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# CONVERGENCE ESTIMATES FOR ABSTRACT SECOND ORDER DIFFERENTIAL EQUATIONS WITH TWO SMALL PARAMETERS AND MONOTONE NONLINEARITIES

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Abstract. In a real Hilbert space $H$ we consider the following perturbed Cauchy problem

$$(\mathrm{P}_{\varepsilon\delta}) \quad \begin{cases} \varepsilon\, u''_{\varepsilon\delta}(t) + \delta\, u'_{\varepsilon\delta}(t) + Au_{\varepsilon\delta}(t) + B(u_{\varepsilon\delta}(t)) = f(t), & t \in (0,T), \\ u_{\varepsilon\delta}(0) = u_0, \quad u'_{\varepsilon\delta}(0) = u_1, \end{cases}$$

where $u_0, u_1 \in H$, $f\colon [0,T] \mapsto H$ and $\varepsilon$, $\delta$ are two small parameters, $A$ is a linear self-adjoint operator, $B$ is a locally Lipschitz and monotone operator. We study the behavior of solutions $u_{\varepsilon\delta}$ to the problem $(\mathrm{P}_{\varepsilon\delta})$ in two different cases:

(i) when $\varepsilon \to 0$ and $\delta \geq \delta_0 > 0$;
(ii) when $\varepsilon \to 0$ and $\delta \to 0$.

We obtain some *a priori* estimates of solutions to the perturbed problem, which are uniform with respect to parameters, and a relationship between solutions to both problems. We establish that the solution to the unperturbed problem has a singular behavior, relative to the parameters, in the neighborhood of $t = 0$. We show the boundary layer and boundary layer function in both cases.

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## 1. Introduction

Let $H$ and $V$ be two real Hilbert spaces endowed with norms $|\cdot|$ and $||\cdot||$, respectively. Denote by $(\,\cdot\,,\,\cdot\,)$ the scalar product in $H$. The framework of our studying will be determined by the following conditions:

(H) *$V \subset H$ densely and continuously, i.e.*

$$||u|| \geq \omega_0\,|u|, \quad \textit{for all } u \in V, \quad \omega_0 > 0.$$

(HA) *$A\colon D(A) = V \mapsto H$ is a linear, self-adjoint and positive definite operator, i.e.*

$$(Au, u) \geq \omega\,|u|^2, \quad \textit{for all } u \in V, \quad \omega > 0.$$

(HB1) *Operator $B\colon D(B) \subseteq H \to H$ is $A^{1/2}$ locally Lipschitz, i. e. $D(A^{1/2}) \subset D(B)$ and, for every $R > 0$, there exists $L(R) \geq 0$ such that*

$$\begin{aligned} |B(u_1) - B(u_2)| \leq L(R)|A^{1/2}(u_1 - u_2)|, &\quad \text{for all } u_i \in D(A^{1/2}),\\ |A^{1/2}u_i| \leq R, &\quad \text{for } i = 1, 2; \end{aligned}$$

(HB2) *Operator $B$ is the Fréchet derivative of some convex and positive functional $\mathcal{B}$ with $D(A^{1/2}) \subset D(\mathcal{B})$.*

(HB3) *Operator $B$ possesses the Fréchet derivative $B'$ in $D(A^{1/2})$ and there exists constant $L_1(R) \geq 0$ such that*

$$\begin{aligned} \big|\big(B'(u_1) - B'(u_2)\big)v\big| \leq L_1(R)\big|A^{1/2}(u_1 - u_2)\big|\,|A^{1/2}v|,&\\ \text{for all } u_1, u_2, v \in D(A^{1/2}),&\\ |A^{1/2}u_i| \leq R, \quad \text{for } i = 1, 2.& \end{aligned}$$

The hypothesis (HB2) implies, in particular, that operator $B$ is monotone and verifies condition

$$\frac{d}{dt}\mathcal{B}(u(t)) = \big(B(u(t)), u'(t)\big), \quad \text{for all } t \in [a, b] \subset \mathbb{R}$$

in the case when $u \in C([a, b], D(A^{1/2})) \cap C^1([a, b], H)$ (see, for example [15]).

Consider the following perturbed Cauchy problem

$$(\mathrm{P}_{\varepsilon\delta}) \qquad \begin{cases} \varepsilon\, u''_{\varepsilon\delta}(t) + \delta\, u'_{\varepsilon\delta}(t) + Au_{\varepsilon\delta}(t) + B(u_{\varepsilon\delta}(t)) = f(t), \quad t \in (0, T),\\ u_{\varepsilon\delta}(0) = u_0, \quad u'_{\varepsilon\delta}(0) = u_1, \end{cases}$$

where $u_0, u_1 \in H$, $f\colon [0, T] \mapsto H$ and $\varepsilon$, $\delta$ are two small parameters.

We study the behavior of solutions $u_{\varepsilon\delta}$ to the problem $(\mathrm{P}_{\varepsilon\delta})$ in two different cases:

(i) $\varepsilon \to 0$ and $\delta \geq \delta_0 > 0$, relative to the following unperturbed system:

$$(\mathrm{P}_{\delta}) \qquad \begin{cases} \delta l'_{\delta}(t) + Al_{\delta}(t) + B(l_{\delta}(t)) = f(t), \quad t \in (0, T),\\ l_{\delta}(0) = u_0, \end{cases}$$

(ii) $\varepsilon \to 0$ and $\delta \to 0$, relative to the following unperturbed problem:

$$(\mathrm{P}_0) \qquad Av(t) + B(v(t)) = f(t), \quad t \in [0, T).$$

The problem $(\mathrm{P}_{\varepsilon\delta})$ is the abstract model of singularly perturbed problems of hyperbolic-parabolic type in the case (i) and of hyperbolic-parabolic-elliptic type in the case (ii). Such kind of problems arises in the mathematical modeling of elasto-plasticity phenomena and abstract results can be applied to singularly perturbed problems of hyperbolic-parabolic-elliptic type with stationary part defined by strongly elliptic operators.

For example in [3], the equation

$$\rho v_{tt} + \gamma v_t = \sigma \Delta v$$

is considered (where $\rho, \gamma, \sigma$ are the mass density per unit area of the membrane, the coefficient of viscosity of the medium, and the tension of the membrane, respectively), which characterizes the vibration of a membrane in a viscous medium, which can be rewritten as

$$\varepsilon^2 u_{tt} + u_t = \Delta u, \quad \text{with } \varepsilon = (\rho\sigma)^{1/2}/\gamma.$$

In the case when the medium is highly viscous ($\gamma \gg 1$), or the density $\rho$ is very small, we have $\varepsilon \to 0$ and the formal "limit" of this equation will be the following first order equation

$$u_t = \Delta u.$$

Without pretending to a complete analysis, let us mention some works dedicated to the study of singularly perturbed Cauchy problems for linear or nonlinear differential equations of second order of type $(\mathrm{P}_{\varepsilon\delta})$. The case when $\delta = 1$ was widely studied by various mathematicians (see, e.g. [4], [5], [8], [10] and the bibliography therein). In [6] the asymptotic behavior of solutions to singular perturbation problems for second order equations, as $\varepsilon \to 0$ and $\delta \to 0$, is studied. In [13] the linear case is considered. In [2], [14], [16], some numerical results about singular behavior of solutions to the problem $(\mathrm{P}_{\varepsilon\delta})$ for some ordinary differential equations and their applicability in modeling of different physical and engineering processes are presented.

