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NODAL SOLUTION FOR A PLANAR PROBLEM WITH FAST INCREASING WEIGHTS

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Abstract. In this paper we prove the existence of a sign-changing solutions for the equation

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = f(u), \quad x \in \mathbb{R}^2,$$

where f has exponential critical growth in the sense of the Trudinger–Moser inequality. In the proof we apply variational methods.

1. Introduction

Consider the nonlinear heat equation

$$v_t - \Delta v = |v|^{p-1} v \quad \text{on } (0,\infty) \times \mathbb{R}^N.$$

If we try to find solutions of the form $v(t,x) = t^{-1/(p-1)} u(t^{-1/2} x)$, a straightforward calculation shows that $u\colon \mathbb{R}^N \to \mathbb{R}$ needs to satisfy

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \frac{1}{p-1}\, u + |u|^{p-1} u \quad \text{in } \mathbb{R}^N.$$

Solutions v with the above profile are called self-similar solutions (see [15], and [8]). Besides providing qualitative properties like global existence, blow-up

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and asymptotic behavior (see e.g. [15], [17], [16]), self-similar solutions (or self-similar variables) are important because they preserve the PDE scaling and so carry simultaneously information about small and large scale behaviors.

For higher dimensions $N \geq 3$, there are results concerning the above equation and its variants obtained by replacing the right-hand side of the equality by more general nonlinearities $f(u)$ (see [2], [8], [21], [20], [7], [14], [9] and references therein). In a large class of such results the authors used variational techniques, in such way that the range of the power p is limited from above by the critical Sobolev exponent $2N/(N-2)$.

In this paper we are interested in the 2-dimensional case, namely the problem

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = f(u), \quad x \in \mathbb{R}^2, \tag{P}$$

where f is such that

(f_0) $f \in C^1(\mathbb{R}, \mathbb{R})$;

(f_1) there exists $\alpha_0 > 0$ such that

$$\lim_{|s| \to +\infty} \frac{f(s)}{e^{\alpha s^2}} = \begin{cases} 0 & \text{if } \alpha > \alpha_0, \\ +\infty, & \text{if } \alpha < \alpha_0. \end{cases}$$

This means that f has critical growth. As it is well known, in dimension two this concept is related with the so callled Trudinger–Moser inequality which appears in the pioneer works [19], [24]. After then, there is a vast literature concerning this kind of critical inequalities. We refer to [10], [1], [11] for bounded domains and to [5], [22], [23] for entire space case. When dealing with the operator $u \mapsto \Delta u + (1/2)(x \cdot \nabla u)$ we need a new Trudinger–Moser type inequality which was estbalished in [13]. There, afer noticing that

$$\operatorname{div}(K(x)\nabla u) = K(x)\left[\Delta u + \frac{1}{2}(x \cdot \nabla u)\right], \quad \text{for } K(x) := e^{|x|^2/4},\ x \in \mathbb{R}^2,$$

the authors introduced the set X as being the closure of the infinitely differentiable radial functions with compact support $C^\infty_{\mathrm{c,rad}}(\mathbb{R}^2)$ with respect to the norm

$$\|u\| := \left(\int_{\mathbb{R}^2} K(x)|\nabla u|^2\,dx\right)^{1/2}.$$

As we shall see in Section 2, the space X has nice properties. In particular, it is well defined the functional $I \in C^1(X, \mathbb{R})$ given by

$$I(u) := \frac{1}{2}\|u\|^2 - \int K(x)F(u)\,dx, \quad \text{where } F(t) := \int_0^t f(\tau)\,d\tau,$$

and its critical points are weak solutions of (P).

We say that a nonzero critical point $w \in X$ of I is a least energy solution if

$$I(w) = \min_{u \in \mathcal{N}} I(u),$$

where $\mathcal{N} := \{u \in X : u \neq 0,\ I'(u)u = 0\}$.

Since we are looking for nodal solutions, instead of the above manifold, we consider the Nehari nodal set

$$\mathcal{M} := \{u \in \mathcal{N} : u^{\pm} \neq 0,\ I'(u^{\pm})u^{\pm} = 0\},$$

where $u^{+}(x) := \max\{u(x), 0\}$ and $u^{-}(x) := \min\{u(x), 0\}$, for all $x \in \mathbb{R}^2$. The main objective is to guarantee that the minimum $c := \min\limits_{u \in \mathcal{M}} I(u)$ is achived at a solution $w \in X$. Notice that the set $\mathcal{M}$ contains all sign-changing radial solutions of (P) and therefore the minimum point w is called *least energy nodal solution*.

