Topological **M**ethods in **N**onlinear **A**nalysis
Volume 54, No. 2A, 2019, 817–831
DOI: 10.12775/TMNA.2019.072

GLOBAL BIFURCATION IN NONLINEAR DIRAC PROBLEMS WITH SPECTRAL PARAMETER IN THE BOUNDARY CONDITION

Ziyatkhan S. Aliyev — Parvana R. Manafova

Abstract. In this paper we consider nonlinear eigenvalue problems for a one-dimensional Dirac equation with spectral parameter in the boundary condition. We investigate local and global bifurcations of nontrivial solutions to these problems. The existence of unbounded continua of nontrivial solutions bifurcating from points and intervals of the line of trivial solutions is shown.

1. Introduction

We consider the nonlinear eigenvalue problem for the Dirac equation

$$\ell w(x) \equiv Bw'(x) - P(x)w(x) = \lambda w(x) + h(x, w(x), \lambda), \quad 0 < x < \pi, \tag{1.1}$$

subject to the boundary conditions

$$U(\lambda, w) = \begin{pmatrix} U_1(w) \\ U_2(\lambda, w) \end{pmatrix}$$

given by

$$U_1(w) := (\sin\alpha, \cos\alpha)\, w(0) = v(0)\cos\alpha + u(0)\sin\alpha = 0, \tag{1.2}$$

2010 *Mathematics Subject Classification.* Primary: 34A30, 34B05, 34K29, 47J10; Secondary: 34B15, 34C10, 34C23, 47J15.

Key words and phrases. Nonlinear Dirac problem; spectral parameter; bifurcation point; eigenvector-function; global continua.

$$(1.3)\qquad \begin{aligned} U_2(\lambda, w) &:= (\lambda\sin\beta + b_1, \lambda\cos\beta + a_1)\, w(\pi) \\ &= (\lambda\cos\beta + a_1)\, v(\pi) + (\lambda\sin\beta + b_1)\, u(\pi) = 0, \end{aligned}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad P(x) = \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix}, \qquad w(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix},$$

$\lambda \in \mathbb{R}$ is a spectral parameter, $p(x)$ and $r(x)$ are real valued, continuous functions on the interval $[0, \pi]$, α, β, a_1 and b_1 are real constants such that $0 \le \alpha, \beta < \pi$ and

$$(1.4)\qquad \sigma = a_1 \sin\beta - b_1 \cos\beta > 0.$$

The function $h\colon [0, \pi] \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is assumed to be continuous and has a representation $h = f + g$, where $f = \binom{f_1}{f_2}$ and $g = \binom{g_1}{g_2}$ are continuous functions on $[0, \pi] \times \mathbb{R}^2 \times \mathbb{R}$ and satisfy the following conditions:

$$(1.5)\qquad |f_1(x, w, \lambda)| \le K|w|, \quad |f_2(x, w, \lambda)| \le M|w|, \quad \text{for } x \in [0, \pi],\ 0 < |w| \le 1,\ \lambda \in \mathbb{R},$$

where K and M are the positive constants;

$$(1.6)\qquad g(x, w, \lambda) = o(|w|) \quad \text{as } |w| \to 0,$$

uniformly in $x \in [0, \pi]$ and $\lambda \in \Lambda$ for every bounded interval $\Lambda \subset \mathbb{R}$ (here $|\cdot|$ denotes a norm in $\mathbb{R}^2$).

Similar problems for nonlinear Sturm–Liouville equation of second and fourth order when the spectral parameter is not involved in the boundary conditions have been considered before by Rabinowitz [21], Schmitt and Smith [23], Chiappinelli [12], Rynne [22], Dai [13], Przybycin [20], Lazer and McKenna [17], Ma and Tompson [19], Aliyev [2], and when the spectral parameter is involved in the boundary condition have been considered by Binding, Browne, Watson [11], Aliyev and Mamedova [3], Aliyev [1]. These authors prove the existence of unbounded continua of nontrivial solutions in $\mathbb{R} \times C^1$ and $\mathbb{R} \times C^3$ bifurcating from eigenvalues or intervals (in $\mathbb{R} \times \{0\}$, which we identify with $\mathbb{R}$) surrounding the eigenvalues of the corresponding linear problem and having usual nodal properties.

In Schmitt and Smith [23] a special case of (1.1)–(1.3), with $a_1 = b_1 = 0$ and $K + M < 1/2$, was considered. But these authors have not been able to investigate the structure and behavior of global continua of nontrivial solutions emanating from bifurcation intervals completely. The reason for this was that at this time the oscillatory properties of the one-dimensional Dirac system were not known. For the first time Aliev and Rzayeva [6] investigated completely the oscillatory properties of eigenvector-functions of the linear one dimensional Dirac system (1.1)–(1.3) with $h \equiv \widetilde{0}$ and $a_1 = b_1 = 0$ (see also [7]), where $\widetilde{0} = \binom{0}{0}$. To study the global bifurcation of solutions of the nonlinear problem (1.1)–(1.3) with $a_1 = b_1 = 0$ in [8] uses oscillatory properties of eigenvector-functions and refined asymptotic formulas for eigenvalues of problem (1.1)–(1.3) with $h \equiv \widetilde{0}$ and $a_1 = b_1 = 0$ which are obtained in [6], [7].

In our recent papers [4], [5] we consider the linear eigenvalue problem obtained from (1.1)–(1.3) by setting $h \equiv \widetilde{0}$, where we show that the eigenvalues of this problem are real, algebraically simple and the values range from $-\infty$ to $+\infty$, and can be numerated in increasing order. Moreover, we study the location of eigenvalues on the real axis, the oscillation properties of eigenvector-functions and the asymptotic behaviors of eigenvalues and eigenvector-functions of this problem.

The purpose of this paper is to study the structure and behavior of global continua of nontrivial solutions of problem (1.1)–(1.3) bifurcating from points and intervals of the line of trivial solutions.

The structure of this paper is follows. In Section 2, by using angular function, we study the oscillatory properties of eigenvector-functions of the linear Dirac system (1.1)–(1.3) with $h \equiv \widetilde{0}$. In Section 3 we define classes of functions that have the obtained oscillation properties of eigenvector-functions of this linear problem by using technique from [11]. Here we reduce problem (1.1)–(1.3) to a nonlinear operator equation, and we apply Rabinowitz's global bifurcation theorem to this equation with $f \equiv \widetilde{0}$. By extending the approximation technique used in [10] and using the basic properties of the angular functions in Section 4 we find bifurcation intervals for problem (1.1)–(1.3). Next we show that the connected components of the set of solutions of this problem bifurcating from these intervals are unbounded and lie in the classes of functions from Section 3.