In what follows we need some notations. For $k \in \mathbb{N}^\star$, $1 \le p \le +\infty$, $(a,b) \subset (-\infty, +\infty)$ and Banach space $X$ we denote by $W^{k,p}(a,b;X)$ the Banach space of all vectorial distributions $u \in D'(a,b;X)$, $u^{(j)} \in L^p(a,b;X)$, $j = 0, 1, \ldots, k$, endowed with the norm

$$\|u\|_{W^{k,p}(a,b;X)} = \begin{cases} \left( \displaystyle\sum_{j=0}^{k} \|u^{(j)}\|^p_{L^p(a,b;X)} \right)^{1/p}, & p \in [1, \infty), \\ \displaystyle\max_{0 \le j \le k} \left\|u^{(j)}\right\|_{L^\infty(a,b;X)}, & p = \infty. \end{cases}$$

If $p = 2$, and $X$ is a Hilbert space, then $W^{k,2}(a,b;X)$ is also a Hilbert space with the inner product

$$(u,v)_{H^k(a,b;X)} = \sum_{j=0}^{k} \int_a^b \Big(u^{(j)}(t), v^{(j)}(t)\Big)_X \, dt.$$

For $s \in \mathbb{R}$, $k \in \mathbb{N}$ and $p \in [1,\infty]$, we define the Banach space

$$W_s^{k,p}(a,b;X) = \big\{f\colon (a,b) \to H \mid f^{(l)}(\,\cdot\,)e^{-st} \in L^p(a,b;X),\ l = 0,\dots,k\big\},$$

with the norm $\|f\|_{W_s^{k,p}(a,b;X)} = \|fe^{-st}\|_{W^{k,p}(a,b;X)}$.

## 2. Existence of solutions to the problems $(\mathrm{P}_{\varepsilon\delta})$ and $(\mathrm{P}_\delta)$

Definition 2.1. Let $T > 0$ and $f \in L^2(0,T;H)$, $A\colon D(A) \subseteq H \to H$, $B\colon D(B) \subseteq H \to H$. The function $u \in L^2(0,T;D(A) \cap D(B))$ with $u' \in L^2(0,T;H)$ and $u'' \in L^2(0,T;H)$ is called strong solution to the Cauchy problem

$$u''(t) + u'(t) + Au(t) + B(u(t)) = f(t), \quad \text{for all } t \in (0,T), \tag{2.1}$$

$$u(0) = u_0, \quad u'(0) = u_1, \tag{2.2}$$

if $u$ satisfies the equality (2.1) in the sense of distributions almost every $t \in (0,T)$ and the initial conditions (2.2) .

Definition 2.2. Let $T > 0$ and $f \in L^2(0,T;H)$, $A\colon D(A) \subseteq H \to H$, $B\colon D(B) \subseteq H \to H$. The function $l \in L^2(0,T;D(A) \cap D(B))$ with $l' \in L^2(0,T;H)$ is called strong solution to the Cauchy problem

$$l'(t) + Al(t) + B(l(t)) = f(t), \quad \text{for all } t \in (0,T), \tag{2.3}$$

$$l(0) = u_0, \tag{2.4}$$

if $v$ verifies the equality (2.3) in the sense of distributions almost every $t \in (0,T)$ and the initial condition (2.4).

Based on the methods from [1], in [11] the following two theorems were established.

Theorem 2.3. *Let $T > 0$. Let us assume that condition*(H) *is fulfilled, the operator $A\colon D(A) \subset H \to H$ satisfies condition* HA) *and the operator $B\colon D(B) \subset H \to H$ satisfies conditions* (HB1), (HB2). *If $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$ and $f \in W^{1,1}(0,T;H)$, then there exists a unique strong solution to problem* (2.1), (2.2), *such that $u \in C^2([0,T];H)$, $A^{1/2}u \in C^1([0,T];H)$, $Au \in C([0,T];H)$. If, in addition, $u_1 \in D(A)$, $f(0) - B(u_0) - Au_0 - u_1 \in D(A^{1/2})$, $f \in W^{2,1}(0,T;H)$ and condition* (HB3) *is fulfilled, then $A^{1/2}u \in W^{2,\infty}(0,T;H)$ and $u \in W^{3,\infty}(0,T;H)$.*

THEOREM 2.4. *Let $T>0$ and assume that condition* (H) *is fulfilled, the operator $A\colon D(A)\subset H\to H$ satisfies condition* (HA) *and the operator $B\colon D(B)\subset H\to H$ satisfies conditions* (HB1), (HB2). *If $u_0\in D(A)$ and $f\in W^{1,1}(0,T;H)$, then there exists a unique strong solution to the problem* (2.3), (2.4), *such that $l\in C^1([0,T];H)$, $Al\in C([0,T];H)$. For this solution the following estimates*

$$
(2.5)\qquad
\begin{aligned}
&\|l\|_{C([0,t];H)}+\|A^{1/2}l\|_{L^2(0,t;H)}\le C\,M_0(t), \qquad \text{for all } t\in[0,T],\\
&\big\|A^{1/2}l\big\|_{C([0,t];H)}+\|l'\|_{C([0,t];H)}+\big\|A^{1/2}l'\big\|_{L^2(0,t;H)}\\
&\qquad\qquad\le C(\omega)\,M_1(t),\quad \text{for all } t\in[0,T],
\end{aligned}
$$

*are valid, where*

$$
\begin{aligned}
M_0(t)&=\big|u_0\big|+\int_0^t\big(|f(s)|+|B(0)|\big)\,ds,\\
M_1(t)&=\big|Au_0\big|+\|f\|_{W^{1,1}(0,t;H)}+|B(0)|+|f(0)|.
\end{aligned}
$$

The problems $(\mathrm{P}_{\varepsilon\delta})$ and $(\mathrm{P}_\delta)$ can be rewritten as follows:

$$
(\mathcal{P}_\mu)\qquad
\begin{cases}
\mu\,U_\mu''(s)+U_\mu'(s)+AU_\mu(s)+B(U_\mu(s))=F(s), \&s\in(0,T/\delta),\\
U_\mu(0)=u_0,\quad U_\mu'(0)=\delta u_1,
\end{cases}
$$

and

$$
(\mathcal{P}_0)\qquad
\begin{cases}
L'(s)+AL(s)+B(L(s))=F(s),\quad s\in(0,T/\delta),\\
L(0)=u_0,
\end{cases}
$$

where $U_\mu(s)=u_{\varepsilon\delta}(\delta\,s)$, $L(s)=l_\delta(s\delta)$, $F(s)=f(s\delta)$ and $\mu=\varepsilon/\delta^2$. Using results obtained in the paper [11] we get the following *a priori* estimates for solutions to the problem $(\mathcal{P}_\mu)$.

LEMMA 2.5. *Let $S>0$. Let us assume that condition* (H) *is fulfilled, the operator $A\colon D(A)\subset H\to H$ satisfies condition* (HA) *and the operator $B$ verifies conditions* (HB1), (HB2). *If $u_0\in D(A), u_1\in D(A^{1/2})$ and $F\in W^{1,1}(0,\infty;H)$, then there exists the constant $C(\omega_0,\omega)>0$ such that for every strong solution $U_\mu$ to the problem* $(\mathrm{P}_\mu)$ *the following estimate holds*:

$$
\big\|A^{1/2}U_\mu\big\|_{C([0,s];H)}+\|U_\mu'\|_{L^2(0,s;H)}+\big(\mathcal{B}(U_\mu(s))\big)^{1/2}\le M_2,
$$

*for all $\mu\in(0,1]$ and for all $s\in[0,S]$,*

$$
\mu\|U_\mu''\|_{C([0,s];H)}+\|U_\mu'\|_{C([0,s];H)}+\big\|A^{1/2}U_\mu'\big\|_{L^2(0,s;H)}\le Ce^{12L^2(M_2)s}M_3,
$$

