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SUMS OF CONVEX COMPACTA AS ATTRACTORS OF HYPERBOLIC IFS'S

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Dedicated to the memory of Professor Ioan I. Vrabie

Abstract. We prove that a finite union of convex compacta in $\mathbb{R}^n$ may be represented as the attractor of a hyperbolic IFS. If such a union is the condensation set for some hyperbolic IFS with condensation, then its attractor can be represented as the attractor of a standard hyperbolic IFS. We illustrate this result with the hyperbolic IFS with condensation, whose attractor is the well-known "The Pythagoras tree" fractal.

1. Introduction

The term *fractal* is usually associated with the attractor of a hyperbolic *Iterated Function System* (see, e.g. [3]). The main ingredients of fractals are self-similarity and fractal dimension.

M. Barnsley [3] has introduced the idea of an *Iterated Function System with condensation*, which means a hyperbolic IFS, accompanied by a constant compact-valued multi-function (condensation). This idea has led to new fractals as attractors of IFS's, mostly related to Cantor sets. However, the computer simulations of such IFS's create more problems than for hyperbolic IFS's.

M. Hata [18] showed that every connected attractor of a hyperbolic IFS must be locally connected. He asked whether there exists a locally connected compact set, which is not attractor of any IFS.

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S. Crovisier and M. Rams [7] constructed a Cantor set in $\mathbb{R}^3$ with the property that any homeomorphism of the ambient space, which preserves this compact set, coincides with the identity map on it. Hence, this compact set can not serve as attractor of any hyperbolic IFS.

E. D'Aniello [8], E. D'Aniello and T. H. Steele [9]–[11] studied attractors of hyperbolic IFS and provided properties according to the nature of the generating functions. In particular, compact subsets of $X = [0, 1]^n$ are found that are not attractors for any system of weak contractions, and hence any hyperbolic system, all these subsets being Cantor sets.

Other examples of non IFS-attractors were studied by T. Banakh and M. Nowak [2], P.F. Duvall and L.S. Husch [12], M. Kulczycki and M. Nowak [20], M. Kwieciński [21], M.J. Sanders [26], [27] and others.

In this connection some questions arise as, for example, the following:

(1) Which compacta can (or can not) serve as attractors of hyperbolic IFS's?
(2) Given a compact set, which may be realized as the attractor of a hyperbolic IFS, what is the minimal possible number of contractions of such an IFS?
(3) Under which conditions the attractor of a hyperbolic IFS with condensation can be realized as the attractor of a standard hyperbolic IFS?

The second question seems to be related also to Borsuk's conjecture [5].

In this paper we formulate some answers to these questions. Section 2 is devoted to some preliminaries from Convex Analysis. In Section 3 we show that given a finite union of convex compacta in $\mathbb{R}^n$ there exists a hyperbolic IFS, whose attractor is exactly this compact set. An algorithm for the construction of convex compact sets in $\mathbb{R}^2$, using the Random Iterative Algorithm (see [3]), called also *Chaos Game*, was described in [17]. As a result, in Section 4 we show that for any IFS with condensation in $\mathbb{R}^n$, whose condensation set is a finite union of convex compacta, there exists a standard hyperbolic IFS with the same attractor. In Section 5 we apply the previous results to a hyperbolic IFS with condensation, whose attractor is the well-known fractal *The Pythagoras tree*. As a consequence, we construct the Pythagoras tree by the CAS *Mathematica*, using only five contractions. In Section 6 we formulate some open questions.

2. Preliminaries

In this paper we will consider the Euclidean space $X = \mathbb{R}^n$. For any $x, y \in X$, $[x, y] = \{(1 - \lambda)x + \lambda y \mid \lambda \in [0, 1]\}$ denotes the segment joining x to y. A set $C \subset X$ is called *convex set*, if for any $x, y \in C$, $[x, y] \subset C$.

Let K be a convex compact set in X. Recall that the *metric projection* onto K is the mapping $\mathrm{Pr}_K \colon X \to K$, $\mathrm{Pr}_K(x) = y$, where $y \in K$ is the unique point such that $\|x - y\| = \min\{\|x - z\| \mid z \in K\}$.

