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TOPOLOGY OF TWISTS, EXTREMISING TWIST PATHS AND MULTIPLE SOLUTIONS TO THE NONLINEAR SYSTEM IN VARIATION $\mathscr{L}[u] = \nabla\mathscr{P}$

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Abstract. In this paper we address questions on the existence and multiplicity of a class of geometrically motivated mappings with certain symmetries that serve as solutions to the nonlinear system in variation:

$$\mathrm{ELS}[(u,\mathscr{P}),\Omega] = \begin{cases} [\nabla u]^t \mathrm{div}[F_\xi \nabla u] - F_s[\nabla u]^t u = \nabla\mathscr{P} & \text{in } \Omega, \\ \det \nabla u = 1 & \text{in } \Omega, \\ u \equiv x & \text{on } \partial\Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain, $F = F(r, s, \xi)$ is a sufficiently smooth Lagrangian, $F_s = F_s(|x|, |u|^2, |\nabla u|^2)$ and $F_\xi = F_\xi(|x|, |u|^2, |\nabla u|^2)$ with F_s and F_ξ denoting the derivatives of F with respect to the second and third variables respectively while $\mathscr{P}$ is an *a priori* unknown hydrostatic pressure resulting from the incompressibility constraint $\det \nabla u = 1$. Among other things, by considering twist mappings u with an $\mathrm{SO}(n)$-valued twist path, we prove the existence of multiple and topologically distinct solutions to ELS for $n \geq 2$ even versus the only (*non*) twisting solution $u \equiv x$ for $n \geq 3$ odd. An extremality analysis for twist paths and those of Lie exponential types and a suitable formulation of a differential operator action on twists relating to ELS are the key ingredients in the proof.

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1. Introduction

The space of continuous self-mappings of a bounded smooth domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) agreeing with the identity on the boundary $\partial\Omega$ has a complex structure and often rich topology. In this paper we consider a nonlinear system in divergence and variational form in a not so non-typical domain geometry that admits a multitude of solutions in the form of such self-mappings. These solutions are rotationally symmetric, whilst the number and form of them exhibit a sharp and stark contrast depending on the parity of the spatial dimension n, specifically, in being even versus odd. Towards this end consider the variational energy integral

$$\mathbb{F}[u,\Omega] = \int_\Omega \mathscr{F}(x,u,\nabla u)\,dx \tag{1.1}$$

where $\mathscr{F} = \mathscr{F}(x,u,\zeta)$ with $(x,u,\zeta) \in \Omega \times \mathbb{R}^n \times \mathbb{M}^{n\times n}$ is a sufficiently regular Lagrangian and the competing mappings $u = (u_1,\ldots,u_n)$ are confined to the space $\mathscr{A}_p = \mathscr{A}_p(\Omega)$ of admissible weakly differentiable *incompressible* Sobolev mappings defined by

$$\mathscr{A}_p(\Omega) := \{u \in W^{1,p}(\Omega,\mathbb{R}^n) : \det\nabla u = 1 \text{ a.e. in } \Omega,\ u = \varphi \text{ on } \partial\Omega\}, \tag{1.2}$$

for a suitable choice of Sobolev exponent $1 \leq p < \infty$. A good motivating source for considering such energies and classes of mappings comes from the nonlinear theory of elasticity where the pair (1.1)–(1.2) together describe a mathematical model for an incompressible hyperelastic material subject to pure displacement boundary conditions with the resulting extremisers, equivalently critical points or solutions to the associated Euler–Lagrange system, and minimisers serving as the equilibrium states and physically stable displacement fields. (For more see [1], [3], [8], [9] and for other motivations see [2], [15], [16], [24], [26].)

The mapping $\varphi \in \mathscr{C}(\partial\Omega,\mathbb{R}^n)$ describing the boundary displacement in (1.2) is taken throughout to be $\varphi \equiv x$ where the last condition in (1.2) then asserts that u agrees with the identity on $\partial\Omega$ in the sense of traces. Furthermore ∇u here denotes the gradient of u, an $n\times n$ matrix-field in Ω, with $\det\nabla u$ denoting its Jacobian determinant. The Euler–Lagrange system (ELS) associated with the variational energy integral (1.1) over the space $\mathscr{A}_p(\Omega)$ is given by ([1])

$$\mathrm{ELS}[(u,\mathscr{P}),\Omega] = \begin{cases} \mathscr{L}[u;\mathscr{F}] = \nabla\mathscr{P} & \text{in } \Omega, \\ \det\nabla u = 1 & \text{in } \Omega, \\ u \equiv \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\mathscr{P} = \mathscr{P}(x)$ is an unknown hydrostatic pressure field corresponding to the pointwise constraint $\det\nabla u = 1$ and the differential operator $\mathscr{L} = \mathscr{L}[u;\mathscr{F}]$

([1]) See the last section for a brief derivation of this system.

takes the explicit form

$$\mathscr{L}[u;\mathscr{F}] = \frac{1}{2}[\nabla u]^t\{\operatorname{div}[\mathscr{F}_\zeta(x,u,\nabla u)] - \mathscr{F}_u(x,u,\nabla u)\}. \tag{1.4}$$

Referring to (1.4) we also point out that the divergence operator "div" in the first term on the right acts row-wise on the matrix field $\mathscr{F}_\zeta(x,u,\nabla u)$ and $[\nabla u]^t$ denotes the transpose of the matrix $[\nabla u]$. For the sake of clarity let us also note that by a (*classical*) solution we hereafter mean a pair $(u,\mathscr{P})$ with u of class $\mathscr{C}(\overline{\Omega},\mathbb{R}^n)\cap\mathscr{C}^2(\Omega,\mathbb{R}^n)$ and $\mathscr{P}$ of class $\mathscr{C}(\overline{\Omega})\cap\mathscr{C}^1(\Omega)$ such that (1.3) holds in a pointwise sense in Ω.

It is known that when $\mathscr{F}=\mathscr{F}(\nabla u)$ and $\Omega\subset\mathbb{R}^n$ is *star-shaped*, then subject to the natural convexity requirements for the application of the direct methods of the calculus of variations, i.e., quasiconvexity of $\mathscr{F}$ everywhere (see [22] or [3], [9]) any solution u to (1.3) is globally minimising: $\mathbb{F}[u,\Omega]=\mathbb{F}[x,\Omega]$ and so subject to the *strict* quasiconvexity of $\mathscr{F}$ at $\zeta=\mathrm{I}$ the only solution to (1.3) is $u\equiv x$ (see [20], [29]). The latter uniqueness raises the question as to how different the situation would be for non star-shaped domains or more generally domains with a non-trivial topology? Are there multiple solutions in such cases?

Now, in order to address this question more profoundly and highlight the role of domain topology, let us proceed by introducing

$$\mathscr{C}_\varphi(\Omega)=\{v\in\mathscr{C}(\overline{\Omega},\overline{\Omega}) : v=\varphi \text{ on } \partial\Omega\},$$

the space of continuous self-mappings of $\overline{\Omega}$ onto itself agreeing with the identity on $\partial\Omega$. The significance of this space for us comes from the embedding $\mathscr{A}_p(\Omega)\subset\mathscr{C}_\varphi(\Omega)$ when $p\geq n$: every u in $\mathscr{A}_p(\Omega)$ has a (precise) representative $u^\star$ in $\mathscr{C}_\varphi(\Omega)$. Now depending on the topology of the domain Ω the space $\mathscr{C}_\varphi(\Omega)$ can have a fairly complex and rich topology itself (see below). Indeed let

$$\pi_0=\{[f] : f\in\mathscr{C}_\varphi(\Omega)\}$$

denote the set of all homotopy classes (or equivalently path-connected components) of the space $\mathscr{C}_\varphi(\Omega)$ taken in the uniform metric. Then considering inverse images gives the decomposition

$$\mathscr{A}_p(\Omega)=\bigcup_{\gamma\in\pi_0}\mathfrak{A}_p^\gamma, \qquad \mathfrak{A}_p^\gamma=\{u\in\mathscr{A}_p(\Omega):\gamma=[u^\star]\}. \tag{1.5}$$

When Ω is homeomorphic to an n-ball then $\mathscr{C}_\varphi(\Omega)$ is easily seen to be connected and so π_0 here is a singleton. On the other hand when Ω is homeomorphic to the product $\mathbb{B}^l\times\mathbb{S}^m$ (with integers $l,m\geq 1$ and $n=m+l$) then (cf. [31], [33])

$$\pi_0=\{[u] : u\in\mathscr{C}_\varphi(\Omega)\}\cong\pi_l[\mathscr{C}(\mathbb{S}^m,\mathbb{S}^m;\deg=+1)], \tag{1.6}$$

where for each $d\in\mathbb{Z}$, $\mathscr{C}(\mathbb{S}^m,\mathbb{S}^m;\deg=d)$ represents the component of the space $\mathscr{C}(\mathbb{S}^m,\mathbb{S}^m)$ containing mappings with Hopf degree d. Although a description of the homotopy groups on the right in (1.6) and the homotopy types of the components $\mathscr{C}(\mathbb{S}^m, \mathbb{S}^m; \deg = d)$ is an outstanding and highly technical problem in topology, one can obtain a good collection of results (for $l, m \geq 1$) that suitably relate to the problem at hand here (see [13], [21], [37], [38] and [31], [33], [39] for more). Indeed, for $m = 1, 3$ and 7, due to $\mathbb{S}^m$ being a Lie or H group (here $\mathbb{S}^m$ can be identified with the group of unit vectors in $\mathbb{C}, \mathbb{H}$ and $\mathbb{O}$, respectively) it follows that the components $\mathscr{C}(\mathbb{S}^m, \mathbb{S}^m; \deg = d)$ have the same homotopy type and so in particular with $d = 0$ and $d = 1$ we get the isomorphisms (^{2})

$$\begin{aligned}\pi_l[\mathscr{C}(\mathbb{S}^m, \mathbb{S}^m; \deg = +1)] &\cong \pi_l[\mathscr{C}(\mathbb{S}^m, \mathbb{S}^m; \deg = 0)] \\ &\cong \pi_l(\mathbb{S}^m) \oplus \pi_{l+m}(\mathbb{S}^m).\end{aligned} \tag{1.7}$$

Interestingly (1.7) remains true for $1 \leq l < m-1$ (the so-called stable range) as can be seen by considering the long exact sequence of the evaluation fibration and taking note of the vanishing of certain homotopy groups along the sequence (see [31], [33]). In particular for the given range $\pi_l(\mathbb{S}^m) \cong 0$ and therefore (1.7) gives $\pi_l[\mathscr{C}(\mathbb{S}^m, \mathbb{S}^m; \deg = +1)] \cong \pi_{l+m}(\mathbb{S}^m)$.

As an important case, if Ω is an n-annulus, then with $l = 1$, $m = n-1$, the above in conjunction with $\pi_1[\mathscr{C}(\mathbb{S}^2, \mathbb{S}^2; \deg = +1)] \cong \pi_1[\mathrm{SO}(3)]$ give (^{3})

$$\begin{aligned}\pi_0 = \{[u] : u \in \mathscr{C}_\varphi(\Omega)\} &\cong \pi_1[\mathscr{C}(\mathbb{S}^m, \mathbb{S}^m); \deg = +1] \\ &\cong \pi_1[\mathrm{SO}(m+1)] \cong \begin{cases} \mathbb{Z} & \text{if } m = 1, \\ \mathbb{Z}_2 & \text{if } m \geq 2. \end{cases}\end{aligned} \tag{1.8}$$

Returning to (1.5) it follows that when $\mathbb{F}$ is bounded from below, sequentially weakly lower semicontinuous and coercive on $W^{1,p}$ (with $p \geq n$) then it admits a minimiser in each $\mathfrak{A}_p^\gamma$ with $\gamma \in \pi_0$. These minimisers are strong local minimisers of $\mathbb{F}$ in that for each such u there exists $\delta = \delta[u] > 0$ such that $\mathbb{F}[u] \leq \mathbb{F}[v]$ for all $v \in \mathscr{A}_p(\Omega)$ satisfying $||u - v||_{L^1} \leq \delta$ (see [31]–[33]). In the case of an n-annulus, by recalling (1.8), this gives the existence of an infinitude of strong local minimisers when $n = 2$ and at least two when $n \geq 3$.

For the purpose of this paper we confine to $\mathscr{F}(x, u, \zeta) = F(r, |u|^2, |\zeta|^2)$ where $F = F(r, s, \xi)$ is a twice continuously differentiable Lagrangian. In this case the variational integral (1.1) (with $|\nabla u|^2 = \mathrm{tr}\{[\nabla u]^t[\nabla u]\} = \mathrm{tr}\{[\nabla u][\nabla u]^t\}$) is

$$\mathbb{F}[u, \Omega] := \int_\Omega F\big(|x|, |u|^2, |\nabla u|^2\big)\, dx. \tag{1.9}$$

Here $\mathscr{F}_\zeta(x, u, \nabla u) = 2F_\xi\big(r, |u|^2, |\nabla u|^2\big)\nabla u$, $\mathscr{F}_u(x, u, \nabla u) = 2F_s\big(r, |u|^2, |\nabla u|^2\big)u$ where F_s and F_ξ denote the derivatives of the Lagrangian F with respect to

(^{2}) The second equivalence in (1.7) is a consequence of the existence of a cross section for the Serre fibration resulting from evaluation at base point: $p \colon \mathscr{C}(\mathbb{S}^m, \mathbb{S}^m; \deg = 0) \to \mathbb{S}^m$ with $p[f] = f(e)$. Here $e \in \mathbb{S}^m$ is the base point and $f \in \mathscr{C}(\mathbb{S}^m, \mathbb{S}^m; \deg = 0)$.

(^{3}) Note incidentally that the latter shows that the first isomorphism in (1.7) does *not* hold for $l = 1$, $m = 2$ as here $\pi_1[\mathscr{C}(\mathbb{S}^2, \mathbb{S}^2; \deg = 0)] \cong \pi_3(\mathbb{S}^2) \cong \mathbb{Z}$. (See also [14], [38], [40].)

the second and third variables respectively. As a result with $\mathscr{L}[u] = \mathscr{L}[u;F]$ the Euler–Lagrange operator (1.4) becomes (4)

$$\mathscr{L}[u] := [\nabla u]^t\big\{\operatorname{div}\big[F_\xi(|x|,|u|^2,|\nabla u|^2)\nabla u\big] - F_s(|x|,|u|^2,|\nabla u|^2)u\big\}. \tag{1.10}$$

We consider the geometric setup in which the domain Ω is a finite n-annulus, for definiteness, $\Omega = \mathbb{X}^n[a,b] := \{x \in \mathbb{R}^n : a < |x| < b\}$ with $0 < a < b < \infty$, and seek multiple solutions for the nonlinear system (1.3)–(1.10) amongst certain classes of self-mappings of the n-annulus onto itself in the form $u = \mathrm{Q}(|x|)v(x)$ for suitable $v \in \mathscr{C}\big(\overline{\mathbb{X}}^n, \overline{\mathbb{X}}^n\big)$ and $\mathrm{Q} \in \mathscr{C}([a,b], \mathrm{SO}(n))$. By a twist mapping (or a twist for brevity) we understand a mapping u of the latter form with $v \equiv x$. Thus a twist in spherical-polar coordinates is represented as $u\colon (r,\theta) \mapsto (r, \mathrm{Q}(r)\theta)$ with $a \le r = |x| \le b$, $\theta = x|x|^{-1}$. The matrix-valued curve $\mathrm{Q} \in \mathscr{C}([a,b], \mathrm{SO}(n))$ here is called the twist *path* (or when $\mathrm{Q}(a) = \mathrm{Q}(b)$ the twist *loop*) associated with the twist u. As can be seen a twist is a homeomorphism whose inverse is again a twist, specifically, if $u = r\mathrm{Q}(r)\theta$ then $u^{-1} = r\mathrm{Q}^{-1}(r)\theta$. Furthermore, subject to the differentiability of the twist path, a twist is incompressible as well as measure-preserving.

In surface topology the significance of twists (also known as *Dehn* twists) and their role as generators of the mapping class group of Riemannan surfaces has a long and rich history ([6]). More recently in geometric analysis and PDEs these two dimensional twists and their higher dimensional counterparts (as above) have proven highly useful in establishing the existence of multiple solutions and multiple equilibria of different topological types (see [17], [18], [25], [31]–[34] as well as [10], [11], [23], [24], [28], [27]).