In the first result we consider the (subcritical) power-type case and prove the following:

Theorem 1.1. *Suppose that* $p > 2$ *and* $f(t) = |t|^{p-2}t$. *Then the problem* (P) *possesses a least energy nodal solution* $w_p \in X$ *such that*

$$c_p := \min_{u \in \mathcal{M}} I(u) = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^2} K(x)|w_p|^p \, dx.$$

The existence of nodal solutions for a power-type concave/convex nonlinearity was obtained in [26] via a fiber map approach. Here, we use a different technique by adapting some ideas from [3] (see also [6]). Actually, it holds for more general nonlinearities with critical (or subcritical) growth. Hence, for our second result, besides (f_0)–(f_1) we suppose that

(f_2) there holds
$$\lim_{t \to 0} \frac{f(t)}{t} = 0;$$

(f_3) there exists $\theta > 2$ such that
$$0 < \theta F(t) \leq f(t)t, \quad \text{for all } t \neq 0;$$

(f_4) the map $t \to f(t)/|t|$ is increasing in $\mathbb{R}\backslash\{0\}$;

(f_5) there exist
$$p > 2 \quad \text{and} \quad \tau > \left[c_p\left(\frac{2\theta}{\theta - 2}\right)\frac{\alpha}{4\pi}\right]^{(p-2)/2}$$
such that $f(t)t \geq \tau|t|^p$, for all $t \in \mathbb{R}$.

The main result of this paper can be stated as follows:

Theorem 1.2. *Suppose that* f *satisfies* (f_0)–(f_5). *Then the problem* (P) *possesses a least energy nodal solution.*

Condition (f_3) is the well-known Ambrosetti–Rabinowitz condition which guarantees that Palais–Smale sequences are bounded. Since f has critical growth, this boundedness is not sufficient to get compactness for the functional. Actually, it is important to use some abstract inequalities proved in [13] as well as the technical condition (f_5). Roughly speaking, it assures that Palais–Smale sequences have small norm and therefore some standard arguments can be applied to recover compactness. The monotonicity condition (f_4) is used to prove some projections properties on the Nehary nodal set $\mathcal{M}$.

The main results of this paper complement those of [12], [13], [26] since we deal with a different class of nonlinearities and we find a nodal solution. They also complement the aforementioned works which study self-similar solutions for the nonlinear heat equation since we consider here the 2-dimensional case.

The paper is organized as follows. In the next section, we present the variational setting of the problem and prove our first theorem. In Section 3 we prove that minimizers of the functional I on $\mathcal{M}$ are critical points. In the final Section 4, we prove Theorem 1.2.

2. Variational framework and the proof of Theorem 1.1

Throughout the paper we write $\int u$ instead of $\int_{\mathbb{R}^2} u(x)\,dx$. In order to present the functional space to deal with our problem we consider $C^\infty_{\mathrm{c,rad}}(\mathbb{R}^2)$ the space of infinitely differentiable radial functions with compact support and denote by X the closure of $C^\infty_{\mathrm{c,rad}}(\mathbb{R}^2)$ with respect to the norm

$$\|u\| := \bigg(\int K(x)|\nabla u|^2\bigg)^{1/2}.$$

For each $s \geq 2$, we also consider the weighted Lebesgue space $L^s_K(\mathbb{R}^2)$ of all the measurable functions $u\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$\|u\|_s := \bigg(\int K(x)|u|^s\bigg)^{1/s} < \infty.$$

According to [13, Lemma 2.1] the space X is compactally embedded into the Lebesgue spaces $L^s_K(\mathbb{R}^2)$ for any $s \in [2,\infty)$. Moreover, the following version of the Trudinger–Moser inequality holds:

Theorem 2.1. *We have that $K(x)|u|^q(e^{\beta u^2}-1) \in L^1(\mathbb{R}^2)$ for any $q \geq 2$, $u \in X$ and $\beta > 0$. Moreover, if $\|u\| \leq M$ and $\beta M^2 < 4\pi$, then there exists $C = C(M,\beta,q) > 0$ such that*

$$\int_{\mathbb{R}^2} K(x)|u|^q(e^{\beta u^2}-1)\,dx \leq C(M,\beta,q).$$

Proof. See [13, Theorem 1.1 and Corollary 1.2]. □

Let $\alpha > \alpha_0$ be given by (f_1) and $q \geq 1$. By using the critical growth of f we obtain

$$\lim_{|t|\to+\infty} \frac{f(t)}{|t|^{q-1}(e^{\alpha t^2}-1)} = 0.$$

This and (f_2) imply that, for any given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\max\{|f(t)t|, |F(t)|\} \le \varepsilon|t|^2 + C_\varepsilon|t|^q\Big(e^{\alpha t^2} - 1\Big), \quad \text{for all } t \in \mathbb{R}. \tag{2.1}$$

This inequality with $q = 2$ and Theorem 2.1 imply that the functional

$$I(u) := \frac{1}{2}\int K(x)|\nabla u|^2\,dx - \int K(x)F(u)\,dx, \quad \text{for all } u \in X,$$

is well defined. By using standard calculations we conclude that $I \in C^1(X, \mathbb{R})$ with the derivative given by

$$I'(u)v = \int K(x)(\nabla u \cdot \nabla v)\,dx - \int K(x)f(u)v\,dx, \quad \text{for all } u,\, v \in X$$

and therefore the critical points of I are weak solutions of (P).

LEMMA 2.2. *Suppose that f satisfies* (f_0)–(f_3). *Then, there exists $\rho > 0$ scuh that, for any $u \in \mathcal{N}$, $\|u\| \ge \rho$ and*

$$I(u) \ge \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2.$$

PROOF. If $u \in \mathcal{N}$, it follows from (f_3) that

$$I(u) = I(u) - \frac{1}{\theta}\,I'(u)u \ge \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2.$$

Suppose, by contradiction, that there exists a sequence $(u_n) \subset \mathcal{N}$ such that $\|u_n\| \to 0$. We may assume that, for some $\beta < 4\pi$, there holds $\alpha\|u_n\|^2 < \beta$. By setting $v_n := u_n/\|u_n\|$, we can use (2.1) with $q > 2$ and the embbeding $X \hookrightarrow L^2_K(\mathbb{R}^2)$ to get

$$\begin{aligned}
\|u_n\|^2 &= \int K(x)f(u_n)u_n \\
&\le \varepsilon\int K(x)u_n^2 + C_\varepsilon\|u_n\|^q\int K(x)|v_n|^q\Big[e^{4\pi\|u_n\|^2|v_n|^2} - 1\Big] \\
&\le \varepsilon C_1\|u_n\|^2 + C_\varepsilon\|u_n\|^q\int K(x)|v_n|^q\Big[e^{\beta|v_n|^2} - 1\Big],
\end{aligned}$$

for some constant $C_1 > 0$. Since $\|v_n\| = 1$, it follows from Theorem 2.1 that

$$(1 - \varepsilon C_1) \le C_\varepsilon\|u_n\|^{q-2}C(1, \beta, q).$$

If $\varepsilon > 0$ is small, the above inequality and $q > 2$ contraditcs $\|u_n\| \to 0$. □

In what follows we prove that the set $\mathcal{M}$ is nonempty.

LEMMA 2.3. *Suppose that f satisfies* (f_0)–(f_2) *and* (f_5). *Then, for any $u \in X$ such that $u^\pm \ne 0$, there exist $t_u, s_u > 0$ such that*

$$I'(t_u u^+ + s_u u^-)u^+ = I'(t_u u^+ + s_u u^-)u^- = 0.$$

Consequently, $t_u u^+ + s_u u^- \in \mathcal{M}$.