LEMMA 2.1. *Let K be a convex compact set in X. The metric projection Pr_K is nonexpansive, i.e. for any $x, y \in X$ one has $\|\mathrm{Pr}_K(x) - \mathrm{Pr}_K(y)\| \leq \|x - y\|$.*

PROOF. Given $x, y \in X$ let denote $\mathrm{Pr}_K(x) = s$ and $\mathrm{Pr}_K(y) = t$. The cases $x = s$, $y = t$ and $s = t$ are obvious.

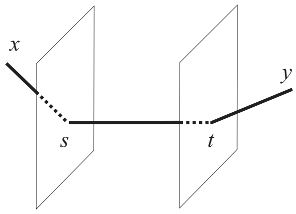

FIGURE 1. The angles $\widehat{xst}$ and $\widehat{yts}$ are obtuse or right.

Consider a 3-dimensional subspace in X, which contains the points x, y, s and t. Let draw two planes, which are perpendicular to the segment $[s, t]$ at its ends. The angles $\widehat{xst}$ and $\widehat{yts}$ are obtuse or right (see Figure 1), otherwise it would be another point on the interval $(s, t) \subset K$, which is closer to x or y than s or t respectively, contradiction. Thus, $\|s - t\| \leq \|x - y\|$. $\square$

Recall that a mapping $f \colon X \to X$ is called a *contraction*, if there exists $\lambda \in [0, 1)$ such that for any $x, y \in X$ one has $\|f(x) - f(y)\| \leq \lambda\|x - y\|$. Denote by $B(x, r)$ the open ball of radius r centered by x and by $\overline{B(x, r)}$ its closure.

Recall some notions from Convex Analysis (see, e.g. [25]). The *affine hull* $\operatorname{aff} C$ of a set $C \subset X$ is the smallest affine set containing C, i.e. the set of all affine combinations of elements of C,

$$\operatorname{aff} C = \left\{ \sum_{i=1}^{m} \lambda_i x_i \;\middle|\; x_i \in C,\ \lambda_i \in \mathbb{R},\ i = 1, \dots, m;\ \sum_{i=1}^{m} \lambda_i = 1;\ m \in \mathbb{N}^* \right\}.$$

Obviously, $C \subset \operatorname{aff} C$.

The *relative interior* of a convex set $C \subset X$, denoted by $\operatorname{ri} C$, is defined as its interior within the affine hull of C, i.e.

$$\operatorname{ri} C = \left\{ x \in \operatorname{aff} C \mid \exists\, r > 0 \text{ such that } \left(\overline{B(x, r)} \cap \operatorname{aff} C\right) \subset C \right\}.$$

The following result is a direct consequence of Theorem 6.2 [25].

THEOREM 2.2 ([25]). *For any convex set C in $\mathbb{R}^n$, containing at least two points, the relative interior $\operatorname{ri} C$ is nonempty, i.e. there exist a point $x \in C$ and a ball $\overline{B(x, r)}$ such that $\left(\overline{B(x, r)} \cap \operatorname{aff} C\right) \subset C$.*

REMARK 2.3. We consider in this paper only the Euclidian space $\mathbb{R}^n$, since Theorem 2.2 is no longer true for convex sets in infinite-dimensional spaces. For example, the Hilbert cube, defined as the Cartesian product of the intervals

$[0, 1/n]$ for $n \in \mathbb{N}^*$, or as the set of all sequences $(x_n)_{n\in\mathbb{N}^*}$ in the Hilbert space l_2 such that $0 \le x_n \le 1/n$ $(n \in \mathbb{N}^*)$, is a convex compact set $\mathcal{C}$ in l_2 and $\operatorname{aff}\mathcal{C} = l_2$. However, the Hilbert cube does not contain any ball in l_2.

3. Convex compacta as attractors of hyperbolic IFS's

Let $X = \mathbb{R}^n$. Denote by $\mathcal{P}_{\mathrm{cp}}(X)$ the space of all nonempty compact subsets of X, endowed with the Pompeiu–Hausdorff metrics.

An *Iterated Function System* (IFS) $\{X; f_1, \ldots, f_m\}$ is defined as a collection of pairwise distinct continuous functions $f_i \colon X \to X$ $(1 \le i \le m)$. If all functions f_i are contractions, one speaks about a *hyperbolic* IFS.

Given an IFS $\{X; f_1, \ldots, f_m\}$ let consider its *Barnsley–Hutchinson operator* $F_* \colon \mathcal{P}_{\mathrm{cp}}(X) \to \mathcal{P}_{\mathrm{cp}}(X)$, defined by $F_*(C) = \bigcup\limits_{i=1}^{m} f_i[C]$, where $f_i[C] = \bigcup\limits_{x\in C} f_i(x)$, for any $C \in \mathcal{P}_{\mathrm{cp}}(X)$ (see e.g. [3], [19]).