One of the main conclusions of the paper is that subject to suitable convexity and monotonicity assumptions on F the pair (1.3)–(1.10) has an infinite family of twist solutions in all even dimensions whilst in odd dimensions this is generally only one, specifically, the non-twisting trivial identity mapping. An example that nicely illustrates this contrast in the behaviour of (1.3)–(1.10) and its twist solutions is $F = h(r,s)\xi$ where h of class $\mathscr{C}^2$ is strictly positive and

$$\begin{aligned} \mathscr{L}[u] &= [\nabla u]^t\big\{\operatorname{div}\big[h(r,|u|^2)\nabla u\big] - h_s(r,|u|^2)|\nabla u|^2 u\} \\ &= [\nabla u]^t\big\{\nabla u\nabla\big[h(r,|u|^2)\big] + h(r,|u|^2)\Delta u - h_s(r,|u|^2)|\nabla u|^2 u\big\}. \end{aligned} \tag{1.11}$$

In Section 4 it is shown that any twist $u = r\mathrm{Q}(r)\theta$ satisfying the PDE $\mathscr{L}[u] = \nabla\mathscr{P}$ must have its twist path satisfying the ODE: $\mathscr{E}_L[\mathrm{Q}] = 0$, namely,

$$\frac{d}{dr}\left\{\int_{\mathbb{S}^{n-1}} r^{n+1}h(r,r^2)\big[\dot{\mathrm{Q}}\theta \otimes \mathrm{Q}\theta - \mathrm{Q}\theta \otimes \dot{\mathrm{Q}}\theta\big]\, d\mathcal{H}^{n-1}(\theta)\right\} = 0. \tag{1.12}$$

(4) Note that the identity mapping $u \equiv x$ is always a solution to this system in view of the vector field $\mathscr{L}[u \equiv x] = \nabla[F_\xi] - F_s x$ with $F_\xi = F_\xi(r,r^2,n)$, $F_s = F_s(r,r^2,n)$ being a gradient field in Ω.

A resolution of this ODE subject to $\mathrm{Q}(a) = \mathrm{Q}(b) = \mathrm{I}_n$ and a refined analysis of (1.11) then has the interesting implication that depending on n being even or odd and with a structural condition on h the twist solutions to (1.3)–(1.11) are:

- n even: There exists $m \in \mathbb{Z}$ and $\mathrm{P} \in \mathrm{O}(n)$ such that $u = u(x; m) = \mathrm{P}\,\mathrm{diag}(\mathrm{R}[\mathscr{G}](r;m), \ldots, \mathrm{R}[\mathscr{G}](r;m))\mathrm{P}^t x$, that is,

$$u = \mathrm{P} \begin{bmatrix} \mathrm{R}[\mathscr{G}](r;m) & 0 & \cdots & 0 & 0 \\ 0 & \mathrm{R}[\mathscr{G}](r;m) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathrm{R}[\mathscr{G}](r;m) & 0 \\ 0 & 0 & \cdots & 0 & \mathrm{R}[\mathscr{G}](r;m) \end{bmatrix} \mathrm{P}^t x. \tag{1.13}$$

  Here

$$\mathscr{G}(r;m) = \frac{2\pi m}{\|\mathsf{H}\|_{L^1(a,b)}} \left[\int_a^r \mathsf{H}(s)\, ds \right] \quad \text{where } \mathsf{H}(s) = \frac{1}{[s^{n+1} h(s, s^2)]}$$

  and each block $\mathrm{R}[\mathscr{G}]$ is an $\mathrm{SO}(2)$ matrix of rotation by angle $\mathscr{G}$ (see (4.7)).
- n odd: $u \equiv x$.

Moving on to the system (1.3)–(1.10) a similar conclusion can be established under suitable convexity and monotonicity assumptions on F (see Section 2) with $\mathscr{G} = \mathscr{G}(r)$ in (1.13) now being the unique solution to a two point boundary value problem relating to F (see Theorem 6.7 and Sections 5 and 6 for details). A surprising outcome, among other things, is that the strong local minimisers resulting from the earlier topological argument cannot be twist mappings here and possess the symmetries one naturally expects – at least in odd dimensions! It would thus be interesting to see what form and symmetries would such strong local minimisers and more generally extremals have if they are not among twist mappings? In another direction the results here can be seen to give a curious characterisation of those twist paths Q for which $u = \mathrm{Q}(|x|)v(x)$ with $v \equiv x$ is a solution to (1.3)–(1.10). It would be interesting to give a similar characterisation for other pairs (Q, v) with v a solution to (1.3)–(1.10) or more generally (1.3)–(1.4) so that the resulting u is a solution to the same system.

2. Preliminaries and formulation of the operator $\mathscr{L}$'s action on twists

Our goal is to seek and classify solutions to the nonlinear system (1.3)–(1.10) in the twist form $u\colon (r,\theta) \mapsto (r, \mathrm{Q}(r)\theta)$. Here we compute and gather together some key identities that will assist us in the subsequent analysis of the system. For the sake of future reference and clarity we assume throughout that $F = F(r,s,\xi)$ is a twice continuously differentiable Lagrangian, that is, $F \in \mathscr{C}^2(U)$ where $U = U(\mathbb{X}^n[a,b]) = [a,b] \times]0,\infty[\times]0,\infty[\subset \mathbb{R}^3$. Moreover, we assume that F is bounded from below, i.e. $F(r,s,\xi) \geq c_0$ for some $c_0 \in \mathbb{R}$ and all $(r,s,\xi) \in U$

and that for every compact set $K \subset]0,\infty[$ there are $c_1 = c_1(K), c_2 = c_2(K) > 0$ such that ([5])

$$|F_\xi(r,s,\zeta^2)\zeta| \le c_2|\zeta|^{p-1}, \quad \text{for all } (r,s,\zeta^2) \in U, \text{ with } s \in K, \tag{2.1}$$

$$c_0 + c_1|\zeta|^p \le F(r,s,\zeta^2) \le c_2|\zeta|^p, \quad \text{for all } (r,s,\zeta^2) \in U, \text{ with } s \in K. \tag{2.2}$$

In particular $\mathbb{F}$ is well-defined and bounded below (yet not necessarily finite everywhere) on $\mathscr{A}_p(\mathbb{X}^n)$. As for convexity all we assume is that $F_\xi > 0$, $F_{\xi\xi} \ge 0$ and that the twice continuously differentiable function $\zeta \mapsto F(r, r^2, n + r^2\zeta^2)$ is uniformly convex in ζ for all $a \le r \le b$ and $\zeta \in \mathbb{R}$. Note that below we write F_r, F_s, F_ξ, $F_{r\xi}$, $F_{s\xi}$, $F_{\xi\xi}$ etc. for the derivatives of F in its respective arguments.

PROPOSITION 2.1. *Let*

$$\mathrm{Q} \in \mathscr{C}([a,b], \mathrm{SO}(n)) \cap \mathscr{C}^2(]a,b[, \mathrm{SO}(n)), \quad v \in \mathscr{C}^2(\overline{\mathbb{X}}^n, \overline{\mathbb{X}}^n), \quad u = \mathrm{Q}(|x|)v(x).$$

Then with $\dot{\mathrm{Q}} = d\mathrm{Q}/dr$, $\ddot{\mathrm{Q}} = d^2\mathrm{Q}/dr^2$ *the following hold*:

(a) $\nabla u = \mathrm{Q}\nabla v + \dot{\mathrm{Q}}v \otimes \theta$,
(b) $|\nabla u|^2 = |\nabla v|^2 + |\dot{\mathrm{Q}}v|^2 + 2\langle \mathrm{Q}^t\dot{\mathrm{Q}}v, \nabla v\,\theta\rangle$,
(c) $\Delta u = 2\dot{\mathrm{Q}}\nabla v\,\theta + \mathrm{Q}\Delta v + \ddot{\mathrm{Q}}v + (n-1)\dot{\mathrm{Q}}v/r$,
(d) $\det\nabla u = \det\nabla v + \langle \mathrm{Q}^t\dot{\mathrm{Q}}v, [\mathrm{cof}\,\nabla v]\theta\rangle$, *whenever* $\det\nabla v(x) \neq 0$.

Furthermore, with the Lagrangian $F = F(r,s,\xi)$ *as described earlier we have*

$$\begin{aligned}\mathrm{div}\big[F_\xi(r,|u|^2,|\nabla u|^2)\nabla u\big] &= F_{\xi\xi}(r,|u|^2,|\nabla u|^2)(\mathrm{Q}\nabla v + \dot{\mathrm{Q}}v\otimes\theta)\\ &\quad\times \big[\nabla(|\nabla v|^2) + \nabla(|\dot{\mathrm{Q}}v|^2) + 2\nabla\langle \mathrm{Q}^t\dot{\mathrm{Q}}v, \nabla v\,\theta\rangle\big]\\ &\quad + F_{s\xi}(r,|u|^2,|\nabla u|^2)(\mathrm{Q}\nabla v + \dot{\mathrm{Q}}v\otimes\theta)\nabla(|v|^2)\\ &\quad + F_{r\xi}(r,|u|^2,|\nabla u|^2)(\mathrm{Q}\nabla v + \dot{\mathrm{Q}}v\otimes\theta)\theta\\ &\quad + F_\xi(r,|u|^2,|\nabla u|^2)\left[2\dot{\mathrm{Q}}\nabla v\,\theta + \mathrm{Q}\Delta v + \ddot{\mathrm{Q}}v + \frac{n-1}{r}\dot{\mathrm{Q}}v\right].\end{aligned}$$

PROOF. The first identity follows by a straightforward differentiation. Indeed proceeding directly we can write

$$\nabla u = \mathrm{Q}\nabla v + \nabla\mathrm{Q}(|x|)v = \mathrm{Q}\nabla v + \dot{\mathrm{Q}}v\otimes\theta = \mathrm{Q}(\nabla v + \mathrm{Q}^t\dot{\mathrm{Q}}v\otimes\theta).$$

Proceeding immediately from this on to (d), using the assumed invertibility of ∇v together with the fact that determinant is a quasiaffine function on $\mathbb{M}^{n\times n}$ (cf. e.g. [22]), as a result of which $\det(\mathrm{I}_n + \zeta\otimes\xi) = 1 + \langle\zeta,\xi\rangle$ for any $\zeta,\xi\in\mathbb{R}^n$, it follows at once that

$$\begin{aligned}\det\nabla u &= \det\mathrm{Q}\times\det(\nabla v + \mathrm{Q}^t\dot{\mathrm{Q}}v\otimes\theta)\\ &= \det\nabla v\big[1 + \langle[\nabla v]^{-1}\mathrm{Q}^t\dot{\mathrm{Q}}v,\theta\rangle\big] = \det\nabla v + \langle\mathrm{Q}^t\dot{\mathrm{Q}}v, [\mathrm{cof}\,\nabla v]\theta\rangle.\end{aligned} \tag{2.3}$$

([5]) The particular example $F(r,s,\xi) = (\xi/s)^{n/2}$ ($n \ge 2$) gives the classical distortion energy and is of great interest (see [2], [15], [24]).

Next for (b), using the description of the Hilbert–Schmidt norm of the matrix field ∇u we can write

$$\begin{aligned}|\nabla u|^2 &= \mathrm{tr}\big\{[\nabla u]^t[\nabla u]\big\} = \mathrm{tr}\big\{([\nabla v]^t\mathrm{Q}^t + \theta\otimes\dot{\mathrm{Q}}v)(\mathrm{Q}[\nabla v] + \dot{\mathrm{Q}}v\otimes\theta)\big\}\\ &= \mathrm{tr}\big\{[\nabla v]^t[\nabla v] + [\nabla v]^t\mathrm{Q}^t\dot{\mathrm{Q}}v\otimes\theta + \theta\otimes[\nabla v]^t\mathrm{Q}^t\dot{\mathrm{Q}}v + (\theta\otimes\dot{\mathrm{Q}}v)(\dot{\mathrm{Q}}v\otimes\theta)\big\}\\ &= |\nabla v|^2 + 2\big\langle \mathrm{Q}^t\dot{\mathrm{Q}}v, \nabla v\,\theta\big\rangle + |\dot{\mathrm{Q}}v|^2.\end{aligned}$$

Likewise for (c) by taking the divergence of ∇u as given by (a), we compute the Laplacian $\Delta u = \mathrm{div}(\mathrm{Q}\nabla v + \dot{\mathrm{Q}}v\otimes\theta) = 2\dot{\mathrm{Q}}\nabla v\theta + \mathrm{Q}\Delta v + \ddot{\mathrm{Q}}v + (n-1)/r\dot{\mathrm{Q}}v$.

The final identity then follows by direct differentiation and use of the chain rule

$$\begin{aligned}\mathrm{div}\big[F_\xi(r,|u|^2,|\nabla u|^2)\nabla u\big] &= F_{\xi\xi}\big(r,|u|^2,|\nabla u|^2\big)\nabla u\nabla\big(|\nabla u|^2\big)\\ &+ F_{s\xi}\big(r,|u|^2,|\nabla u|^2\big)\nabla u\nabla\big(|u|^2\big) + F_{r\xi}\big(r,|u|^2,|\nabla u|^2\big)\nabla u\theta\\ &+ F_\xi\big(r,|u|^2,|\nabla u|^2\big)\Delta u.\end{aligned} \tag{2.4}$$

Noting that $|v|^2 = |u|^2$ we now have all the identities to complete the expression (2.4) and the result follows. □

Proposition 2.2. *Let*

$$\mathrm{Q}\in\mathscr{C}([a,b],\mathrm{SO}(n))\cap\mathscr{C}^2]a,b[,\mathrm{SO}(n)),\quad v\in\mathscr{C}^2(\overline{\mathbb{X}}^n,\overline{\mathbb{X}}^n),\quad u=\mathrm{Q}(|x|)v(x).$$

Then, referring to the Euler–Lagrange operator $\mathscr{L}$ *as given by* (1.10)*, we have*

$$\begin{aligned}\mathscr{L}[u] = \big[(\nabla v)^t\mathrm{Q}^t + \theta\otimes\dot{\mathrm{Q}}v\big]\bigg\{ & F_{\xi\xi}\big(r,|u|^2,|\nabla u|^2\big)\big(\mathrm{Q}\nabla v + \dot{\mathrm{Q}}v\otimes\ \big)\\ &\times\big[\nabla\big(|\nabla v|^2\big) + \nabla\big(|\dot{\mathrm{Q}}v|^2\big) + 2\nabla\big\langle\mathrm{Q}^t\dot{\mathrm{Q}}v,\nabla v\,\theta\big\rangle\big]\\ &+ F_{s\xi}\big(r,|u|^2,|\nabla u|^2\big)\big(\mathrm{Q}\nabla v + \dot{\mathrm{Q}}v\otimes\theta)\nabla(|v|^2\big)\\ &+ F_{r\xi}\big(r,|u|^2,|\nabla u|^2\big)\big(\mathrm{Q}\nabla v + \dot{\mathrm{Q}}v\otimes\theta\big)\theta\\ &+ F_\xi\big(r,|u|^2,|\nabla u|^2\big)\bigg[2\dot{\mathrm{Q}}\nabla v\,\theta + \mathrm{Q}\Delta v + \ddot{\mathrm{Q}}v + \frac{n-1}{r}\dot{\mathrm{Q}}v\bigg]\\ &- F_s\big(r,|u|^2,|\nabla u|^2\big)\mathrm{Q}(r)v\bigg\}.\end{aligned}$$

Proof. The result is a direct consequence of the definition of $\mathscr{L}[u]$ (1.10) and the relevant identities in Proposition 2.1. □

Corollary 2.3. *Let* u *be a twist with a twice continuously differentiable twist path* Q*, specifically,* $\mathrm{Q}\in\mathscr{C}([a,b],\mathrm{SO}(n))\cap\mathscr{C}^2(]a,b[,\mathrm{SO}(n))$. *Then the following hold*:

(a) $\nabla u = \mathrm{Q} + r\dot{\mathrm{Q}}\theta\otimes\theta$,
(b) $|\nabla u|^2 = n + r^2\big|\dot{\mathrm{Q}}\theta\big|^2$,
(c) $\Delta u = \big[(n+1)\dot{\mathrm{Q}} + r\ddot{\mathrm{Q}}\big]\theta$,

(d) $\det \nabla u = \det\big(\mathrm{Q} + r\dot{\mathrm{Q}}\theta \otimes \theta\big) = 1.$

As a result, if $F = F(r, s, \xi)$ *is a Lagrangian as described earlier, then*

$$
\begin{aligned}
\operatorname{div}&[F_\xi(r, |u|^2, |\nabla u|^2)\nabla u] \\
&= F_{\xi\xi}(r, r^2, |\nabla u|^2)(\mathrm{Q} + r\dot{\mathrm{Q}}\theta \otimes \theta)(2r|\dot{\mathrm{Q}}\theta|^2\theta + r^2\nabla|\dot{\mathrm{Q}}\theta|^2) \\
&\quad + [2rF_{s\xi}(r, r^2, |\nabla u|^2) + F_{r\xi}(r, r^2, |\nabla u|^2)](\mathrm{Q}\theta + r\dot{\mathrm{Q}}\theta) \\
&\quad + F_\xi(r, r^2, |\nabla u|^2)[(n+1)\dot{\mathrm{Q}} + r\ddot{\mathrm{Q}}]\theta.
\end{aligned}
$$