2. Preliminaries

If $h \equiv 0$, then from (1.1)–(1.3) we get the following linear one-dimensional Dirac system

$$\begin{aligned} &\ell w(x) = \lambda w(x), \quad 0 < x < \pi, \\ &U(\lambda, w) = \widetilde{0}. \end{aligned} \tag{2.1}$$

It follows from [4, Lemma 2.1 and Theorem 3.2] that eigenvalues of the boundary value problem (2.1) are real, algebraically simple and the values range from $-\infty$ to $+\infty$, and can be numerated in increasing order.

In order to study the bifurcation of solutions of nonlinear problem (1.1)–(1.3) we consider a more general linear problem

$$\begin{aligned} &\widetilde{\ell} w(x) \equiv Bw'(x) - \widetilde{P}(x)w(x) = \lambda w(x), \quad 0 < x < \pi, \\ &U(\lambda, w) = \widetilde{0}, \end{aligned} \tag{2.2}$$

where

$$\widetilde{P}(x) = \begin{pmatrix} p(x) & q(x) \\ s(x) & r(x) \end{pmatrix},$$

$q(x)$ and $s(x)$ are real-valued continuous functions on the interval $[0, \pi]$.

Remark 2.1. In view of [8, Remark 2.1] without loss of generality we can assume that $s(x) \equiv q(x)$.

Lemma 2.2. *The eigenvalues of the boundary value problem* (2.2) *are real, simple and form a countable set without finite limit points.*

The proof of this lemma is similar to that of [4, Lemma 2.1].

It should be noted that there exists a unique solution

$$w(x, \lambda) = \begin{pmatrix} u(x, \lambda) \\ v(x, \lambda) \end{pmatrix}$$

of the Dirac equation

$$\widetilde{\ell} w(x) = \lambda w(x), \quad 0 < x < \pi,$$

satisfying the initial condition

$$u(0, \lambda) = \cos\alpha, \quad v(0, \lambda) = -\sin\alpha; \tag{2.3}$$

moreover, for each fixed $x \in [0, \pi]$ the functions $u(x, \lambda)$ and $v(x, \lambda)$ are entire functions of λ. The proof of this assertion is similar to that of [18, Chapter 1, § 1, Theorem 1.1] with obvious modifications.

Let us introduce the boundary condition

$$U_2(w) := (\sin\gamma,\, \cos\gamma)\, w(\pi) = v(\pi)\cos\gamma + u(\pi)\sin\gamma = 0, \tag{2.4}$$

where $\gamma \in [0, \pi)$.

Along with problem (2.2) consider the following boundary value problem

$$\begin{aligned} &\widetilde{\ell} w(x) = \lambda w(x), \quad 0 < x < \pi, \\ &U(w) = \widetilde{0}, \end{aligned} \tag{2.5}$$

where $U(w) = \begin{pmatrix} U_1(w) \\ U_2(w) \end{pmatrix}$ (see (1.2) and (2.4)).

The problem (2.5) has been considered in [7], where the authors study the oscillation properties of the eigenvector-functions of this problem. The eigenvalues $\lambda_k = \lambda_k(\alpha, \gamma)$, $k \in \mathbb{Z}$, of problem (2.5) are real, algebraically simple and the values range from $-\infty$ to $+\infty$, and can be numerated in increasing order on the real axis.

To study the oscillatory properties of eigenvector-functions of problem (2.2), we introduce the Prüfer angular variable

$$\theta(x, \lambda) = \cot^{-1}(u(x, \lambda)/v(x, \lambda)) \tag{2.6}$$

(see [9, Chapter 8, § 3]), or more precisely,

$$\theta(x,\lambda) = \arg\{u(x,\lambda) + iv(x,\lambda)\}. \tag{2.7}$$

We recall that u, v have fixed initial values for $x = 0$, and all λ, given by (2.3). We define initially

$$\theta(0,\lambda) = -\alpha, \tag{2.8}$$

in view of (2.3). For other x and λ, $\theta(x,\lambda)$ is given by (2.7) (or (2.6)) except for an arbitrary multiple of 2π, since u and v cannot vanish simultaneously. This multiple of 2π is to be fixed so that $\theta(x,\lambda)$ satisfies (2.7) and is continuous in x and λ. Since the (x,λ)-region, namely, $0 \le x \le \pi$, $-\infty < \lambda < +\infty$, is simply-connected, this defines $\theta(x, \lambda)$ uniquely.

Remark 2.3. From (2.7) it is obvious that the zeros of the functions $u(x,\lambda)$ and $v(x,\lambda)$ are the same as the occasions on which $\theta(x, \lambda)$ is an odd or even multiple of $\pi/2$, respectively.

Remark 2.4. In virtue of [7, Theorem 1] the eigenvalues $\lambda_k(\alpha,\gamma)$, $k \in \mathbb{Z}$, of problem (2.5) can be numbered in increasing order on the real axis so that the angular function $\theta(x, \lambda_k(\alpha,\gamma))$ at $x = \pi$ satisfy the condition

$$\theta(\pi, \lambda_k(\alpha,\gamma)) = -\gamma + k\pi.$$

The next lemma follows from [7, Lemmas 1–3 and Theorem 2] and is useful in the sequel.

Lemma 2.5. (a) *$\theta(x, \lambda)$ satisfies the differential equation, with respect to x,*

$$\theta' = \lambda + p\cos^2\theta + r\sin^2\theta + \frac{1}{2}(q+s)\sin 2\theta. \tag{2.9}$$

(b) *If $\lambda > \lambda^*$, then as x increases, θ cannot tend to a multiple of $\pi/2$ from above, and as x decreases, θ cannot tend to a multiple of $\pi/2$ from below. If $\lambda < \lambda^*$, then as x increases, θ cannot tend to a multiple of $\pi/2$ from below, and as x decreases, θ cannot tend to a multiple of $\pi/2$ from above, where $\lambda^* = \lambda_0(\alpha,\alpha)$.*

(c) *As λ increases, for fixed x, the function θ is increasing; in particular, $\theta(\pi,\lambda)$ is a strictly increasing function of λ.*

