
&\quad + \int_{t-2\lambda}^{t-\lambda} c(t + h - s)\|g(s)\|_F\, ds + \int_{t-2\lambda}^{t-\lambda} c(t - s)\|g(s)\|_F\, ds \\
&\quad + \int_{t-\lambda}^{t+h} c(t + h - s)\|g(s)\|_F\, ds + \int_{t-\lambda}^{t} c(t - s)\|g(s)\|_F\, ds \\
&\leq \frac{\varepsilon}{\|\xi\|_E}\|\xi\|_E + \frac{\varepsilon}{c(\lambda)\|g\|_{L^{r'}([0,T];F)} T^{1/r}}\, c(\lambda)\|g\|_{L^{r'}([0,T];F)} T^{1/r} \\
&\quad + 2\|c\|_{L^r([0,T],\mathbb{R}_+)} \|g\|_{L^{r'}([t-2\lambda,t-\lambda];F)} + \|c\|_{L^r([0,T],\mathbb{R}_+)} \|g\|_{L^{r'}([t-\lambda,t+h];F)} \\
&\quad + \|c\|_{L^r([0,T],\mathbb{R}_+)} \|g\|_{L^{r'}([t-\lambda,t];F)} \leq 6\varepsilon.
\end{aligned}$$

The last inequality make sense only if $\xi \neq 0$. If $\xi = 0$,

$$\|S(t + h) - S(t)\|_{\mathcal{L}(E)} \|\xi\|_E = 0.$$

Thus, in any case, the continuity of the map $\mathcal{F}(\xi, g)(\,\cdot\,)$ follows. □

Proposition 3.4. *If $A\colon D(A) \subset E \to E$ satisfies* (A1) *and* (A2), *then for each bounded subset B of E and each subset G in $L^{r'}([0,T];F)$ such that*

$\{\|g\|_F^{r'}, g \in G\}$ is uniformly integrable, the set $\mathcal{F}(B \times G)$ is relatively compact in $C([\delta, T]; E)$ for each $\delta \in (0, T)$. If, in addition, B is relatively compact in E, then $\mathcal{F}(B \times G)$ is relatively compact in $C([0, T]; E)$.

The proof is quite similar to that of Theorem 8.4.1 [14], see also Remark 8.4.1 in [14].

Proof. First of all, the operator $\mathcal{F}\colon E \times L^{r'}([0, T]; F) \to C([0, T]; E)$ maps bounded subsets in $E \times L^{r'}([0, T]; F)$ into bounded subsets in $C([0, T]; E)$. Indeed, by (A2), for $(\xi, g) \in E \times L^{r'}([0, T]; F)$, we have

$$\begin{aligned}\|\mathcal{F}(\xi, g)(t)\|_E &\leq \|S(t)\xi\|_E + \int_0^t \|S(t-s)g(s)\|_E \, ds \\ &\leq \|\xi\|_E + \int_0^t c(t-s)\|g(s)\|_F \, ds \\ &\leq \|\xi\|_E + \|c\|_{L^r([0,T];\mathbb{R}_+)}\|g\|_{L^{r'}([0,T];F)}\end{aligned}$$

for every $t \in [0, T]$.

Now we prove that the set $\mathcal{F}(B \times G)(t)$ is relatively compact for every $t \in (0, T]$. To this aim, let $t \in (0, T]$ and $\lambda > 0$ with $t - \lambda \geq 0$. For every $(\xi, g) \in B \times G$ we have

$$\mathcal{F}(\xi, g)(t) = S(t)\xi + S(\lambda)\int_0^{t-\lambda} S(t-\lambda-s)g(s)\, ds + \int_{t-\lambda}^t S(t-s)g(s)\, ds.$$

Since $S(\lambda)$ is compact and the fact that the operator $\mathcal{F}$ maps bounded subsets in $E \times L^{r'}([0, T]; F)$ into bounded subsets in $C([0, T]; E)$ we deduce that the operator $P_\lambda\colon \mathcal{F}(B, G)(t) \to E$ defined by

$$P_\lambda(\mathcal{F}(\xi, g)(t)) = S(t)\xi + S(\lambda)\int_0^{t-\lambda} S(t-\lambda-s)g(s)\, ds, \quad (\xi, g) \in B \times G,$$

maps the set $\mathcal{F}(B, G)(t)$ into a relatively compact set in E. In addition, for every $(\xi, g) \in B \times G$ we have

$$\begin{aligned}&\|P_\lambda(\mathcal{F}(\xi, g)(t)) - \mathcal{F}(\xi, g)(t)\|_E \\ &\qquad\leq \int_{t-\lambda}^t \|S(t-s)g(s)\|_E \, ds \leq \|c\|_{L^r([0,T];\mathbb{R}_+)}\|g\|_{L^{r'}([t-\lambda,t];F)}.\end{aligned}$$