Consequently, the action of $\mathscr{L}$ *as defined by* (1.10) *on* u *can be written as*

$$
\begin{aligned}
\mathscr{L}[u] &= [\nabla u]^t\big\{\operatorname{div}\big[F_\xi\big(|x|, |u|^2, |\nabla u|^2\big)\nabla u\big] - F_s\big(|x|, |u|^2, |\nabla u|^2\big)u\big\} \\
&= (\mathrm{Q}^t + r\theta \otimes \dot{\mathrm{Q}}\theta) \\
&\quad \times \big[F_{\xi\xi}\big(r, r^2, |\nabla u|^2\big)\big(\mathrm{Q} + r\dot{\mathrm{Q}}\theta \otimes \theta\big)\big(2r|\dot{\mathrm{Q}}\theta|^2\theta + r^2\nabla|\dot{\mathrm{Q}}\theta|^2\big) \\
&\quad + 2rF_{s\xi}\big(r, r^2, |\nabla u|^2\big)\big(\mathrm{Q}\theta + r\dot{\mathrm{Q}}\theta\big) + F_{r\xi}\big(r, r^2, |\nabla u|^2\big)\big(\mathrm{Q}\theta + r\dot{\mathrm{Q}}\theta\big) \\
&\quad + F_\xi\big(r, r^2, |\nabla u|^2\big)\big[(n+1)\dot{\mathrm{Q}} + r\ddot{\mathrm{Q}}\big]\theta - rF_s\big(r, r^2, |\nabla u|^2\big)\mathrm{Q}\theta\big].
\end{aligned} \tag{2.5}
$$

PROOF. The proof is a direct consequence of Propositions 2.1 and 2.2 upon setting $v \equiv x$ and noting $\big\langle \mathrm{Q}^t\dot{\mathrm{Q}}\theta, \theta\big\rangle = 0$ and $|u|^2 = |r\mathrm{Q}(r)\theta|^2 = r^2$. □

3. Analysis of extremality: Twist paths and Lie exponentials $\mathrm{Q} = \exp\{\mathscr{G}(r)\mathrm{H}\}$

In this section we look at extremality conditions for general classes of curves on $\mathrm{SO}(n)$, initially disregarding any connection with the variational energy integral (1.9) and its twist extremals (see (3.2) below) before specialising the results and conclusions to the case of twist paths and loops at hand. For the sake of future reference we introduce the Sobolev class of weakly differentiable admissible loops at identity (with $p \geq 1$ fixed) by setting

$$
\mathscr{B}_p = \mathscr{B}_p(a, b) := \big\{\mathrm{Q} \in W^{1,p}(a, b; \mathrm{SO}(n)) : \mathrm{Q}(a) = \mathrm{Q}(b) = \mathbf{I}_n\big\}. \tag{3.1}
$$

We denote by $\exp\{\,\cdot\,\}$ the Lie exponential on $\mathbf{G} = \mathrm{SO}(n)$ whose domain is the Lie algebra $\mathfrak{g} = \mathfrak{so}(n)$ of skew-symmetric matrices. Any Lie group is parallelisable with a trivial tangent bundle $\mathbf{G} \times \mathfrak{g}$. In case of $\mathrm{SO}(n)$ for left invariant vector fields $X, Y, Z \in \mathfrak{so}(n)$ the Lie bracket is given by $[X, Y] = XY - YX$ and the bi-invariant metric is induced by the Killing form $B(X, Y) = (n-2)\mathrm{tr}(XY)$.

PROPOSITION 3.1. *Suppose* $L = L(r, \eta, \zeta)$ *is a twice continuously differentiable Lagrangian and* $\mathrm{Q} \in \mathscr{C}^2([a, b]; \mathrm{SO}(n))$ *with* $\mathrm{Q}(a) = \mathrm{Q}(b) = \mathrm{I}_n$ *is an extremal of the integral*

$$
\mathbb{L}[\mathrm{Q}, (a, b)] := \int_a^b L\big(r, \mathrm{Q}, \dot{\mathrm{Q}}\big)\, dr, \qquad \dot{\mathrm{Q}} = \frac{d\mathrm{Q}}{dr}, \tag{3.2}
$$

considered over $\mathscr{B}_p(a,b)$. *Then* $\mathscr{E}_L[\mathrm{Q};(a,b)]=0$ *on* $]a,b[$ *where* $\mathscr{E}_L$ *is the second-order Euler–Lagrange operator*

$$\mathscr{E}_L[\mathrm{Q};(a,b)] = -\frac{d}{dr}\big[L_\zeta \mathrm{Q}^t - \mathrm{Q}L_\zeta^t\big] + L_\eta \mathrm{Q}^t - \mathrm{Q}L_\eta^t + L_\zeta \dot{\mathrm{Q}}^t - \dot{\mathrm{Q}}L_\zeta^t. \tag{3.3}$$

Here $L_\eta = L_\eta\big(r,\mathrm{Q},\dot{\mathrm{Q}}\big)$ *and* $L_\zeta = L_\zeta\big(r,\mathrm{Q},\dot{\mathrm{Q}}\big)$ *with the subscripts denoting the derivatives of* L *with respect to the second and third arguments, respectively.*

Proof. Let Q be as described and pick a compactly supported skew-symmetric matrix field $\mathrm{H} \in \mathscr{C}_0^\infty(]a,b[,\mathbb{M}^{n\times n})$. Let $\mathrm{Q}_\varepsilon = \mathrm{Q}_\varepsilon(r)$ in $\mathscr{C}^1([a,b]\times[-\ell,\ell],\mathrm{SO}(n))$ with $|\varepsilon|\le\ell$ and $\ell>0$ sufficiently small denote the one parameter family of variations of the extremal Q verifying

$$\begin{cases} \mathrm{Q}_0(r) = \mathrm{Q}(r) & \text{for all } r\in[a,b],\\ d\mathrm{Q}_\varepsilon/d\varepsilon|_{\varepsilon=0} = \mathrm{HQ} & \text{for all } r\in[a,b],\\ \mathrm{Q}_\varepsilon(a)=\mathrm{I}_n,\ \mathrm{Q}_\varepsilon(b)=\mathrm{I}_n & \text{for all } \varepsilon\in[-\ell,\ell]. \end{cases} \tag{3.4}$$

The pull-back $\mathrm{Q}^t\mathrm{HQ}$ takes values in $\mathfrak{so}(n)$ and so HQ is a [*tangent*] vector field. Taking "d/dr" from the second line in (3.4) gives $d\dot{\mathrm{Q}}_\varepsilon/d\varepsilon|_{\varepsilon=0} = \mathrm{H}\dot{\mathrm{Q}} + \dot{\mathrm{H}}\mathrm{Q}$ (with dots denoting d/dr). Now proceeding forward and on to the first variation of the $\mathbb{L}$-energy (3.2), by definition we have that Q is an extremal if and only if we have:

$$\begin{aligned} \delta\mathbb{L}[\mathrm{Q}](\mathrm{H}) &= \frac{d}{d\varepsilon}\,\mathbb{L}[\mathrm{Q}_\varepsilon,(a,b)]\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}\bigg\{\int_a^b L\big(r,\mathrm{Q}_\varepsilon,\dot{\mathrm{Q}}_\varepsilon\big)\,dr\bigg\}\bigg|_{\varepsilon=0}\\ &= \int_a^b \bigg\{\bigg\langle L_\eta\big(r,\mathrm{Q},\dot{\mathrm{Q}}\big), \frac{d\mathrm{Q}_\varepsilon}{d\varepsilon}\bigg|_{\varepsilon=0}\bigg\rangle + \bigg\langle L_\zeta\big(r,\mathrm{Q},\dot{\mathrm{Q}}\big), \frac{d\dot{\mathrm{Q}}_\varepsilon}{d\varepsilon}\bigg|_{\varepsilon=0}\bigg\rangle\bigg\}dr = 0. \end{aligned} \tag{3.5}$$

Focusing on the integral on the second line and writing $L_\eta = L_\eta(r,\mathrm{Q},\dot{\mathrm{Q}})$ and $L_\zeta = L_\zeta(r,\mathrm{Q},\dot{\mathrm{Q}})$ for brevity respectively we have

$$\begin{aligned} \delta\,\mathbb{L}[\mathrm{Q}](\mathrm{H}) &= \int_a^b \big\{\langle L_\eta,\mathrm{HQ}\rangle + \big\langle L_\zeta,\big(\mathrm{H}\dot{\mathrm{Q}}+\dot{\mathrm{H}}\mathrm{Q}\big)\big\rangle\big\}\,dr\\ &= \int_a^b \big\{\big\langle L_\eta\mathrm{Q}^t + L_\zeta\dot{\mathrm{Q}}^t,\mathrm{H}\big\rangle + \big\langle L_\zeta\mathrm{Q}^t,\dot{\mathrm{H}}\big\rangle\big\}\,dr\\ &= \int_a^b \bigg\langle -\frac{d}{dr}\big(L_\zeta\mathrm{Q}^t\big) + L_\eta\mathrm{Q}^t + L_\zeta\dot{\mathrm{Q}}^t,\mathrm{H}\bigg\rangle\,dr = 0, \end{aligned} \tag{3.6}$$

where in deducing the third line we have used the integration by parts formula. The conclusion now follows by noting the arbitrariness of the skew-symmetric matrix field $\mathrm{H}\in\mathscr{C}_0^\infty(]a,b[,\mathbb{M}^{n\times n})$ and upon invoking the fundamental lemma of the calculus of variations. □

Remark 3.2. If instead of $d\mathrm{Q}_\varepsilon/d\varepsilon|_{\varepsilon=0} = \mathrm{HQ}$ in the second line in (3.4) we set $d\mathrm{Q}_\varepsilon/d\varepsilon|_{\varepsilon=0} = \mathrm{QH}$ (note that QH is clearly a (*tangent*) vector field) then a similar argument gives the Euler–Lagrange $\mathscr{F}_L[\mathrm{Q};(a,b)] = 0$ where

$$\mathscr{F}_L[\mathrm{Q};(a,b)] = -\frac{d}{dr}\big[\mathrm{Q}^t L_\zeta - L_\zeta^t \mathrm{Q}\big] + \mathrm{Q}^t L_\eta - L_\eta^t \mathrm{Q} + \dot{\mathrm{Q}}^t L_\zeta - L_\zeta^t \dot{\mathrm{Q}}. \tag{3.7}$$

In view of the identity established in Lemma 3.3 below the two Euler–Lagrange equations are easily seen to be *equivalent*.

Lemma 3.3. $\mathscr{F}_L[\mathrm{Q};(a,b)] = \mathrm{Q}^t \mathscr{E}_L[\mathrm{Q};(a,b)]\mathrm{Q}$.

Proof. Below for brevity we write $\mathscr{E}_L = \mathscr{E}_L[\mathrm{Q};(a,b)]$, $\mathscr{F}_L = \mathscr{F}_L[\mathrm{Q};(a,b)]$ and suppress the arguments in L_η, L_ζ. Starting from

$$\begin{aligned}\mathscr{F}_L &= -\frac{d}{dr}\big[\mathrm{Q}^t L_\zeta - L_\zeta^t \mathrm{Q}\big] + \mathrm{Q}^t L_\eta - L_\eta^t \mathrm{Q} + \dot{\mathrm{Q}}^t L_\zeta - L_\zeta^t \dot{\mathrm{Q}} \\ &= \mathrm{Q}^t\Big\{ - \mathrm{Q}\,\frac{d}{dr}\big[\mathrm{Q}^t L_\zeta - L_\zeta^t \mathrm{Q}\big]\mathrm{Q}^t + L_\eta \mathrm{Q}^t - \mathrm{Q} L_\eta^t + \mathrm{Q}\dot{\mathrm{Q}}^t L_\zeta \mathrm{Q}^t - \mathrm{Q} L_\zeta^t \dot{\mathrm{Q}}\mathrm{Q}^t\Big\}\mathrm{Q}.\end{aligned} \tag{3.8}$$

Now, focusing on the first term in the brackets on the right on the second line, using orthogonality, we can write

$$\begin{aligned}\frac{d}{dr}\big[\mathrm{Q}^t L_\zeta - L_\zeta^t \mathrm{Q}\big] &= \frac{d}{dr}\big[\mathrm{Q}^t(L_\zeta \mathrm{Q}^t - \mathrm{Q} L_\zeta^t)\mathrm{Q}\big] \\ &= \dot{\mathrm{Q}}^t\big(L_\zeta \mathrm{Q}^t - \mathrm{Q} L_\zeta^t\big)\mathrm{Q} + \mathrm{Q}^t \frac{d}{dr}\big[L_\zeta \mathrm{Q}^t - \mathrm{Q} L_\zeta^t\big]\mathrm{Q} + \mathrm{Q}^t(L_\zeta \mathrm{Q}^t - \mathrm{Q} L_\zeta^t)\dot{\mathrm{Q}}.\end{aligned}$$

Hence substituting this expansion back in (3.8) gives

$$\begin{aligned}\mathscr{F}_L =& -\frac{d}{dr}\big[\mathrm{Q}^t L_\zeta - L_\zeta^t \mathrm{Q}\big] + \mathrm{Q}^t L_\eta - L_\eta^t \mathrm{Q} + \dot{\mathrm{Q}}^t L_\zeta - L_\zeta^t \dot{\mathrm{Q}} \\ =& \,\mathrm{Q}^t\Big\{ - \mathrm{Q}\dot{\mathrm{Q}}^t(L_\zeta \mathrm{Q}^t - \mathrm{Q} L_\zeta^t)\mathrm{Q}\mathrm{Q}^t - \mathrm{Q}\mathrm{Q}^t \frac{d}{dr}\big[L_\zeta \mathrm{Q}^t - \mathrm{Q} L_\zeta^t\big]\mathrm{Q}\mathrm{Q}^t \\ & - \mathrm{Q}\mathrm{Q}^t\big(L_\zeta \mathrm{Q}^t - \mathrm{Q} L_\zeta^t\big)\dot{\mathrm{Q}}\mathrm{Q}^t + L_\eta \mathrm{Q}^t - \mathrm{Q} L_\eta^t + \mathrm{Q}\dot{\mathrm{Q}}^t L_\zeta \mathrm{Q}^t - \mathrm{Q} L_\zeta^t \dot{\mathrm{Q}}\mathrm{Q}^t\Big\}\mathrm{Q}.\end{aligned}$$

Upon noting the identities $\mathrm{Q}\dot{\mathrm{Q}}^t \mathrm{Q} L_\zeta^t = -\dot{\mathrm{Q}} L_\zeta^t$ and $-L_\zeta \mathrm{Q}^t \dot{\mathrm{Q}}\mathrm{Q}^t = L_\zeta \dot{\mathrm{Q}}^t$, both resulting from skew-symmetry, this after cancellations simplifies to

$$\begin{aligned}\mathscr{F}_L =& \,\mathrm{Q}^t\Big\{ - \mathrm{Q}\dot{\mathrm{Q}}^t(L_\zeta \mathrm{Q}^t - \mathrm{Q} L_\zeta^t) - \frac{d}{dr}\big[L_\zeta \mathrm{Q}^t - \mathrm{Q} L_\zeta^t\big] \\ & - \big(L_\zeta \mathrm{Q}^t - \mathrm{Q} L_\zeta^t\big)\dot{\mathrm{Q}}\mathrm{Q}^t + L_\eta \mathrm{Q}^t - \mathrm{Q} L_\eta^t + \mathrm{Q}\dot{\mathrm{Q}}^t L_\zeta \mathrm{Q}^t - \mathrm{Q} L_\zeta^t \dot{\mathrm{Q}}\mathrm{Q}^t\Big\}\mathrm{Q} \\ =& \,\mathrm{Q}^t\Big\{ - \frac{d}{dr}\big[L_\zeta \mathrm{Q}^t - \mathrm{Q} L_\zeta^t\big] + L_\eta \mathrm{Q}^t - \mathrm{Q} L_\eta^t + L_\zeta \dot{\mathrm{Q}}^t - \dot{\mathrm{Q}} L_\zeta^t\Big\}\mathrm{Q} = \mathrm{Q}^t \mathscr{E}_L \mathrm{Q},\end{aligned}$$

which is the desired conclusion. □

An important class of twist paths that arise as solutions to the above Euler–Lagrange equations are those in the Lie exponential form $\mathrm{Q}(r) = \exp\{\mathscr{G}(r)\mathrm{H}\}$

for suitable choices of profile curves $\mathscr{G} \in \mathscr{C}^2[a,b]$ and matrices $\mathrm{H} \in \mathfrak{so}(n)$ (cf. Sections 4–6 below). In this event a basic calculation shows that the extremality condition in Proposition 3.1 leads to the nonlinear ODE for the pair $(\mathscr{G}, \mathrm{H})$:

$$\begin{aligned}(3.9)\quad \mathscr{E}_L[\exp\{\mathscr{G}\mathrm{H}\};(a,b)] &= -\frac{d}{dr}\big[L_\zeta \exp\{-\mathscr{G}\mathrm{H}\} - \exp\{\mathscr{G}\mathrm{H}\}L_\zeta^t\big] \\ &+ L_\eta \exp\{-\mathscr{G}\mathrm{H}\} - \exp\{\mathscr{G}\mathrm{H}\}L_\eta^t + \dot{\mathscr{G}}\big(L_\zeta \exp\{-\mathscr{G}\mathrm{H}\}\mathrm{H}^t - \mathrm{H}\exp\{\mathscr{G}\mathrm{H}\}L_\zeta^t\big) = 0.\end{aligned}$$