For the function

$$\Phi(\lambda) = \frac{\lambda\cos\beta + a_1}{\lambda\sin\beta + b_1}$$

we have

$$\Phi'(\lambda) = \frac{-\sigma}{(\lambda\sin\beta + b_1)^2}.$$

Since $\sigma > 0$ (see (1.4)), it follows that for $\beta = 0$ the function $\Phi(\lambda)$ is strictly decreasing on the interval $(-\infty, +\infty)$, and we have $\lim\limits_{\lambda\to\pm\infty} \Phi(\lambda) = \mp\infty$; for $\beta \ne 0$

the function $\Phi(\lambda)$ is decreasing on each of the intervals $(-\infty, -b_1/\sin\beta)$ and $(-b_1/\sin\beta, +\infty)$, and we have

$$\lim_{\lambda\to -b_1/\sin\beta-0} \Phi(\lambda) = -\infty, \qquad \lim_{\lambda\to -b_1/\sin\beta+0} \Phi(\lambda) = +\infty, \qquad \lim_{\lambda\to\pm\infty} \Phi(\lambda) = \cot\beta.$$

Following arguments of [11, p. 251] and taking into account Remark 2.4, we define a continuous function $\varrho(\lambda)$, $\lambda \in \mathbb{R}$, as follows:

$$\varrho(\lambda) = -\cot^{-1}\Phi(\lambda) \quad \text{for } \beta = 0,$$

$$\varrho(\lambda) = \begin{cases} -\cot^{-1}\Phi(\lambda) & \text{if } \lambda \in (-\infty, -b_1/\sin\beta], \\ -\cot^{-1}\Phi(\lambda) - \pi & \text{if } \lambda \in (-b_1/\sin\beta, +\infty), \end{cases} \quad \text{for } \beta \neq 0.$$

It is obvious that

$$\cot\varrho(\lambda) = -\Phi(\lambda) = -\frac{\lambda\cos\beta + a_1}{\lambda\sin\beta + b_1}, \qquad \varrho\left(-\frac{b_1}{\sin\beta}\right) = -\pi.$$

It follows from the above consideration that the function $\varrho(\lambda)$ is strictly decreasing on $\mathbb{R}$. Moreover, we have

$$\lim_{\lambda\to-\infty} \varrho(\lambda) = -\beta, \qquad \lim_{\lambda\to+\infty} \varrho(\lambda) = -\beta - \pi. \tag{2.10}$$

THEOREM 2.6. *The eigenvalues λ_k, $k \in \mathbb{Z}$, of problem (2.2) can be numbered in ascending order on the real axis so that the corresponding angular function $\theta(x, \lambda_k)$ at $x = \pi$ satisfy the condition*

$$\theta(\pi, \lambda_k) = \varrho(\lambda_k) + k\pi. \tag{2.11}$$

PROOF. By virtue of statement (c) of Lemma 2.5, $\theta(\pi, \lambda)$ is a strictly increasing function of λ, $\lambda \in \mathbb{R}$. Moreover, by Remark 2.1 it follows from [7, (10)] that

$$\theta(\pi, \lambda) \to -\infty \quad \text{as } \lambda \to -\infty, \qquad \theta(\pi, \lambda) \to +\infty \quad \text{as } \lambda \to +\infty. \tag{2.12}$$

Note that the eigenvalues of problem (2.2) are the roots of the equation

$$(\lambda\cos\beta + a_1)\, v(\pi, \lambda) + (\lambda\sin\beta + b_1)\, u(\pi, \lambda) = 0. \tag{2.13}$$

Equation (2.13) can be expressed in the following equivalent form:

$$\theta(\pi, \lambda) = \varrho(\lambda) + k\pi, \quad k \in \mathbb{Z}. \tag{2.14}$$

It follows from relations (2.10), (2.12) and the properties of continuity and strict monotonicity of functions $\theta(\pi, \lambda)$ and $\varrho(\lambda)$ that for each $k \in \mathbb{Z}$ there exists a unique root $\lambda = \lambda_k$ of equation (2.14), i.e.

$$\theta(\pi, \lambda_k) = \varrho(\lambda_k) + k\pi, \quad k \in \mathbb{Z},$$

where $\ldots < \lambda_{-k} < \ldots < \lambda_{-1} < \lambda_0 < \lambda_1 < \ldots < \lambda_k < \ldots$ □

Remark 2.7. By virtue of Theorem 2.6 and the definition of the function $\varrho(\lambda)$ when numbering the eigenvalues of problem (2.2) will be proceed from the following consideration: the zero sequence number will be assigned to eigenvalue which is contained in $(\lambda_{-1}(\alpha,0),\lambda_0(\alpha,0)]$ and is closest to $\lambda_0(\alpha,0)$.

3. The classes S_k^ν, $k\in\mathbb{Z}$, $\nu\in\{+,-\}$, and global bifurcation of solutions of problem (1.1)–(1.3) in the case $f\equiv 0$

Consider $E=C\big([0,\pi];\mathbb{R}^2\big)\cap\{w:U_1(w)=0\}$ with the usual norm

$$\|w\|=\max_{x\in[0,\pi]}|u(x)|+\max_{x\in[0,\pi]}|v(x)|;$$

then E is a Banach space. Let S be the subset of E given by

$$S=\{w\in E:|u(x)+|v(x)|>0,\ \text{for all } x\in[0,\pi]\}$$

with metric inherited from E. For each $w=\binom{u}{v}\in S$ we define $\theta(w,\,\cdot\,)$ to be continuous function on $[0,\pi]$ satisfying

$$\cot(w,x)=\frac{u(x)}{v(x)},\qquad \theta(w,0)=-\alpha$$

(see, e.g. [8], [11]). It is apparent that $\theta\colon S\times[0,\pi]\to\mathbb{R}$ is continuous. From (2.8) and (2.11) we have

$$\theta(w_k,0)=-\alpha,\quad \theta(w_k,\pi)=\varrho(\lambda_k)+k\pi,\quad k\in\mathbb{Z}, \tag{3.1}$$

where $w_k(x)$ is an eigenfunction corresponding to the eigenvalue λ_k of problem (2.2).

For each $k\in\mathbb{Z}$ and each $\lambda\in\mathbb{R}$ let $S^+_{k,\,\lambda}$ be the set of functions $w=\binom{u}{v}\in S$ satisfying the following conditions:

(i) $\theta(w,\pi)=\varrho(\lambda)+k\pi$;
(ii) the function $u(x)$ is positive in a neighbourhood of $x=0$;
(iii) if $\beta=0$, $k>0$ or $\beta=0$, $k=0$, $\varrho(\lambda)\ge-\alpha$ or $\beta\ne0$, $k>1$ or $\beta\ne0$, $k=0,1$, $\varrho(\lambda)\ge-\alpha$, then for fixed w, as x increases from 0 to π, the function θ cannot tend to a multiple of $\pi/2$ from above, and as x decreases, the function θ cannot tend to a multiple of $\pi/2$ from below; if $\beta=0$, $k<0$ or $\beta=0$, $k=0$, $\varrho(\lambda)<-\alpha$ or $\beta\ne0$, $k<1$ or $\beta\ne0$, $k=0,1$, $\varrho(\lambda)<-\alpha$, then for fixed w, as x increases, the function θ cannot tend to a multiple of $\pi/2$ from below, and as x decreases, the function θ cannot tend to a multiple of $\pi/2$ from above.

