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EXISTENCE OF SOLUTIONS FOR FRACTIONAL p-KIRCHHOFF TYPE EQUATIONS WITH A GENERALIZED CHOQUARD NONLINEARITY

Wenjing Chen

Abstract. In this article, we establish the existence of solutions to the fractional p-Kirchhoff type equations with a generalized Choquard nonlinearity without assuming the Ambrosetti–Rabinowitz condition.

1. Introduction and statement of main result

In this work, we consider the following fractional p-Laplacian generalized Choquard equation

$$M(\|u\|_W^p)\big[(-\Delta)_p^s u + V(x)|u|^{p-2}u\big] = \lambda(\mathcal{I}_\mu * F(u))f(u), \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $1 < ps < N$, $M\colon \mathbb{R}_0^+ \to \mathbb{R}^+$ is a Kirchhoff function,

$$\|u\|_W = \bigg([u]_{s,p}^p + \int_{\mathbb{R}^N} V(x)|u|^p\,dx\bigg)^{1/p} \tag{1.2}$$

with

$$[u]_{s,p} = \bigg(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\bigg)^{1/p},$$

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the potential function $V\colon \mathbb{R}^N \to \mathbb{R}^+$ is continuous, $f \in C(\mathbb{R},\mathbb{R})$ and $F \in C(\mathbb{R},\mathbb{R})$ with $F(u) = \int_0^u f(t)\,dt$, here $\mathcal{I}_\mu(x) = |x|^{-\mu}$ is the Riesz potential of order $\mu \in (0, ps)$, and $(-\Delta)_p^s$ is the fractional p-Laplacian operator which, up to a normalization constant, is defined as

$$(-\Delta)_p^s \varphi(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dy, \quad x \in \mathbb{R}^N,$$

along functions $\varphi \in C_0^\infty(\mathbb{R}^N)$, where $B_\varepsilon(x)$ denotes the ball of $\mathbb{R}^N$ centered at $x \in \mathbb{R}^N$ and radius $\varepsilon > 0$.

On the one hand, this paper is motivated by some works that has been focused on the study of the Kirchhoff type problems. To be precise, in the classical Laplacian operator case, it is related to the stationary analogue of the equation

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2\right)\triangle u = g(x,u), \tag{1.3}$$

proposed by Kirchhoff [16] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Equation (1.3) received much attention only after Lions [21] proposed an abstract framework to the problem. Fiscella and Valdinoci [15] first proposed a stationary fractional Kirchhoff variational model as follows

$$\begin{cases} M\left(\displaystyle\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\right)(-\Delta)^s u(x) & \\ \qquad\qquad\qquad\qquad = \lambda f(x,u) + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded set, $2^* = 2N/(N-2s)$, $N > 2s$ with $s \in (0,1)$. M and f are two continuous functions under some suitable assumptions. In [15], the authors first provided a detailed discussion about the physical meaning underlying the fractional Kirchhoff problems and their applications. They supposed that $M\colon \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing and continuous function, and there exists $m_0 > 0$ such that $M(t) \geq m_0 = M(0)$ for all $t \in \mathbb{R}^+$. Based on the truncated skill and mountain pass theorem, they obtained the existence of a non-negative solution to problem (1.4) for any $\lambda > \lambda^* > 0$, where λ^* is an appropriate threshold. Autuori et al. [5] established the existence and the asymptotic behavior of non-negative solutions to problem (1.4) under different assumptions on M, the Kirchhoff function M can be zero at zero, that is, the problem is degenerate case.

Moreover, there is a lot of literature concerning the existence and multiplicity of solutions for the fractional p-Laplacian Kirchhoff type problems. Xiang et al.