Now aiming to establish the existence of multiple solutions to this ODE, one possible approach is to use variational methods and proceed by extremising the restriction of the $\mathbb{L}$-energy (3.2) to the subclass of Lie exponential twist paths over the space of profile curves satisfying suitable Dirichlet boundary conditions in line with $\mathrm{Q}(a) = \mathrm{Q}(b) = \mathrm{I}_n$, i.e. in line with $\mathrm{Q} \in \mathscr{B}_p(a,b)$. An advantage here is that, whilst $\pi_1[\mathrm{SO}(n)] \cong \mathbb{Z}_2$ (for $n \geq 3$) and so the minimisation of $\mathbb{L}$ can lead to the existence of at most two minimisers – one in each homotopy class of closed curves based at I_n in $\mathrm{SO}(n)$ – by considering the restricted energy, one obtains an infinitude of distinct minimisers by considering an infinitude of distinct boundary conditions on $(\mathscr{G}, \mathrm{H})$, that is, $\mathscr{G}(a)\mathrm{H}, \mathscr{G}(b)\mathrm{H} \in \exp^{-1}\{\mathrm{I}_n\}$, thanks to the pre-image $\exp^{-1}\{\mathrm{I}_n\} \subset \mathfrak{so}(n)$ being an infinite set! The task is then to discuss the relation these minimisers bear to the ODE (3.9). Towards this end the restriction of the $\mathbb{L}$-energy to the space of profiles $\mathscr{G} = \mathscr{G}(r)$ (with H fixed), that is, $\mathbb{I}_{\mathrm{H}}[\mathscr{G}] = \mathbb{L}\,[\exp\{\mathscr{G}\mathrm{H}\},(a,b)]$ is seen to be

$$(3.10)\qquad \mathbb{I}_{\mathrm{H}}[\mathscr{G}] = \int_a^b L\big(r, \exp\{\mathscr{G}(r)\mathrm{H}\}, \dot{\mathscr{G}}(r)\mathrm{H}\exp\{\mathscr{G}(r)\mathrm{H}\}\big)\,dr.$$

Proposition 3.4. *Suppose $L = L(r,\eta,\zeta)$ is a twice continuously differentiable Lagrangian and that $\mathscr{G} \in \mathscr{C}^2[a,b]$ is an extremal of the integral $\mathbb{I}_{\mathrm{H}}$ as in* (3.10). *Then $\mathscr{I}_L[\mathscr{G}] = 0$ on $]a,b[$ where*

$$(3.11)\qquad \mathscr{I}_L[\mathscr{G}] = -\frac{d}{dr}\langle L_\zeta, \mathrm{H}\exp\{\mathscr{G}\mathrm{H}\}\rangle + \big\langle L_\eta - \dot{\mathscr{G}}\mathrm{H}L_\zeta, \mathrm{H}\exp\{\mathscr{G}\mathrm{H}\}\big\rangle.$$

Proof. Towards this end pick $\mathscr{H} \in \mathscr{C}_c^\infty(a,b)$ and for ε sufficiently small consider the one parameter family of Lie exponentials $\mathrm{Q}_\varepsilon(r) = \exp\{(\mathscr{G}(r) + \varepsilon\mathscr{H}(r))\mathrm{H}\}$. Then $\mathrm{Q}_0(r) = \exp\{\mathscr{G}(r)\mathrm{H}\} = \mathrm{Q}(r)$ and a straightforward differentiation gives $\dot{\mathrm{Q}}_\varepsilon(r) = \big(\dot{\mathscr{G}}(r) + \varepsilon\dot{\mathscr{H}}(r)\big)\mathrm{H}\exp\{(\mathscr{G}(r) + \varepsilon\mathscr{H}(r))\mathrm{H}\}$. With these assumptions in place the vanishing of the first variation of energy at Q amounts to $\delta\mathbb{I}_{\mathrm{H}}[\mathscr{G}](\mathscr{H}) = d/d\varepsilon\mathbb{L}[\mathrm{Q}_\varepsilon]|_{\varepsilon=0} = 0$ and so

$$\begin{aligned}\frac{d}{d\varepsilon}\int_a^b L\big(r, \mathrm{Q}_\varepsilon, \dot{\mathrm{Q}}_\varepsilon\big)\Big|_{\varepsilon=0} &= \int_a^b \langle L_\eta, \mathscr{H}\mathrm{H}\mathrm{Q}\rangle + \big\langle L_\zeta, \dot{\mathscr{H}}\mathrm{H}\mathrm{Q} + \dot{\mathscr{G}}\mathrm{H}\mathscr{H}\mathrm{H}\mathrm{Q}\big\rangle \\ &= \int_a^b \Big\{-\frac{d}{dr}\langle L_\zeta, \mathrm{H}\mathrm{Q}\rangle + \langle L_\eta + \dot{\mathscr{G}}\mathrm{H}^t L_\zeta, \mathrm{H}\mathrm{Q}\rangle\Big\}\mathscr{H} = 0.\end{aligned}$$

The conclusion now follows from the arbitrariness of $\mathscr{H}$. □

REMARK 3.5. To see that the profile curve $\mathscr{G}$ of an extremising twist path of the $\mathbb{L}$-energy (3.2) given in the Lie exponential form $\mathrm{Q}(r) = \exp\{\mathscr{G}(r)\mathrm{H}\}$ satisfies the ODE (3.11) it suffices to take the matrix inner product of (3.9) with H. Then by virtue of $\dot{\mathrm{Q}} = \dot{\mathscr{G}}\mathrm{HQ}$, a basic calculation gives,

$$
\begin{aligned}
(3.12)\quad & \langle \mathscr{E}_L[\mathrm{Q} = \exp\{\mathscr{G}\mathrm{H}\}], \mathrm{H}\rangle \\
&= -\frac{d}{dr}\big\langle \big[L_\zeta \mathrm{Q}^t - \mathrm{Q}L_\zeta^t\big], \mathrm{H}\big\rangle + \big\langle L_\eta \mathrm{Q}^t - \mathrm{Q}L_\eta^t, \mathrm{H}\big\rangle + \big\langle L_\zeta \dot{\mathrm{Q}}^t - \dot{\mathrm{Q}}L_\zeta^t, \mathrm{H}\big\rangle \\
&= -2\frac{d}{dr}\langle L_\zeta, \mathrm{HQ}\rangle + 2\langle L_\eta, \mathrm{HQ}\rangle + 2\dot{\mathscr{G}}\big\langle \mathrm{H}^t L_\zeta, \mathrm{HQ}\big\rangle,
\end{aligned}
$$

from which the claim follows at once. Note however that the reverse implication is *not* in general true due to the more restrictive type of variations taken to arrive at (3.11) compared to those used to get (3.3). Interestingly, however, we discuss a number of cases where the two ODEs are *equivalent*. (See below for more.)

The connection between the ongoing discussion on extremality of twist paths on the one hand and the question of multiple solutions to the nonlinear system (1.3) on the other becomes more transparent when we consider restricting the variational energy integral (1.9) to the class of twist mappings. Towards this end let us set $\mathbb{L}[\mathrm{Q}, (a, b)] = \mathbb{F}[u = r\mathrm{Q}\theta, \mathbb{X}^n]$. Then is it plain that

$$
\begin{aligned}
(3.13)\qquad \mathbb{L}[\mathrm{Q}, (a, b)] &= \int_a^b L\big(r, \mathrm{Q}, \dot{\mathrm{Q}}\big)\, dr \\
&= \int_a^b \int_{\mathbb{S}^{n-1}} F\big(r, r^2|\mathrm{Q}\theta|^2, n + r^2|\dot{\mathrm{Q}}\theta|^2\big) r^{n-1}\, d\mathcal{H}^{n-1}(\theta)\, dr,
\end{aligned}
$$

where upon comparing with (3.2), and noting $\langle \mathrm{Q}\theta, \mathrm{Q}\theta\rangle = 1$, the Lagrangian $L = L(r, \eta, \zeta)$ here is given by the spherical integral

$$
(3.14)\qquad L(r, \eta, \zeta) = \int_{\mathbb{S}^{n-1}} F\big(r, r^2, n + r^2|\zeta\theta|^2\big)\, r^{n-1}\, d\mathcal{H}^{n-1}(\theta).
$$

Naturally in view of L being independent of the η variable here we have $L_\eta \equiv 0$ and so referring to the Euler–Lagrange operator $\mathscr{E}_L$ in (3.3) it follows that

$$
(3.15)\qquad \mathscr{E}_L[\mathrm{Q}; (a, b)] = -\frac{d}{dr}\big[L_\zeta \mathrm{Q}^t - \mathrm{Q}L_\zeta^t\big] + L_\zeta \dot{\mathrm{Q}}^t - \dot{\mathrm{Q}}L_\zeta^t.
$$

Now a further reference to the description of the Lagrangian L in (3.14) gives

$$
(3.16)\qquad L_\zeta(r, \dot{\mathrm{Q}}) = 2\int_{\mathbb{S}^{n-1}} r^{n+1} F_\xi\big(r, r^2, n + r^2|\dot{\mathrm{Q}}\theta|^2\big)\dot{\mathrm{Q}}\theta \otimes \theta\, d\mathcal{H}^{n-1}(\theta),
$$

and so in particular we have $L_\zeta \dot{\mathrm{Q}}^t - \dot{\mathrm{Q}}L_\zeta^t \equiv 0$. In summary, returning to (3.15) and substituting for L_ζ using (3.16), after a basic manipulation and taking into account the necessary cancellations we obtain the following.

Corollary 3.6. *The Euler–Lagrange equation associated with the* $\mathbb{L}$*-energy* (3.13) *over the space of admissible twist loops* $\mathscr{B}_p(a,b)$ *is given by*

$$\int_{\mathbb{S}^{n-1}} \frac{d}{dr}\big\{r^{n+1}F_\xi\big(r,r^2,n+r^2|\dot{\mathrm{Q}}\theta|^2\big)\big[\dot{\mathrm{Q}}\theta\otimes\mathrm{Q}\theta-\mathrm{Q}\theta\otimes\dot{\mathrm{Q}}\theta\big]\big\}d\mathcal{H}^{n-1}(\theta)=0. \tag{3.17}$$

Next referring to the energy integral $\mathbb{I}_{\mathrm{H}}$ in (3.10) and the formulation of its associated Euler–Lagrange operator $\mathscr{I}_L$ in Proposition 3.4, it is seen by virtue of (3.13), that firstly, $\mathbb{I}_{\mathrm{H}}[\mathscr{G}]=\mathbb{L}[\exp\{\mathscr{G}\mathrm{H}\},(a,b)]=\mathbb{F}[r\exp\{\mathscr{G}\mathrm{H}\}\theta,\mathbb{X}^n]$, that is,

$$\mathbb{I}_{\mathrm{H}}[\mathscr{G}]=\int_a^b\int_{\mathbb{S}^{n-1}} F\big(r,r^2,n+r^2\dot{\mathscr{G}}^2|\mathrm{H}\exp\{\mathscr{G}\mathrm{H}\}\theta|^2\big)r^{n-1}\,d\mathcal{H}^{n-1}(\theta)\,dr, \tag{3.18}$$

which precisely describes the restriction of the variational energy integral (1.9) to the subclass of twist mapping with a Lie exponential twist path, and secondly and as a result that we have the following.

Corollary 3.7. *The Euler–Lagrange equation associated with the integral* $\mathbb{I}_{\mathrm{H}}$ *in* (3.18) *is given by*

$$\int_{\mathbb{S}^{n-1}} \frac{d}{dr}\big\{r^{n+1}F_\xi(r,r^2,n+r^2\dot{\mathscr{G}}^2|\mathrm{H}\theta|^2)\dot{\mathscr{G}}\big\}|\mathrm{H}\theta|^2d\mathcal{H}^{n-1}(\theta)=0. \tag{3.19}$$

Proof. Here we have

$$(3.11)=-\frac{d}{dr}\langle L_\zeta,\mathrm{H}\exp\{\mathscr{G}\mathrm{H}\}\rangle-\big\langle\dot{\mathscr{G}}\mathrm{H}L_\zeta,\mathrm{H}\exp\{\mathscr{G}\mathrm{H}\}\big\rangle=\Big\langle-\frac{d}{dr}\,L_\zeta,\mathrm{H}\exp\{\mathscr{G}\mathrm{H}\}\Big\rangle$$

and so (3.19) follows by substituting for L_ζ using (3.16). □

4. Analysis of extremality: A totally integrable case and the ODE (3.17) vs. the PDE (1.3)

Before proceeding on to the system (1.3) and dealing with the implications of the Euler–Lagrange equations (3.17) and (3.19) we pause briefly to discuss an important special case. Here we take the integrand $F(r,s,\xi)=h(r,s)\xi$ for some positive $h\in\mathscr{C}^2([a,b]\times[0,\infty[)$ with the resulting variational integral (1.9) being a *weighted* form of the classical Dirichlet energy and compute among other things the two Euler-Lagrange operators $\mathscr{E}_L$ and $\mathscr{I}_L$ as formulated in Corollaries 3.6 and 3.7. We then proceed on to solving the ODEs (3.17) and (3.19) by taking advantage of their totally integrable structures before moving on to the system (1.3) and characetrising all its twisting solutions. To this end, referring to (3.14), it is first seen that the Lagrangian L of the $\mathbb{L}$-energy (3.13) here becomes

$$\begin{aligned} L(r,\mathrm{Q},\dot{\mathrm{Q}}) &= \int_{\mathbb{S}^{n-1}} F\big(r,r^2|\mathrm{Q}\theta|^2,n+r^2|\dot{\mathrm{Q}}\theta|^2\big)r^{n-1}\,d\mathcal{H}^{n-1}(\theta) \\ &= \int_{\mathbb{S}^{n-1}} h\big(r,r^2\big)\big(n+r^2|\dot{\mathrm{Q}}\theta|^2\big)\,r^{n-1}\,d\mathcal{H}^{n-1}(\theta) \\ &= \omega_n r^{n-1}h\big(r,r^2\big)\big(n^2+r^2|\dot{\mathrm{Q}}|^2\big). \end{aligned} \tag{4.1}$$

Thus upon noting $F_\xi = h(r,s)$ and using (3.16) or proceeding directly from the last equation in (4.1) we obtain

$$L\big(r, \dot{\mathrm{Q}}\big) = 2\int_{\mathbb{S}^{n-1}} r^{n+1} h\big(r, r^2\big)\dot{\mathrm{Q}}\theta \otimes \theta \, d\mathcal{H}^{n-1}(\theta) = 2\omega_n r^{n+1} h\big(r, r^2\big)\dot{\mathrm{Q}}. \tag{4.2}$$

Now as regards to the Euler–Lagrange operator $\mathscr{E}_L$ in (3.3), by substituting for L_ζ from (4.2) and recalling the identity $\dot{\mathrm{Q}}\mathrm{Q}^t + \mathrm{Q}\dot{\mathrm{Q}}^t = 0$, we have that

$$\begin{aligned}\mathscr{E}_L[\mathrm{Q}] &= -\frac{d}{dr}\big[L_\zeta \mathrm{Q}^t - \mathrm{Q}L_\zeta^t\big] + L_\zeta \dot{\mathrm{Q}}^t - \dot{\mathrm{Q}}L_\zeta^t \\ &= -\frac{d}{dr}\big\{2\omega_n r^{n+1} h\big(r,r^2\big)\big[\dot{\mathrm{Q}}\mathrm{Q}^t - \mathrm{Q}\dot{\mathrm{Q}}^t\big]\big\} \\ &= -\frac{d}{dr}\big\{4\omega_n r^{n+1} h\big(r,r^2\big)\dot{\mathrm{Q}}\mathrm{Q}^t\big\} \\ &= -\mathrm{Q}\frac{d}{dr}\Big\{4\omega_n r^{n+1} h\big(r,r^2\big)\mathrm{Q}^t\dot{\mathrm{Q}}\Big\}\mathrm{Q}^t.\end{aligned} \tag{4.3}$$

The last equation here is the product $\mathrm{Q}\mathscr{F}_L[\mathrm{Q}]\mathrm{Q}^t$ with the operator $\mathscr{F}_L$ as in (3.7) (see Remark 3.2). Likewise for the Euler–Lagrange operator $\mathscr{I}_L$ in (3.11), writing $\mathrm{Q} = \exp\{\mathscr{G}\mathrm{H}\}$, $\dot{\mathrm{Q}} = \dot{\mathscr{G}}\mathrm{HQ}$ and noting $\big\langle \mathrm{H}^2\mathrm{Q}, \mathrm{HQ}\big\rangle = 0$ we obtain

$$\begin{aligned}\mathscr{I}_L[\mathscr{G}] &= -\frac{d}{dr}\langle L_\zeta, \mathrm{H}\exp\{\mathscr{G}\mathrm{H}\}\rangle + \big\langle L_\eta - \dot{\mathscr{G}}\mathrm{H}L_\zeta, \mathrm{H}\exp\{\mathscr{G}\mathrm{H}\}\big\rangle \\ &= -2\omega_n\Big\{\frac{d}{dr}\big\langle r^{n+1}h\big(r,r^2\big)\dot{\mathscr{G}}\mathrm{HQ}, \mathrm{HQ}\big\rangle + \big\langle r^{n+1}h\big(r,r^2\big)\dot{\mathscr{G}}^2\mathrm{H}^2\mathrm{Q}, \mathrm{HQ}\big\rangle\Big\} \\ &= -2\omega_n\frac{d}{dr}\big\{r^{n+1}h\big(r,r^2\big)\dot{\mathscr{G}}\big\}|\mathrm{H}|^2,\end{aligned} \tag{4.4}$$

agreeing with (3.19) in this context. An interesting outcome here is that unlike the operator $\mathscr{E}_L = \mathscr{E}_L[\mathrm{Q}]$ in (4.3), the operator $\mathscr{I}_L = \mathscr{I}_L[\mathscr{G}]$ in (4.4) is linear. Moreover, by direct verification it is seen that any solution to $\mathscr{I}_L[\mathscr{G}] = 0$ (for H fixed) corresponds to a solution $\mathrm{Q} = \exp\{\mathscr{G}\mathrm{H}\}$ to $\mathscr{E}_L[\mathrm{Q}] = 0$ which is a reverse to the implication discussed in Remark 3.5.