Let $S^-_{k,\lambda}=-S^+_{k,\,\lambda}$ and $S_{k,\lambda}=S^+_{k,\lambda}\cup S^-_{k,\lambda}$. It follows from (3.1), Remark 2.7 and statement (b) of Lemma 2.5 that for each $\lambda\in\mathbb{R}$ the sets $S^+_{k,\lambda}$, $S^-_{k,\lambda}$ and $S_{k,\lambda}$, $k\in\mathbb{Z}$, are nonempty.

From now on ν will denote an element of $\{+,-\}$ that is, either $\nu = +$ or $\nu = -$.

Remark 3.1. It follows from the definition of the sets $S^{\nu}_{k,\lambda}$, $k \in \mathbb{Z}$, that for each $\lambda \in \mathbb{R}$ these sets are disjoint and open in E. Furthermore, if $w \in \partial S^{\nu}_{k,\lambda}$, then there exists $t \in [0,\pi]$ such that $|w(t)| = 0$, i.e. $u(t) = v(t) = 0$ (see [8]).

Now we define the sets S_k and S^{ν}_k, $k \in \mathbb{Z}$, as follows:

$$S_k = \bigcup_{\lambda \in \mathbb{R}} S_{k,\lambda} \quad \text{and} \quad S^{\nu}_k = \bigcup_{\lambda \in \mathbb{R}} S^{\nu}_{k,\lambda}.$$

In view of Remark 3.1 the sets S_k and S^{ν}_k, $k \in \mathbb{Z}$, are disjoint and open in E. Moreover, if $w \in \partial S^{\nu}_k$, $k \in \mathbb{Z}$, then there exists a point $t \in [0,\pi]$ such that $|w(t)| = 0$, i.e. $u(t) = v(t) = 0$.

Lemma 3.2 ([8, Lemma 2.8]). *If $(\lambda, w) \in \mathbb{R} \times E$ is a solution of problem* (1.1)–(1.3) *and $w \in \partial S^{\nu}_k$, then $w \equiv \widetilde{0}$ (more precisely, $u \equiv 0$ and $v \equiv 0$).*

Let $\widehat{E} = E \oplus \mathbb{R}$ be the Banach space with the norm

$$\|\widehat{w}\| = \left\| \begin{pmatrix} w \\ \eta \end{pmatrix} \right\| = \|w\| + |\eta|.$$

Let us define the operator L by

$$L(\widehat{w}) = L\begin{pmatrix} w \\ \eta \end{pmatrix} = \begin{pmatrix} \ell(w) \\ a_1 v(\pi) + b_1 u(\pi) \end{pmatrix}$$

with the domain

$$D(L) = \left\{ \widehat{w} = \begin{pmatrix} w \\ \eta \end{pmatrix} \in \widehat{E} : w \in C^1\big([0,\pi];\mathbb{R}^2\big), \right.$$
$$\left. \eta = -(v(\pi)\cos\beta + u(\pi)\sin\beta) \right\}.$$

Obviously, the operator L is well defined in $\widehat{E}$. Then linear problem (2.1) takes the form

$$L\widehat{w} = \lambda \widehat{w}, \tag{3.2}$$

i.e. the eigenvalues λ_k, $k \in \mathbb{Z}$, of problem (2.1) and the operator L coincide, and between the eigenvector-functions, there is an one-to-one correspondence

$$w_k = \begin{pmatrix} u_k \\ v_k \end{pmatrix} \leftrightarrow \widehat{w}_k = \begin{pmatrix} w_k \\ \eta_k \end{pmatrix} = \begin{pmatrix} u_k \\ v_k \\ \eta_k \end{pmatrix},$$
$$\eta_k = -(v_k(\pi)\cos\beta + u_k(\pi)\sin\beta).$$

It is obvious that L is a closed (nonself-adjoint) operator in $\widehat{E}$ with compact resolvent.

We define the operators $F\colon \mathbb{R}\times\widehat{E}\to\widehat{E}$ and $G\colon \mathbb{R}\times\widehat{E}\to\widehat{E}$ as follows:

$$F(\lambda,\widehat{w}) = F\bigg(\lambda, \begin{pmatrix} w \\ \eta \end{pmatrix}\bigg) = \begin{pmatrix} f(x,w,\lambda) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1(x,w,\lambda) \\ f_2(x,w,\lambda) \\ 0 \end{pmatrix},$$

$$G(\lambda,\widehat{w}) = G\bigg(\lambda, \begin{pmatrix} w \\ \eta \end{pmatrix}\bigg) = \begin{pmatrix} g(x,w,\lambda) \\ 0 \end{pmatrix} = \begin{pmatrix} g_1(x,w,\lambda) \\ g_2(x,w,\lambda) \\ 0 \end{pmatrix},$$

where $\eta = -(v(\pi)\cos\beta + u(\pi)\sin\beta)$. Then problem (1.1)–(1.3) reduces to the nonlinear problem

$$L\widehat{w} = \lambda\widehat{w} + F(\lambda,\widehat{w}) + G(\lambda,\widehat{w}). \tag{3.3}$$

i.e., there is a one-to-one correspondence

$$(\lambda, w) \leftrightarrow (\lambda,\widehat{w})$$

between solutions of these problems. If $\lambda = 0$ is not an eigenvalue of the linear problem (2.1), then L^{-1} exists and $L^{-1}\colon \widehat{E}\to\widehat{E}$. We define the operators $\widehat{L}\colon \widehat{E}\to\widehat{E}$, $\widehat{F}\colon \mathbb{R}\times\widehat{E}\to\widehat{E}$ and $\widehat{G}\colon \mathbb{R}\times\widehat{E}\to\widehat{E}$ as follows:

$$\widehat{L} = L^{-1}, \quad \widehat{F} = L^{-1}F \quad \text{and} \quad \widehat{G} = L^{-1}G.$$

Then problem (3.3) (or (1.1)–(1.3)) can be written in the following equivalent form:

$$\widehat{w} = \lambda\widehat{L}\widehat{w} + \widehat{F}(\lambda,\widehat{w}) + \widehat{G}(\lambda,\widehat{w}). \tag{3.4}$$