Remark 3.1. It is worth noting that multivalued contractions sometimes are called also *Nadler contractions* (see e.g. [4], [16]), since S.B. Nadler [24] was the first who studied these mappings.

There are various definitions of attractor. In ordinary dynamics (e.g. iterations of mappings) by an attractor one usually means an invariant set, which is dynamically indivisible and whose basin – the set of attracted points – is a large set. The basin must contain a neighbourhood of the attractor. At the same time, set-valued multi-functions need not be continuous or even semicontinuous, so for the basin to contain a neighbourhood is not an adequate demand.

In the case of compact spaces in [1] (see also [22]) the following definition has been proposed: A is an attractor, if it is invariant and there exists a closed neighborhood V of A such that $\bigcap\limits_{n\ge 0} f^n[V]$ is contained in A.

In [23] the invariance $f[A] = A$ is relaxed up to the condition $f[A] \supset A$ with the assumption that A attracts any bounded subset of a neighbourhood of A.

In this context we will say that a compact set $A \in \mathcal{P}_{\mathrm{cp}}(X)$ is *attractor* for an IFS with the Barnsley–Hutchinson operator F_*, if:

- $F_*(A) \supset A$;
- there is a closed neighbourhood $\overline{V}_\delta = \{x \in X \mid \exists\, y \in A,\ \|x - y\| \le \delta\}$ of A such that $\bigcap\limits_{n\ge 0} F_*^n(\overline{V}_\delta) \subset A$.

Both inclusions are, in fact, equalities (see [15]).

Theorem 3.2 ([19]). *Any hyperbolic* IFS $\{X; f_1, \ldots, f_m\}$ *has a nonempty compact attractor A and this attractor is the unique fixed point of the corresponding Barnsley-Hutchinson operator F_*, i.e. $F_*(A) = \bigcup\limits_{i=1}^{m} f_i[A] = A$.*

THEOREM 3.3. *Any convex compact set in* $\mathbb{R}^n$ *can be represented as the attractor of a hyperbolic* IFS.

PROOF. Let K be a convex compact set in $\mathbb{R}^n$. If K consists of at least two points, then by Theorem 2.2 there exist an affine subspace $L = \operatorname{aff} K \subset \mathbb{R}^n$, a point $x_0 \in K$ and a ball $\overline{B(x_0, r)}$ such that $\big(\overline{B(x_0, r)} \cap L\big) \subset K \subset L$.

Fix $\varrho \in (0, r)$ and denote $\lambda = \varrho/r \in (0, 1)$. Let $\mathrm{Pr}_L \colon \mathbb{R}^n \to L$ be the orthogonal projection onto the subspace L, let $g_1 \colon \mathbb{R}^n \to \overline{B(x_0, r)}$ be the metric projection onto the ball $\overline{B(x_0, r)}$, and let $g_2 \colon \mathbb{R}^n \to \mathbb{R}^n$, $g_2(x) = \lambda x + (1 - \lambda)x_0$, be the similitude with ratio λ and center x_0.

Define the mapping $f \colon \mathbb{R}^n \to \mathbb{R}^n$ by $f = g_2 \circ g_1 \circ \mathrm{Pr}_L$. Due to the structure of f we have

$$f[\mathbb{R}^n] = f[\overline{B(x_0, r)} \cap L] = g_2[\overline{B(x_0, r)} \cap L] = \overline{B(x_0, \varrho)} \cap L \subset K.$$

It is easy to check that f is a λ-contraction.

Consider the covering $\bigcup_{x \in K} B(x, \varrho)$ of the compact set K by open balls of radius ϱ. Among these balls there exists a finite subcovering $\bigcup_{i=1}^{p} B(x_i, \varrho)$ of K. Fix the centers of balls $x_1, \dots, x_p \in K$.

Define the mappings $\varphi_i \colon \mathbb{R}^n \to K$ $(1 \le i \le p)$ by

$$\varphi_i = \mathrm{Pr}_K \circ T_i \circ f, \tag{3.1}$$

where $T_i \colon \mathbb{R}^n \to \mathbb{R}^n$, $T_i(x) = x + (x_i - x_0)$, is the translation by the vector $(x_i - x_0) \in K$, and $\mathrm{Pr}_K \colon \mathbb{R}^n \to K$ is the metric projection onto the convex compact set K. Since f is a contraction, by Lemma 2.1 all mappings φ_i $(1 \le i \le p)$ are contractions as well.