Thus, by the uniform integrability of the set $\{\|g\|_F^{r'}, g \in G\}$ we obtain

$$\lim_{\lambda \downarrow 0} \|P_\lambda(\mathcal{F}(\xi, g)(t)) - \mathcal{F}(\xi, g)(t)\|_E = 0,$$

uniformly for $g \in G$, it follows that $\mathcal{F}(B \times G)(t)$ is relatively compact in E for each $t \in (0, T]$.

Now we prove that the set $\mathcal{F}(B \times G)$ is equicontinuous. By the uniform integrability of the set $\{\|g\|_F^{r'}, g \in G\}$, for every $\varepsilon > 0$ there exists $\gamma(\varepsilon) > 0$ such that

$$\int_D \|g(s)\|_F^{r'}\,ds \leq \frac{\varepsilon^{r'}}{\|c\|^{r'}_{L^r([0,T];\mathbb{R}_+)}}$$

for every measurable subset $D \subset E$ such that $\mu(D) \leq \gamma(\varepsilon)$ and uniformly with respect to $g \in G$, where μ denotes the Lebesgue measure on $\mathbb{R}$. Let $\varepsilon > 0$, $t \in (0,T]$, and let us fix $\lambda = \lambda(\varepsilon) > 0$ such that $t - \lambda > 0$ and $2\lambda \leq \gamma(\varepsilon)$. Since $\mathcal{F}(B \times G)(t-\lambda)$ is relatively compact in E, for each $\varepsilon > 0$ there exists a finite family $\{(\xi_1, g_1), \ldots, (\xi_J, g_J)\}$ in $B \times G$ such that for every $(\xi, g) \in B \times G$ there exists $i \in \mathcal{J} = \{1, \ldots, J\}$ such that

$$\|\mathcal{F}(\xi,g)(t-\lambda) - \mathcal{F}(\xi_i,g_i)(t-\lambda)\|_E \leq \varepsilon.$$

On the other hand, the family $\{\mathcal{F}(\xi_1,g_1), \ldots, \mathcal{F}(\xi_J,g_J)\}$ is equicontinuous at t, so being a finite family of continuous functions in $[0,T]$. Hence, for the very same $\varepsilon > 0$, there exists $\delta(\varepsilon) \in (0,\lambda]$ such that

$$\|\mathcal{F}(\xi_i,g_i)(t+h) - \mathcal{F}(\xi_i,g_i)(t)\|_E \leq \varepsilon.$$

for every $i = 1, \ldots, J$ and every $h \in \mathbb{R}$ with $|h| \leq \delta(\varepsilon)$. We then have

$$\begin{aligned}\|\mathcal{F}(\xi,g)(t+h) - \mathcal{F}(\xi,g)(t)\|_E &\leq \|\mathcal{F}(\xi,g)(t+h) - \mathcal{F}(\xi_i,g_i)(t+h)\|_E \\ &\quad + \|\mathcal{F}(\xi_i,g_i)(t+h) - \mathcal{F}(\xi_i,g_i)(t)\|_E + \|\mathcal{F}(\xi_i,g_i)(t) - \mathcal{F}(\xi,g)(t)\|_E.\end{aligned}$$

As before notice that, for every $(\xi,g) \in B \times G$,

$$\mathcal{F}(\xi,g)(t+h) = S(\lambda+h)\mathcal{F}(\xi,g)(t-\lambda) + \int_{t-\lambda}^{t+h} S(t+h-s)\,g(s)\,ds.$$