*for all $\mu\in(0,1]$ and for all $s\in[0,S]$,*

$$
\|AU_\mu\|_{C([0,s];H)}\le CM_4e^{(6L^2(M_2)+1)s},
$$

*for all* $\mu \in (0,1/2]$ *and for all* $s \in [0,S]$, *where*

$$M_2 = |A^{1/2}u_0| + |u_1| + |\mathcal{B}(u_0)|^{1/2} + \|F\|_{W^{1,1}(0,\infty;H)},$$
$$M_3 = |Au_0| + |A^{1/2}u_1| + |B(u_0)| + |\mathcal{B}(u_0)|^{1/2} + \|F\|_{W^{1,1}(0,\infty;H)},$$
$$M_4 = (L(M_2)+1)M_1.$$

## 3. Relationship between solutions to the problems $(\mathrm{P}_{\varepsilon\delta})$ and $(\mathrm{P}\delta)$ in the linear case

In what follows, for all $\mu > 0$, denote by

$$K(s,\tau,\mu) = \frac{1}{2\sqrt{\pi}\mu}\big(K_1(s,\tau,\mu) + 3K_2(s,\tau,\mu) - 2K_3(s,\tau,\mu)\big),$$

where

$$K_1(s,\tau,\mu) = \exp\bigg\{\frac{3s-2\tau}{4\mu}\bigg\}\lambda\bigg(\frac{2s-\tau}{2\sqrt{\mu s}}\bigg),$$
$$K_2(s,\tau,\mu) = \exp\bigg\{\frac{3s+6\tau}{4\mu}\bigg\}\lambda\bigg(\frac{2s+\tau}{2\sqrt{\mu s}}\bigg),$$
$$K_3(s,\tau,\mu) = \exp\bigg\{\frac{\tau}{\mu}\bigg\}\lambda\bigg(\frac{s+\tau}{2\sqrt{\mu s}}\bigg), \qquad \lambda(s) = \int_s^\infty e^{-\eta^2}\,d\eta.$$

The properties of kernel $K(t,\tau,\varepsilon)$ are collected in the following lemma.

Lemma 3.1 ([9]). *The function* $K(t,\tau,\varepsilon)$ *possesses the following properties*:

(a) $K \in C([0,\infty)\times[0,\infty)) \cap C^2((0,\infty)\times(0,\infty))$;

(b) $K_t(t,\tau,\varepsilon) = \varepsilon K_{\tau\tau}(t,\tau,\varepsilon) - K_\tau(t,\tau,\varepsilon)$, *for all* $t > 0$ *and all* $\tau > 0$;

(c) $\varepsilon K_\tau(t,0,\varepsilon) - K(t,0,\varepsilon) = 0$, *for all* $t \ge 0$;

(d) *For all* $\tau \ge 0$

$$K(0,\tau,\varepsilon) = \frac{1}{2\varepsilon}\exp\bigg\{-\frac{\tau}{2\varepsilon}\bigg\};$$

(e) *For every* $t > 0$ *fixed and every* $q, s \in \mathbb{N}$ *there exist constants* $C_1(q,s,t,\varepsilon) > 0$ *and* $C_2(q,s,t) > 0$ *such that*

$$\big|\partial_t^s\partial_\tau^q K(t,\tau,\varepsilon)\big| \le C_1(q,s,t,\varepsilon)\exp\{-C_2(q,s,t)\tau/\varepsilon\}, \quad \text{for all } \tau > 0;$$

*Moreover, for* $\gamma \in \mathbb{R}$ *there exist* $C_1$, $C_2$ *and* $\varepsilon_0$, *all of them positive and depending on* $\gamma$, *such that the following estimates are fulfilled*:

$$\int_0^\infty e^{\gamma\tau}\big|K_t(t,\tau,\varepsilon)\big|\,d\tau \le C_1\,\varepsilon^{-1}\,e^{C_2 t}, \quad \text{for all } \varepsilon \in (0,\varepsilon_0], \text{ for all } t \ge 0,$$
$$\int_0^\infty e^{\gamma\tau}\big|K_\tau(t,\tau,\varepsilon)\big|\,d\tau \le C_1\,\varepsilon^{-1}\,e^{C_2 t}, \quad \text{for all } \varepsilon \in (0,\varepsilon_0], \text{ for all } t \ge 0,$$
$$\int_0^\infty e^{\gamma\tau}\big|K_{\tau\tau}(t,\tau,\varepsilon)\big|\,d\tau \le C_1\,\varepsilon^{-2}\,e^{C_2 t}, \quad \text{for all } \varepsilon \in (0,\varepsilon_0], \text{ for all } t \ge 0;$$

(f) $K(t,\tau,\varepsilon) > 0$, *for all* $t \ge 0$ *and for all* $\tau \ge 0$;

(g) *For every continuous function* $\varphi\colon [0,\infty) \to H$ *with* $|\varphi(t)| \le M \exp\{\gamma\, t\}$ *the following equality is true*:

$$\lim_{t\to 0} \left| \int_0^\infty K(t,\tau,\varepsilon)\varphi(\tau)\, d\tau - \int_0^\infty e^{-\tau}\varphi(2\varepsilon\tau)\, d\tau \right| = 0,$$

*for every* $\varepsilon \in \big(0, (2\,\gamma)^{-1}\big)$;

(h) $\displaystyle\int_0^\infty K(t,\tau,\varepsilon) d\tau = 1,$ *for all* $t \ge 0$,

(i) *Let* $\gamma > 0$ *and* $q \in [0,1]$. *There exist* $C_1, C_2$ *and* $\varepsilon_0$ *all of them positive and depending on* $\gamma$ *and* $q$, *such that the following estimates are fulfilled*:

$$\int_0^\infty K(t,\tau,\varepsilon)\, e^{\gamma\tau} |t-\tau|^q\, d\tau \le C_1\, e^{C_2 t}\, \varepsilon^{q/2},$$

*for all* $\varepsilon \in (0,\varepsilon_0]$, *and for allt* $> 0$. *If* $\gamma \le 0$ *and* $q \in [0,1]$, *then*

$$\int_0^\infty K(t,\tau,\varepsilon)\, e^{\gamma\tau}\, |t-\tau|^q\, d\tau \le C\, \varepsilon^{q/2} \left(1+\sqrt{t}\right)^q,$$

*for all* $\varepsilon \in (0,1]$ *and for all* $t \ge 0$;

(j) *Let* $p \in (1,\infty]$ *and* $f\colon [0,\,\infty) \to H$, $f(t) \in W^{1,p}_\gamma(0,\infty;H)$. *If* $\gamma > 0$, *then there exist* $C_1$, $C_2$ *and* $\varepsilon_0$ *all of them positive and depending on* $\gamma$ *and* $p$, *such that*

$$\left| f(t) - \int_0^\infty K(t,\tau,\varepsilon) f(\tau)\, d\tau \right| \le C_1\, e^{C_2 t}\, \|f'\|_{L^p_\gamma(0,\infty;H)} \varepsilon^{(p-1)/2p},$$

*for all* $\varepsilon \in (0,\varepsilon_0]$, *and for all* $t \ge 0$. *If* $\gamma \le 0$, *then*

$$\left| f(t) - \int_0^\infty K(t,\tau,\varepsilon) f(\tau)\, d\tau \right| \\ \le C(\gamma,p)\, \|f'\|_{L^p_\gamma(0,\infty;H)} \left(1+\sqrt{t}\right)^{(p-1)/p} \varepsilon^{(p-1)/2p},$$

*for all* $\varepsilon \in (0,1]$ *and for all* $t \ge 0$.