Since L has the compact resolvent in $\widehat{E}$, we can regard $\widehat{L}$ as a completely continuous operator in $\widehat{E}$. Hence $\widehat{F}\colon \mathbb{R}\times\widehat{E}\to\widehat{E}$ and $\widehat{G}\colon \mathbb{R}\times\widehat{E}\to\widehat{E}$ are completely continuous. Moreover, by virtue of (1.6), we have

$$\widehat{G}(\lambda,\widehat{w}) = o(\|\widehat{w}\|) \quad \text{as } \|\widehat{w}\|\to 0, \tag{3.5}$$

uniformly in $\lambda\in\Lambda$ for every bounded interval $\Lambda\subset\mathbb{R}$. Let

$$\widehat{S}_k^\nu = \{\widehat{w}\in\widehat{E} : w\in S_k^\nu\}, \quad \widehat{S}_k = \{\widehat{w}\in\widehat{E} : w\in S_k\}, \quad k\in\mathbb{Z}.$$

We denote by $\widehat{\mathcal{C}}$ the closure in $\mathbb{R}\times\widehat{E}$ of the set of nontrivial solutions of problem (3.4). We suppose that

$$f \equiv \widetilde{0} \tag{3.6}$$

(in effect, we suppose that the nonlinearity h itself satisfies (1.6)). In this case problem (1.1)–(1.3) is equivalent to the following problem

$$\widehat{w} = \lambda\widehat{L}\widehat{w} + \widehat{G}(\lambda,\widehat{w}). \tag{3.7}$$

Note that problem (3.7) is of the form (0.1) of [21] (see also [14]). The linearization of (3.7) at $\widehat{w} = \widehat{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is the linear problem

$$\widehat{w} = \lambda \widehat{L} \widehat{w}. \tag{3.8}$$

It is obvious that problem (3.8) is equivalent to the linear problem (2.1) (also to problem (3.2)). For problem (3.7) we the have the following global bifurcation result.

Theorem 3.3. *Suppose that* (3.6) *holds. Then for each $k \in \mathbb{Z}$ and each ν there exists a continuum $\widehat{\mathcal{C}}_k^\nu$ of solutions of problem* (3.7) *which contains* $(\lambda_k, \widehat{0})$*, lies in* $\big(\mathbb{R} \times \widehat{S}_k^\nu\big) \cup \big\{\big(\lambda_k, \widehat{0}\big)\big\}$ *and is unbounded in* $\mathbb{R} \times \widehat{E}$.

Proof. If $\lambda = 0$ is not an eigenvalue of problem (3.2) the proof of this statement is similar to that of [21, Theorem 2.3] with use the above arguments and relation (3.5). If $\lambda = 0$ is an eigenvalue of (3.2), then replacing ℓ by $\ell + \varepsilon$ (we can obviously choose ε in a way that the number zero will not be an eigenvalue of the new linear problem), and passing to a limit using the already established result and the completely continuity of $\widehat{L}$ and $\widehat{G}$, completes the proof of this theorem. $\square$

Since there exists an isomorphism $(\lambda, \widehat{w}) \leftrightarrow (\lambda, w)$ between solutions of problem (3.7) and (1.1)–(1.3), Theorem 3.3 yields the following result.

Theorem 3.4. *Suppose that* (3.6) *holds. Then for each $k \in \mathbb{Z}$ and each ν there exists a continuum $\mathcal{C}_k^\nu$ of solutions of problem* (1.1)–(1.3) *which contains* $\big(\lambda_k, \widetilde{0}\big)$*, lies in* $(\mathbb{R} \times S_k^\nu) \cup \{(\lambda_k, \widetilde{0})\}$ *and is unbounded in* $\mathbb{R} \times E$.

4. Global bifurcation of solutions of problem (1.1)–(1.3) in the general case

We say that $(\lambda, \widetilde{0})$ is a bifurcation point of problem (1.1)–(1.3) with respect to the set $\mathbb{R} \times S_k^\nu$, $k \in \mathbb{Z}$, if any small neighbuorhood of this point there is a solution of this problem which is contained in $\mathbb{R} \times S_k^\nu$.

To study the bifurcation of solutions of problem (1.1)–(1.3) we consider the following approximation problem

$$\begin{aligned} &\ell w(x) = \lambda w(x) + f(x, |w(x)|^\varepsilon w(x), \lambda) + g(x, w(x), \lambda), \quad 0 < x < \pi, \\ &U(\lambda, w) = \widetilde{0}, \end{aligned} \tag{4.1}$$

where $\varepsilon \in (0, 1]$. By virtue of (1.5) for any $\varepsilon \in (0, 1]$ we have

$$f(x, |w(x)|^\varepsilon w(x), \lambda) = o(|w|) \quad \text{as } |w| \to 0, \tag{4.2}$$

uniformly in $x \in [0, \pi]$ and $\lambda \in \Lambda$. Then, because of Theorem 3.4, for each $k \in \mathbb{Z}$ and each ν there exists an unbounded continuum $\mathcal{C}_{k,\varepsilon}^\nu$ of solutions of problem (4.1) such that

$$(\lambda_k, \widetilde{0}) \in \mathcal{C}_{k,\varepsilon}^\nu \subset (\mathbb{R} \times S_k^\nu) \cup \big\{(\lambda_k, \widetilde{0})\big\}.$$

LEMMA 4.1. *For each $k \in \mathbb{Z}$ and each ν and for sufficiently small $\tau > 0$ there exists a solution (λ_τ, w_τ) of the nonlinear problem* (1.1)–(1.3) *which satisfies the conditions $w_\tau \in S_k^\nu$ and $\|w_\tau\| = \tau$.*

PROOF. It follows from the above reasoning that for any $\varepsilon \in (0,1]$ there exists a solution

$$\big(\lambda_{\tau,\varepsilon}, w_{\tau,\,\varepsilon}\big) = \left(\lambda_{\tau,\varepsilon}, \begin{pmatrix} u_{\tau,\varepsilon} \\ v_{\tau,\varepsilon} \end{pmatrix}\right)$$

of problem (4.1) such that $w_{\tau,\varepsilon} \in S_k^\nu$ and $\|w_{\tau,\varepsilon}\| = \tau$.