Thus we have

$$\begin{aligned}&\|\mathcal{F}(\xi,g)(t+h) - \mathcal{F}(\xi_i,g_i)(t+h)\|_E \\ &\qquad \leq \bigg\| S(\lambda+h)\mathcal{F}(\xi,g)(t-\lambda) + \int_{t-\lambda}^{t+h} S(t+h-s)\,g(s)\,ds \\ &\qquad\quad - S(\lambda+h)\mathcal{F}(\xi_i,g_i)(t-\lambda) - \int_{t-\lambda}^{t+h} S(t+h-s)\,g_i(s)\,ds \bigg\|_E \\ &\qquad \leq \big\|\mathcal{F}(\xi,g)(t-\lambda) - \mathcal{F}(\xi_i,g_i)(t-\lambda)\big\|_E \\ &\qquad\quad + \int_{t-\lambda}^{t+h} \|S(t+h-s)(g(s)-g_i(s))\|_E\,ds \\ &\qquad \leq \varepsilon + \|c\|_{L^r([0,T];\mathbb{R}_+)}\big(\|g\|_{L^{r'}([t-\lambda,t+h];F)} + \|g_i\|_{L^{r'}([t-\lambda,t+h];F)}\big) \leq 3\varepsilon.\end{aligned}$$

Analogously, we have

$$
\begin{aligned}
&\|\mathcal{F}(\xi,g)(t)-\mathcal{F}(\xi_i,g_i)(t)\|_E \\
&\qquad\le \bigg\| S(\lambda)\mathcal{F}(\xi,g)(t-\lambda)+\int_{t-\lambda}^{t} S(t-s)\,g(s)\,ds \\
&\qquad\qquad - S(\lambda)\mathcal{F}(\xi_i,g_i)(t-\lambda)-\int_{t-\lambda}^{t} S(t-s)\,g_i(s)\,ds\bigg\|_E \\
&\qquad\le \|\mathcal{F}(\xi,g)(t-\lambda)-\mathcal{F}(\xi_i,g_i)(t-\lambda)\|_E \\
&\qquad\qquad + \int_{t-\lambda}^{t} \|S(t-s)(g(s)-g_i(s))\|_E\,ds \\
&\qquad\le \varepsilon+\|c\|_{L^r([0,T];\mathbb{R}_+)}\big(\|g\|_{L^{r'}([t-\lambda,t+h];F)}+\|g_i\|_{L^{r'}([t-\lambda,t+h];F)}\big)\le 3\varepsilon.
\end{aligned}
$$

In conclusion, we have obtained

$$
\|\mathcal{F}(\xi,g)(t+h)-\mathcal{F}(\xi,g)(t)\|_E\le 7\varepsilon
$$

for every $(\xi,g)\in B\times G$ and $h\in\mathbb{R}$ with $|h|\le\delta(\varepsilon)$. Thus, $\mathcal{F}(B\times G)$ is an equicontinuous family of maps on $(0,T]$ and, by the Ascoli–Arzelà Theorem, we have that it is relatively compact in $C([\delta,T];E)$ for every $\delta\in(0,T)$. Moreover, if $B\subset E$ is a relatively compact set then the set $\mathcal{F}(B\times G)(0)=B$ is relatively compact and the family $\mathcal{F}(B\times G)$ is equicontinuous at 0 as well. Again by the Ascoli–Arzelà theorem, we have the relative compactness of the family $\mathcal{F}(B\times G)$ in $C([0,T];E)$. □

4. Approximating problems

In this section, we introduce a family of approximating problems. For $n\in\mathbb{N}$, we consider the following semilinear problem

$$
(\mathrm{P}_n)\qquad
\begin{cases}
u'(t)=Au(t)+S\Big(\dfrac{1}{n}\Big)f(t,u(t)), & \text{for a.e. } t\in[0,T],\\[2mm]
u(0)=u_0\in E,
\end{cases}
$$

where $A\colon[0,T]\to E$, A satisfies (A1), (A2) and f satisfies (f1)–(f4). We will prove that there exists $n_0>0$ such that for every $n>n_0$ problems (P_n) admits at least one mild solution.