We claim that the hyperbolic IFS $\{\mathbb{R}^n; \varphi_1, \dots, \varphi_p\}$ has K as attractor. It is sufficient to show that for the corresponding Barnsley–Hutchinson operator F_* we have

$$F_*(K) = \bigcup_{i=1}^{p} \varphi_i[K] = K. \tag{3.2}$$

Indeed, by construction $\bigcup_{i=1}^{p} \varphi_i[K] \subset K$.

Conversely, assume $x \in K$. There exists i such that $x \in \big(B(x_i, \varrho) \cap L\big)$. It follows that there exists $y = T_i^{-1}(x) \in \big(B(x_0, \varrho) \cap L\big)$. Denote

$$z = g_2^{-1}(y) = \frac{1}{\lambda}\, y + \frac{\lambda - 1}{\lambda}\, x_0 \in \big(B(x_0, r) \cap L\big) \subset K.$$

As a result, $\varphi_i(z) = x$. It implies that $K \subset \bigcup_{i=1}^{p} \varphi_i[K]$.

Note that, due to the structure of φ_i (use of the metric projection Pr_K), for any compact set V such that $K \subset V$ one has

$$\bigcup_{i=1}^{p} \varphi_i[V] = \bigcup_{i=1}^{p} \varphi_i[K] = K. \tag{3.3}$$

By Theorem 3.2 the compact set K, which verifies (3.2), is the attractor of the IFS $\{\mathbb{R}^n; \varphi_1, \dots, \varphi_p\}$. $\square$

REMARK 3.4. The proof of Theorem 3.3 shows that, given a convex compact set K, the value of the radius r of the ball from Theorem 2.2 can affect the number of contractions in the required IFS: a possible increase of r and an optimization of the covering of the compact set K may decrease the number of required contractions.

THEOREM 3.5. *A finite union of convex compacta in* $\mathbb{R}^n$ *can be represented as attractor of a hyperbolic* IFS.

PROOF. Let the compact set $K = K_1 \cup \ldots \cup K_m$ be a finite union of convex compacta K_i $(1 \le i \le m)$. By Theorem 3.3 for each convex compact set K_i $(1 \le i \le m)$ there exist contractions $\varphi_{ij} \colon \mathbb{R}^n \to K_i$ $(1 \le j \le p_i)$, which have by analogy to (3.1) the structure $\varphi_{ij} = \mathrm{Pr}_{K_i} \circ T_{ij} \circ f_i$, where $\mathrm{Pr}_{K_i} \colon \mathbb{R}^n \to K_i$ is the metric projection onto the convex compact set K_i. As a result, $\bigcup_{j=1}^{p_i} \varphi_{ij}[K_i] = K_i$.

Due to the structure of contractions φ_{ij} (use of the metric projections Pr_{K_i}), for any i $(1 \le i \le m)$ and any compact set V such that $K_i \subset V$, by analogy to (3.3) it follows that

$$\bigcup_{j=1}^{p_i} \varphi_{ij}[V] = \bigcup_{j=1}^{p_i} \varphi_{ij}[K_i] = K_i. \tag{3.4}$$

Consider the hyperbolic IFS, consisting of all these contractions $\{\mathbb{R}^n; \varphi_{11}, \dots, \varphi_{1p_1}, \varphi_{21}, \dots, \varphi_{2p_2}, \dots, \varphi_{m1}, \dots, \varphi_{mp_m}\}$. Let F_* be the corresponding Barnsley–Hutchinson operator. We claim that

$$\bigcup_{i=1}^{m} \bigcup_{j=1}^{p_i} \varphi_{ij}[K] = K.$$

Indeed, since $K_i \subset K$ for any i $(1 \le i \le m)$, using (3.4) we obtain

$$F_*(K) = \bigcup_{i=1}^{m} \bigcup_{j=1}^{p_i} \varphi_{ij}[K] = \bigcup_{i=1}^{m} \left(\bigcup_{j=1}^{p_i} \varphi_{ij}[K] \right) = \bigcup_{i=1}^{m} K_i = K.$$

Therefore, K is the attractor of this IFS. $\square$

REMARK 3.6. Note that each contraction $\varphi_{ij} \colon \mathbb{R}^n \to K_i$ mentioned in the proof of Theorem 3.5 has the property: $\varphi_{ij}[\mathbb{R}^n] \subset K_i \subset K$.