Obviously, $\big(\lambda_{\tau,\varepsilon}, w_{\tau,\varepsilon}\big)$ solves the nonlinear problem

$$\begin{aligned} &\ell w(x) = \lambda w(x) + P_{\tau,\varepsilon}(x)w(x) + g(x, w(x), \lambda), \quad 0 < x < \pi, \\ &U(\lambda, w) = \widetilde{0}. \end{aligned} \tag{4.3}$$

where

$$P_{\tau,\varepsilon}(x) = \begin{pmatrix} \varphi_{\tau,\varepsilon}(x) & \psi_{\tau,\varepsilon}(x) \\ \phi_{\tau,\,\varepsilon}(x) & \omega_{\tau,\varepsilon}(x) \end{pmatrix}$$

and the functions $\varphi_{\tau,\varepsilon}(x)$, $\psi_{\tau,\varepsilon}(x)$, $\phi_{\tau,\varepsilon}(x)$ and $\tau_{\tau,\varepsilon}(x)$ are determined as follows:

$$\begin{aligned} \varphi_{\tau,\varepsilon}(x) &= \frac{f_1(x, |w_{\tau,\,\varepsilon}(x)|^\varepsilon u_{\tau,\varepsilon}(x), |w_{\tau,\,\varepsilon}(x)|^\varepsilon v_{\tau,\varepsilon}(x), \lambda_{\tau,\varepsilon})\, u_{\tau,\varepsilon}(x)}{u^2_{\tau,\varepsilon}(x) + v^2_{\tau,\varepsilon}(x)}, \\ \psi_{\tau,\varepsilon}(x) &= \frac{f_1(x, |w_{\tau,\varepsilon}(x)|^\varepsilon u_{\tau,\varepsilon}(x), |w_{\tau,\varepsilon}(x)|^\varepsilon v_{\tau,\varepsilon}(x), \lambda_{\tau,\varepsilon}) v_{\tau,\varepsilon}(x)}{u^2_{\tau,\varepsilon}(x) + v^2_{\tau,\varepsilon}(x)}, \\ \phi_{\tau,\varepsilon}(x) &= \frac{f_2(x, |w_{\tau,\varepsilon}(x)|^\varepsilon u_{\tau,\varepsilon}(x), |w_{\tau,\varepsilon}(x)|^\varepsilon v_{\tau,\varepsilon}(x), \lambda_{\tau,\varepsilon})\, u_{\tau,\varepsilon}(x)}{u^2_{\tau,\varepsilon}(x) + v^2_{\tau,\varepsilon}(x)}, \\ \omega_{\tau,\varepsilon}(x) &= \frac{f_2(x, |w_{\tau,\varepsilon}(x)|^\varepsilon u_{\tau,\varepsilon}(x), |w_{\tau,\varepsilon}(x)|^\varepsilon v_{\tau,\varepsilon}(x), \lambda_{\tau,\varepsilon})\, v_{\tau,\varepsilon}(x)}{u^2_{\tau,\varepsilon}(x) + v^2_{\tau,\varepsilon}(x)}. \end{aligned} \tag{4.4}$$

By (1.6) the linearization of the nonlinear eigenvalue problem (4.3) at $w = 0$ is the linear eigenvalue problem

$$\begin{aligned} &\ell w(x) = \lambda w(x) + P_{\tau,\varepsilon}(x)w(x), \quad 0 < x < \pi, \\ &U(\lambda, w) = \widetilde{0}. \end{aligned} \tag{4.5}$$

By [16, Chapter 4, § 2, Theorem 2.1] and Theorem 3.4 the point $(\lambda_{k,\tau,\varepsilon}, \widetilde{0})$ is an only bifurcation point of problem (4.3) with respect to the set $\mathbb{R} \times S_k^\nu$, and this point corresponds to a continuous branch of nontrivial solutions of this problem, where $\lambda_{k,\tau,\varepsilon}$ is the kth eigenvalue of the linear problem (4.5). Hence to each small $\tau > 0$ we can assign a small $\rho_{\tau,\varepsilon} > 0$ such that

$$\lambda_{\tau,\varepsilon} \in (\lambda_{k,\tau,\varepsilon} - \rho_{\tau,\varepsilon}, \lambda_{k,\tau,\varepsilon} + \rho_{\tau,\varepsilon}). \tag{4.6}$$

In view of (1.5) it follows from (4.4) that

$$|\varphi_{\tau,\varepsilon}(x)|, |\psi_{\tau,,\varepsilon}(x)| \le M, \quad |\phi_{\tau,\varepsilon}(x)|, |\omega_{\tau,,\varepsilon}(x)| \le K, \quad x \in [0, \pi]. \tag{4.7}$$

Let $w_{k,\tau,\varepsilon}$, $k \in \mathbb{Z}$, be the eigenvector-function corresponding to the eigenvalue $\lambda_{k,\tau,\varepsilon}$ of problem (4.5). In virtue of (2.9) we have

$$\theta'_k(x) = \lambda_k + \frac{1}{2}\{p(x) + r(x)\} + \frac{1}{2}\{p(x) - r(x)\}\cos 2\,\theta_k(x), \tag{4.8}$$

$$\begin{aligned}\theta'_{k,\tau,\varepsilon}(x) = \lambda_{k,\tau,\varepsilon} &+ \{p(x) + r(x) + \varphi_{\tau,\varepsilon}(x) + \omega_{\tau,\varepsilon}(x)\} \\ &+ \frac{1}{2}\{p(x) + \varphi_{\tau,\varepsilon}(x) - r(x) - \omega_{\tau,\varepsilon}(x)\}\cos 2\theta_{\tau,\varepsilon}(x) \\ &+ \frac{1}{2}\{\psi_{\tau,\varepsilon}(x) + \phi_{\tau,\varepsilon}(x)\}\sin 2\theta_{\tau,\varepsilon},\end{aligned} \tag{4.9}$$

where $\theta_k(x) = \theta(w_k, x)$ and $\theta_{k,\tau,\varepsilon}(x) = \theta(w_{k,\tau,\varepsilon}, x)$. Moreover, it follows from (2.8) and (2.11) that

$$\begin{gathered}\theta_k(0) = \theta_{k,\tau,\varepsilon}(0) = -\alpha, \qquad \theta_k(\pi) = \varrho(\lambda_k) + k\pi, \\ \theta_{k,\tau,\varepsilon}(\pi) = \varrho(\lambda_{k,\tau,\varepsilon}) + k\pi.\end{gathered} \tag{4.10}$$