LEMMA 3.2 ([9]). *Let* $B = 0$. *Assume that* $A\colon D(A) \subset H \to H$ *is a linear, self-adjoint, positive definite operator and* $F \in L^\infty_\gamma(0,\infty;H)$ *for some* $\gamma \ge 0$. *If* $U_\mu$ *is the strong solution to the problem* $(\mathcal{P}_\mu)$ *with* $U_\mu \in W^{2,\infty}_\gamma(0,\infty;H) \cap L^\infty_\gamma(0,\infty;H)$, $AU_\mu \in L^\infty_\gamma(0,\infty;H)$, *then for every* $0 < \mu < (4\gamma)^{-1}$ *the function* $W_\mu$, *defined by*

$$W_\mu(s) = \int_0^\infty K(s,\tau,\mu)\, U_\mu(\tau)\, d\tau,$$

*is the strong solution in* $H$ *to the problem*

$$\begin{cases} W'_\mu(s) + AW_\mu(s) = F_0(s,\mu) & \text{for a.e. } s > 0 \text{ in } H, \\ W_\mu(0) = \varphi_\mu. \end{cases}$$

$$F_0(s,\mu) = \frac{1}{\sqrt{\pi}}\bigg[2\exp\bigg\{\frac{3s}{4\mu}\bigg\}\lambda\bigg(\sqrt{\frac{s}{\mu}}\bigg) - \lambda\bigg(\frac{1}{2}\sqrt{\frac{s}{\mu}}\bigg)\bigg]\delta u_1 + \int_0^\infty K(s,\tau,\mu)F(\tau)\,d\tau,$$
$$\varphi_\mu = \int_0^\infty e^{-\tau}\,U_\mu(2\mu\tau)\,d\tau.$$

## 4. Behavior of solutions to the problem $(\mathrm{P}_{\varepsilon\delta})$, when $\varepsilon \to 0$ and $\delta \geq \delta_0 > 0$

Using results obtained in the paper [11] we get the relationship between the solutions to the problems $(\mathrm{P}_{\varepsilon\delta})$ and $(\mathrm{P}_\delta)$ in the case $\delta \geq \delta_0 > 0$, presented in the following two theorems.

Theorem 4.1. *Let $T > 0$, $\delta \geq \delta_0 > 0$ and $p > 1$. Let us assume that condition* (H) *is fulfilled, the operator $A$ satisfies condition* (HA) *and the operator $B$ verifies conditions* (HB1), (HB2). *If $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$ and $f \in W^{1,p}(0,T;H)$, then there exist constants $C = C(T,p,\delta_0,\omega_0,\omega,L(\mathbf{m})) > 0$, $\varepsilon_0 = \varepsilon_0(\omega_0,\omega,L(\mathbf{m}))$, $\varepsilon_0 \in (0,1)$, such that*

$$\begin{aligned}&\|u_{\varepsilon\delta} - l_\delta\|_{C([0,T];H)} + \big\|A^{1/2}u_{\varepsilon\delta} - A^{1/2}l_\delta\big\|_{L^2(0,T;H)}\\ &\qquad\leq C\,\varepsilon^\beta\big(|Au_0| + |A^{1/2}u_1| + |B(u_0)| + |\mathcal{B}(u_0)|^{1/2} + \|f\|_{W^{1,p}(0,T;H)}\big),\end{aligned}$$

*for all $\varepsilon \in (0,\varepsilon_0]$, where $u_{\varepsilon\delta}$ and $l_\delta$ are strong solutions to problems $(\mathrm{P}_{\varepsilon\delta})$ and $(\mathrm{P}_\delta)$ respectively, $\beta = \min\{1/4,\ (p-1)/2p\}$,*

$$\mathbf{m}(T,\delta_0,u_0,u_1,f) = C\big(|A^{1/2}u_0| + |\mathcal{B}(u_0)|^{1/2} + |u_1| + ||f||_{W^{1,p}(0,T;H)}\big).$$

Theorem 4.2. *Let $T > 0$, $\delta \geq \delta_0 > 0$ and $p > 1$. Let us assume that condition* (H) *is fulfilled, the operator $A$ satisfies condition* (HA) *and the operator $B$ satisfies conditions* (HB1)–(HB3). *If $u_0, Au_0, B(u_0), u_1 f(0) \in D(A)$ and $f \in W^{2,p}(0,T;H)$, then there exist constants $C = C(T,p,\delta_0,\omega_0,\omega,L(\mathbf{m}), L_1(\mathbf{m_1}), \|B'(0)\|) > 0$, $\varepsilon_0 = \varepsilon_0(\omega_0,\omega,L(\mathbf{m})), \varepsilon_0 \in (0,1)$, such that*

$$\begin{aligned}&\|u'_{\varepsilon\delta} - l'_\delta + H_{\varepsilon\delta}e^{-\delta^2 t/\varepsilon}\|_{C([0,T];H)} + \big\|A^{1/2}\big(u'_{\varepsilon\delta} - l'_\delta + H_{\varepsilon\delta}e^{-\delta^2 t/\varepsilon}\big)\big\|_{L^2(0,T;H)}\\ &\qquad\leq C\,\varepsilon^\beta\big(|Au_0| + |Au_1| + |\mathcal{B}(u_0)|^{1/2} + |AH_{\varepsilon\delta}| + \|f\|_{W^{2,p}(0,T;H)} + 1\big)^3,\end{aligned}$$

*for all $\varepsilon \in (0,\varepsilon_0]$, where $u_{\varepsilon\delta}$ and $l_\delta$ are strong solutions to problems $(\mathrm{P}_{\varepsilon\delta})$ and $(\mathrm{P}_\delta)$, respectively,*

$$H_{\varepsilon\delta} = \delta^{-1}f(0) - u_1 - \delta^{-1}Au_0 - \delta^{-1}B(u_0), \qquad \beta = \min\{1/4, (p-1)/2p\},$$
$$\mathbf{m}_1 = C(\mathbf{m} + |Au_0|).$$

## 5. Behaviour of solutions to the problem $(\mathrm{P}_{\varepsilon\delta})$, when $\varepsilon \to 0$ and $\delta \to 0$

For the case $\varepsilon \to 0$, $\delta \to 0$ and in the linear case ($B = 0$) in [12] the following theorem was proved.

Theorem 5.1. *Let $T > 0$ and $p > 1$. Let $B = 0$. Let us assume that condition* (H) *is fulfilled, the operator $A$ satisfies condition* (HA). *If $u_0 \in V$, $u_1 \in H$, $f \in W^{1,p}(0,T;H)$, then there exist constants $C = C(p,T,\omega_0,\omega) > 0$ and $\varepsilon_0 = \varepsilon_0(\omega_0,\omega)$, $\varepsilon_0 \in (0,1)$, such that*

$$\|u_{\varepsilon\delta} - v - h_\delta\|_{C([0,T];H)} \le C\big(\big|A^{1/2}u_0\big| + |u_1| + \|f\|_{W^{1,p}(0,T;H)}\big)\Theta(\varepsilon,\delta),$$

*for all $\varepsilon \in (0,\varepsilon_0]$ and for all $\delta \in (0,1]$, where $u_{\varepsilon\delta}$ and $v$ are strong solutions to the problems $(\mathrm{P}_{\varepsilon\delta})$ and $(\mathrm{P}_0)$, respectively,*

$$\Theta(\varepsilon,\delta) = \frac{\varepsilon^\beta}{\delta^{1+1/p}} + \sqrt{\delta}, \qquad \beta = \min\{1/4,\ (p-1)/2p\}.$$

*The function $h_\delta$ is the solution to the problem*

$$\begin{cases} \delta h_\delta'(t) + Ah_\delta(t) = 0, & t \in (0,T), \\ h_\delta(0) = u_0 - A^{-1}f(0), \end{cases}$$

*and*

$$|h_\delta(t)| \le \big|u_0 - A^{-1}f(0)\big|e^{-\delta t/\omega}, \quad t \in [0,T].$$

*If, in adition, $u_1 \in D(A^{1/2})$, then*

$$||u_{\varepsilon\delta} - v - h_\delta||_{C([0,T];H)} \le C\big(|A^{1/2}u_0| + |A^{1/2}u_1| + ||f||_{W^{1,p}(0,T;H)}\big)\Theta_1(\varepsilon,\delta),$$

*for all $\varepsilon \in (0,\varepsilon_0]$ and for all $\delta \in (0,1]$, and*

$$\Theta_1(\varepsilon,\delta) = \frac{\varepsilon^{(p-1)/(2p)}}{\delta^{2+1/p}} + \sqrt{\delta}.$$

The main result of this paper valid for the nonlinear case is presented in the following theorem.