Lemma 4.1. *If $A\colon D(A)\subset E\to E$ satisfies* (A1), (A2) *and $f\colon[0,T]\times E\to F$ satisfies* (f1)–(f4)*, then there exists $n_0\in\mathbb{N}$ such that for every $n>n_0$ problems (P_n) have at least one mild solution $u_n\in C([0,T];E)$, satisfying $\|u_n(t)\|_E<R_0$ for every $t\in[0,T]$.*

Proof. For $n\in\mathbb{N}$ and $R\in(r_0,R_0)$, $R\ge\|u_0\|_E$, define

$$
Q_1=\{q\in C([0,T];E):\|q(t)\|_E\le R \text{ for all } t\in[0,T]\}
$$

and the operator $\Sigma_n\colon Q_1 \times [0,1] \to C([0,T];E)$ as

$$\Sigma_n(q,\lambda)(t) = \lambda\, S(t)u_0 + \lambda \int_0^t S(t-\tau)S\left(\frac{1}{n}\right) f(\tau, q(\tau))\,d\tau, \quad t \in [0,T].$$

Reasoning as in Proposition 3.3 it is possible to prove that the operator Σ_n is well defined. Moreover, a fixed point $q = \Sigma_n(q,1)$ is a mild solution of the problem (P_n). We will prove the existence of such fixed points using the Leray–Schauder continuation principle (see Theorem 1.2). In what follows, we denote by $B_R = \{v \in E : \|v\|_E \le R\}$. Now, we divide the proof in several steps.

Step 1. For each $n \in \mathbb{N}$, the operator $\Sigma_n\colon Q_1 \times [0,1] \to C([0,T];E)$ is continuous.

Let $\{q_k\} \subset Q_1$ and $\{\lambda_k\} \subset [0,1]$ two convergent sequences $q_k \to q$ in $C[0,T];E)$ and $\lambda_k \to \lambda$ in $[0,1]$. By (f2) we have that

$$\|f(t,q_k(t)) - f(t,q(t))\|_F \to 0 \quad \text{for all } t \in [0,T],$$

hence, by (A2) (iii) it follows

$$\left\|S(t)S\left(\frac{1}{n}\right)(f(t,q_k(t)) - f(t,q(t)))\right\|_E \le c\left(\frac{1}{n}\right)\|f(t,q_k(t)) - f(t,q(t))\|_F \to 0$$

for all $t \in [0,T]$. Moreover, by (A2) (iii) and (f3) we get

$$\left\|S(t)S\left(\frac{1}{n}\right)f(t,q_k(t))\right\|_E \le c\left(\frac{1}{n}\right)\nu_{B_R}(t) \quad \text{for a.e. } t \in [0,T].$$

Thus, by the Lebesgue's dominated convergence theorem we conclude that, for every $t \in [0,T]$,

$$\begin{aligned}
\|\Sigma_n(q_k,\lambda_k)(t) &- \Sigma_n(q,\lambda)(t)\|_E \le |\lambda_k - \lambda|\|S(t)u_0\|_E \\
&+ |\lambda_k - \lambda| \int_0^t \left\|S(t-\tau)S\left(\frac{1}{n}\right)f(\tau,q(\tau))\right\|_E d\tau \\
&+ \lambda_k \int_0^t \left\|S(t-\tau)S\left(\frac{1}{n}\right)(f(\tau,q_k(\tau)) - f(\tau,q(\tau))\right\|_E d\tau \\
\le{}& |\lambda_k - \lambda|\|u_0\|_E + |\lambda_k - \lambda|\, c\left(\frac{1}{n}\right)\|\nu_{B_R}\|_{L^{r'}([0,T],\mathbb{R}_+)} \\
&+ \lambda_k \int_0^T \left\|S(t-\tau)S\left(\frac{1}{n}\right)(f(\tau,q_k(\tau)) - f(\tau,q(\tau))\right\|_E d\tau \to 0
\end{aligned}$$

Hence $\Sigma_n(q_k,\lambda_k) \to \Sigma_n(q,\lambda)$ in $C([0,T];E)$, obtaining the continuity of the operator Σ_n.

Step 2. For every $n \in \mathbb{N}$, the operator Σ_n send $Q_1 \times [0,1]$ into a relatively compact set of $C([0,T];E)$.