Integrating both sides of (4.8) and (4.9) from 0 to π and using (4.10) we obtain

$$\begin{aligned}\rho(\lambda_k) + k\pi + \alpha = \lambda_k\pi &+ \frac{1}{2}\int_0^\pi \{p(x) + r(x)\}\,dx \\ &+ \frac{1}{2}\int_0^\pi \{p(x) - r(x)\}\cos 2\theta_k(x)\,dx, \\ \rho(\lambda_{k,\tau,\varepsilon}) + k\pi + \alpha = \lambda_{k,\tau,\varepsilon}\pi &+ \frac{1}{2}\int_0^\pi \{p(x) + r(x) + \varphi_{\tau,\varepsilon}(x) + \omega_{\tau,\,\varepsilon}(x)\}\,dx \\ &+ \frac{1}{2}\int_0^\pi \{p(x) + \varphi_{\tau,\varepsilon}(x) - r(x) - \omega_{\tau,\varepsilon}(x)\}\cos 2\theta_{k,\tau,\varepsilon}(x)\,dx \\ &+ \frac{1}{2}\int_0^\pi \{\psi_{\tau,\varepsilon}(x) + \phi_{\tau,\varepsilon}(x)\}\sin 2\theta_{k,\tau,\varepsilon}(x)\,dx,\end{aligned}$$

respectively. Subtracting the first equality from the second equality we obtain

$$\begin{aligned}\rho(\lambda_{k,\tau,\varepsilon}) - \rho(\lambda_k) = (\lambda_{k,\tau,\varepsilon} - \lambda_k)\,\pi &+ \frac{1}{2}\int_0^\pi \{\varphi_{\tau,\,\varepsilon}(x) + \omega_{\tau,\varepsilon}(x)\}\,dx \\ &+ \frac{1}{2}\int_0^\pi \{p(x) + \varphi_{\tau,\varepsilon}(x) - r(x) - \omega_{\tau,\varepsilon}(x)\}\cos 2\theta_{k,\tau,\varepsilon}(x)\,dx \\ &+ \frac{1}{2}\int_0^\pi \{\psi_{k,\tau,\varepsilon}(x) + \phi_{\tau,\varepsilon}(x)\}\sin 2\theta_{k,\tau,\varepsilon}(x)\,dx \\ &- \frac{1}{2}\int_0^\pi \{p(x) - r(x)\}\cos 2\theta_k(x)\,dx.\end{aligned} \tag{4.11}$$

It follows from [15, Lemma 4.3] that, for sufficiently large $|k|$, the following relations hold:

$$\int_0^\pi \{p(x) + \varphi_{\tau,\varepsilon}(x) - r(x) - \omega_{\tau,\varepsilon}(x)\}\cos 2\theta_{k,\tau,\varepsilon}(x)\,dx = O\Big(\frac{1}{k}\Big), \tag{4.12}$$

$$\int_0^\pi \{\psi_{\tau,\varepsilon}(x) + \phi_{\tau,\varepsilon}(x)\} \sin 2\theta_{k,\tau,\varepsilon}\, dx = O\biggl(\frac{1}{k}\biggr), \tag{4.13}$$

$$\int_0^\pi \{p(x) - r(x)\} \cos 2\theta_k(x)\, dx = O\biggl(\frac{1}{k}\biggr). \tag{4.14}$$

Since the function $\varrho(\lambda)$ is strictly decreasing on $\mathbb{R}$ it follows from (2.10) that

$$|\rho(\lambda_{k,\tau,\varepsilon}) - \rho(\lambda_k)| < \pi. \tag{4.15}$$

Using (4.7) and (4.12)–(4.15) from (4.11) we obtain

$$|\lambda_{k,\tau,\varepsilon} - \lambda_k| \le \frac{1}{2}(M+K) + 1 + c_k \tag{4.16}$$

where $c_k = O(1/k)$. Then by virtue of (4.16) it follows from (4.6) that

$$\lambda_{\tau,\,\varepsilon} \in \bigl[\lambda_k - \widetilde{c}_k - \rho_0, \lambda_k + \widetilde{c}_k + \rho_0\bigr], \tag{4.17}$$

where $\widetilde{c}_k = (M+K)/2 + 1 + c_k$, $\rho_0 = \sup\limits_{\tau,\varepsilon} \rho_{\tau,\varepsilon} > 0$.

Since $\|w_{\tau,\varepsilon}\| = \tau$ for $0 < \varepsilon \le 1$, $f, g \in C([0,\pi] \times \mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2)$ and $\lambda_{\tau,\varepsilon} \in [\lambda_k - \widetilde{c}_k - \rho_0, \lambda_k + \widetilde{c}_k + \rho_0]$ for $0 < \varepsilon \le 1$ (see (4.17)) it follows from (4.1) that the set $\{w_{\tau,\varepsilon} \in E : 0 < \varepsilon \le 1\}$ is bounded in $C^1([0,\pi]; \mathbb{R}^2)$. Then the set $\{w_{\tau,\varepsilon} \in E : 0 < \varepsilon \le 1\}$ is precompact in E by the Arzelà–Ascoli theorem.

Let $\{\varepsilon_n\}_{n=1}^\infty \subset (0,1)$ be a sequence such that $\varepsilon_n \to 0$ and $(\lambda_{\tau,\varepsilon_n}, w_{\tau,\varepsilon_n}) \to (\lambda_\tau, w_\tau)$ in $\mathbb{R} \times E$. Taking the limit in (4.1) we see that (λ_τ, w_τ) is a solution of problem (1.1)–(1.3). Note that $w_\tau \in \overline{S_k^\nu} = S_k^\nu \cup \partial S_k^\nu$. Since $\|w_\tau\| = \tau$ it follows from Lemma 3.2 that $w_\tau \in S_k^\nu$. □

Corollary 4.2. *The set of bifurcation points of problem* (1.1)–(1.3) *with respect to the set* $\mathbb{R} \times S_k^\nu$ *is nonempty.*

Lemma 4.3. *Let* $\{\varepsilon_n\}_{n=1}^\infty \subset (0,1)$ *be a sequence converging to* 0. *If* $(\lambda_{\varepsilon_n}, w_{\varepsilon_n})$ *in* $\mathbb{R} \times S_k^\nu$ *is a solution of problem* (4.1) *corresponding to* $\varepsilon = \varepsilon_n$ *and sequence* $\bigl\{(\lambda_{\varepsilon_n}, w_{\varepsilon_n})\bigr\}_{n=1}^\infty$ *converges to* $(\zeta, \widetilde{0})$ *in* $\mathbb{R} \times E$, *then* $\zeta \in I_k$, *where* $I_k = [\lambda_k - \widetilde{c}_k, \lambda_k + \widetilde{c}_k]$.

The proof of this lemma is similar to that of [8, Lemma 5.3].

Corollary 4.4. *If* $(\lambda, \widetilde{0})$ *is a bifurcation point of problem* (1.1)–(1.3) *with respect to the set* S_k^ν, *then* $\lambda \in I_k$.