Theorem 5.2. *Let $T > 0$ and $p \ge 2$. Let us assume that condition* (H) *is fulfilled, the operator $A$ satisfies condition* (HA), *the operator $B$ verifies conditions* (HB1), (HB2). *If $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$ and $f \in W^{1,p}(0,T;H) \cap R(A+B)$, then there exist constants $C = C(T,p,\omega_0,\omega,L(\mathbf{m})) > 0$, $C_0 = C_0(T,L(\mathbf{m})) > 0$, $\varepsilon_0 = \varepsilon_0(\omega_0,\omega,L(\mathbf{m}))$, $\varepsilon_0 \in (0,1)$, such that*

$$\begin{aligned} &||u_{\varepsilon\delta} - v - h_\delta||_{C([0,T];H)} \\ &\quad \le C\left(|Au_0| + \big|A^{1/2}u_1\big| + \big|B(u_0)\big| + \big|\mathcal{B}(u_0)\big|^{1/2} + \|f\|_{W^{1,p}(0,T;H)}\right)\Theta(\varepsilon,\delta), \end{aligned} \tag{5.1}$$

*for all* $\varepsilon \in (0, \varepsilon_0]$ *and for all* $\delta \in (0, 1]$, *where* $u_{\varepsilon\delta}$ *and* $v$ *are strong solutions to the problems* $(\mathrm{P}_{\varepsilon\delta})$ *and* $(\mathrm{P}_0)$, *respectively, the function* $h_\delta$ *is the solution to the problem*

$$\begin{cases} \delta h'_\delta(t) + Ah_\delta(t) + B(l_\delta(t)) - B(v(t)) = 0, & t \in (0,T), \\ h_\delta(0) = u_0 - (A+B)^{-1} f(0), \end{cases} \tag{5.2}$$

$$\Theta(\varepsilon, \delta) = \frac{\varepsilon^{1/4}}{\delta^{7/4+1/p}} + \sqrt{\delta}$$

*and*

$$\mathbf{m}(T, u_0, u_1, f) = C\big(\big|A^{1/2}u_0\big| + |\mathcal{B}(u_0)|^{1/2} + |u_1| + ||f||_{W^{1,p}(0,T;H)}\big).$$

Proof. During the proof, we will denote by $C$ all constants depending on $T, p$, $\omega_0, \omega$ and $L(\mathbf{m})$ that may vary from line to line and let

$$\mathcal{M}_1 = |Au_0| + \big|A^{1/2}u_1\big| + |B(u_0)| + |\mathcal{B}(u_0)|^{1/2} + \|f\|_{W^{1,p}(0,T;H)}.$$

Consider the function $f \in W^{1,p}(0,T;H)$. Define on $[0,\infty)$ the function $\widetilde{f}$ as follows:

$$\widetilde{f}(t) = \begin{cases} f(t), & 0 \le t \le T, \\ \dfrac{2T - t}{T} f(T), & T \le t \le 2T, \\ 0, & t \ge 2T, \end{cases}$$

and get

$$\|\widetilde{f}\|_{W^{1,p}(0,\infty;H)} \le C(p,T)||f||_{W^{1,p}(0,T;H)}, \quad C(p,T) = \bigg(\frac{1}{p+1}\bigg)^{1/p} T + 3. \tag{5.3}$$

If we denote by $\widetilde{U}_\mu$ the unique strong solution to the problem $(\mathcal{P}_\mu)$, defined on $(0,\infty)$ instead of $(0,S)$ with $S = T/\delta$ and $\widetilde{f}$ instead of $f$, then, from Lemma 2.5, it follows that $\widetilde{U}_\mu \in W^{2,\infty}_\gamma(0,\infty;H) \cap W^{1,2}_\gamma(0,\infty;V)$, $A^{1/2}\widetilde{U}_\mu \in L^\infty_\gamma(0,\infty;H)$, $A\widetilde{U}_\mu \in L^\infty_\gamma(0,\infty;H)$ with $\gamma = \gamma(\omega_0, \omega, L(\mu))$.

Moreover, the estimate (5.3) implies that

$$\begin{cases} \|\widetilde{F}\|_{L^p(0,\infty;H)} \le C(p,T)\, \delta^{-1/p} \|f\|_{L^p(0,T;H)}, \\ \qquad\qquad \text{for } p \in (1,\infty), \ \text{for all } \delta \in (0,1], \\ \|\widetilde{F}'\|_{L^p(0,\infty;H)} \le C(p,T)\, \delta^{1-1/p} \|f'\|_{L^p(0,T;H)}, \\ \qquad\qquad \text{for } p \in (1,\infty), \ \text{for all } \delta \in (0,1], \\ \|\widetilde{F}\|_{W^{1,p}(0,\infty;H)} \le C(p,T)\, \delta^{-1/p} \|f\|_{W^{1,p}(0,T;H)}, \\ \qquad\qquad \text{for } p \in (1,\infty), \text{for all } \delta \in (0,1]. \end{cases} \tag{5.4}$$

Due to these estimates and Lemma 2.5, the following estimates

$$\big\|A^{1/2}\widetilde{U}_\mu\big\|_{C([0,\,s];H)} + \big\|\widetilde{U}'_\mu\big\|_{L^2(0,\,s;H)} \le C\delta^{-1/p}, \tag{5.5}$$

$$\big\|\widetilde{U}'_\mu\big\|_{C([0,s];H)} + \big\|A^{1/2}\widetilde{U}'_\mu\big\|_{L^2(0,s;H)} + \big\|A\widetilde{U}_\mu\big\|_{C([0,s];H)} \le C\mathcal{M}_1\delta^{-1/p}, \tag{5.6}$$

for all $\delta \in (0,1]$ and all $s \in [0,S]$, are valid.

By Lemma 3.2, the function $W_\mu$, defined by

$$W_\mu(s) = \int_0^\infty K(s,\tau,\mu)\,\widetilde{U}_\mu(\tau)\,d\tau, \tag{5.7}$$

is the strong solution in $H$ to the problem

$$\begin{cases} W'_\mu(s) + AW_\mu(s) = \widetilde{F}_0(s,\mu) \quad \text{for a.e. } s > 0 \text{ in } H,\\ W_\mu(0) = \varphi_\mu, \end{cases} \tag{5.8}$$

for every $\varepsilon \in (0,\varepsilon_0]$, where

$$\begin{aligned} \widetilde{F}_0(s,\mu) &= \delta f_0(s,\mu)u_1 \\ &\qquad + \int_0^\infty K(s,\tau,\mu)\,\widetilde{F}(\tau)\,d\tau - \int_0^\infty K(s,\tau,\mu)\,B(\widetilde{U}_\mu(\tau))\,d\tau,\\ f_0(s,\mu) &= \frac{1}{\sqrt{\pi}}\bigg[2\exp\Big\{\frac{3s}{4\mu}\Big\}\lambda\Big(\sqrt{\frac{s}{\mu}}\Big) - \lambda\Big(\frac{1}{2}\sqrt{\frac{s}{\mu}}\Big)\bigg],\\ \varphi_\mu &= \int_0^\infty e^{-\tau}\,\widetilde{U}_\mu(2\mu\tau)\,d\tau. \end{aligned}$$

Using properties (f), (h), (j) from Lemma 3.1, and (5.5), we obtain that

$$\big\|\widetilde{U}_\mu - W_\mu\big\|_{C([0,\,s];\,H)} \le C\,\mu^{1/4}\,\delta^{-1/p}\sqrt{1+\sqrt{s}} \le C\,\frac{\varepsilon^{1/4}}{\delta^{3/4+1/p}}, \tag{5.9}$$

for all $\varepsilon \in (0,\varepsilon_0]$, for all $\delta \in (0,1]$, for all $s \in [0,S]$.