Now as our first task we aim at resolving the ODE (3.17) subject to identity boundary conditions, hence obtaining all the extremising twist paths associated with the energy integral (3.13) in $\mathscr{B}_2(a,b)$ with the choice $F = h(r,s)\xi$. This, upon referring to (4.3) amounts to solving

$$\begin{aligned}\mathscr{E}_L[\mathrm{Q}] = 0 &\Leftrightarrow \frac{d}{dr}\Big\{\int_{\mathbb{S}^{n-1}} r^{n+1}h\big(r,r^2\big)\big[\dot{\mathrm{Q}}\theta \otimes \mathrm{Q}\theta - \mathrm{Q}\theta \otimes \dot{\mathrm{Q}}\theta\big]\Big\} = 0, \\ &\Leftrightarrow \frac{d}{dr}\big\{r^{n+1}h\big(r,r^2\big)\dot{\mathrm{Q}}\mathrm{Q}^t\big\} = 0.\end{aligned} \tag{4.5}$$

Integrating (4.5) once gives $r^{n+1}h(r,r^2)\dot{\mathrm{Q}}\mathrm{Q}^t = \mathrm{H}$ where H is a constant $n \times n$ skew-symmetric matrix. This by noting the boundary condition $\mathrm{Q}(a) = \mathrm{I}_n$ on twist paths as required by $\mathrm{Q} \in \mathscr{B}_2(a,b)$ [see (3.1) with $p = 2$] has the general solution $\mathrm{Q} = \mathrm{Q}(r)$ given by the Lie exponential

$$\begin{aligned}\mathrm{Q}(r) &= \exp\{\mathscr{G}(r)\mathrm{H}\}, \quad a \le r \le b,\\ \mathscr{G}(r) &= \bigg\{\int_a^r \frac{s^{-(n+1)}\,ds}{h(s,s^2)}\bigg\}\bigg\{\int_a^b \frac{s^{-(n+1)}\,ds}{h(s,s^2)}\bigg\}^{-1}.\end{aligned} \tag{4.6}$$

We see from the above that $\mathscr{G}(a) = 0$ and $\mathscr{G}(b) = 1$ so the boundary condition $\mathrm{Q}(a) = \mathrm{I}_n$ for the twist path is immediately seen to be satisfied. Depending on whether the dimension n is even or odd, the skew-symmetric matrix H can be orthogonally diagonalised and written as $\mathrm{H} = \mathrm{P}\,\mathrm{diag}(c_1\mathrm{J}, \dots, c_k\mathrm{J})\mathrm{P}^t$ when $n = 2k$, and $\mathrm{H} = \mathrm{P}\,\mathrm{diag}(c_1\mathrm{J}, \dots, c_{k-1}\mathrm{J}, 0)\mathrm{P}^t$ when $n = 2k-1$. Here $\mathrm{P} \in \mathrm{O}(n)$ and the scalars $c_1, \dots, c_k$ are all real – in fact, the eigenvalues of H are seen to be $\pm ic_j$ with $1 \le j \le k$ when $n = 2k$, and $0, \pm ic_j$ with $1 \le j \le k-1$ when $n = 2k-1$. Furthermore, here and for future reference, the 2×2 matrices J and R are given respectively by

$$\mathrm{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \mathrm{R}[\pi/2], \qquad \mathrm{R}[t] = \{t\mathrm{J}\} = \exp\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \tag{4.7}$$

both lying in the special orthogonal group $\mathrm{SO}(2)$. It is thus seen that

$$\mathrm{Q}(b) = \mathrm{I}_n \iff \exp\{\mathscr{G}(b)\mathrm{H}\} = \mathrm{I}_n \Leftrightarrow \exp\{\mathrm{H}\} = \mathrm{I}_n \tag{4.8}$$

and plainly this last identity holds if and only if $c_j \in 2\pi\mathbb{Z}$ for all $1 \le j \le k$. This therefore characterises all solutions to $\mathscr{E}_L[\mathrm{Q}] = 0$ in $\mathscr{B}_2(a,b)$ as $\mathrm{Q}(r) = \exp\{\mathscr{G}(r)\mathrm{H}\}$ with $\mathscr{G}$ as in (4.6) and H satisfying (4.8) as just described.

Now, moving forward onto evaluating the action of the differential operator $\mathscr{L}$ on the twist map u with twist path $\mathrm{Q} = \mathrm{Q}(r)$ we first note that here

$$\begin{aligned}\mathscr{L}[u] &= [\nabla u]^t\big\{\mathrm{div}\big[h\big(r,|u|^2\big)\nabla u\big] - h_s\big(r,|u|^2\big)|\nabla u|^2 u\big\}\\ &= [\nabla u]^t\big\{\nabla u\nabla\big[h(r,|u|^2)\big] + h\big(r,|u|^2\big)\Delta u - h_s\big(r,|u|^2\big)|\nabla u|^2 u\big\},\end{aligned} \tag{4.9}$$

and so upon differentiation, substitution for u and noting $|u|^2 = r^2$ we can write

$$\begin{aligned}\mathscr{L}[u] =& \big(\mathrm{Q}^t + r\theta\otimes\dot{\mathrm{Q}}\theta\big)\big\{\big[h_r\big(r,r^2\big) + 2rh_s\big(r,r^2\big)\big](\mathrm{Q} + r\dot{\mathrm{Q}})\\ &+ h(r,r^2)\big[(n+1)\dot{\mathrm{Q}} + r\ddot{\mathrm{Q}}\big] - rh_s\big(r,r^2\big)\big(n + r^2|\dot{\mathrm{Q}}\theta|^2\big)\mathrm{Q}\big\}\theta.\end{aligned} \tag{4.10}$$

Expanding (4.5) by direct differentiation and using $\mathscr{F}_L[\mathrm{Q}] = \mathrm{Q}^t\mathscr{E}_L[\mathrm{Q}]\mathrm{Q} = 0$ the above simplifies to

$$\begin{aligned}\mathscr{L}[u] =& \big[h_r\big(r,r^2\big) + 2rh_s\big(r,r^2\big)\big]\theta\\ &+ \big[r^2h_r\big(r,r^2\big) + r^3h_s\big(r,r^2\big) + (n+1)rh\big(r,r^2\big)\big]|\dot{\mathrm{Q}}\theta|^2\theta\\ &+ \big[r^2h(r,r^2)\big\langle\dot{\mathrm{Q}}\theta,\ddot{\mathrm{Q}}\theta\big\rangle - nrh_s(r,r^2)\big]\theta - rh(r,r^2)\mathrm{Q}^t\,\dot{\mathrm{Q}}\,\dot{\mathrm{Q}}^t\,\mathrm{Q}\theta.\end{aligned} \tag{4.11}$$

Referring to (1.3) we next need to verify $\mathscr{L}[u] = \nabla\mathscr{P}$. Clearly the first two terms in (4.11) form $\nabla h(|x|,|x|^2)$ whilst upon substituting $\mathrm{Q}(r) = \exp\{\mathscr{G}(r)\mathrm{H}\}$

from the solution to $\mathscr{E}_L[\mathrm{Q}] = 0$ with $\dot{\mathrm{Q}} = \dot{\mathscr{G}}\mathbf{H}\mathrm{Q}$ and $\ddot{\mathrm{Q}} = \big(\ddot{\mathscr{G}}\mathrm{H} + \dot{\mathscr{G}}^2\mathrm{H}^2\big)\mathrm{Q}$ it is plain that $\big\langle \dot{\mathrm{Q}}\theta, \ddot{\mathrm{Q}}\theta\big\rangle = \dot{\mathscr{G}}\ddot{\mathscr{G}}|\mathrm{H}\theta|^2$ and $\mathrm{Q}^t\dot{\mathrm{Q}}\dot{\mathrm{Q}}^t\mathrm{Q} = -\dot{\mathscr{G}}^2\mathrm{H}^2$. Therefore

$$
\begin{aligned}
(4.12)\qquad \mathscr{L}[u = r\exp\{\mathscr{G}(r)\mathrm{H}\}\theta] &= \nabla h\big(|x|, |x|^2\big)\\
&+ \big[r^2 h_r\big(r, r^2\big) + r^3 h_s\big(r, r^2\big) + (n+1) r h\big(r, r^2\big)\big]\dot{\mathscr{G}}^2|\mathrm{H}\theta|^2\theta\\
&+ [r^2 h(r, r^2)\dot{\mathscr{G}}\ddot{\mathscr{G}}|\mathrm{H}\theta|^2 - n r h_s(r, r^2)]\theta + r h(r, r^2)\dot{\mathscr{G}}^2\mathrm{H}^2\theta.
\end{aligned}
$$

Now an application of Lemma 4.1 (see below) to the vector field $\mathscr{L}[u]$ as given above and noting $\dot{\mathscr{B}}/r - 2\mathscr{A} \not\equiv 0$ if and only if $r h_r(r, r^2) + 2(n+1)h(r, r^2) + 4r^2 h_s(r, r^2) \not\equiv 0$ on $]a, b[$ (cf. Lemma 4.1 for notation) leads to

$$
(4.13)\qquad \operatorname{curl}\big(\mathscr{L}[u = r\exp\{\mathscr{G}(r)\mathrm{H}\}\theta] - \nabla h\big) = 0 \iff \mathrm{H}^2 = -c^2\,\mathrm{I}_n.
$$

This therefore leads to the conclusion $|c_1|^2 = \ldots = |c_k|^2 = c^2$ when $n = 2k$, and $|c_1| = \ldots = |c_{k-1}| = 0$ when $n = 2k - 1$. Finally, setting $c = 2m\pi$ with $m \in \mathbb{Z}$ ($m = 0$ when n odd) the boundary condition $\mathrm{Q}(b) = \mathrm{I}_n$ is also seen to be satisfied. In conclusion, and summarising, we see that here the *reduced* Euler–Lagrange equation (the ODE) versus the *full* Euler–Lagrange equation (the PDE) associated with the choice $F = h(r, s)\xi$ have the following contrasting consequences:

(ODE I) From the formulation of the Euler–Lagrange operator $\mathscr{E}_L$ in (4.3) and the resulting ODE (4.5) it follows that here all extremising twist paths are of Lie exponential form, specifically,

$$
(4.14)\qquad \mathscr{E}_L[\mathrm{Q}] = 0 \iff \mathrm{Q}(r) = \exp\{\mathscr{G}(r)\mathrm{H}\}\mathrm{Q}_a \qquad \mathrm{H}^t = -\mathrm{H}.
$$

Here $\mathrm{Q}_a = \mathrm{Q}(a)$ and the profile curve $\mathscr{G} = \mathscr{G}(r)$ is as described by (4.6). If additionally Q is to lie in $\mathscr{B}_2(a, b)$ then $\mathrm{Q}_a = \mathrm{I}_n$ and the skew-symmetric matrix H must be further restricted to (4.8).

(PDE) Here a twist solution $u = r\mathrm{Q}(r)\theta$ to the system $\mathscr{L}[u] = \nabla\mathscr{P}$ in $\mathbb{X}^n$ with $u = x$ on $\partial\mathbb{X}^n$ (cf. (1.3)) must have an extremising twist path of Lie exponential form $\mathrm{Q}(r) = \exp\{\mathscr{G}(r)\mathrm{H}\}$ satisfying $\mathrm{Q}(a) = \mathrm{Q}(b) = \mathrm{I}_n$. Now $\mathscr{L}[u]$ simplifies to (4.12) and so subject to $r h_r(r, r^2) + 2(n+1)h(r, r^2) + 4r^2 h_s(r, r^2) \not\equiv 0$ on $]a, b[$ we have

$$
(4.15)\qquad \operatorname{curl}\big(\mathscr{L}[u] - \nabla h\big) = 0 \iff \mathbf{H} = 2\pi m \times \begin{cases} \mathrm{P}\mathrm{J}_n\mathrm{P}^t & \text{for } n \text{ even},\\ 0 & \text{for } n \text{ odd}. \end{cases}
$$

Hence $u \equiv x$ (for n odd) and $u = r\,\mathrm{P}\exp\{2\pi m\mathscr{G}(r)\mathrm{J}_n\}\mathrm{P}^t\theta$ (for n even). Here $\mathrm{J}_n = \mathrm{diag}(\mathrm{J}, \ldots, \mathrm{J})$ with J as in (4.7). (6)

(6) When $n = 2k$, (4.13) gives $c_1, \ldots, c_k \in \{\pm c\}$. Adjusting $\mathrm{P} \in \mathrm{O}(n)$ in an obvious way if necessary we can arrange and assume that indeed $c_1 = \ldots = c_k = c$.

(ODE II) As a further remark note that by considering the strengthened form of the ODE (3.17)–(4.3) that is obtained by discarding the spherical integral and instead assuming (for all $a < r < b$, $\theta \in \mathbb{S}^{n-1}$):

$$\frac{d}{dr}\big\{r^{n+1}h\big(r,r^2\big)\big[\dot{\mathrm{Q}}\theta\otimes\mathrm{Q}\theta-\mathrm{Q}\theta\otimes\dot{\mathrm{Q}}\theta\big]\big\}=0, \tag{4.16}$$

we have from (4.14) that any solution here has the form $\mathrm{Q}(r) = \exp\{\mathscr{G}(r)\mathrm{H}\}$ and upon noting $\mathrm{QH} = \mathrm{HQ}$ and invoking Proposition 7.1 in [27] that

$$\begin{aligned}(4.16)\ \Leftrightarrow\ &\frac{d}{dr}\big\{r^{n+1}h(r,r^2)\dot{\mathscr{G}}(r)[\mathrm{HQ}\theta\otimes\mathrm{Q}\theta-\mathrm{Q}\theta\otimes\mathrm{HQ}\theta]\big\}=0\\ \Leftrightarrow\ &\frac{d}{dr}\big\{r^{n+1}h(r,r^2)\dot{\mathscr{G}}(r)\big\}[\mathrm{HQ}\theta\otimes\mathrm{Q}\theta-\mathrm{Q}\theta\otimes\mathrm{HQ}\theta]\\ &+\big\{r^{n+1}h(r,r^2)\dot{\mathscr{G}}(r)^2\big\}[\mathrm{H}^2\mathrm{Q}\theta\otimes\mathrm{Q}\theta-\mathrm{Q}\theta\otimes\mathrm{H}^2\mathrm{Q}\theta]=0.\end{aligned}$$

In conclusion (4.16) $\Leftrightarrow$ $\mathrm{Q}[\mathrm{H}^2\theta\otimes\theta-\theta\otimes\mathrm{H}^2\theta]\mathrm{Q}^t=0$ $\Leftrightarrow$ $\mathrm{H}^2=-c^2\mathrm{I}_n$. It is thus seen that this strengthened version of (3.17) imposes the same restriction on the twist paths Q as does the curl-free condition in the PDE. (Note however that here $rh_r(r,r^2)+2(n+1)h(r,r^2)+4r^2h_s(r,r^2)\not\equiv 0$ is not needed.) This stronger form of (3.17) and its curious implications will be discussed further in the next section.

Lemma 4.1. *Let $\mathscr{A}=\mathscr{A}(r)$, $\mathscr{B}=\mathscr{B}(r)\in\mathscr{C}^1(]a,b[)$ and let $\mathbb{F}\in\mathbb{M}^{n\times n}$ be a constant symmetric matrix. Consider the vector field $U\in\mathscr{C}^1(\mathbb{X}^n,\mathbb{R}^n)$ given by*

$$U_{\mathrm{F}}(x)=\mathscr{A}(|x|)\langle\mathrm{F}x,x\rangle x+\mathscr{B}(|x|)\mathrm{F}x. \tag{4.17}$$

Then $\operatorname{curl}U_{\mathrm{F}}=\big(\dot{\mathscr{B}}/r-2\mathscr{A}\big)[\mathrm{F}x\otimes x-x\otimes\mathrm{F}x]$. *Furthermore, if $\dot{\mathscr{B}}/r-2\mathscr{A}\not\equiv 0$ in $]a,b[$ then*

$$U_{\mathrm{F}}\ \text{is curl-free in}\ \mathbb{X}^n\ \Leftrightarrow\ \exists\alpha\in\mathbb{R}:\mathrm{F}=\alpha\mathrm{I}_n. \tag{4.18}$$

In this event $U_{\mathrm{F}}=\nabla\mathscr{P}$ where $\mathscr{P}=\mathscr{P}(|x|)$ satisfies $d\mathscr{P}/dr=\alpha r(r^2\mathscr{A}+\mathscr{B})$.