First of all notice that $\Sigma_n(Q_1 \times [0,1])(0)$ is a compact set, since it coincides with u_0. Moreover, by (f3) and (3.3), there exist a function $\nu_{B_R} \in L^{r'}([0,T];\mathbb{R}_+)$

such that

$$\left\| S\left(\frac{1}{n}\right) f(t,q(t)) \right\|_F \le M\nu_{B_R}(t), \quad \text{for a.e. } t \in [0,T] \text{ and for every } q \in Q_1,$$

implying that the set $\{S(1/n)f(\,\cdot\,,q(\,\cdot\,)),\ q \in Q_1\}$ is a family of maps in $L^{r'}([0,T],F)$ such that $\{\|S(1/n)f(\,\cdot\,,q(\,\cdot\,))\|_F^{r'},\ q \in Q_1\}$ is uniformly integrable. Therefore, observing that $\Sigma_n(q,\lambda) = \lambda\mathcal{F}(u_0, S(1/n)f((\,\cdot\,),q(\,\cdot\,)))$ for every (q,λ) in $Q_1 \times [0,1]$ we obtain, by Proposition 3.4, that the set $\Sigma_n(Q_1 \times [0,1])$ is relatively compact in $C([0,T];E)$.

Step 3. $\Sigma_n(Q_1 \times \{0\}) \subset \operatorname{int}(Q_1)$.

Since $\Sigma_n(Q_1 \times \{0\}) \equiv 0$ the conclusion follows trivially.

Step 4. The operator $\Sigma_n(\,\cdot\,,\lambda)$ has no fixed points on ∂Q_1 for every $\lambda \in [0,1]$ and $n > n_0$, where n_0 is from (f4).

By contradiction, assume that there exists $\overline{\lambda} \in [0,1]$, $\overline{u} \in Q_1$ and $t_0 \in [0,T]$ such that $\overline{u} = \Sigma_n(\overline{u},\overline{\lambda})$ and $\|\overline{u}(t_0)\|_E = R$. Since $\overline{\lambda} = 0$ implies $\overline{u} \equiv 0$ and $\overline{\lambda} = 1$, gives the existence of at least one fixed point $\overline{u} = \Sigma_n(\overline{u},1)$, we may assume $\overline{\lambda} \in (0,1)$. Notice that $t_0 \neq 0$. Indeed, if $t_0 = 0$ we have

$$R = \|\overline{u}(0)\|_E = \|\Sigma_n(\overline{u},\overline{\lambda})(0)\|_E = \overline{\lambda}\|u_0\|_E < R.$$

Hence, there exists $\delta > 0$ such that $r_0 < \|\overline{u}(t)\|_E \le R$ for every $t \in [t_0-\delta, t_0]$ and $\|\overline{u}(t_0-\delta)\|_E < R$. Denoting by $g_n(t) = S(1/n)f(t,\overline{u}(t))$, $t \in [0,T]$, we consider the linear problem

$$\begin{cases} u'(t) = Au(t) + g_n(t), & \text{for a.e. } t \in [0,T], \\ u(0) = u_0 \in E. \end{cases} \tag{4.1}$$

By the fact that $\|\overline{u}(t)\|_E \le R$ for every $t \in [0,T]$, by (A2) and (f3), we have that

$$\|g_n(t)\|_E = \left\| S\left(\frac{1}{n}\right) f(t,\overline{u}(t)) \right\|_E \le c\left(\frac{1}{n}\right)\nu_{B_R}(t) \quad \text{for a.e. } \ t \in [0,T],$$

obtaining that $g_n \in L^{r'}([0,T];E)$. We denote by $u \in C([0,T],E)$ the unique mild solution of (4.1), i.e.

$$u(t) = S(t)u_0 + \int_0^t S(t-s)g_n(s)\,ds, \quad t \in [0,T].$$

By Theorems 2.8 and 2.10, we have that u is the unique integral solution of (4.1), i.e.

$$\|u(t)-x\|_E \le \|u(s)-x\|_E + \int_s^t [u(\tau)-x, g_n(\tau)-Ax]_+\,d\tau$$

for each $x \in D(A)$ and $0 \le s \le t \le T$. Since E is reflexive with a strictly convex dual, bearing in mind (2.2), we have that

$$\|u(t)-x\|_E \le \|u(s)-x\|_E + \int_s^t \frac{1}{\|u(\tau)-x\|_E}\,\langle J_E(u(\tau)-x), g_n(\tau)-Ax\rangle\,d\tau$$

for each $x \in D(A)$ and $0 \leq s \leq t \leq T$. By the definition of the operator $\Sigma_n(\,\cdot\,,\overline{\lambda})$ and the fact that $\overline{u}$ is a fixed point of it, for every $t \in [0,T]$, we obtain

$$\begin{aligned} u(t) &= S(t)u_0 + \int_0^t S(t-\tau)g_n(\tau)\,ds \\ &= S(t)u_0 + \int_0^t S(t-\tau)S\bigg(\frac{1}{n}\bigg)f(\tau,\overline{u}(\tau))\,d\tau = \frac{\overline{u}(t)}{\overline{\lambda}}. \end{aligned}$$