in [39] investigated the existence of solutions for Kirchhoff type problems involving the fractional p-Laplacian by variational methods, where the nonlinearity is subcritical and the Kirchhoff function is non-degenerate. Combining the mountain pass theorem with the Ekeland variational principle, Xiang et al. in [40] established the existence of two solutions for a degenerate fractional p-Laplacian Kirchhoff equation in $\mathbb{R}^N$ with concave-convex nonlinearity. By the same methods as in [40], Pucci et al. in [30] obtained the existence of two solutions for a nonhomogenous Schrödinger–Kirchhoff type equation involving the fractional p-Laplacian in $\mathbb{R}^N$ on a nondegenerate situation. Furthermore, nonexistence and multiplicity of solutions for a nonhomogeneous fractional p-Kirchhoff type problem involving critical exponent in $\mathbb{R}^N$ were studied in [41]. The existence of infinitely many solutions was proved in [31], [36] by using Krasnosel'skiĭ's genus theory under degenerate frameworks. Recently, Song and Shi considered infinitely many solutions for the degenerate p-fractional Kirchhoff equations with the critical Sobolev–Hardy nonlinearities in [34], [35]. Xiang, Radulescu and Zhang obtained the existence of nontrivial radial solutions for a fractional Choquard–Kirchhoff-type problem involving an external magnetic potential and a critical nonlinearity in [38]. The local existence and blow-up of solutions for a diffusion model of Kirchhoff-type driven by a nonlocal integro-differential operator were studied in [37].

On the other hand, we mention some results about the Choquard equation, consider the following Choquard or the nonlinear Schrödinger–Newton equation

$$-\Delta u + V(x)u = (\mathcal{I}_\mu * u^2)u + \lambda f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

which was elaborated by Pekar [29] in the framework of quantum mechanics. The first investigation for the existence and symmetry of solutions to (1.5) went back to the works of Lieb [19]. Equations of type (1.5) have been extensively studied, see e.g. [1], [25], [26] and references therein. Moroz and van Schaftingen in [26] considered the existence of ground-states for a generalized Choquard equation. The existence, multiplicity and concentration of solutions for a generalized quasilinear Choquard equation were studied by Alves and Yang in [2], [3]. We refer to [28] for a good survey of the Choquard equation.

In the setting of the fractional Choquard equations,

$$(-\Delta)^s u + V(x)u = (\mathcal{I}_\mu * F(u))f(u) \quad \text{in } \mathbb{R}^N, \tag{1.6}$$

Wu [42] investigated existence and stability of solutions to (1.6) with $f(u) = u$ and $\mu \in (N-2s, N)$. Subsequently, D'Avenia and Squassina in [12] studied the existence, regularity and asymptotic behavior of solutions to (1.6) with $f(u) = u^p$ and $V(x) \equiv \text{const}$. In particular, they claimed the nonexistence of solutions as $q \in ((2N-\mu)/N, (2N-\mu)/(N-2s))$. If $V(x) = 1$ and f satisfies Berestycki–Lions type assumptions, the existence of ground state solutions for a fractional

Choquard equation has been established in [33]. Very recently, Ambrosio studied the concentration phenomena of solutions for a fractional Choquard equation with mangetic field in [4].

Recently, Belchior et al. in [6] applied the mountain pass theorem without (PS) condition and a characterization of the infimum more suitable to the Nehari manifold naturally attached to the problem to study the existence of ground state, regularity and polynomial decay for the following fractional Choquard equation

$$(-\Delta)_p^s u + A|u|^{p-2}u = (\mathcal{I}_\mu * F(u))f(u) \quad \text{in } \mathbb{R}^N, \tag{1.7}$$

where A is a positive constant, f is a C^1 positive function on $(0,+\infty)$,

$$\lim_{t\to 0}\frac{|f(t)|}{t^{p-1}} = 0, \qquad \lim_{t\to+\infty}\frac{f(t)}{t^{q-1}} = 0$$

for some $p < q < (2N-\mu)p/(2(N-ps))$, and

$$f'(t)t^2 - (p-1)f(t)t > 0 \quad \text{for all } t > 0. \tag{1.8}$$

An example of function f satisfying these hypotheses is given by

$$f(t) = |t|^{q_1-1}t^+ + |t|^{q_2-1}t^+, \quad \text{where } p < q_1 < q_2 < \frac{(N-\mu)p}{N-ps}$$

and $t^+ = \max\{t, 0\}$. From (1.8), f satisfies the Ambrosetti–Rabinowitz condition ((AR) for short):

$$pF(t) < tf(t) \quad \text{for all } t > 0, \tag{1.9}$$

and the function $f(t)/t^{p-1}$ is increasing. It is well known that the (AR)-condition is quite natural and important not only to ensure that an Euler–Lagrangian functional has the mountain pass geometry structure, but also to ensure that the Palais–Smale sequence of the functional is bounded. However, there are many functions which are superlinear at infinity, but do not satisfy the (AR)-condition, for example, the function $f(t) = |t|^{p-2}t\log(1+|t|)$. Thus, many researchers have tried to drop the (AR)-condition for elliptic equations involving the p-Laplacian, see [14], [17], [18], [22] and references therein. In particular, Lee et al. in [17] considered the existence of nontrivial weak solutions for the quasilinear Choquard equation, where the nonlinearity f does not satisfy the (AR)-condition.