Proof. Given $q > 2$, it follows from $I'(tu^+ + sv^-)(tu^+) = I'(tu^+)(tu^+)$ and (2.1) that

$$I'(tu^+ + sv^-)(tu^+) \geq \frac{t^2}{2}\|v^+\|^2 - \frac{t^2}{2}\varepsilon \int K(x)(v^+)^2 - t^q C_\varepsilon \int K(x)(v^+)^q (e^{\alpha(tv^+)^2} - 1).$$

Recalling that $X \hookrightarrow L^2_K(\mathbb{R}^2)$ and using Theorem 2.1, we obtain $C_1 > 0$ and $C_2 = C_2(\varepsilon, \|v^+\|, q) > 0$ such that

$$I'(tu^+ + su^-)(tu^+) \geq \frac{t^2}{2}(1 - \varepsilon C_1)\|v^+\|^2 - C_2 t^q \|v^+\|^q,$$

for any $0 \leq t < \sqrt{(4\pi)/(2\alpha\|v^+\|^2)}$. By picking $\varepsilon > 0$ small and using $q > 2$, we obtain $t_* > 0$ such that

$$I'(tu^+ + su^-)(tu^+) > 0, \quad \text{for all } s \geq 0,\ t \in [0, t_*]. \tag{2.2}$$

On the other hand, by integrating the inequality in (f_5), we conclude that

$$F(t) \geq \frac{\tau}{p}|t|^p, \quad \text{for all } t \in \mathbb{R}. \tag{2.3}$$

Hence,

$$I'(tu^+ + su^-)(tu^+) \geq \frac{t^2}{2}\|v^+\|^2 - \tau\frac{t^p}{p}\int K(x)(v^+)^p.$$

Recalling that $p > 2$ we obtain $t^* > t_*$ such that

$$I'(tu^+ + su^-)(tu^+) < 0, \quad \text{for all } s \geq 0,\ t \in [t^*, +\infty). \tag{2.4}$$

In the same way, starting from $I'(tu^+ + su^-)(su^-) = I'(su^-)(su^-)$, we obtain $s^* > s_* > 0$ such that

$$\begin{aligned} I'(tu^+ + su^-)(su^-) &> 0, \quad \text{for all } t \geq 0,\ s \in [0, s_*], \\ I'(tu^+ + su^-)(su^-) &< 0, \quad \text{for all } t \geq 0,\ s \in [s^*, +\infty). \end{aligned}$$

The result follows from the above inequalities, (2.2)–(2.4) and a version of the Intermediate Value Theorem proved by Miranda in [18].

We are ready to present the proof of our first main result.

Proof of Theorem 1.1. Supose that $f(t) = |t|^{p-2}t$. Then the associated functional is

$$I_p(u) := \frac{1}{2}\|u\|^2 - \frac{1}{p}\int K(x)|u|^p, \quad u \in X.$$

Let $\mathcal{N}_p$ and $\mathcal{M}_p$ be the Nehari manifold and the Nehari nodal set of I_p, respectively, and set

$$c_p := \min_{u \in \mathcal{M}_p} I_p(u).$$

We are going to show that c_p is attained in a (least energy) nodal solution of the problem (P).

Let $(u_n) \subset \mathcal{M}_p$ be such that $I_p(u_n) \to c_p$. By Lemma 2.2, (u_n) is bounded in X and, up to a subsequence, $u_n \rightharpoonup u$ weakly in X. Moreover, since the map $w \mapsto w^\pm$ is continuous from $L^s(\mathbb{R}^2)$ to $L^s(\mathbb{R}^2)$ (see [6, Lemma 2.3]), we also have that $u_n^\pm \rightharpoonup u^\pm$ weakly in X. By Lemma 2.2, we have that

$$\rho \le \|u_n^\pm\|^2 = \int K(x)|u_n^\pm|^p.$$

Taking the limit and recalling that the embedding $X \hookrightarrow L^p_K(\mathbb{R}^2)$ is compact, we conclude that $\int K(x)|u^\pm|^p \ge \rho > 0$ and therefore $u^\pm \ne 0$. We can now use Lemma 2.3 to obtain t_u, $s_u > 0$ such that $w_p := t_u u^+ + s_u u^- \in \mathcal{M}_p$. By using the weak convergence and the compact embedding again we get

$$c_p \le I_p(w_p) \le \liminf_{n\to+\infty} I_p(u_n) = c_p.$$

Moreover,

$$c_p = I_p(w_p) - \frac{1}{p} I_p'(w_p)w_p = \left(\frac{1}{2} - \frac{1}{p}\right) \int K(x)|w_p|^p.$$