4. Attractors of IFS's with condensation

A constant compact-valued multi-function $f_0 \colon X \to \mathcal{P}_{\mathrm{cp}}(X)$, $f_0(x) \equiv K$ for some $K \in \mathcal{P}_{\mathrm{cp}}(X)$ and any $x \in X$, is called [3] a *condensation* with the compact set K as *condensation set.* A condensation is a super-contraction with the contractivity factor equal to 0. A hyperbolic *Iterated Function System with Condensation* (IFSC) $\{X; f_0, \dots, f_m\}$ consists of a condensation f_0 and contractions $f_1, \dots, f_m$.

M. Barnsley [3] has obtained a formula for the attractor of a hyperbolic IFS with condensation. Namely, let $\{X; f_1, \dots, f_m\}$ be a hyperbolic IFS with attractor A, and with F_* as the corresponding Barnsley–Hutchinson operator. Let f_0 be a condensation with $K \in \mathcal{P}_{\mathrm{cp}}(X)$ as its image. In this case the attractor A_c of the IFS with condensation $\{X; f_0, \dots, f_m\}$ is described as

$$A_c = A \cup \left(\bigcup_{n \geq 0} F_*^n(K) \right), \quad \text{where } F_*^0(K) = K. \tag{4.1}$$

THEOREM 4.1. *For any hyperbolic* IFS *with condensation in* $\mathbb{R}^n$*, whose condensation set is a finite union of convex compacta, there exists a standard hyperbolic* IFS *with the same attractor.*

PROOF. Let $\{\mathbb{R}^n; f_0, \dots, f_m\}$ be a hyperbolic IFSC with the condensation f_0 and let the condensation set K be a finite union of convex compacta. Denote by A_c its attractor.

For the hyperbolic IFS $\{\mathbb{R}^n; f_1, \dots, f_m\}$ denote by F_* its Barnsley–Hutchinson operator and by A its attractor, i.e. $F_*(A) = A$. Hence, the attractor A_c is given by (4.1).

By Theorem 3.5 there exists a hyperbolic IFS $\{\mathbb{R}^n; \varphi_1, \dots, \varphi_p\}$, having K as attractor. According to Remark 3.6, we can choose the contractions of this IFS such that $\varphi_i[\mathbb{R}^n] \subset K$ $(1 \leq i \leq p)$. Let Φ_* be the Barnsley–Hutchinson operator of this hyperbolic IFS. Hence,

$$\Phi_*(K) = K \quad \text{and} \quad \Phi_*(M) \subset K \quad \text{for any } M \in \mathcal{P}_{\mathrm{cp}}(\mathbb{R}^n). \tag{4.2}$$

We claim that the hyperbolic IFS $\{\mathbb{R}^n; f_1, \dots, f_m, \varphi_1, \dots, \varphi_p\}$ and the IFSC $\{\mathbb{R}^n; f_0, \dots, f_m\}$ have the same attractor A_c. It is sufficient to show that

$$(F_* \cup \Phi_*)[A_c] = F_*[A_c] \cup \Phi_*[A_c] = \left(\bigcup_{i=1}^{m} f_i[A_c] \right) \cup \left(\bigcup_{j=1}^{p} \varphi_j[A_c] \right) = A_c.$$

From (4.1) and (4.2) we have

$$\begin{aligned}(F_* \cup \Phi_*)[A_c] &= F_*[A_c] \cup \Phi_*[A_c] \\ &= \left(F_*(A) \cup \left(\bigcup_{n \geq 0} F_*^{n+1}(K) \right) \right) \cup \Phi_* \left(A \cup K \cup \left(\bigcup_{n \geq 1} F_*^n(K) \right) \right)\end{aligned}$$

$$= \left(A \cup \left(\bigcup_{n\geq 1} F_*^n(K)\right)\right) \cup \Phi_*(K) \cup \Phi_*\left(A \cup \left(\bigcup_{n\geq 1} F_*^n(K)\right)\right)$$

$$= \left(A \cup \left(\bigcup_{n\geq 1} F_*^n(K)\right)\right) \cup K = \left(A \cup \left(\bigcup_{n\geq 0} F_*^n(K)\right)\right) = A_c.$$