For each $k \in \mathbb{Z}$ and each ν, let $\widetilde{\mathcal{D}}_k^\nu \subset \mathcal{C}$ denote the union of all connected components $\mathcal{D}_{k,\,\lambda}^\nu$ of $\mathcal{C}$ emanating from bifurcation points $(\lambda, \widetilde{0}) \in I_k \times \{\widetilde{0}\}$ with respect to the set $\mathbb{R} \times S_k^\nu$. It follows from Corollaries 4.2 and 4.4 that $\widetilde{\mathcal{D}}_k^\nu \neq \emptyset$. Note that $\mathcal{D}_k^\nu = \widetilde{\mathcal{D}}_k^\nu \cup (I_k \times \{\widetilde{0}\})$ is a connected subset of $\mathbb{R} \times E$, but $\widetilde{\mathcal{D}}_k^\nu$ is not necessarily connected in $\mathbb{R} \times E$.

Theorem 4.5. *For each* $k \in \mathbb{Z}$ *and each* ν *the connected component* $\mathcal{D}_k^\nu$ *of* $\mathcal{C}$ *lies in* $(\mathbb{R} \times S_k^\nu) \cup (I_k \times \{\widetilde{0}\})$ *and is unbounded in* $\mathbb{R} \times E$.

The proof of this theorem is similar to that of [2, Theorem 1.3] with the use Lemmas 4.1, 4.3 and Corollaries 4.2, 4.4.

Since between solutions of problem (1.1)–(1.3) and (3.3) there exists an isomorphism $(\lambda, w) \leftrightarrow (\lambda, \widehat{w})$ Theorem 4.5 yields the following result.

Theorem 4.6. *For each $k \in \mathbb{Z}$ and each ν the connected component $\widehat{\mathcal{D}}_k^\nu = \{\widehat{w} \in \widehat{E} : w \in \mathcal{D}_k^\nu\}$ of $\widehat{\mathcal{C}}$ lies in $(\mathbb{R} \times \widehat{S}_k^\nu) \cup (I_k \times \{\widehat{0}\})$ and is unbounded in $\mathbb{R} \times \widehat{E}$.*

References

[1] Z.S. Aliyev, *Some global results for nonlinear fourth order eigenvalue problems*, Cent. Eur. J. Math. **12** (2014), no. 12, 1811–1828.

[2] Z.S. Aliyev, *Global bifurcation of solutions of certain nonlinear eigenvalue problems for ordinary differential equations of fourth order*, Sb. Math. **207** (2016), no. 12, 1625–1649.

[3] Z.S. Aliyev and G.M. Mamedova, *Some global results for nonlinear Sturm–Liouville problems with spectral parameter in the boundary condition*, Ann. Polon. Math. **115** (2015), no. 1, 75–87.

[4] Z.S. Aliyev and P.R. Manafova, *Oscillation theorems for the Dirac operator with spectral parameter in the boundary condition*, Electron. J. Qual. Theory Differ. Equ. **115** (2016), 1–10.

[5] Z.S. Aliyev and P.R. Manafova, *Oscillation properties for the Dirac equation with spectral parameter in the boundary condition*, Bull. Malays. Math. Sci. Soc., DOI: 10.1007/s40840-019-00749-1.

[6] Z.S. Aliyev and H.Sh. Rzayeva, *Oscillation properties for the equation of the relativistic quantum theory*, Appl. Math. Comput. **271** (2015), 308–316.

[7] Z.S. Aliyev and H.Sh. Rzayeva, *On the oscillation of eigenvector functions of the one-dimensional Dirac operator*, Doklady Math. **94** (2016), no. 1, 401–405.

[8] Z.S. Aliyev and H.Sh. Rzayeva, *Global bifurcation for nonlinear Dirac problems*, Electron. J. Qual. Theory Differ. Equ. **46** (2016), 1–14.

[9] F.V. Atkinson, *Discrete and Continuous Boundary Problems*, Academic Press, New York-London, 1964.

[10] H. Berestycki, *On some nonlinear Sturm–Liouville problems*, J. Differential Equations **26** (1977), 375–390.

[11] P.A. Binding, P.J. Browne and B.A. Watson, *Spectral problem for nonlinear Sturm–Liouville equations with eigenparameter dependent boundary conditions*, Canad. J. Math. **52** (2000), no. 2, 248–264.

[12] R. Chiappinelli, *On eigenvalues and bifurcation for nonlinear Sturm–Liouville operators*, Boll. Unione Mat. Ital. **A4** (1985), 77–83.

[13] G. Dai, *Global bifurcation from intervals for Sturm–Liouville problems which are not linearizable*, Electron. J. Qual. Theory Differ. Equ. **65** (2013), 1–7.

[14] E.N. Dancer, *On the structure of solutions of nonlinear eigenvalue problems*, Indiana Univ. Math. J. **23** (1974), 1069–1076.

[15] N.B. Kerimov, *A boundary value problem for the Dirac system with a spectral parameter in the boundary conditions*, Differ. Equ. **38** (2002), no. 2, 164–174.

[16] M.A. Krasnosel'skiĭ, *Topological Methods in the Theory of Nonlinear Integral Equations*, Macmillan, New York, 1965.

[17] A.C. Lazer and P.J. McKenna, *Global bifurcation and a theorem of Tarantello*, J. Math. Anal. Appl. **181** (1994), no. 3, 648–655.

[18] B.M. Levitan and I.S. Sargsjan, *Introduction to Spectral Theory; Selfadjoint Ordinary Differential Operators*, Transl. Math. Monogr., vol. 39, Amer. Math. Soc., Providence, R.I., 1975.

[19] R. Ma and B. Thompson, *Nodal solutions for a nonlinear fourth-order eigenvalue problem*, Acta Math. Sinica (Engl. Ser.) **24** (2008), no. 1, 27–34.

[20] J. Przybycin, *Some applications of bifurcation theory to ordinary differential equations of the fourth order*, Ann. Polon. Math. **53** (1991), no. 2, 153–160.

[21] P.H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal. **7** (1971), 487–513.

[22] B.P. Rynne, *Bifurcation from zero or infinity in Sturm–Liouville problems which are not linearizable*, J. Math. Anal. Appl. **228** (1998), 141–156.

[23] K. Schmitt and H.L. Smith, *On eigenvalue problems for nondifferentiable mappings*, J. Differential Equations **33** (1979), no. 3, 294–319.

Manuscript received December 7, 2018

accepted April 5, 2019

Ziyatkhan S. Aliyev
Department of Mathematical Analysis
Baku State University
Baku AZ1148, AZERBAIJAN
and
Department of Differential Equations
Institute of Mathematics and Mechanics
NAS Azerbaijan
Baku AZ1141, AZERBAIJAN

E-mail address: z_aliyev@mail.ru

Parvana R. Manafova
Department of Mathematical Analysis
Baku State University
Baku AZ1148, AZERBAIJAN

E-mail address: manafova.pervane@bk.ru