It remains to prove that w_p is a critical point of I_p. This will be done in the next section (see Proposition 3.2) in a more general setting. $\square$

3. The deformation argument

For each $u \in X$ with $u^\pm \ne 0$, let us consider $h^u : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ given by

$$h^u(t,s) := I(tu^+ + su^-), \qquad \text{for all } (t,s) \in \mathbb{R}^+ \times \mathbb{R}^+$$

and denote by $\Phi^u \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^2$ its gradient, that is,

$$\Phi^u(t,s) := \big(\Phi_1^u(t,s), \Phi_2^u(t,s)\big) = \big(I'(tu^+ + su^-)u^+, I'(tu^+ + su^-)u^-\big),$$

for every $(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+$.

The next result is a version of [3, Lemma 2,1] and states that, when dealing with the Nehari nodal set, the map h^u has the same properties of the usual fiber maps.

Lemma 3.1. *Suppose that f satisfies* (f$_0$)–(f$_2$) *and* (f$_4$)*. If $u \in \mathcal{M}$, then*

$$h^u(t,s) < h^u(1,1) = I(u), \quad \textit{for all } (s,t) \in (\mathbb{R}^+ \times \mathbb{R}^+) \setminus \{(1,1)\}.$$

Moreover, $\det(\Phi^u)'(1,1) > 0$.

Proof. Let $u \in \mathcal{M}$ and notice that $0 = I'(u)u^\pm = I'(u^+ + u^-)u^\pm$. Hence,

$$\Phi^u(1,1) = \left(\frac{\partial h^u}{\partial t}(1,1), \frac{\partial h^u}{\partial s}(1,1)\right) = (0,0)$$

and we conclude that $(1,1)$ is a critical point of h^u. Given $t, s \ge 0$, we infer from (2.3) that

$$h^u(t,s) \le \frac{t^2}{2}\|u^+\|^2 + \frac{s^2}{2}\|u^-\|^2 - \frac{t^p}{p}\tau \int K(x)|u^+|^p - \frac{s^p}{p}\tau \int K(x)|u^-|^p.$$

Since $p > 2$, it follows that

$$\lim_{|(t,s)|\to+\infty} h^u(t,s) = -\infty,$$

and therefore h^u attains its maximun value at some point $(\bar{t},\bar{s}) \in \mathbb{R}^+ \times \mathbb{R}^+$.

We first prove that $\bar{t},\bar{s} > 0$. Suppose, by contradiction, that $\bar{s} = 0$. Thus, $I'(\bar{t}u^+)\bar{t}u^+ = 0$, or equivalently,

$$\|u^+\|^2 = \int \frac{K(x)f(\bar{t}u^+)}{\bar{t}}\, u^+.$$

This and $I'(u^+)u^+ = 0$ provides

$$\int_{\{u>0\}} K(x)\bigg[\frac{f(tu^+)}{tu^+} - \frac{f(u^+)}{u^+}\bigg](u^+)^2 = 0,$$

and it follows from (f_4) that $\bar{t} = 1$. Since Lemma 2.2 provides $I(u^-) > 0$, we get

$$h^u(\bar{t},0) = h^u(1,0) = I(u^+) < I(u^+) + I(u^-) = I(u) = h^u(1,1),$$

which is absurd because $(\bar{t},0)$ is a global maximum point. The same argument proves that $\bar{t} > 0$.

Since $(1,1)$ and $(\bar{t},\bar{s})$ are both critical points of h^w, we have that $I'(\bar{t}u^+)\bar{t}u^+ = I'(\bar{s}u^-)\bar{s}u^- = 0$ and $I'(u^+)u^+ = I'(u^-)u^- = 0$. Hence, we can argue as above to conclude that $\bar{t} = \bar{s} = 1$.