Hence, the IFS $\{\mathbb{R}^n; f_1, \dots, f_m, \varphi_1, \dots, \varphi_p\}$ and the IFSC $\{\mathbb{R}^n; f_0, \dots, f_m\}$ have the same attractor A_c. $\square$

THEOREM 4.2. *Let a convex compact set K in $\mathbb{R}^n$ be the attractor of a hyperbolic* IFS*, consisting of p contractions. Let K be also the condensation set for a hyperbolic* IFSC*, consisting of a condensation and m contractions. Then the attractor of this* IFSC *can be represented as the attractor of a standard hyperbolic* IFS*, consisting of at most $m+p$ contractions.*

PROOF. Instead of contractions of the given hyperbolic IFS consider their compositions with the metric projection onto the convex compact set K. By analogy with Theorems 3.3 and 4.1 we show that the new hyperbolic IFS, consisting of all these $(m+p)$ contractions, has K as attractor. $\square$

REMARK 4.3. Theorem 4.2 gives an algorithm to construct attractors of such hyperbolic IFS's with condensation, using the Random Iterative Algorithm.

REMARK 4.4. In general case, given a hyperbolic IFS with condensation, if we substitute the constant compact-valued multi-function with some contractions, which generate the condensation set as attractor, then the attractor of the new "extended" hyperbolic IFS does not necessarily coincide with the attractor of the given IFS with condensation.

EXAMPLE 4.5. Consider the IFS with condensation $\left\{\mathbb{R}^2; f_0, f_1\right\}$, where f_0 is the condensation and f_1 is the similitude with ratio $1/2$ and the center in $(0,1)$ as follows:

$$f_0\colon (x,y) \mapsto K = [0,1] \times \{0\}, \qquad f_1\colon (x,y) \mapsto \left(\frac{x}{2}, \frac{1+y}{2}\right).$$

The attractor of this IFS with condensation is the set of parallel segments

$$A = \bigcup_{n=0}^{+\infty} \left([0, 2^{-n}] \times \{1 - 2^{-n}\}\right) \bigcup \{(0,1)\}$$

(see Figure 2 (left)). The condensation set $K = [0,1] \times \{0\}$ is a convex compact set, which can be represented as the attractor of the hyperbolic IFS $\{\mathbb{R}^2; \varphi_1, \varphi_2\}$, where

$$\varphi_1 : (x,y) \mapsto \left(\frac{x}{2}, \frac{y}{2}\right), \quad \varphi_2 : (x,y) \mapsto \left(\frac{1+x}{2}, \frac{y}{2}\right).$$

The new "extended" IFS $\left\{\mathbb{R}^2; \varphi_1, \varphi_2, f_1\right\}$ consists of three similitudes with ratio $1/2$ and the centers in $(0,0)$,$(1,0)$ and $(0,1)$, respectively. It is known [3] that its

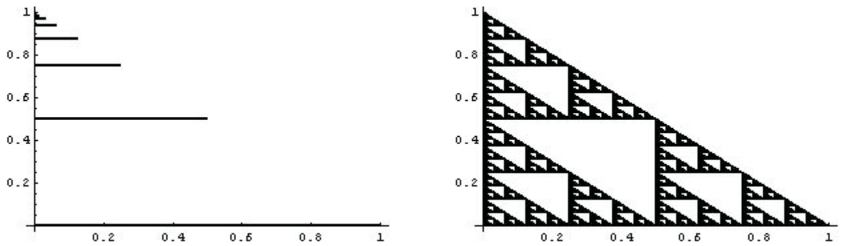

FIGURE 2. Attractors.

attractor is the *Sierpiński triangle* (see Figure 2 (right)). Hence, the attractor of this "extended" IFS does not coincide with the attractor of the given IFS with condensation.

On the other hand, we can represent K as well as the attractor of another hyperbolic IFS $\{\mathbb{R}^2; \psi_1, \psi_2\}$, where

$$\psi_1\colon (x,y) \mapsto \left(\frac{x}{2}, 0\right), \qquad \psi_2\colon (x,y) \mapsto \left(\frac{1+x}{2}, 0\right).$$

It is easy to check that $\psi_1[A] \cup \psi_2[A] \cup f_1[A] = A$. Thus, the new "extended" IFS $\{\mathbb{R}^2; \psi_1, \psi_2, f_1\}$ has the same attractor A as the given IFS with condensation.