Now, considering $x = 0 \in D(A)$ and observing that $\|u(s)\|_E > 0$ for every $s \in [t_0-\delta, t_0]$, it follows that

$$\begin{aligned} 0 &< \frac{\|\overline{u}(t_0)\|_E - \|\overline{u}(t_0-\delta)\|_E}{\overline{\lambda}} = \|u(t_0)\|_E - \|u(t_0-\delta)\|_E \\ &\leq \int_{t_0-\delta}^{t_0} \frac{1}{\|u(\tau)\|_E}\,\langle J_E(u(\tau)), g_n(\tau)\rangle\,d\tau \\ &= \int_{t_0-\delta}^{t_0} \frac{1}{\|u(\tau)\|_E}\bigg\langle J_E(u(\tau)), S\bigg(\frac{1}{n}\bigg)f(\tau,\overline{u}(\tau))\bigg\rangle\,d\tau \\ &= \int_{t_0-\delta}^{t_0} \frac{1}{\|u(\tau)\|_E}\bigg\langle J_E\bigg(\frac{\overline{u}(\tau)}{\overline{\lambda}}\bigg), S\bigg(\frac{1}{n}\bigg)f(\tau,\overline{u}(\tau))\bigg\rangle\,d\tau \\ &= \int_{t_0-\delta}^{t_0} \frac{1}{\overline{\lambda}\,\|u(\tau)\|_E}\bigg\langle J_E(\overline{u}(\tau)), S\bigg(\frac{1}{n}\bigg)f(\tau,\overline{u}(\tau))\bigg\rangle\,d\tau, \end{aligned}$$

where the last equality follows from the Proposition 2.1 part (c). Now, by (f4) for every $n > n_0$ we get the contradiction

$$0 < \frac{\|\overline{u}(t_0)\|_E - \|\overline{u}(t_0-\delta)\|_E}{\overline{\lambda}} \leq 0.$$

By Theorem 1.2, for every $n > n_0$, we obtain the existence of a fixed point $u = \Sigma_n(u,1)$. Thus for every $n > n_0$, we get a mild solution of (P_n). $\square$

5. Proof of Theorem 3.2

By Lemma 4.1, we know that there exists $n_0 > 0$ such that problems (P_n) have at least one mild solution for every $n > n_0$. We consider now the set of these mild solutions. More precisely, by the characterization introduced in Lemma 4.1, we consider the set

$$M_1 = \big\{u_n \in C([0,T];E) \cap Q_1 : u_n = \Sigma_n(u_n,1),\ n > n_0\big\}.$$

Let $u_n \in M_1$. By the fact that $u_n \in Q_1$, we have that $\|u_n(t)\|_E \leq R$ for every $t \in [0,T]$. Thus, by (f3) and (3.3), there exist a function $\nu_{B_R} \in L^{r'}([0,T];\mathbb{R}_+)$

$$\bigg\|S\bigg(\frac{1}{n}\bigg)f(t,u_n(t))\bigg\|_F \leq M\nu_{B_R}(t), \quad \text{for a.e. } t \in [0,T],$$

implying that the set $G_1 = \{S(1/n)f(\,\cdot\,,u_n(\,\cdot\,)),\ n > n_0\}$ is a family of maps in $L^{r'}([0,T],F)$ such that $\big\{\|S(1/n)f(\,\cdot\,,u_n(\,\cdot\,))\|_F^{r'},\ n > n_0\big\}$ is uniformly integrable.

Hence, applying Proposition 3.4 with $B = \{u_0\}$ and $G = G_1$, we obtain the relative compactness of M_1 in $C([0,T];E)$, i.e. without loss of generality, we can assume that for $\{u_n\} \subset M_1$ there exists $u^* \in C([0,T];E)$ such that $\{u_n\}$ converges to u^* in $C([0,T];E)$. We will prove that u^* is a mild solution of problem (3.2).