Motivated by the above results, in the present paper, we are interested in the existence of solutions for the fractional p-Kirchhoff type equation (1.1) with a generalized Choquard nonlinearity without assuming the Ambrosetti–Rabinowitz condition. We first give the following assumptions on the potential function V and the Kirchhoff function M.

(V) $V\colon \mathbb{R}^N \to \mathbb{R}^+$ is a continuous function and there exists $V_0 > 0$ such that $\inf_{\mathbb{R}^N} V \geq V_0$.

(M$_1$) $M\colon \mathbb{R}_0^+ \to \mathbb{R}^+$ is a continuous function and there exists $m_0 > 0$ such that $\inf_{t\geq 0} M(t) = m_0$.

(M$_2$) There exists $\theta \in [1, (2N-\mu)/N)$ such that

$$M(t)t \leq \theta \mathscr{M}(t), \quad \text{for all } t \geq 0, \quad \text{where } \mathscr{M}(t) = \int_0^t M(\tau)\,d\tau.$$

A typical example is $M(t) = m_0 + bt^{\theta-1}$, where $b \geq 0$, $t \geq 0$.

Moreover, we impose the following assumption on the nonlinearity $f\colon \mathbb{R} \to \mathbb{R}$ that

(F$_1$) $F \in C^1(\mathbb{R}, \mathbb{R})$.

(F$_2$) There exist a constant $c_0 > 0$ and $p < q_1 \leq q_2 < (N-\mu)p/(N-ps)$ such that

$$|f(t)| \leq c_0(|t|^{q_1-1} + |t|^{q_2-1}), \quad \text{for all } t \in \mathbb{R}.$$

(F$_3$) $\lim_{|u(x)|\to\infty} F(u(x))/|u(x)|^{p\theta} = \infty$ uniformly for $x \in \mathbb{R}^N$.

(F$_4$) There exist $c_1 \geq 0$, $r_0 \geq 0$ and $\kappa > N/(ps)$ such that

$$|F(t)|^\kappa \leq c_1|t|^{\kappa p}\mathscr{F}(t) \quad \text{for all } t \in \mathbb{R} \text{ and } |t| \geq r_0,$$

where

$$\mathscr{F}(t) = \frac{1}{p\theta}f(t)t - \frac{1}{2}F(t) \geq 0.$$

The main result is as follows.

THEOREM 1.1. *Let* $0 < \mu < ps < N$, *and* (V), (M$_1$)–(M$_2$), (F$_1$)–(F$_4$) *hold. Then problem* (1.1) *has a nontrivial weak solution for any* $\lambda > 0$.

The paper is organized as follows. In Section 2, we give some definitions and preliminaries. Section 3 is devoted to prove Theorem 1.1, we obtain the existence of solution to problem (1.1) by the mountain pass theorem.

2. Preliminaries

We introduce some useful notations. The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined by

$$W^{s,p}(\mathbb{R}^N) = \big\{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\big\},$$

where $[u]_{s,p}$ denotes the Gagliardo norm defined by

$$[u]_{s,p} = \bigg(\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\bigg)^{1/p},$$

and $W^{s,p}(\mathbb{R}^N)$ is equipped with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \big(\|u\|_p^p + [u]_{s,p}^p\big)^{1/p},$$

where and hereafter we denote by $\|\cdot\|_q$ the norm of Lebesgue space $L^q(\mathbb{R}^N)$. As it is well-known, $W^{s,p}(\mathbb{R}^N) = \big(W^{s,p}(\mathbb{R}^N), \|u\|_{W^{s,p}(\mathbb{R}^N)}\big)$ is a uniformly convex Banach space. Let $L^p(\mathbb{R}^N, V)$ denote the Lebesgue space of real-valued functions, with $V(x)|u|^p \in L^1(\mathbb{R}^N)$, equipped with norm

$$\|u\|_{p,V} = \bigg(\int_{\mathbb{R}^N} V(x)|u|^p\,dx\bigg)^{1/p} \quad \text{for all } u \in L^p\big(\mathbb{R}^N, V\big).$$