Proof. Fix F as described and write $U=U_{\mathrm{F}}$. Then, for indices $1\le i,j\le n$, and with "dot" denoting d/dr as before, a straightforward differentiation gives

$$\begin{aligned}U_{i,j}=\bigg(\dot{\mathscr{A}}(r)\langle\mathrm{F}x,x\rangle\,\frac{x_ix_j}{r}+2\mathscr{A}(r)x_i[\mathrm{F}x]_j+\mathscr{A}(r)\langle\mathrm{F}x,x\rangle\delta_{ij}\bigg)&\\ +\frac{\dot{\mathscr{B}}(r)}{r}[\mathrm{F}x]_ix_j+\mathscr{B}(r)\mathrm{F}_{ij},&\end{aligned}$$

and in a similar way

$$\begin{aligned}U_{j,i}=\bigg(\dot{\mathscr{A}}(r)\langle\mathrm{F}x,x\rangle\,\frac{x_jx_i}{r}+2\mathscr{A}(r)[\mathrm{F}x]_ix_j+\mathscr{A}(r)\langle\mathrm{F}x,x\rangle\,\delta_{ji}\bigg)&\\ +\frac{\dot{\mathscr{B}}(r)}{r}\,x_i[\mathrm{F}x]_j+\mathscr{B}(r)\mathrm{F}_{ji}.&\end{aligned}$$

Thus $\operatorname{curl} U = [\nabla U] - [\nabla U]^t = 2\mathscr{A}(x \otimes \mathrm{F}x - \mathrm{F}x \otimes x) + \dot{\mathscr{B}}/r(\mathrm{F}x \otimes x - x \otimes \mathrm{F}x) = (\dot{\mathscr{B}}/r - 2\mathscr{A})(\mathrm{F}x \otimes x - x \otimes \mathrm{F}x)$ as claimed. Now, if $(\dot{\mathscr{B}}/r - 2\mathscr{A}) \not\equiv 0$, then it follows from U_{F} being curl-free in $\mathbb{X}^n$ that $\mathrm{F}\theta \otimes \theta - \theta \otimes \mathrm{F}\theta \equiv 0$ for all unit vectors θ. This immediately gives $\mathrm{F} = \alpha \mathrm{I}_n$ for some $\alpha \in \mathbb{R}$. Conversely, if $\mathrm{F} = \alpha \mathrm{I}_n$, then $U_{\mathrm{F}} = \alpha(r^2\mathscr{A}(r) + \mathscr{B}(r))x$ is clearly a gradient field in $\mathbb{X}^n$ with the choice of $\mathscr{P}$ as given in the lemma and thus curl free. □

5. Extremising twist paths as scaled geodesics on the Lie group SO(n)

One of the main features of the Euler–Lagrange equation (3.17) is the presence of the spherical integral which, unlike the case with the weighted Dirichlet energy considered in the last section [see (4.5)], prevents one from reducing the equation to a directly integrable ODE in the radial variable and thus obtaining an explicit representation of the solutions as in (4.6). Motivated by the discussion in the previous section we start here by first considering solutions to (3.17) in the form $\mathrm{Q}(r) = \exp\{\mathscr{G}(r)\mathrm{H}\}$ where $\mathscr{G} = \mathscr{G}(r)$ is a suitable function in $\mathscr{C}^2[a,b]$ and H is the constant $n \times n$ skew-symmetric matrix with $\mathrm{H} = \mathrm{P}\mathrm{J}_n\mathrm{P}^t$. Here and below $\mathrm{J}_n = \mathrm{diag}(\mathrm{J}, \dots, \mathrm{J})$ when n is even and $\mathrm{J}_n = \mathrm{diag}(\mathrm{J}, \dots, \mathrm{J}, 0)$ when n is odd. Then starting with the n even case where $\left|\dot{\mathrm{Q}}\theta\right|^2 = \dot{\mathscr{G}}^2|\mathrm{H}\theta|^2 = \dot{\mathscr{G}}^2$ and writing $F_\xi = F_\xi\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big)$ for short it is readily seen that

$$\begin{aligned}\mathrm{LHS}\,(3.17) &= \frac{d}{dr}\left\{r^{n+1}F_\xi\dot{\mathscr{G}}\int_{\mathbb{S}^{n-1}}[\mathrm{HQ}\theta \otimes \mathrm{Q}\theta - \mathrm{Q}\theta \otimes \mathrm{HQ}\theta]\right\}d\mathcal{H}^{n-1}(\theta)\\ &= \frac{d}{dr}\big\{r^{n+1}F_\xi\dot{\mathscr{G}}\omega_n\big[\mathrm{HQQ}^t - \mathrm{Q}(\mathrm{HQ})^t\big]\big\} = \frac{d}{dr}\big\{r^{n+1}F_\xi\dot{\mathscr{G}}\big\}(2\omega_n\mathrm{H}).\end{aligned}$$

As such in even dimensions a twist path $\mathrm{Q}(r) = \exp\{\mathscr{G}(r)\mathrm{H}\}$ is a solution to the Euler–Lagrange equation (3.17) provided that the angle of rotation function $\mathscr{G}$ satisfies the second order ODE

$$\frac{d}{dr}\big[r^{n+1}F_\xi\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big)\dot{\mathscr{G}}\big] = 0, \qquad a < r < b. \tag{5.1}$$

In odd dimensions and by contrast it is a trivial matter to see that $\mathscr{G}(r) \equiv 0$ and hence $\mathrm{Q} \equiv \mathrm{I}_n$ is a solution to (3.17).

Now rather than following the route leading to (4.15) based on an analysis and verification of the PDE (1.3)–(1.10) and the curl-free condition on the vector field $\mathscr{L}[u = r\mathrm{Q}(r)\theta]$, in what follows we focus instead on the the ODE (3.17) and show that by a natural strengthening of (3.17) and invoking an interesting observation regarding geodesics on SO(n), the twist paths $\mathrm{Q} = \mathrm{Q}(r)$ serving as solutions here, must have exactly the form and structure alluded to above. It is readily seen that a stronger condition implying (3.17) is the strengthened ODE:

$$\frac{d}{dr}\big\{r^{n+1}F_\xi\big(r, r^2, n + r^2|\dot{\mathrm{Q}}\theta|^2\big)\big[\dot{\mathrm{Q}}\theta \otimes \mathrm{Q}\theta - \mathrm{Q}\theta \otimes \dot{\mathrm{Q}}\theta\big]\big\} = 0, \quad a < r < b, \tag{5.2}$$

for all $\theta \in \mathbb{S}^{n-1}$. That $\mathrm{Q} = \exp\{\mathscr{G}\mathrm{H}\}$ with $\mathscr{G}$ satisfying (5.1) is still a solution to this stronger form of (3.17) follows by noting that here $\dot{\mathrm{Q}}\theta \otimes \mathrm{Q}\theta = \dot{\mathscr{G}}\mathrm{HQ}\theta \otimes \mathrm{Q}\theta$ and $\ddot{\mathrm{Q}}\theta \otimes \mathrm{Q}\theta = \ddot{\mathscr{G}}\mathrm{HQ}\theta \otimes \mathrm{Q}\theta - \dot{\mathscr{G}}^2\mathrm{Q}\theta \otimes \mathrm{Q}\theta$. Hence, for n even, by substitution and a straightforward differentiation starting from (3.17) we have

$$
\begin{aligned}
\text{(5.3)} \qquad \text{LHS (5.2)} &= \frac{d}{dr}\big\{r^{n+1}F_\xi\big(r, r^2, n + r^2|\dot{\mathrm{Q}}\theta|^2\big)\big[\dot{\mathrm{Q}}\theta \otimes \mathrm{Q}\theta - \mathrm{Q}\theta \otimes \dot{\mathrm{Q}}\theta\big]\big\} \\
&= \bigg\{\frac{d}{dr}\big[r^{n+1}F_\xi\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big)\big]\dot{\mathscr{G}} + r^{n+1}F_\xi\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big)\ddot{\mathscr{G}}\bigg\} \\
&\quad \times (\mathrm{HQ}\theta \otimes \mathrm{Q}\theta - \mathrm{Q}\theta \otimes \mathrm{HQ}\theta) \\
&= \frac{d}{dr}\big[r^{n+1}F_\xi\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big)\dot{\mathscr{G}}\big](\mathrm{HQ}\theta \otimes \mathrm{Q}\theta - \mathrm{Q}\theta \otimes \mathrm{HQ}\theta) = 0,
\end{aligned}
$$

as claimed. Now moving forward note that for a twist path $\mathrm{Q} \in \mathscr{C}^1([a,b], \mathrm{SO}(n))$ the integral $I(\mathrm{Q},\theta) = ||\dot{\mathrm{Q}}\theta||_{L^1(a,b)}$ represents the Euclidean length of the curve $\gamma_\theta \in \mathscr{C}^1([a,b], \mathbb{S}^{n-1})$ given by $\gamma_\theta(r) = \mathrm{Q}(r)\theta$. Evidently for n even and $\mathrm{Q} = \exp\{\mathscr{G}\mathrm{H}\}$ with $\mathrm{H} = \mathrm{PJ}_n\mathrm{P}^t$ these lengths are independent of θ. We are now in a position to prove a structure theorem for such Q.

Theorem 5.1. *Assume that* $\mathrm{Q} \in \mathscr{C}^1([a,b], \mathrm{SO}(n)) \cap \mathscr{C}^2(]a,b[, \mathrm{SO}(n))$ *verifying* $\mathrm{Q}(a) = \mathrm{I}_n$ *and* $\mathrm{Q}(b) = \mathrm{I}_n$ *satisfies* (5.2). *Assume that the lengths* $I(\mathrm{Q},\theta)$ *of the family of curves* $\gamma_\theta(r) = \mathrm{Q}(r)\theta$ *are independent of* $\theta \in \mathbb{S}^{n-1}$. *Then*:

(a) *n even*: *There exists* $m \in \mathbb{Z}$ *and* $\mathrm{P} \in \mathrm{O}(n)$ *so that* Q *admits the representation* $\mathrm{Q}(r) = \mathrm{Q}(r;m) = \exp\{\mathscr{G}(r;m)\mathrm{PJ}_n\mathrm{P}^t\}$, *that is*

$$
\text{(5.4)} \qquad \mathrm{Q}(r) = \mathrm{P}\begin{bmatrix} \mathrm{R}[\mathscr{G}](r;m) & 0 & \dots & 0 & 0 \\ 0 & \mathrm{R}[\mathscr{G}](r;m) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathrm{R}[\mathscr{G}](r;m) & 0 \\ 0 & 0 & \dots & 0 & \mathrm{R}[\mathscr{G}](r;m) \end{bmatrix}\mathrm{P}^t.
$$

Here $\mathrm{J}_n = \mathrm{diag}(\mathrm{J}, \dots, \mathrm{J})$ *and* $\mathscr{G} = \mathscr{G}(r;m) \in \mathscr{C}^2[a,b]$ *is the unique solution to the two point boundary value problem* ([7])

$$
\text{(5.5)} \qquad \mathbb{BVP}[\mathscr{G};m] := \begin{cases} \dfrac{d}{dr}\big[r^{n+1}F_\xi\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big)\dot{\mathscr{G}}\big] = 0, & a < r < b, \\ \mathscr{G}(a) = 0, \\ \mathscr{G}(b) = 2m\pi. \end{cases}
$$

(b) *n odd*: $\mathrm{Q} \equiv \mathrm{I}_n$ *corresponding to* $m = 0$ *and* $\mathscr{G} \equiv 0$ *in* (5.5).

Proof. Since $I(\mathrm{Q},\theta) = 0$ implies $|\dot{\mathrm{Q}}\theta| = 0$ and hence $\mathrm{Q} \equiv \mathrm{I}_n$, in the rest of the proof we assume $I(\mathrm{Q},\theta) > 0$. Now we start by observing that, if Q is

---

([7]) Note that for even n any matrix $\mathrm{H} = \mathrm{PJ}_n\mathrm{P}^t$ is a *skew-symmetric* square root of $-\mathrm{I}_n$. For odd n there is no such root.

a solution to (5.2) for every θ, then it also satisfies the equation

$$\frac{d}{dr}\left[r^{n+1}F_\xi(r,r^2,n+r^2|\dot{\mathrm{Q}}\theta|^2)\dot{\mathrm{Q}}\theta\right]+r^{n+1}F_\xi\big(r,r^2,n+r^2|\dot{\mathrm{Q}}\theta|^2\big)\big|\dot{\mathrm{Q}}\theta\big|^2\mathrm{Q}\theta=0. \tag{5.6}$$

Indeed starting from the left and writing $F_\xi = F_\xi\big(r,r^2,n+r^2|\dot{\mathrm{Q}}\theta|^2\big)$ we have by virtue of $\big[\dot{\mathrm{Q}}\theta\otimes\mathrm{Q}\theta\big]\mathrm{Q}\theta=\dot{\mathrm{Q}}\theta$ and $\big\langle\dot{\mathrm{Q}}\theta,\mathrm{Q}\theta\big\rangle=0$,

$$\begin{aligned}\frac{d}{dr}\big\{r^{n+1}F_\xi\dot{\mathrm{Q}}\theta\big\} &= \frac{d}{dr}\big\{r^{n+1}F_\xi\big[\dot{\mathrm{Q}}\theta\otimes\mathrm{Q}\theta-\mathrm{Q}\theta\otimes\dot{\mathrm{Q}}\theta\big]\mathrm{Q}\theta\big\}\\ &=\frac{d}{dr}\big\{r^{n+1}F_\xi\big[\dot{\mathrm{Q}}\theta\otimes\mathrm{Q}\theta-\mathrm{Q}\theta\otimes\dot{\mathrm{Q}}\theta\big]\big\}\mathrm{Q}\theta\\ &\quad+\big\{r^{n+1}F_\xi\big[\dot{\mathrm{Q}}\theta\otimes\mathrm{Q}\theta-\mathrm{Q}\theta\otimes\dot{\mathrm{Q}}\theta\big]\big\}\dot{\mathrm{Q}}\theta=-r^{n+1}F_\xi\big|\dot{\mathrm{Q}}\theta\big|^2\mathrm{Q}\theta,\end{aligned} \tag{5.7}$$

where in deducing the last equality we have used (5.2). Let us now introduce the integral $\mathscr{F}(r,\theta) := \int_a^r |\dot{\mathrm{Q}}(s)\theta|\,ds$ with $a \le r \le b$ and $\theta\in\mathbb{S}^{n-1}$. Then, testing (5.5) against $\mathscr{F}$ and using (5.6) by way of differentiating and then taking the inner product with $\dot{\mathrm{Q}}\theta$, we can write with $F_\xi=F_\xi\big(r,r^2,n+r^2\big|\dot{\mathrm{Q}}\theta\big|^2\big)$ and upon noting $\dot{\mathscr{F}}^2=|\dot{\mathrm{Q}}\theta|^2$,

$$\begin{aligned}\frac{d}{dr}\big\{r^{n+1}F_\xi|\dot{\mathrm{Q}}\theta|\big\} &= \frac{d}{dr}\big\{r^{n+1}F_\xi\big\}\big|\dot{\mathrm{Q}}\theta\big| + r^{n+1}F_\xi\,\frac{\big\langle\ddot{\mathrm{Q}}\theta,\dot{\mathrm{Q}}\theta\big\rangle}{\big|\dot{\mathrm{Q}}\theta\big|}\\ &= -r^{n+1}F_\xi\big\langle\dot{\mathrm{Q}}\theta,\mathrm{Q}\theta\big\rangle\big|\dot{\mathrm{Q}}\theta\big|=0,\end{aligned}$$

where the last identity uses the skew-symmetry of $\dot{\mathrm{Q}}\mathrm{Q}^t$. Note that this argument shows that, as a function of r, $r^{n+1}F_\xi|\dot{\mathrm{Q}}\theta|$ is a positive constant on any interval on which $|\dot{\mathrm{Q}}\theta|$ is non-zero and so a basic continuity argument implies that either $\big|\dot{\mathrm{Q}}\theta\big|\equiv 0$ on $[a,b]$ or $\big|\dot{\mathrm{Q}}\theta\big|>0$ on $[a,b]$. Furthermore, it also shows that $\mathscr{F}(r,\theta)$ is a [*non-zero*] solution to the ODE in (5.5) for every fixed $\theta\in\mathbb{S}^{n-1}$.