In order to check that $\det(\Phi^u)'(1,1) > 0$, we first notice that

$$(\Phi^u)'(t,s) = \begin{pmatrix} g_1'(t) & 0 \\ 0 & g_2'(s) \end{pmatrix},$$

where

$$g_1(t) := \Phi_1^u(t,s) = I'(tu^+)u^+ = t\|u^+\|^2 - \int K(x)f(tu^+)u^+,$$

and $g_2(s) := \Phi_2^u(t,s)$. Since $u^+ \in \mathcal{N}$, it follows from the definition of $g_1(t)$ and (f_4) that

$$g_1'(1) = \|u^+\|^2 - \int K(x)f'(u^+)(u^+)^2 = \int K(x)\left[f(u^+)u^+ - f'(u^+)(u^+)^2\right] < 0.$$

Analagously $g_2'(1) < 0$, and therefore we conclude that

$$\det(\Phi^u)'(1,1) = g_1'(1)g_2'(1) > 0. \qquad \square$$

We now use a deformation argument to show that the set $\mathcal{M}$ is a natural constraint for the functional I. The proof is adapted from [4, Proposition 3.1].

PROPOSITION 3.2. *Suppose that f satisfies* (f_0)–(f_2) *and* (f_4)*. If $w \in \mathcal{M}$ is such that*

$$I(w) = c := \min_{u\in\mathcal{M}} I(u), \tag{3.1}$$

then $I'(w) = 0$.

PROOF. Suppose, by contradiction, that the result is false. Then, there exist $\delta, \lambda > 0$ such that $\|I'(v)\| > \lambda$ whenever $\|v - w\| < 3\delta$. Setting $g(t,s) := tw^+ + sw^-$, we can use Lemma 3.1 to obtain $D \subset \mathbb{R}^2$ such that $(1,1) \in D$ and

$$\alpha := \max_{(t,s)\in\partial D} I(g(t,s)) = \max_{(t,s)\in\partial D} h^w(t,s) < c. \tag{3.2}$$

For $\varepsilon < \min\{(c-\alpha)/2, \lambda\delta/8\}$ and $S := B_\delta(w)$, it follows from [25, Lemma 2.3] that there exists $\eta \in C([0,1] \times X, X)$ verifying

(i) $\eta(1,u) = u$, if $u \notin I^{-1}([c-2\varepsilon, c+2\varepsilon])$;
(ii) $\eta(1, I^{c+\varepsilon} \cap S) \subset I^{c-\varepsilon}$;
(iii) $I(\eta(1,u)) \le I(u)$, for any $u \in X$.

By Lemma 3.1, (ii) and (iii) it follows that

$$\max_{(t,s)\in D} I(\eta(1, g(t,s))) < c. \tag{3.3}$$

It follows from the definition of Φ^w and $w \in \mathcal{M}$ that $\Phi^w(t,s) = 0$ if, and only if, $(t,s) = (1,1) \in D$. Thus, from the definition of the Brouwer degree and Lemma 3.1, we get

$$\deg(\Phi^w, D, 0) = \operatorname{sgn}\det(\Phi^w)'(1,1) = 1.$$

We set $h(t,s) := \eta(1, g(t,s))$,

$$\Psi(t,s) := (t^{-1}I'(h(t,s))h(t,s)^+, s^{-1}I'(h(t,s))h(t,s)^-) \tag{3.4}$$

and notice that, by the choice of $\varepsilon > 0$, (3.2) and (i), we have that that $g = h$ on ∂D. So, the definition of Φ^w and (3.4) imply that $\Phi^w = \Psi$ on ∂D, from which we obtain

$$\deg(\Psi, D, 0) = \deg(\Phi^w, D, 0) = 1.$$

Thus, there exists $(t,s) \in D$ such that $h(t,s) \in \mathcal{M}$, which contradicts (3.3). This contradiction proves that $I'(w) = 0$ and we conclude the proof. □

4. Proof of Theorem 1.2

We devote this section to the proof of our second main result. The main idea is to consider the minimization problem defined in (3.1). The first step of the proof is obtaining an estimative of the number c.

LEMMA 4.1. *Suppose that f satisfies* (f_0)–(f_2) *and* (f_5). *Then the numer c defined in* (3.1) *satisfies*

$$c \le \frac{c_p}{\tau^{2/(p-2)}},$$

where $c_p > 0$ comes from Theorem 1.1.