Denote by $R(s,\mu) = \widetilde{L}(s) - W_\mu(s)$, where $\widetilde{L}$ is the strong solution to the problem $(\mathcal{P}_0)$ with $\widetilde{f}$ instead of $f$, $T = \infty$ and $W_\mu$ is the strong solution of (5.8). Then, due to Theorem 2.4, $R(\,\cdot\,,\mu) \in W^{1,\infty}_\gamma(0,\infty;H)$ and $R$ is the strong solution in $H$ to the problem

$$\begin{cases} R'(s,\mu) + AR(s,\mu) + B\big(\widetilde{L}(s)\big) - B(W_\mu(s)) = \mathcal{F}(s,\mu) \quad \text{for a.e. } t > 0,\\ R(0,\mu) = R_0, \end{cases}$$

where $R_0 = u_0 - W_\mu(0)$ and

$$\begin{aligned} \mathcal{F}(s,\mu) = \widetilde{F}(s) &- \int_0^\infty K(s,\tau,\mu)\widetilde{F}(\tau)\,d\tau - \delta f_0(s,\mu)\,u_1 \\ &- B(W_\mu(s)) + \int_0^\infty K(s,\tau,\mu)\,B(\widetilde{U}_\mu(\tau))\,d\tau. \end{aligned} \tag{5.10}$$

Taking the inner product in $H$ by $R$ and then integrating, we obtain

$$|R(s,\mu)|^2 + 2\int_0^s \big|A^{1/2}R(\xi,\mu)\big|^2\,d\xi + 2\int_0^s \big(B\big(\widetilde{L}(\xi)\big) - B(W_\mu(\xi)), \widetilde{L}(\xi) - W_\mu(\xi)\big)\,d\xi \leq |R(0,\mu)|^2 + 2\int_0^s |\mathcal{F}(\xi,\mu)||R(\xi,\mu)|\,d\xi$$

for all $s \geq 0$. Using the property of monotonicity of the operator $B$, we obtain

$$|R(s,\mu)|^2 + 2\int_0^s \big|A^{1/2}R(\xi,\mu)\big|^2\,d\xi \leq |R(0,\mu)|^2 + 2\int_0^s |\mathcal{F}(\xi,\mu)||R(\xi,\mu)|\,d\xi$$

for all $s \geq 0$. Applying Lemma of Brézis (see, e.g. [7]), we get

$$|R(s,\mu)| + \bigg(\int_0^s \big|A^{1/2}R(\xi,\mu)\big|^2\,d\xi\bigg)^{1/2} \leq |R(0,\mu)| + \int_0^s |\mathcal{F}(\xi,\mu)|\,d\xi, \tag{5.11}$$

for all $s \geq 0$. Using (5.5), we obtain

$$|R_0| \leq \int_0^\infty e^{-\tau}\big|\widetilde{U}_\mu(2\mu\tau) - u_0\big|\,d\tau \leq \int_0^\infty e^{-\tau}\int_0^{2\mu\tau}\big|\widetilde{U}_\mu'(\xi)\big|\,d\xi\,d\tau \leq C\,\mu^{1/2}\,\delta^{-1/p} = C\,\frac{\varepsilon^{1/2}}{\delta^{1+1/p}}, \tag{5.12}$$

for all $\varepsilon \in (0,\varepsilon_0]$ and all $\delta \in (0,1]$. In what follows we will estimate $\big|\mathcal{F}(t,\varepsilon)\big|$. Using the property (j) from Lemma 3.1, (5.3) and (5.4), we have

$$\bigg|\widetilde{F}(s) - \int_0^\infty K(s,\tau,\mu)\widetilde{F}(\tau)\,d\tau\bigg| \leq C(T)\big\|\widetilde{F}'\big\|_{L^p(0,\infty;\,H)}(1+\sqrt{s})^{1-1/p}\mu^{1/2-1/2p} \leq C\,\frac{\varepsilon^{1/2-1/2p}}{\delta^{1/2-1/2p}}, \tag{5.13}$$

for $\varepsilon \in (0,\varepsilon_0]$, all $\delta \in (0,1]$ and all $s \in [0,S]$. Since $e^{\xi}\lambda(\sqrt{\xi}) \leq C$, for all $\xi \geq 0$, the estimates

$$\int_0^s \exp\Big\{\frac{3\xi}{4\mu}\Big\}\lambda\bigg(\sqrt{\frac{\xi}{\mu}}\bigg)\,d\xi \leq C\mu\int_0^\infty e^{-\xi/4}\,d\xi \leq C\mu, \quad \text{for all } s \geq 0,$$

$$\int_0^s \lambda\bigg(\frac{1}{2}\sqrt{\frac{\xi}{\mu}}\bigg)\,d\xi \leq \mu\int_0^\infty \lambda\bigg(\frac{1}{2}\sqrt{\xi}\bigg)\,d\xi \leq C\mu, \quad \text{for all } s \geq 0,$$

hold. Then

$$\bigg|\delta\int_0^s f_0(\xi,\mu)\,u_1\,d\xi\bigg| \leq C\,\delta\mu|u_1| \leq C\,\frac{\varepsilon}{\delta}, \tag{5.14}$$

for all $\varepsilon \in (0,\varepsilon_0]$, all $\delta \in (0,1]$ and all $s \geq 0$. In what follows we will estimate the difference

$$I(s,\varepsilon) = \int_0^\infty K(s,\tau,\mu)B(\widetilde{U}_\mu(\tau))\,d\tau - B(W_\mu(s)) = I_1(s,\varepsilon) + I_2(s,\varepsilon), \tag{5.15}$$

where, due to the property (h) from Lemma 3.1, we have

$$I_1(s,\varepsilon) = \int_0^\infty K(s,\tau,\mu)\big(B\big(\widetilde{U}_\mu(\tau)\big) - B\big(W_\mu(\tau)\big)\big)\,d\tau,$$

$$I_2(s,\varepsilon) = \int_0^\infty K(s,\tau,\mu)\Big(B(W_\mu(\tau)) - B(W_\mu(s))\Big)\,d\tau.$$

Using properties (f), (h), (i), from Lemma 3.1, condition (HB1), (5.6) and (5.7), for $I_1(s,\varepsilon)$ we deduce the following estimates

$$\begin{aligned}(5.16)\qquad &\big|A^{1/2}\widetilde{U}_\mu(s) - A^{1/2}W_\mu(s)\big|\\ &\le \int_0^\infty K(s,\tau,\mu)\,|s-\tau|^{1/2}\bigg|\int_\tau^s |A^{1/2}\widetilde{U}_\mu'(\xi)|^2\,d\xi\bigg|^{1/2} d\tau\\ &\le C\mu^{1/4}(1+\sqrt{s})^{1/2}\,\big\|A^{1/2}\widetilde{U}_\mu'\big\|_{L^2(0,s;H)}C\mathcal{M}_1\,\frac{\varepsilon^{1/4}}{\delta^{3/4+1/p}},\end{aligned}$$

for $\varepsilon\in(0,\varepsilon_0]$, all $\delta\in(0,1]$ and all $s\ge 0$,

$$\begin{aligned}(5.17)\quad |I_1(s,\varepsilon)| &\le L(\mathbf{m})\int_0^\infty K(s,\tau,\mu)\big|A^{1/2}\widetilde{U}_\mu(\tau) - A^{1/2}W_\mu(\tau)\big|\,d\tau\\ &\le C\mathcal{M}_1\,\frac{\varepsilon^{1/4}}{\delta^{3/4+1/p}},\end{aligned}$$