Let $W^{s,p}_V(\mathbb{R}^N)$ denote the completion of $C_0^\infty(\mathbb{R}^N)$, with respect to the norm

$$\|u\|_W = \big([u]^p_{s,p} + \|u\|^p_{p,V}\big)^{1/p}.$$

The embedding $W^{s,p}_V(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N)$ is continuous for any $\nu \in [p, Np/(N-ps)]$ by [13, Theorem 6.7], namely there exists a positive constant C_ν such that

$$\|u\|_\nu \le C_\nu \|u\|_W \quad \text{for all } u \in W^{s,p}_V(\mathbb{R}^N). \tag{2.1}$$

Next, we recall the Hardy–Littlewood–Sobolev inequality.

Theorem 2.1 ([20, Theorem 4.3]). *Assume that $1 < r,\ t < \infty$, $0 < \mu < N$ and*

$$\frac{1}{r} + \frac{1}{t} + \frac{\mu}{N} = 2.$$

Then there exists $C(N,\mu,r,t) > 0$ such that

$$\iint_{\mathbb{R}^{2N}} \frac{|g(x)|\cdot|h(y)|}{|x-y|^\mu}\,dx\,dy \le C(N,\mu,r,t)\|g\|_r\|h\|_t$$

for all $g \in L^r(\mathbb{R}^N)$ and $h \in L^t(\mathbb{R}^N)$.

In particular, $F(t) = |t|^{q_1}$ for some $q_1 > 0$, by the Hardy–Littlewood–Sobolev inequality, the integral

$$\iint_{\mathbb{R}^{2N}} \frac{F(u(x))F(u(y))}{|x-y|^\mu}\,dx\,dy$$

is well defined if $F \in L^t(\mathbb{R}^N)$, for some $t > 1$, satisfying $2/t + \mu/N = 2$, that is $t = 2N/(2N-\mu)$. Hence, by the fractional Sobolev embedding theorem, if $u \in W^{s,p}_V(\mathbb{R}^N)$, we must require that $tq_1 \in [p, Np/(N-ps)]$. Thus, for the subcritical case, we must assume

$$\widetilde{p}_{\mu,s} = \frac{(N-\mu/2)p}{N} < q_1 \le q_2 < \frac{(N-\mu/2)p}{N-ps} = p^*_{\mu,s}.$$

Hence, $\widetilde{p}_{\mu,s}$ is called the lower critical exponent and $p^*_{\mu,s}$ is said to be the upper critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality.

Equation (1.1) has a variational structure and its associated energy functional $\mathcal{J}_\lambda\colon W^{s,p}_V(\mathbb{R}^N) \to \mathbb{R}$ is defined by

$$\mathcal{J}_\lambda(u) = \Phi(u) - \lambda\Psi(u).$$

with

$$\Phi(u) := \frac{1}{p}\mathscr{M}(\|u\|_W^p) \quad \text{and} \quad \Psi(u) := \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{F(u(x))F(u(y))}{|x-y|^\mu}\,dx\,dy.$$

We have that $\mathcal{J}_\lambda$ is of class $C^1\big(W_V^{s,p}(\mathbb{R}^N),\mathbb{R}\big)$ (see Lemmas 2.2 and 2.4). We say that $u \in W_V^{s,p}(\mathbb{R}^N)$ is a weak solution of problem (1.1), if

$$M(\|u\|_W^p)\bigg[\langle u,\varphi\rangle_{s,p} + \int_{\mathbb{R}^N} V|u|^{p-2}u\varphi\,dx\bigg] = \lambda\int_{\mathbb{R}^N}(\mathcal{I}_\mu * F(u))f(u)\varphi\,dx,$$

for all $\varphi \in W_V^{s,p}(\mathbb{R}^N)$, where

$$\langle u,\varphi\rangle_{s,p} = \iint_{\mathbb{R}^{2N}} \frac{\big[|u(x)-u(y)|^{p-2}(u(x)-u(y))\big]\cdot\big[\varphi(x)-\varphi(y)\big]}{|x-y|^{N+ps}}\,dx\,dy.$$

Clearly, the critical points of $\mathcal{J}_\lambda$ are exactly the weak solutions of problem (1.1).