Proof. Let $w_p \in X$ be the solution given by Theorem 1.1. Since $w^{\pm} \neq 0$, by Lemma 2.3, we obtain t_{w_p}, $s_{w_p} > 0$ such that $t_{w_p} w_p^+ + s_{w_p} w_p^- \in \mathcal{M}$. Hence, by (2.3) and $I_p'(w_p^{\pm})w_p^{\pm} = 0$, we get

$$\begin{aligned} c &\leq I(t_{w_p} w_p^+ + s_{w_p} w_p^-) \\ &\leq \frac{t_{w_p}^2}{2}\|w_p^+\|^2 + \frac{s_{w_p}^2}{2}\|w_p^-\|^2 - \frac{\tau}{p} t_{w_p}^p \int K(x)|w_p^+|^p - \frac{\tau}{p} s_{w_p}^p \int K(x)|w_p^-|^p \\ &= \left[\frac{t_{w_p}^2}{2} - \tau \frac{t_{w_p}^p}{p}\right] \int K(x)|w_p^+|^p + \left[\frac{s_{w_p}^2}{2} - \tau \frac{s_{w_p}^p}{p}\right] \int K(x)|w_p^-|^p \\ &\leq \max_{s \geq 0} \left[\frac{s^2}{2} - \tau \frac{s^p}{p}\right] \int K(x)|w_p|^p. \end{aligned}$$

Recalling the value of c_p given in the statement of Theorem 1.1, a straightforward calculation provides

$$c \leq \max_{s \geq 0} \left[\frac{s^2}{2} - \tau \frac{s^p}{p}\right] \left(\frac{1}{2} - \frac{1}{p}\right)^{-1} c_p \leq \frac{c_p}{\tau^{2/(p-2)}},$$

and we have done. $\square$

It can be proved that any function $u \in X$ decays as $|x|^{-1/2} e^{-|x|^2/8}$ at infinity. Hence, we have the following techical result whose proof can be found in [13, Lemma 4.4]:

Lemma 4.2. *Suppose $G \in C(\mathbb{R}, \mathbb{R})$ satisfies*

$$G(t) \leq c_1 t^4 (e^{\alpha t^2} - 1), \quad \textit{for all } t \in \mathbb{R},$$

with $c_1, \alpha > 0$. Then there exists $c_2, c_3 > 0$ such that, for any radial function $u \in X$ and $R > 1$, there holds

$$\int_{B_R(0)^c} K(x) G(u)\, dx \leq \frac{c_2}{R} \|u\|^4 (e^{\alpha c_3^2 \|u\|^2} - 1).$$

In our next result, we use the above lemma to obtain a compactness property for sequences with small norm in X.

Lemma 4.3. *Suppose that f satisfies (f_0)–(f_2). If $(u_n) \subset X$ and there exists $0 < \delta < (4\pi)/\alpha$ such that*

$$\limsup_{n \to +\infty} \|u_n\|^2 < \delta,$$

then, up to a subsequence, $u_n \rightharpoonup u$ weakly in X,

$$\lim_{n \to +\infty} \int K(x) F(u_n) = \int K(x) F(u), \tag{4.1}$$

$$\lim_{n \to +\infty} \int K(x) f(u_n) u_n = \int K(x) f(u) u. \tag{4.2}$$

PROOF. The first statement is a direct consequence of the boundedness of (u_n) in X. We shall prove (4.1) since the other convergence follows in the same way. For any given $\varepsilon > 0$ and $R > 0$, we can use (2.1) with $q = 4$ and Lemma 4.2 to get

$$\int_{B_R(0)^c} K(x)F(u_n)\,dx \leq \varepsilon \int K(x)|u_n|^2 + \frac{c_2}{R}\|u_n\|^4\Big(e^{\alpha c_3^2\|u_n\|^2} - 1\Big) \leq c_4\varepsilon + \frac{c_5}{R},$$

from which we conclude that

$$\limsup_{R\to+\infty} \int_{B_R(0)^c} K(x)F(u_n)\,dx \leq c_4\varepsilon. \tag{4.3}$$