First of all notice that $u_n(t) \xrightarrow{E} u^*(t)$ for every $t \in [0,T]$. Moreover, by the continuity of $S(1/n)$ for every $n \in \mathbb{N}$ we have, for every $t \in [0,T]$, that

$$S\left(\frac{1}{n}\right) f(t, u_n(t)) \xrightarrow{F} f(t, u^*(t)), \quad \text{for a.e. } t \in [0,T],$$

moreover, the convergence is dominated

$$\left\| S\left(\frac{1}{n}\right) f(t, u_n(t)) \right\|_F \le M \nu_{B_R}(t) \quad \text{for a.e. } t \in [0,T],$$

where $M > 0$ is defined in (3.3). Thus we get

$$u_n(t) = S(t)u_0 + \int_0^t S(t-\tau) S\left(\frac{1}{n}\right) f(\tau, u_n(\tau))\, d\tau \\ \to S(t)u_0 + \int_0^t S(t-\tau) f(\tau, u^*(\tau))\, d\tau.$$

By the uniqueness of the limit, we obtain the claimed result. Moreover, for every $t \in [0,T]$, it holds

$$\|u^*(t)\|_E = \lim_{n\to\infty} \|u_n(t)\|_E \le R < R_0. \qquad \square$$

6. Applications

In this section, we show how the above abstract existence result directly applies to partial differential equations of parabolic type. Consider the following nonlinear heat equation

$$\text{(6.1)} \qquad \begin{aligned} &u_t = \Delta u + g(t,x,u(t,x)) && \text{for } (t,x) \in \,]0,T[\, \times \Omega, \\ &u(t,x) = 0 && \text{for } (t,x) \in \,]0,T[\, \times \partial\Omega, \\ &u(0,x) = u_0(x) && \text{for } x \in \Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^k$, $2 \le k < 2pq/(p-q)$, with $2 \le q < p < \infty$, ($k \ge 2$, in the case $p = q$), is a bounded domain with C^2-boundary and $g\colon [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is such that

(g1) g is a continuous function;
(g2) there exists $b > 0$ and $a \in L^q(\Omega; \mathbb{R}_+)$ such that

$$|g(t,x,v)| \le a(x) + b|v|^{p/q}, \quad \text{for every } (t,x,v) \in [0,T] \times \Omega \times \mathbb{R}$$

with $2 \le q \le p < \infty$;
(g3) $v\, g(t,x,v) \le 0$ for every $v \in \mathbb{R}$.

As expected, the problem (6.1) can be rewritten as an abstract evolution equation of the form (3.2) satisfying the hypotheses of Theorem 3.2 with $E = L^p(\Omega;\mathbb{R})$ and $F = L^q(\Omega;\mathbb{R})$, $2 \le q \le p < \infty$.

In fact, by Example 3.1, we have that the Laplace operator subjected to the Dirichlet boundary conditions $A\colon D(A) \subset E \to E$ defined as

$$D(A) = W_0^{1,p}(\Omega,\mathbb{R}) \cap W^{2,p}(\Omega,\mathbb{R}), \qquad Aw = \Delta w,$$

satisfies conditions (A1) and (A2) with $c(t) = (4\pi t)^{-k(1/q-1/p)/2}$. Notice that $0 < k(1/q - 1/p)/2 < 1$, provided $2 \le k < 2pq/(p-q)$, $2 \le q < p < \infty$ and $k(1/q-1/p)/2 = 0$ for $p = q$, hence the function $c \in L^1([0,T];\mathbb{R}_+)$. By (g1) and the Vainberg Theorem (see Theorem 1.1) we have that the Nemytskiĭ operator $f\colon [0,T] \times E \to F$ defined as $f(t,u)(x) = g(t,x,u(x))$ maps the space E into F and is continuous. Moreover, by (g2), we get

$$\|f(t,u)\|_q^q = \int_\Omega |g(t,x,u(x))|^q\,dx \le C\big(\|a\|_q^q + b\|v\|_p^p\big),$$

where $C > 0$ is a suitable constant. Hence for every bounded subset D of E, we have that $\|f(t,u)\|_F \le C_1$, for every $u \in D$, with $C_1 > 0$ another suitable constant. So assumption (f3) is satisfied with $\nu_D \in L^\infty([0,T];\mathbb{R}_+)$. We recall that, for every $w \in E$ with $\|w\|_p > 0$, we have

$$\langle J_E(w), v\rangle = \frac{1}{\|w\|_p^{p-2}} \int_\Omega |w(\xi)|^{p-2} w(\xi) v(\xi)\,d\xi,$$

see e.g. Example 1.4.4 in [13].