for $\varepsilon\in(0,\varepsilon_0]$, all $\delta\in(0,1]$ and all $s\ge 0$,

$$\begin{aligned}(5.18)\qquad \big|B(W_\mu(s)) - B(W_\mu(\tau))\big| &\le L(\mathbf{m})\,\big|A^{1/2}W_\mu(s) - A^{1/2}W_\mu(\tau)\big|\\ &\le L(\mathbf{m})\big|A^{1/2}W_\mu(s) - A^{1/2}\widetilde{U}_\mu(s)\big|\\ &\quad + L(\mathbf{m})\,\big|A^{1/2}\widetilde{U}_\mu(\tau) - A^{1/2}W_\mu(\tau)\big|\\ &\quad + L(\mathbf{m})\,\big|A^{1/2}\widetilde{U}_\mu(s) - A^{1/2}\widetilde{U}_\mu(\tau)\big|\\ &\le C\,\mathcal{M}_1\,\frac{\varepsilon^{1/4}}{\delta^{3/4+1/p}} + L(\mathbf{m})\,\bigg|\int_\tau^s |A^{1/2}\widetilde{U}_\mu'(\xi)|\,d\xi\bigg|,\end{aligned}$$

for $\varepsilon\in(0,\varepsilon_0]$, all $\delta\in(0,1]$ and all $s,\tau\ge 0$. Using the last estimate, (5.6) and properties (h), (i) from Lemma 3.1, for $I_2(t,\varepsilon)$ we get the estimate

$$\begin{aligned}(5.19)\quad |I_2(t,\varepsilon)| &\le C\,\mathcal{M}_1\,\frac{\varepsilon^{1/4}}{\delta^{3/4+1/p}}\\ &\quad + L(\mathbf{m})\int_0^\infty K(s,\tau,\mu)\,|s-\tau|^{1/2}\bigg|\int_\tau^s |A^{1/2}\widetilde{U}_\mu'(\xi)|^2\,d\xi\bigg|^{1/2} d\tau\\ &\le C\mathcal{M}_1\,\frac{\varepsilon^{1/4}}{\delta^{3/4+1/p}},\end{aligned}$$

for $\varepsilon\in(0,\varepsilon_0]$, all $\delta\in(0,1]$ and all $s\ge 0$. From (5.15), using (5.16) and (5.19), for $I(t,\varepsilon)$ we get the estimate

$$(5.20)\qquad |I(t,\varepsilon)| \le C\mathcal{M}_1\,\frac{\varepsilon^{1/4}}{\delta^{3/4+1/p}},\quad \varepsilon\in(0,\varepsilon_0],$$

for all $\delta \in (0,1]$, all $s \geq 0$ and $p \geq 2$. Using (5.13), (5.14) and (5.20), from (5.10) we obtain

$$(5.21)\qquad \int_0^s \big|\mathcal{F}(\tau,\varepsilon)\big|\,d\tau \leq C\,\mathcal{M}_1\,\frac{\varepsilon^{1/4}}{\delta^{7/4+1/p}},\quad \text{for all } \varepsilon \in (0,\varepsilon_0 \text{ and all } s \in [0,S].$$

From (5.11), using (5.12) and (5.21) we get the estimate

$$(5.22)\qquad \|R\|_{C([0,\,s];H)} + \big\|A_0^{1/2}R\big\|_{L^2(0,s;H)} \leq C\,\mathcal{M}_1\,\frac{\varepsilon^{1/4}}{\delta^{7/4+1/p}},$$

for all $\varepsilon \in (0,\varepsilon_0]$, all $\delta \in (0,1]$ and all $s \in [0,S]$. Consequently, from (5.9) and (5.22), we deduce

$$(5.23)\qquad \big\|\widetilde{U}_\mu - \widetilde{L}\big\|_{C([0,s];H)} \leq \big\|\widetilde{U}_\mu - W_\mu\big\|_{C([0,s];H)} + ||R||_{C([0,s];H)} \leq C\,\mathcal{M}_1\,\frac{\varepsilon^{1/4}}{\delta^{7/4+1/p}},$$

for all $\varepsilon \in (0,\varepsilon_0]$, all $\delta \in (0,1]$ and all $s \in [0,S]$. Since $U_\mu(s) = \widetilde{U}_\mu(s)$, $L(s) = \widetilde{L}(s)$, for all $s \in [0,S]$, $U_\mu(s) = u_{\varepsilon\delta}(\delta\, s)$ and $L(s) = l_\delta(\delta\, s)$, from (5.23) we get

$$(5.24)\qquad \|u_{\varepsilon\delta} - l_\delta\|_{C([0,T];H)} \leq C\,\mathcal{M}_1\,\frac{\varepsilon^{1/4}}{\delta^{7/4+1/p}},$$

for all $\varepsilon \in (0,\varepsilon_0]$, all $\delta \in (0,1]$ and for $p \geq 2$.

In what follows, let us denote by $R_1(t,\delta) = l_\delta(t) - v(t) - h_\delta(t)$, where $l_\delta$ is the solution to the problem $(\mathrm{P}_\delta)$, $v$ is the solution to the problem $(\mathrm{P}_0)$ and $h_\delta$ is the solution to the problem (5.2). In this case we deduce that $R_1$ is the solution to the system

$$\begin{cases} \delta R_1'(t,\delta) + AR_1(t,\delta) = -\delta v'(t), & t \in (0,T), \\ R_1(0,\delta) = 0. \end{cases}$$

Taking the inner product in (H) by $R_1$ and integrating on $(0,t)$ we obtain

$$\delta|R_1(t,\delta)|^2 + 2\int_0^t \big(AR_1(\tau,\delta), R_1(\tau,\delta)\big)\,d\tau = -2\delta\int_0^t \big(v_\delta'(\tau), R_1(\tau,\delta)\big)d\tau,\quad \text{for } t \in (0,T).$$

Using condition (HA) and (2.5), we get

$$\delta|R_1(t,\delta)|^2 + 2\int_0^t \big(AR_1(\tau,\delta), R_1(\tau,\delta)\big)\,d\tau \leq \frac{\delta^2}{\omega}\int_0^t |v_\delta'(\tau|^2\,d\tau + \int_0^t \big(AR_1(\tau,\delta), R_1(\tau,\delta)\big)\,d\tau,$$

for $t \in (0,T)$, and consequently

$$(5.25)\qquad |R_1(t,\delta)| \leq \sqrt{\delta}C\mathcal{M}_1,\quad \text{for } t \in (0,T),\ \delta \in (0,1].$$

Thus, the estimate (5.1) is a simple consequence of (5.24) and (5.25). □

## 6. Example

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with $C^1$ boundary $\partial\Omega$. In the real Hilbert space $L^2(\Omega)$ we consider the following boundary-value problem:

$$\begin{cases} \varepsilon\, \partial_t^2 u_{\varepsilon\delta} + \delta\, \partial_t u_{\varepsilon\delta} + A u_{\varepsilon\delta} + b|u_{\varepsilon\delta}|^q u_{\varepsilon\delta} = f(x,t), & (x,t) \in \Omega \times (0,T), \\ u_{\varepsilon\delta}(x,0) = u_0(x), \quad \partial_t\, u_{\varepsilon\delta}(x,0) = u_1(x), & x \in \overline{\Omega}, \\ u_{\varepsilon\delta}\big|_{\partial\Omega} = 0, & t \in [0,T), \end{cases} \tag{6.1}$$

where $\varepsilon$ and $\delta$ are small positive parameters, $u_{\varepsilon\delta}, f\colon [0,T) \to L^2(\Omega)$ and the operator $A$ is defined as follows:

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega),$$

$$Au(x) = -\sum_{i,j=1}^{n} \partial_{x_i}\big(a_{ij}(x)\partial_{x_j} u(x)\big) + a(x)\, u(x), \quad u \in D(A),$$

$$a_{ij} \in C^1(\overline{\Omega}), \quad a \in C(\overline{\Omega}), \quad a(x) \ge 0, \quad a_{ij}(x) = a_{ji}(x), \quad x \in \overline{\Omega}, \tag{6.2}$$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\,\xi_j \ge a_0|\xi|^2, \quad x \in \overline{\Omega}, \quad \xi = (\xi_i)_{i=1}^n \in \mathbb{R}^n, \quad a_0 > 0. \tag{6.3}$$

If we consider the operator $B$ defined as

$$D(B) = L^2(\Omega) \cap L^{2(q+1)}(\Omega), \quad Bu = b|u|^q u,$$

then, for $b > 0$ the operator $B$ is a Fréchet derivative of the convex and positive functional $\mathcal{B}$, which is defined as follows

$$D(\mathcal{B}) = L^{q+2}(\Omega) \cap L^2(\Omega), \qquad \mathcal{B}u = \frac{b}{q+2}\int_\Omega |u(x)|^{q+2}\, dx,$$

and the Fréchet's derivative of operator $B$ is defined by the relationships

$$D\big(B'(u)\big) = \{v \in L^2(\Omega) : u^q v \in L^2(\Omega)\}, \qquad B'(u)v = b(q+1)|u|^q v.$$

For $b > 0$ and

$$\begin{cases} q \in [0, 2/(n-2)] & \text{if } n > 2, \\ q \in [0, \infty) & \text{if } n = 1, 2, \end{cases} \tag{6.4}$$

the operator $B$ verifies condition (HB1). For $b > 0$ and

$$\begin{cases} q \in [1, 2/(n-2)] & \text{if } \ n > 2, \\ q \in [1, \infty) & \text{if } n = 1, 2, \end{cases} \tag{6.5}$$

the operator $B$ verifies conditions (HB3). In this case the corresponding unperturbed problems are:

$$
\begin{cases} \delta\, \partial_t l_\delta + A l_\delta + b|l_\delta|^q l_\delta = f(x,t), & (x,t) \in \Omega \times (0,T), \\ l_\delta(x,0) = u_0(x), & x \in \overline{\Omega}, \\ l_\delta\big|_{\partial\Omega} = 0, & t \in [0,T), \end{cases} \tag{6.6}
$$

$$
\begin{cases} Av + b|v|^q v = f(x,t), & (x,t) \in \Omega \times [0,T), \\ v\big|_{\partial\Omega} = 0, & t \in [0,T). \end{cases} \tag{6.7}
$$

From Theorems 4.1, 4.2 and 5.2, we obtain the following theorems.

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with $C^1$ boundary $\partial\Omega$. Let $T > 0$, $\delta \geq \delta_0 > 0$, $p > 1$, $b > 0$, $q$ verifies* (6.4) *and* (6.2)–(6.3) *are fulfilled. If $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$ and $f \in W^{1,p}(0,T; L^2(\Omega))$, then there exist constants $C = C(T, p, \delta_0, \omega_0, \omega, n, q, b, \Omega, \mathbf{m})) > 0$, $\varepsilon_0 = \varepsilon_0(\omega_0, \omega, n, q, b, \Omega, \mathbf{m})$, $\varepsilon_0 \in (0,1)$, such that*

$$
\|u_{\varepsilon\delta} - l_\delta\|_{C([0,T];L^2(\Omega))} \leq C\,\varepsilon^\beta \big(\|u_0\|_{H^2(\Omega)} + \|u_1\|_{H_0^1(\Omega)} + \|f\|_{W^{1,\,p}(0,\,T;,L^2(\Omega))}\big),
$$

*for all $\varepsilon \in (0, \varepsilon_0]$, where $u_{\varepsilon\delta}$ and $l_\delta$ are strong solutions to problems* (6.1) *and* (6.6)*, respectively $\beta = \min\{1/4,\ (p-1)/2p\}$,*

$$
\mathbf{m} = C\big(\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|f\|_{W^{1,\,p}(0,T;L^2(\Omega))}\big).
$$

Theorem 6.2. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with $C^1$ boundary $\partial\Omega$. Let $T > 0$, $\delta \geq \delta_0 > 0$, $p > 1$, $b > 0$, $q$ verifies* (6.4)–(6.5) *and* (6.2)–(6.3) *are fulfilled. If $u_0, \Delta u_0, |u_0|^{q+1}, u_1, f(0) \in H^2(\Omega) \cap H_0^1(\Omega)$ and $f \in W^{2,p}(0,T; L^2(\Omega))$, then there exist constants $C = C(T, p, \delta_0, \omega_0, \omega, n, q, b, \Omega, \mathbf{m}, \mathbf{m_1}) > 0$, $\varepsilon_0 = \varepsilon_0(\omega_0, \omega, n, q, b, \Omega, \mathbf{m})$, $\varepsilon_0 \in (0,1)$, such that*

$$
\begin{aligned}
&\|u'_{\varepsilon\delta} - l'_\delta + H_{\varepsilon\delta} e^{-\delta^2 t/\varepsilon}\|_{C([0,\,T]\,;L^2(\Omega))} \\
&\quad \leq C\varepsilon^\beta \big(\|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^2(\Omega)} + \|H_{\varepsilon\delta}\|_{H^2(\Omega)} + \|f\|_{W^{2,\,p}(0,T;L^2(\Omega))} + 1\big)^3,
\end{aligned}
$$

*for all $\varepsilon \in (0, \varepsilon_0]$, where $u_{\varepsilon\delta}$ and $l_\delta$ are strong solutions to problems* (6.1) *and* (6.6)*, respectively,*

$$
H_{\varepsilon\delta} = \delta^{-1} f(0) - u_1 - \delta^{-1} A u_0 - \delta^{-1} b|u_0|^q u_0, \quad \beta = \min\{1/4, (p-1)/2p\},
$$
$$
\mathbf{m_1} = C\big(\mathbf{m} + \|u_0\|_{H^2(\Omega)}\big).
$$

Theorem 6.3. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with $C^1$ boundary $\partial\Omega$. Let $T > 0$, $p \geq 2$, $b > 0$, $q$ verifies* (6.4) *and* (6.2)–(6.3) *are fulfilled. If $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$ and $f \in W^{1,p}(0,T; L^2(\Omega)) \cap L^{q+1}(\Omega)$, then there* *exist constants* $C = C(T, p, \omega_0, \omega, n, q, b, \Omega, \mathbf{m}) > 0$*,* $C_0 = C_0(T, L(\mathbf{m})) > 0$*,* $\varepsilon_0 = \varepsilon_0(\omega_0, \omega, n, q, b, \Omega, \mathbf{m}))$*,* $\varepsilon_0 \in (0,1)$*, such that*

$$\|u_{\varepsilon\delta} - v - h_\delta\|_{C([0,T];H)} \\ \leq C\big(\|u_0\|_{H^2(\Omega)} + \|u_1\|_{H_0^1(\Omega)} + \|f\|_{W^{1,p}(0,T;L^2(\Omega))}\big)\,\Theta(\varepsilon,\delta),$$

*for all* $\varepsilon \in (0, \varepsilon_0]$ *and all* $\delta \in (0,1]$*, where* $u_{\varepsilon\delta}$ *and* $v$ *are strong solutions to the problems* (6.1) *and* (6.7)*, respectively, the function* $h_\delta$ *is the solution to the problem*

$$\begin{cases} \delta h_\delta'(t) + Ah_\delta(t) + B(l_\delta(t)) - B\big(v(t)\big) = 0, \quad t \in (0,T), \\ h_\delta(0) = u_0 - (A+B)^{-1} f(0), \end{cases}$$

$$\Theta(\varepsilon,\delta) = \frac{\varepsilon^{1/4}}{\delta^{7/4+1/p}} + \sqrt{\delta}.$$

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