In order to estimate the integral in the ball we recall that, since the embedding $X \hookrightarrow L^2_K(\mathbb{R}^2)$ is compact, there exists $\psi_2 \in L^2(B_R(0))$ such that, for almost every $x \in B_R(0)$, there holds

$$K(x)F(u_n) \leq K(x)|u_n|^2 + c_6K(x)|u_n|\Big(e^{\alpha|u_n|^2} - 1\Big) \leq \psi_2(x)^2 + c_7|u_n|e^{\alpha|u_n|^2},$$

where we have used (2.1) with $\varepsilon = q = 1$. We now claim that

$$g_n(x) := |u_n|e^{\alpha|u_n|^2} \to |u|e^{\alpha|u|^2},$$

stongly in $L^1(B_R(0))$. If this is true, it follows from the pointwise convergence $F(u_n(x)) \to F(u(x))$ almost everywhere in $B_R(0)$, the last inequality and the Lebesgue Theorem that

$$\lim_{n\to+\infty} \int_{B_R(0)} K(x)F(u_n)\,dx = \int_{B_R(0)} K(x)F(u)\,dx.$$

Since $R > 0$ is arbitrary, this and (4.3) imply (4.1).

It remains to check that g_n converges in $L^1(B_R(0))$. For any $s > 1$, we have that

$$\Big(e^{\alpha|u_n|^2}\Big)^s \leq e^{\alpha s\delta(u_n^2/\|u_n\|^2)}, \quad \text{for a.e. } x \in B_R(0).$$

Since $\delta < (4\pi)/\alpha$, we can pick $s > 1$ sufficiently close to 1 in such way that $\alpha s\delta < 4\pi$. Thus, it follows from the above inequality and the classical Trudinger–Moser inequality (see [19], [24]) that the sequence $\big(e^{\alpha u_n^2}\big)$ is bounded in $L^s(B_R(0))$. Since we also have pointwise convergence, we may assume that

$$e^{\alpha u_n^2} \rightharpoonup e^{\alpha u^2} \quad \text{weakly in } L^s(B_R(0)).$$

If we denote by $s' > 1$ the conjugated exponent of s, we have that $|u_n| \to |u|$ in $L^s(B_R(0))$ for any $s > 1$. These two convergences and Hölder's inequality imply that g_n strongly converges to $|u|e^{\alpha|u|^2}$ in $L^1(B_R(0))$. $\square$

We are ready to present the proof of the main result of this paper.

Proof of Theorem 1.2. Let $(u_n) \subset \mathcal{M}$ be such that $I(u_n) \to c$ as $n \to +\infty$. Since $I'(u_n)u_n = 0$, it follows from (f_3) that

$$c + o_n(1) = I(u_n) - \frac{1}{\theta} I'(u_n)u_n \geq \bigg(\frac{1}{2} - \frac{1}{\theta}\bigg)\|u_n\|^2.$$

Hence, we infer from Lemma 4.1 that

$$\limsup_{n\to+\infty} \|u_n\|^2 \leq \bigg(\frac{2\theta}{\theta-2}\bigg)c \leq \bigg(\frac{2\theta}{\theta-2}\bigg)\frac{c_p}{\tau^{2/(p-2)}}.$$

By (f_5),

$$\tau > \bigg[c_p\bigg(\frac{2\theta}{\theta-2}\bigg)\frac{\alpha}{4\pi}\bigg]^{(p-2)/2}$$

and therefore we conclude that $\limsup_{n\to+\infty} \|u_n\|^2 < 4\pi/\alpha$. So, we can use Lemma 4.3 to obtain $u \in X$ such that (4.1)–(4.2) hold.

As in the proof of Theorem 1.1, we have that $u_n^\pm \rightharpoonup u^\pm$ weakly in X. Recalling that $(u_n^\pm) \subset \mathcal{N}$, we can use the last inequality, Lemmas 2.2 and 4.3 to conclude that

$$\rho \leq \|u_n^\pm\|^2 = \int K(x)f(u_n^\pm)u_n^\pm + o_n(1) = \int K(x)f(u^\pm)u^\pm,$$

and therefore $u^\pm \neq 0$. By Lemma 2.3, there exist t_u, $s_u > 0$ such that $w := t_u u^+ + s_u u^- \in \mathcal{M}$. From (4.1) and Lemma 3.1 we get

$$c \leq I(w) = I(t_u u^+ + s_u u^-) \leq \liminf_{n\to+\infty} I(t_u u_n^+ + s_u u_n^-) \leq \liminf_{n\to+\infty} I(u_n^+ + u_n^-) = c,$$

and therefore the infimum of I over $\mathcal{M}$ is attained at $w \in \mathcal{M}$. Proposition 3.2 implies that w is a least energy nodal solution of (P). □

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