Now this solution satisfies the end-point conditions $\mathscr{F}(a)=0$ and $\mathscr{F}(b)=I(\mathrm{Q},\theta)>0$ where the latter by assumption is independent of θ. We next aim to show that these together imply that $\mathscr{F}(r,\theta)$ is independent of θ. To this end we first note that solutions to (5.5) are extremisers in their Dirichlet class of the energy integral

$$\Gamma[\mathscr{G}]=\int_a^b F\big(r,r^2,n+r^2\dot{\mathscr{G}}^2(r)\big)r^{n-1}\,dr. \tag{5.8}$$

It is straightforward to verify that the functional Γ is strictly convex (due to the assumptions on F: $F_\xi>0$ and F being uniformly convex in ξ). Therefore, using standard results, solutions to (5.5) are the unique minimisers of this energy functional with respect to their own boundary conditions. This implies that as $\mathscr{F}(r,\theta)$ solves the ODE in (5.5) for all θ and the end-point conditions on $\mathscr{F}$, i.e. at $r=a$ and $r=b$ are independent of θ, by the stated uniqueness of minimisers, the function $\mathscr{F}(r,\theta)$ must also be independent of θ. Now returning to the ODE in (5.5) it follows after integrating once that any solution $\mathscr{G}=\mathscr{G}(r)$ satisfies

$$r^{n+1}F_\xi\big(r,r^2,n+r^2\dot{\mathscr{G}}^2\big)\dot{\mathscr{G}}\equiv c,\quad a<r<b, \tag{5.9}$$

for a suitable constant $c\in\mathbb{R}$. Thus as $F_\xi>0$, all non-zero solutions to (5.5), in particular $\mathscr{F}$, are (*strictly*) monotone and hence invertible. Let $\mathscr{F}^{-1}(s)=r(s)$ and $\mathrm{Q}(r(s))=\mathrm{L}(s)$ for $\mathrm{L}\in\mathscr{C}^2(]0,l[,\mathrm{SO}(n))\cap\mathscr{C}([0,l],\mathrm{SO}(n))$ where $l=\mathscr{F}(b)$. Then writing $\mathrm{Q}(r)=\mathrm{L}(\mathscr{F}(r))$ we have $\dot{\mathrm{Q}}=\mathrm{L}'\dot{\mathscr{F}}$ (where prime denotes d/ds). Hence starting from (5.6) we can write, with $F_\xi=F_\xi\big(r,r^2,n+r^2\dot{\mathscr{F}}^2\big)$ for short,

$$\frac{d}{dr}\big[r^{n+1}F_\xi\dot{\mathrm{Q}}\theta\big]+r^{n+1}F_\xi\big|\dot{\mathrm{Q}}\theta\big|^2\mathrm{Q}\theta=0. \tag{5.10}$$

This upon substitution and a change of variables with $d/dr=\dot{\mathscr{F}}d/ds$ gives

$$\frac{d}{ds}\bigg[r^{n+1}F_\xi\dot{\mathscr{F}}\,\frac{d\,\mathrm{L}}{ds}\,\theta\bigg]+r^{n+1}F_\xi\dot{\mathscr{F}}\bigg|\frac{d\,\mathrm{L}}{ds}\,\theta\bigg|^2\mathrm{L}\,\theta=c\big[\mathrm{L}''+|\mathrm{L}'\theta|^2\mathrm{L}\big]\theta=0, \tag{5.11}$$

that is the geodesic equation on the unit sphere for $\gamma(s)=\mathrm{L}(s)\theta$. We need to solve this for $\mathrm{L}=\mathrm{L}(s)$ subject to $|\mathrm{L}'\theta|^2=\big|\dot{\mathrm{Q}}\theta\big|^2/\dot{\mathscr{F}}^2=1$.

Indeed by taking the exponential $\mathrm{L}(s)=\exp\{s\mathrm{A}\}$ for a constant $n\times n$ skew-symmetric matrix A we have $[\mathrm{A}^2+\mathrm{I}_n]\mathrm{L}=0$. For n odd this has no solution (with $I(\mathrm{Q},\theta)>0$) whilst for n even it gives $\mathrm{A}=\mathrm{P}\mathrm{J}_n\mathrm{P}^t$. It now follows at once that Q has the form described in the theorem, that is, for n odd $\mathrm{Q}(r)\equiv\mathrm{I}_n$ and for n even firstly $\mathrm{L}(s)=\mathrm{P}\,\mathrm{diag}(\mathrm{R}[s],\dots,\mathrm{R}[s])\mathrm{P}^t$ for $0\le s\le l$ with $l=2m\pi$ so that $\mathrm{L}(0)=\mathrm{L}(l)=\mathrm{I}_n$ and then $\mathrm{Q}(r)=\mathrm{L}(\mathscr{F}(r))$ where $\mathscr{F}$ is a solution to (5.5) with $\mathscr{F}(a)=0$, $\mathscr{F}(b)=2m\pi$. □

6. The nonlinear system (1.3)–(1.10) as a system in variation and the multiple twist solutions

Let us begin this section by illustrating as to why the system (1.3) is in variational form and how it arises as the Euler–Lagrange system (ELS) associated with the energy $\mathbb{F}$ over the space $\mathscr{A}_p(\Omega)$. Towards this end we use the method of Lagrange multipliers and consider the unconstrained energy functional (see [3], [5], [8])

$$\mathbb{E}[u,\Omega]:=\int_\Omega\mathscr{F}(x,u,\nabla u)\,dx-\int_\Omega 2\mathscr{P}(x)[\det\nabla u-1]\,dx, \tag{6.1}$$

where $\mathscr{P}$ is a suitable Lagrange multiplier. Note in particular that here $\mathbb{E}[u,\Omega]=\mathbb{F}[u,\Omega]$ when $u\in\mathscr{A}_p(\Omega)$. Now fix $u\in\mathscr{A}_p(\Omega)$ of class $\mathscr{C}^2$ and for $\phi\in\mathscr{C}^\infty_c(\Omega,\mathbb{R}^n)$ and $\varepsilon\in\mathbb{R}$ put $u_\varepsilon=u+\varepsilon\phi$. We proceed on to examining the first-order condition $d/d\varepsilon(\mathbb{E}[u_\varepsilon,\Omega])|_{\varepsilon=0}=0$ where

$$\frac{d}{d\varepsilon}\,\mathbb{E}[u_\varepsilon,\Omega]\bigg|_{\varepsilon=0}=\frac{d}{d\varepsilon}\int_\Omega\big\{\mathscr{F}(x,u_\varepsilon,\nabla u_\varepsilon\big)-2\mathscr{P}(x)\big[\det\nabla u_\varepsilon-1\big]\big\}\,dx\bigg|_{\varepsilon=0}. \tag{6.2}$$

Now as

$$\frac{d}{d\varepsilon}[\det(\mathrm{E}+\varepsilon\mathrm{F})]_{\varepsilon=0} = \frac{d}{d\varepsilon}\big[\det\mathrm{E} + \varepsilon\langle\mathrm{cof}\, E, \mathrm{F}\rangle + O(\varepsilon^2)\big]_{\varepsilon=0} = \langle\mathrm{cof}\, E, \mathrm{F}\rangle,$$

evaluating the derivative, using the divergence theorem and writing for brevity $\mathscr{F}_u = \mathscr{F}_u(x,u,\nabla u)$, $\mathscr{F}_\zeta = \mathscr{F}_\zeta(x,u,\nabla u)$ where $\mathscr{F}_u$, $\mathscr{F}_\zeta$ denote the derivatives of $\mathscr{F}$ with respect to the second and third arguments respectively, leads to

$$\begin{aligned}(6.3)\quad \frac{d}{d\varepsilon}\mathbb{E}[u_\varepsilon,\Omega]\bigg|_{\varepsilon=0} &= \int_\Omega \{\langle\mathscr{F}_u,\phi\rangle + \langle\mathscr{F}_\zeta,\nabla\phi\rangle - 2\mathscr{P}\langle\mathrm{cof}\,\nabla u,\nabla\phi\rangle\}\,dx\\ &= \int_\Omega \langle\mathscr{F}_u - \mathrm{div}\,\mathscr{F}_\zeta + 2\,\mathrm{cof}\,\nabla u\nabla\mathscr{P} + 2\mathscr{P}\mathrm{div}\,\mathrm{cof}\,\nabla u,\phi\rangle\,dx\\ &= \int_\Omega \langle\mathscr{F}_u - \mathrm{div}\mathscr{F}_\zeta + 2\,\mathrm{cof}\,\nabla u\nabla\mathscr{P},\phi\rangle\,dx = 0.\end{aligned}$$

Note that the last line here uses the Piola identity (see, e.g. [22], [35]) whilst the divergence operator as before acts row-wise on the matrix field $\mathscr{F}_\zeta(x,u,\nabla u)$. Now the arbitrariness of $\phi \in \mathscr{C}^\infty_c(\Omega,\mathbb{R}^n)$ gives $\mathscr{F}_u - \mathrm{div}\mathscr{F}_\zeta + 2\,\mathrm{cof}\,\nabla u\nabla\mathscr{P} = 0$ or alternatively upon using $[\mathrm{cof}\,\nabla u]^{-1} = [\nabla u]^t$ (recall that $\det\nabla u \equiv 1$) that

$$(6.4)\quad \mathscr{L}[u;\mathscr{F}] = [2\,\mathrm{cof}\,\nabla u]^{-1}\{\mathrm{div}\,\mathscr{F}_\zeta - \mathscr{F}_u\} = [\nabla u]^t\{\mathrm{div}\,\mathscr{F}_\zeta - \mathscr{F}_u\}/2 = \nabla\mathscr{P}.$$

Remark 6.1. For $\mathbb{F}$ as in (1.9) with Lagrangian F satisfying the assumptions set at the start of Section 2 (cf. also (2.1)) the above derivation should be slightly adjusted. Here we take $u \in \mathscr{A}_p(\Omega)$ of class $\mathscr{C}^2$ with $\big(|x|,|u|^2,|\nabla u|^2\big) \in U = U(\Omega)$ for all $x \in \Omega$ and for $\phi \in \mathscr{C}^\infty_c(\Omega,\mathbb{R}^n)$ and $\varepsilon \in \mathbb{R}$ put $u_\varepsilon = u + \varepsilon\phi$. Then, by a basic compactness argument for ε sufficiently small, $\big(|x|,|u_\varepsilon|^2,|\nabla u_\varepsilon|^2\big) \in U$ for all $x \in \Omega$ and therefore $d/d\varepsilon(\mathbb{E}[u_\varepsilon,\Omega])|_{\varepsilon=0} = 0$ which then leads to (1.10). Note that for a twist $u = r\mathrm{Q}(r)\theta$ we have $\big(|x|,|u|^2,|\nabla u|^2\big) = \big(r,r^2,n+r^2|\dot{\mathrm{Q}}\theta|^2\big) \in U$.

We proceed by re-working Corollary 2.3 for twist paths $\mathrm{Q}(r) = \exp\{\mathscr{G}(r)\mathrm{A}\}$ with $\mathscr{G} = \mathscr{G}(r)$ a sufficiently regular profile curve and $\mathrm{A} \in \mathfrak{so}(n)$ fixed.

Proposition 6.2. *Let $u = r\exp\{\mathscr{G}(r)\mathrm{A}\}\theta$ be a twist of class $\mathscr{C}^2\big(\overline{\mathbb{X}}^n,\overline{\mathbb{X}}^n\big)$ with $\mathscr{G} \in \mathscr{C}^2[a,b]$ and $\mathrm{A} \in \mathfrak{so}(n)$. Then with $\theta^\star = \mathrm{A}\theta$, $\theta^{\star\star} = \mathrm{A}\theta^\star$ we have*

(a) $[\nabla u] = \exp\{\mathscr{G}\mathrm{A}\}\big(\mathrm{I}_n + r\dot{\mathscr{G}}\theta^\star\otimes\theta\big)$,
(b) $[\nabla u]^t = \big(\mathrm{I}_n + r\dot{\mathscr{G}}\theta\otimes\theta^\star\big)\exp\{-\mathscr{G}\mathrm{A}\}$,
(c) $|\nabla u|^2 = tr[\nabla u][\nabla u]^t = tr[\nabla u]^t[\nabla u] = n + r^2\dot{\mathscr{G}}^2|\theta^\star|^2$,
(d) $\Delta u = \exp\{\mathscr{G}\mathrm{A}\}\big[(n+1)\dot{\mathscr{G}}\theta^\star + r\ddot{\mathscr{G}}\theta^\star + \dot{\mathscr{G}}^2\theta^{\star\star}\big]$,

Proof. These follow from Corollary 2.3 upon noting that for $\mathrm{Q} = \exp\{\mathscr{G}\mathrm{A}\}$ we have $\dot{\mathrm{Q}} = \dot{\mathscr{G}}\mathrm{A}\mathrm{Q}$, $\ddot{\mathrm{Q}} = (\ddot{\mathscr{G}}\mathrm{A} + \dot{\mathscr{G}}^2\mathrm{A}^2)\mathrm{Q}$ and $|\dot{\mathrm{Q}}\theta|^2 = \dot{\mathscr{G}}^2\langle\mathrm{A}\theta,\mathrm{A}\theta\rangle = \dot{\mathscr{G}}^2|\theta^\star|^2$. We also recall that here A, Q commute while $\langle\theta,\theta^\star\rangle = 0$ by virtue of A being skew-symmetric. On passing we point out that $\det\nabla u = \det(\mathrm{I}_n + r\dot{\mathscr{G}}\theta^\star\otimes\theta) = 1$. □

Note that restricting the variational integral (6.1) to the subclass of twists $u = r\exp\{\mathscr{G}(r)\mathrm{A}\}\theta$ leads to the energy integral $\mathbb{I}_{\mathrm{A}}[\mathscr{G}]$ as in (3.18) with the Euler–Lagrange equation (3.19) for $\mathscr{G}$. The ODE (3.19) here is certainly implied by the PDE (6.4) (see below) but in general not vice versa! Remarkably the differential operator action $\mathscr{L}[u]$ here admits the following convenient formulation.

Theorem 6.3. *With the Lagrangian $F = F(r,s,\xi)$ as before, u, $\mathscr{G}$, A as in the previous proposition and $\theta^\star = \mathrm{A}\theta$, $\theta^{\star\star} = \mathrm{A}\theta^\star$ we have*

$$\begin{aligned}(6.5)\qquad \mathscr{L}[u] - \nabla F_\xi =& \frac{1}{r^{n-1}}\frac{d}{dr}\big[r^{n+1}F_\xi(r,r^2,n+r^2\dot{\mathscr{G}}^2|\theta^\star|^2)\dot{\mathscr{G}}^2\big]|\theta^\star|^2\theta \\ &- r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}F_\xi\big(r,r^2,n+r^2\dot{\mathscr{G}}^2|\theta^\star|^2\big)|\theta^\star|^2\theta \\ &- rF_s\big(r,r^2,n+r^2\dot{\mathscr{G}}^2|\theta^\star|^2\big)\theta \\ &+ \frac{1}{r^n}\frac{d}{dr}\big[r^{n+1}F_\xi\big(r,r^2,n+r^2\dot{\mathscr{G}}^2|\theta^\star|^2\big)\dot{\mathscr{G}}\big]\theta^\star \\ &+ r\dot{\mathscr{G}}^2F_\xi\big(r,r^2,n+r^2\dot{\mathscr{G}}^2|\theta^\star|^2\big)\theta^{\star\star}.\end{aligned}$$

Here $\nabla F_\xi = \nabla\big[F_\xi\big(|x|,|x|^2,n+\dot{\mathscr{G}}^2(|x|)|\mathrm{A}x|^2\big)\big]$.

Proof. Starting from the description of $\mathscr{L}[u]$ in Proposition 2.2 and Corollary 2.3 it follows upon invoking the identities formulated in Proposition 6.2 that

$$\begin{aligned}(6.6)\qquad \mathscr{L}[u] =& (\mathrm{I}_n + r\dot{\mathscr{G}}\theta\otimes\theta^\star)\big\{F_{\xi\xi}(r,r^2,n+r^2\dot{\mathscr{G}}^2|\theta^\star|^2) \\ &\times\big(\mathrm{I}_n + r\dot{\mathscr{G}}\theta^\star\otimes\theta\big)\big(2r\dot{\mathscr{G}}^2|\theta^\star|^2\theta + r^2\nabla[\dot{\mathscr{G}}^2|\theta^\star|^2]\big) \\ &+ \big[2rF_{s\xi}\big(r,r^2,n+r^2\dot{\mathscr{G}}^2|\theta^\star|^2\big) \\ &+ F_{r\xi}\big(r,r^2,n+r^2\dot{\mathscr{G}}^2|\theta^\star|^2\big)\big]\big(\theta + r\dot{\mathscr{G}}\theta^\star\big) \\ &+ F_\xi\big(r,r^2,n+r^2\dot{\mathscr{G}}^2|\theta^\star|^2\big)\big[(n+1)\dot{\mathscr{G}}\theta^\star + r\ddot{\mathscr{G}}\theta^\star + r\dot{\mathscr{G}}^2\theta^{\star\star}\big] \\ &- rF_s(r,r^2,n+r^2\dot{\mathscr{G}}^2|\theta^\star|^2)\theta\big\}.\end{aligned}$$

We now proceed on to evaluating the individual terms in this expansion. Indeed

$$\big(\mathrm{I}_n + r\dot{\mathscr{G}}\theta\otimes\theta^\star\big)\big(\mathrm{I}_n + r\dot{\mathscr{G}}\theta^\star\otimes\theta\big) = \mathrm{I}_n + r\dot{\mathscr{G}}\big(\theta^\star\otimes\theta + \theta\otimes\theta^\star\big) + r^2\dot{\mathscr{G}}^2|\theta^\star|^2\theta\otimes\theta$$

and, by direct differentiation,

$$(6.7)\qquad r^2\nabla\big[\dot{\mathscr{G}}^2|\theta^\star|^2\big] = 2\big(r^2\dot{\mathscr{G}}\ddot{\mathscr{G}} - r\dot{\mathscr{G}}^2\big)|\theta^\star|^2\theta - 2r\,\dot{\mathscr{G}}^2\theta^{\star\star}.$$