5. The Pythagoras tree

We illustrate Theorem 4.1 with a hyperbolic IFS with condensation, whose attractor is the Pythagoras tree. The well-known fractal called *The Pythagoras tree* is a plane fractal, which was invented by the Dutch mathematics teacher Albert E. Bosman in 1942. This fractal, constructed from squares, can be represented as the attractor of an Iterated Function System with condensation. Given a right triangle, this IFS is determined by a constant compact-valued mapping with the "hypotenuse's square" as condensation set, together with two affine contractions, which map this square onto the other two squares related to the given right triangle. The shape of fractal depends on the shape of this triangle.

EXAMPLE 5.1. Figure 3 (left) represents the classical Pythagoras tree, which was created manually by A.E. Bosman [6]. Figure 3 (right) represents the Pythagoras tree, obtained by computer simulation as the attractor of the IFS with condensation $\{\mathbb{R}^2; f_0, f_1, f_2\}$, consisting of the condensation f_0 and two contractions f_1, f_2 as follows:

$$f_0\colon (x,y) \mapsto K = [-0.5, 0.5] \times [0,1];$$
$$f_1\colon (x,y) \mapsto (0.64x - 0.48y - 0.18, 0.48x + 0.64y + 1.24);$$
$$f_2 : (x,y) \mapsto (0.36x + 0.48y + 0.32, -0.48x + 0.36y + 1.24).$$

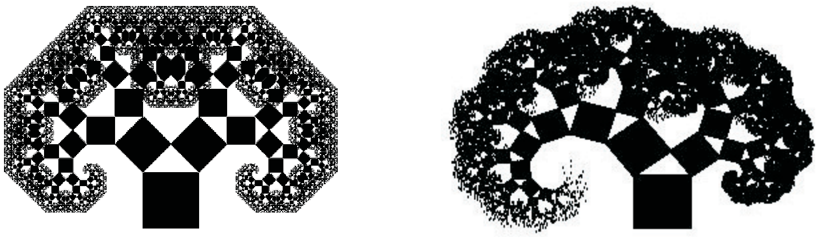

FIGURE 3. Pythagoras trees.

It is known [3] that any square can be represented as the attractor of a hyperbolic IFS, consisting of four affine contractions (e.g. four similitudes with ratio 1/2 and centers in the vertices of the square).

According to Theorem 4.2 one can construct the Pythagoras tree as the attractor of a hyperbolic IFS, consisting of six contractions. In fact, the Pythagoras tree can be constructed using a smaller number of contractions.

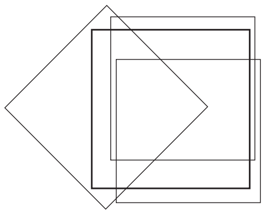

FIGURE 4. Covering of square with three smaller squares.

It is known (see e.g. [13]) that a square can not be covered with two smaller squares. In Figure 4 it is shown that a square can be covered with three smaller squares (see also [13]).

THEOREM 5.2. *A square can be obtained as attractor of a hyperbolic* IFS, *consisting of three contractions.*

PROOF. The given square can be covered with three smaller squares. Take three affine contractions such that each of them maps the basic square to one of these small squares. Afterwards, take the composition of each of these three contractions with the metric projection onto the basic square. By Theorem 3.2 the attractor of the hyperbolic IFS, consisting of these three new contractions, coincides with the given square. □

THEOREM 5.3. *The Pythagoras tree can be obtained as the attractor of a hyperbolic* IFS, *consisting of at most five contractions.*

PROOF. It follows from Theorems 4.2 and 5.2. □

REMARK 5.4. The Pythagoras tree from Figure 3 (right) is constructed as the attractor of a hyperbolic IFS, which consists of five contractions, using the Random Iterative Algorithm. Another application of the Random Iterative Algorithm to construct the Pythagoras tree was described in [14].

6. Open questions

In this context some questions arise and may be of independent interest:

(1) Is it possible to obtain the Pythagoras tree as attractor with only four contractions?
(2) Is it true that any image of a square, generated by a contraction, can be covered with a smaller square?
(3) Is it true that a square can be covered with the union of its two images, generated by two contractions?

All numerical calculations and graphic objects have been done using the Computer Algebra System *Mathematica*.

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