Let $w \in E$ with $\|w\|_p > 0$ and $t \in [0,T]$. Denoting by $\Omega^+ = \{x \in \Omega : w(\xi) > 0\}$ and by $\Omega^- = \{x \in \Omega : w(\xi) < 0\}$ and considering the function $\operatorname{sign}\colon \mathbb{R} \to \mathbb{R}$ defined as

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x > 0,\\ 0 & \text{if } x = 0,\\ -1 & \text{if } x < 0,\end{cases}$$

we have that

$$\begin{aligned}
&\bigg\langle J_E(w), S\Big(\frac{1}{n}\Big) f(t,w)\bigg\rangle \\
&\quad= \frac{1}{\|w\|_p^{p-2}} \int_\Omega |w(\xi)|^{p-2} w(\xi) S\Big(\frac{1}{n}\Big) g(t,\xi,w(\xi))\,d\xi \\
&\quad= \frac{1}{\|w\|_p^{p-2}} \int_\Omega |w(\xi)|^{p-1} \operatorname{sign}(w(\xi)) S\Big(\frac{1}{n}\Big) g(t,\xi,w(\xi))\,d\xi \\
&\quad= \frac{1}{\|w\|_p^{p-2}} \int_{\Omega^+} |w(\xi)|^{p-1} S\Big(\frac{1}{n}\Big) g(t,\xi,w(\xi))\,d\xi \\
&\qquad - \frac{1}{\|w\|_p^{p-2}} \int_{\Omega^-} |w(\xi)|^{p-1} S\Big(\frac{1}{n}\Big) g(t,\xi,w(\xi))\,d\xi
\end{aligned}$$

$$= -\frac{1}{\|w\|_p^{p-2}} \int_{\Omega^+} |w(\xi)|^{p-1} S\left(\frac{1}{n}\right)(-g(t,\xi,w(\xi)))\,d\xi$$
$$- \int_{\Omega^-} |w(\xi)|^{p-1} S\left(\frac{1}{n}\right) g(t,\xi,w(\xi))\,d\xi.$$

The semigroup $\{S(t)\}$ generated by the Laplace operator subjected to the Dirichlet boundary conditions is a positive semigroup in $L^p(\Omega;\mathbb{R})$ with $p \geq 2$, i.e. for each $t > 0$ we have $S(t)\varphi \geq 0$ almost everywhere in Ω, provided $\varphi \in L^p(\Omega;\mathbb{R})$ is a nonnegative function, see [14, Lemma 7.2.3]. Moreover, by (g3),

$$-g(t,\xi,w(\xi)) \geq 0 \quad \text{for a.e. } \xi \in \Omega_+,$$
$$g(t,\xi,w(\xi)) \geq 0 \quad \text{for a.e. } \xi \in \Omega_-.$$

Thus, we have that

$$S\left(\frac{1}{n}\right)(-g(t,\xi,w(\xi))) \geq 0 \quad \text{for a.e. } \xi \in \Omega_+,$$
$$S\left(\frac{1}{n}\right)g(t,\xi,w(\xi)) \geq 0 \quad \text{for a.e. } \xi \in \Omega_-.$$

Hence we get

$$\left\langle J_E(w), S\left(\frac{1}{n}\right)f(t,w)\right\rangle \leq 0$$

for every $n \in N$, for almost every $t \in [0,T]$ and for every $w \in L^p(\Omega;\mathbb{R})$. Thus A and f satisfy all the hypotheses of Theorem 3.2. Hence there exists a solution $u \in C([0,T];L^p(\Omega;\mathbb{R}))$ of problem (6.1).

Remark 6.1. A trivial example of a superlinear function that satisfies all the conditions (g1)–(g3) is $g(t,x,u) = -u^3(\sin(u)+2)$. Indeed, all the required assumptions are satisfied for instance for $2 \leq q < \infty$, $p = 3q$ and $2 \leq k < 3q$.

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