Therefore, by noting the identities $\langle\theta^{\star\star},\theta\rangle = -|\theta^\star|^2$, $\langle\theta^{\star\star},\theta^\star\rangle = \langle\theta^\star,\theta\rangle = 0$, it follows after substitution and taking into account the necessary and relevant cancellations that

$$\begin{aligned}&\big(\mathrm{I}_n + r\dot{\mathscr{G}}\theta\otimes\theta^\star\big)\big(\mathrm{I}_n + r\dot{\mathscr{G}}\theta^\star\otimes\theta\big)\big(2r\dot{\mathscr{G}}^2|\theta^\star|^2\theta + r^2\nabla\big[\dot{\mathscr{G}}^2|\theta^\star|^2\big]\big) \\ &= \big[2r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}|\theta^\star|^2 + 2r^3\dot{\mathscr{G}}^3\big(\dot{\mathscr{G}} + r\ddot{\mathscr{G}}\big)|\theta^\star|^4\big]\theta + 2r^2\dot{\mathscr{G}}^2\big(\dot{\mathscr{G}} + r\ddot{\mathscr{G}}\big)|\theta^\star|^2\theta^\star - 2r\dot{\mathscr{G}}^2\theta^{\star\star}.\end{aligned}$$

Likewise

$$\big(\mathrm{I}_n + r\dot{\mathscr{G}}\theta\otimes\theta^\star\big)\big(\theta + r\dot{\mathscr{G}}\theta^\star\big) = \theta + r\dot{\mathscr{G}}\theta^\star + r^2\dot{\mathscr{G}}^2|\theta^\star|^2\theta$$

and again from the above identities

$$\begin{aligned}&\big(\mathrm{I}_n + r\dot{\mathscr{G}}\theta\otimes\theta^\star\big)\big[(n+1)\dot{\mathscr{G}}\theta^\star + r\ddot{\mathscr{G}}\theta^\star + r\dot{\mathscr{G}}^2\theta^{\star\star}\big]\\&\qquad = (n+1)\dot{\mathscr{G}}\theta^\star + r\big(\ddot{\mathscr{G}}\theta^\star + \dot{\mathscr{G}}^2\theta^{\star\star}\big) + r(n+1)\dot{\mathscr{G}}^2|\theta^\star|^2\theta + r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}|\theta^\star|^2\theta.\end{aligned}$$

Now, returning to (6.6), by multiplying through and making the relevant substitutions, we have

$$\begin{aligned}\mathscr{L}[u] = {}& F_{\xi\xi}\big[[2r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}|\theta^\star|^2 + 2r^3\dot{\mathscr{G}}^3\big(\dot{\mathscr{G}} + r\ddot{\mathscr{G}}\big)|\theta^\star|^4]\theta + 2r^2\dot{\mathscr{G}}^2\big(\dot{\mathscr{G}} + r\ddot{\mathscr{G}}\big)|\theta^\star|^2\theta^\star\big]\\&- 2rF_{\xi\xi}\dot{\mathscr{G}}^2\theta^{\star\star} + [2rF_{s\xi} + F_{r\xi}]\big(\theta + r\dot{\mathscr{G}}\theta^\star + r^2\dot{\mathscr{G}}^2|\theta^\star|^2\theta\big) - rF_s\theta\\&+ F_\xi\big[(n+1)\dot{\mathscr{G}}\theta^\star + r\big(\ddot{\mathscr{G}}\theta^\star + \dot{\mathscr{G}}^2\theta^{\star\star}\big) + r(n+1)\dot{\mathscr{G}}^2|\theta^\star|^2\theta + r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}|\theta^\star|^2\theta\big].\end{aligned}$$

Note that here, for the sake of convenience, we have abbreviated the arguments in F_s, F_ξ, $F_{r\xi}$, $F_{s\xi}$, $F_{\xi\xi}$ by writing $F_{\xi\xi} = F_{\xi\xi}\big(r, r^2, n + r^2\dot{\mathscr{G}}^2|\theta^\star|^2\big)$ and similarly for the other derivatives. This last equation, upon regrouping and rearranging terms in $\theta, \theta^\star$ and $\theta^{\star\star}$ respectively gives

$$\begin{aligned}\mathscr{L}[u] = {}& (F_{r\xi} + 2rF_{s\xi})\theta + 2F_{\xi\xi}\big[r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}|\theta^\star|^2\theta - r\dot{\mathscr{G}}^2\theta^{\star\star}\big]\\&+ \big\{2F_{\xi\xi}r^3\dot{\mathscr{G}}^3\big(\dot{\mathscr{G}} + r\ddot{\mathscr{G}}\big)|\theta^\star|^4 + F_\xi\big[r(n+1)\dot{\mathscr{G}}^2|\theta^\star|^2 + r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}|\theta^\star|^2\big]\\&+ [2rF_{s\xi} + F_{r\xi}]r^2\dot{\mathscr{G}}^2|\theta^\star|^2\big\}\theta - rF_s\theta\\&+ \big\{2F_{\xi\xi}r^2\dot{\mathscr{G}}^2\big(\dot{\mathscr{G}} + r\ddot{\mathscr{G}}\big)|\theta^\star|^2 + [2rF_{s\xi} + F_{r\xi}]r\dot{\mathscr{G}}\\&+ F_\xi\big[(n+1)\dot{\mathscr{G}} + r\ddot{\mathscr{G}}\big]\big\}\theta^\star + rF_\xi\dot{\mathscr{G}}^2\theta^{\star\star}.\end{aligned}$$

The conclusion now follows by rearranging terms, forming the differentials in line with (6.5) and noting

$$\nabla F_\xi = (F_{r\xi} + 2rF_{s\xi})\theta + 2F_{\xi\xi}\big[r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}|\theta^\star|^2\theta - r\dot{\mathscr{G}}^2\theta^{\star\star}\big]. \qquad \square$$

Remark 6.4. Returning now to the discussion preceding the theorem, by taking the dot product of (6.6) with $\theta^\star$ and using $\langle\theta, \theta^\star\rangle = \langle\theta^{\star\star}, \theta^\star\rangle = 0$, $|\theta^\star|^2 = |\mathrm{H}\theta|^2$, from $\mathscr{L}[u] = \nabla\mathscr{P}$ we have

$$\begin{aligned}0 &= \int_{\mathbb{S}^{n-1}} \langle\nabla\mathscr{P} - \nabla F_\xi, \theta^\star\rangle = \int_{\mathbb{S}^{n-1}} \langle\mathscr{L}[u] - \nabla F_\xi, \theta^\star\rangle\\&= \frac{1}{r^n}\frac{d}{dr}\int_{\mathbb{S}^{n-1}} \big[r^{n+1}F_\xi\big(r, r^2, n + r^2\dot{\mathscr{G}}^2|\theta^\star|^2\big)\dot{\mathscr{G}}\big]|\theta^\star|^2,\end{aligned} \tag{6.8}$$

which is precisely the Euler–Lagrange equation $\mathscr{I}_L[\mathscr{G}] \equiv 0$ with the Lagrangian L as in (3.14) (see (3.19) in Corollary 3.7). Note that the first identity in (6.8) follows from an application of Lemma 6.5 below.

Lemma 6.5. *Let $f \in \mathscr{C}^1(\mathscr{V})$ for some open neighbourhood $\mathscr{V} \supset \mathbb{S}^{n-1}$ and let* $\mathrm{H} \in \mathfrak{so}(n)$. *Then $\langle\nabla f, \theta^\star\rangle$ with $\theta^\star = \mathrm{H}\theta$ has a zero mean over the unit sphere.*

If $\mathscr{V} \supset\supset \mathcal{B}$ (with $\mathcal{B}$ the unit n-ball) and f where of class $\mathscr{C}^2$ then a straightforward application of the divergence theorem would give

$$\int_{\mathbb{S}^{n-1}} \langle \nabla f, \theta^\star \rangle \, d\mathcal{H}^{n-1}(\theta) = \int_{\mathcal{B}} -\mathrm{div}(\mathrm{H}\nabla f) \, dx = \int_{\mathcal{B}} \langle \mathrm{H}, \nabla^2 f \rangle \, dx = 0 \tag{6.9}$$

in view of $\langle \mathrm{H}, \nabla^2 f \rangle \equiv 0$. For the general case one can take a cut-off function $\psi \in \mathscr{C}_0^\infty(\mathscr{V})$ such that $\psi \equiv 1$ in a small neighbourhood of $\mathbb{S}^{n-1}$ in $\mathscr{V}$. Then, using a standard mollifier (ρ_ε) and applying the above to $\rho_\varepsilon \star f\psi$, gives

$$\int_{\mathbb{S}^{n-1}} \langle \nabla f, \theta^\star \rangle = \int_{\mathbb{S}^{n-1}} \langle \nabla [f\psi], \theta^\star \rangle = \lim_{\varepsilon \searrow 0} \int_{\mathbb{S}^{n-1}} \langle \nabla [\rho_\varepsilon \star f\psi], \theta^\star \rangle = 0 \tag{6.10}$$

by virtue of the local uniform convergence $\nabla[\rho_\varepsilon \star f\psi] \to \nabla[f\psi]$ as $\varepsilon \searrow 0$ in $\mathscr{V}$. Below we give a different argument purely restricting to the sphere.

Proof. Write $\nabla f = \nabla_T f + \nabla_N f = (\mathrm{I}_n - \theta \otimes \theta)\nabla f + \langle \nabla f, \theta \rangle \theta$ where $\nabla_T f$, $\nabla_N f$ stand for the tangential and normal components of ∇f. Then clearly $\langle \nabla_N f, \theta^\star \rangle = \langle \nabla f, \theta \rangle \langle \theta, \theta^\star \rangle = 0$ while by an application of the divergence theorem on the unit sphere

$$\int_{\mathbb{S}^{n-1}} \langle \nabla_T f, \theta^\star \rangle = \int_{\mathbb{S}^{n-1}} -f \, \mathrm{div}_T \theta^\star = 0 \tag{6.11}$$

in view of the vector field $\theta^\star$ being divergence-free, that is,

$$\mathrm{div}_T \theta^\star = \mathrm{div}_T(\mathrm{H}\theta) = tr\, \mathrm{H} = 0. \qquad \square$$

The above proposition shows that if the pair $\mathscr{G} = \mathscr{G}(r)$ and $\mathrm{A} \in \mathfrak{so}(n)$ are such that the expression on the right in (6.5) is a gradient field in the annulus $\mathbb{X}^n$ then the twist u with the Lie exponential type twist path $\mathrm{Q}(r) = \exp\{\mathscr{G}(r)\mathrm{H}\}$ serves as a solution to the nonlinear system (1.3). The next proposition and the subsequent theorem it leads to give an infinite number of such solutions in the case of n even. This complements the explicit solutions constructed in Section 4 for the special Lagrangian $F = h\big(|x|, |u|^2\big)|\nabla u|^2$.

Proposition 6.6. *For $n \geq 2$ even, let $\mathscr{G} \in \mathscr{C}^2[a,b]$ be a solution to* (5.1), $\mathrm{J}_n = \mathrm{diag}(\mathrm{J}, \ldots, \mathrm{J})$ *with* J *as in* (4.7) *and* $\mathrm{P} \in \mathrm{O}(n)$ *be arbitrary. Then*

$$\begin{aligned}\mathscr{L}\big[r\mathrm{P}\exp\{\mathscr{G}(r)\mathrm{J}_n\}\mathrm{P}^t\theta\big] = \frac{d}{dr}\big[F_\xi\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big)\big]\theta \\ - r\big[\dot{\mathscr{G}}^2 F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2) + F_s\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big)\big]\theta.\end{aligned}$$

In particular $\mathscr{L}\big[r\,\mathrm{P}\exp\{\mathscr{G}(r)\mathrm{J}_n\}\mathrm{P}^t\theta\big]$ is a gradient field in $\mathbb{X}^n[a,b]$.

Proof. We make use of the action formulation (6.5) in Proposition 6.2 by writing $\mathrm{A} = \mathrm{H} = \mathrm{P}\mathrm{J}_n\mathrm{P}^t$. In this case basic calculations give $\mathrm{H}^2 = \mathrm{P}\mathrm{J}_n^2\mathrm{P}^t = -\mathrm{I}_n$, while $|\theta^\star|^2 = 1$, $\theta^{\star\star} = -\theta$ and $\langle\theta^\star, \theta\rangle = 0$. As a result (6.5) can be rewritten as

$$\begin{aligned}\mathscr{L}\big[r\,\mathrm{Pexp}\{\mathscr{G}(r)\mathrm{J}_n\}\mathrm{P}^t\theta\big] = {} & \frac{d}{dr}\big[F_\xi\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big)\big]\theta \\ & + \frac{1}{r^{n-1}}\frac{d}{dr}\big[r^{n+1}F_\xi\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big)\dot{\mathscr{G}}^2\big]\theta \\ & - r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}F_\xi\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big)\theta - rF_s\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big)\theta \\ & + \frac{1}{r^n}\frac{d}{dr}\big[r^{n+1}F_\xi\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big)\dot{\mathscr{G}}\big]\theta^\star \\ & - r\,\dot{\mathscr{G}}^2F_\xi\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big)\theta.\end{aligned}$$

Now, by assumption $\mathscr{G}$ satisfies (5.1) and so

$$\frac{d}{dr}\big[r^{n+1}F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}\big] = 0.$$

As a result referring to the second term in the expression on the right we have

$$\begin{aligned}&\frac{1}{r^{n-1}}\frac{d}{dr}\big[r^{n+1}F_\xi\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big)\dot{\mathscr{G}}^2\big] \\ &\quad = r^2F_\xi\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big)\dot{\mathscr{G}}\ddot{\mathscr{G}} + \frac{\dot{\mathscr{G}}}{r^{n-1}}\frac{d}{dr}\big[r^{n+1}F_\xi\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big)\dot{\mathscr{G}}\big] \\ &\quad = r^2F_\xi\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big)\dot{\mathscr{G}}\ddot{\mathscr{G}}.\end{aligned}$$

Finally, a further reference to the ODE satisfied by $\mathscr{G}$ and taking into account the resulting cancellations gives

$$\begin{aligned}\mathscr{L}\big[r\mathrm{Pexp}\{\mathscr{G}(r)\mathrm{J}_n\}\mathrm{P}^t\theta\big] &= -r\big[\dot{\mathscr{G}}^2F_\xi\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big) \\ + F_s\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big)\big]\theta &+ \frac{d}{dr}\big[F_\xi\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big)\big]\theta = -\nabla G + \nabla F_\xi.\end{aligned}$$

Here $G = G(r)$ satisfies

$$\dot{G}(r) = r\big[\dot{\mathscr{G}}^2F_\xi\big(r, r^2, n + r^2\dot{\mathscr{G}}^2\big) + F_s(r, r^2, n + r^2\dot{\mathscr{G}}^2)\big]$$

and $F_\xi = F_\xi\big(r, r^2, r^2\dot{\mathscr{G}}\big)$. In summary $\mathscr{L}\big[r\exp\{\mathscr{G}(r)\mathrm{H}\}\theta\big] = \nabla\mathscr{P}$ where $\mathscr{P} = \mathscr{P}(|x|)$ is given, up to a constant, by $\mathscr{P} = F_\xi - G$. □

Theorem 6.7. *For $n \geq 2$ even and $F = F(r, s, \xi)$ as described earlier the system* (1.3)–(1.10) *has an infinite family of twisting solutions*

$$u = u(x; m) = r\exp\big\{\mathscr{G}(r; m)\mathrm{H}\big\}\theta = r\,\mathrm{P}\,\mathrm{diag}\big(\mathcal{R}[\mathscr{G}](r; m), \ldots, \mathcal{R}[\mathscr{G}](r; m)\big)\mathrm{P}^t\theta$$

(with $m \in \mathbb{Z}$), where $\mathscr{G} = \mathscr{G}(r; m) \in \mathscr{C}^2[a, b]$ is the unique solution to the two point boundary value problem (5.5), $\mathrm{H} = \mathrm{P}\mathrm{J}_n\mathrm{P}^t$ *and $\mathrm{P} \in \mathrm{O}(n)$ is arbitrary.*

Proof. With the above propositions at our disposal it remains to prove that for each $m \in \mathbb{Z}$ the boundary value problem (5.5) has a unique solution

$\mathscr{G} = \mathscr{G}(r; m)$ in $\mathscr{C}^2[a, b]$. To this end we note that solutions to (5.5) are minimisers of (5.8) over the Dirichlet class

$$\mathscr{J}_p(a, b) = \{\mathscr{G} \in W^{1,p}(a, b) : \mathscr{G}(a) = 0, \mathscr{G}(b) = 2m\pi\}.$$

As this energy is coercive and sequentially weakly lower semicontinuous on $W^{1,p}$ the existence of a minimiser follows by an application of the direct methods of the calculus of variations. The $\mathscr{C}^2$ regularity of the minimiser then follows by invoking the classical Tonelli–Hilbert–Weierstrass differentiability theorem (cf. e.g. [7, pp. 55–61]). Finally, the uniqueness of minimiser and solution to (5.5) is a consequence of the uniform convexity of the function $\xi \mapsto F(r, r^2, n + r^2\xi^2)$ for $a \leq r \leq b$ and $\xi \in \mathbb{R}$ and the fact that solutions to the Euler–Lagrange equation (5.5) are minimisers of the energy (5.8) in their own Dirichlet class. □

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