LEMMA 2.2 ([30, Lemma 2]). *Let* (V) *and* (M_1) *hold. Then* Φ *is of class* $C^1(W_V^{s,p}(\mathbb{R}^N),\mathbb{R})$ *and*

$$\langle\Phi'(u),\varphi\rangle = M(\|u\|_W^p)\bigg[\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)|u(x)|^{p-2}u(x)\varphi(x)\,dx\bigg],$$

for all $u,\varphi \in W_V^{s,p}(\mathbb{R}^N)$. *Moreover,* Φ *is weakly lower semi-continuous in* $W_V^{s,p}(\mathbb{R}^N)$.

The next result is stated in [2], its proof is included for the readers' convenience.

LEMMA 2.3. *Assume* (F_2) *holds, then there exists* $K>0$ *such that*

$$|\mathcal{I}_\mu * F(v)| \le K \quad \text{for } v \in W_V^{s,p}(\mathbb{R}^N). \tag{2.2}$$

PROOF. By the assumption (F_2) and note that $p<q_1\le q_2<(N-\mu)p/(N-ps) < Np/(N-ps)$, using (2.1), we have

$$\begin{aligned}
|\mathcal{I}_\mu * F(v)| &= \bigg|\int_{\mathbb{R}^N} \frac{F(v)}{|x-y|^\mu}\,dy\bigg| \\
&\le \bigg|\int_{|x-y|\le 1} \frac{F(v)}{|x-y|^\mu}\,dy\bigg| + \bigg|\int_{|x-y|\ge 1} \frac{F(v)}{|x-y|^\mu}\,dy\bigg| \\
&\le c_0\int_{|x-y|\le 1} \frac{|v|^{q_1}+|v|^{q_2}}{|x-y|^\mu}\,dy + c_0\int_{|x-y|\ge 1}(|v|^{q_1}+|v|^{q_2})\,dy \\
&\le c_0\int_{|x-y|\le 1} \frac{|v|^{q_1}+|v|^{q_2}}{|x-y|^\mu}\,dy + C(\|v\|_W^{q_1}+\|v\|_W^{q_2}) \\
&\le c_0\int_{|x-y|\le 1} \frac{|v|^{q_1}+|v|^{q_2}}{|x-y|^\mu}\,dy + C.
\end{aligned}$$

Moreover, choosing

$$t_1 \in \left(\frac{N}{N-\mu}, \frac{Np}{(N-ps)q_1} \right) \quad \text{and} \quad t_2 \in \left(\frac{N}{N-\mu}, \frac{Np}{(N-ps)q_2} \right),$$

using the Hölder inequality and (2.1), we find

$$\begin{aligned}
&\int_{|x-y|\le 1} \frac{|v|^{q_1}+|v|^{q_2}}{|x-y|^{\mu}}\,dy \\
&\quad \le \left(\int_{|x-y|\le 1} |v|^{q_1 t_1}\,dy \right)^{1/t_1} \left(\int_{|x-y|\le 1} |x-y|^{-\mu t_1/(t_1-1)}\,dy \right)^{(t_1-1)/t_1} \\
&\qquad + \left(\int_{|x-y|\le 1} |v|^{q_2 t_2}\,dy \right)^{1/t_2} \left(\int_{|x-y|\le 1} |x-y|^{-\mu t_2/(t_2-1)}\,dy \right)^{(t_2-1)/t_2} \\
&\quad \le C\big(\|v\|_W^{q_1}+\|v\|_W^{q_2}\big) \left[\left(\int_{r\le 1} r^{N-1-\mu t_1/(t_1-1)}\,dy \right)^{(t_1-1)/t_1} \right. \\
&\qquad \left. + \left(\int_{r\le 1} r^{N-1-\mu t_2/(t_2-1)}\,dy \right)^{(t_2-1)/t_2} \right] \le C. \qquad \square
\end{aligned}$$

Lemma 2.4. *Let* (V) *and* (F_1)–(F_2) *hold. Then* Ψ *and* Ψ' *are weakly strongly continuous on* $W_V^{s,p}(\mathbb{R}^N)$.

Proof. Let $\{u_n\}$ be a sequence in $W_V^{s,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $W_V^{s,p}(\mathbb{R}^N)$ as $n\to\infty$. Then $\{u_n\}$ is bounded in $W_V^{s,p}(\mathbb{R}^N)$, and then there exists a subsequence denoted by itself, such that

$$u_n \to u \quad \text{in } L^{q_1}(\mathbb{R}^N)\cap L^{q_2}(\mathbb{R}^N) \quad \text{and} \quad u_n \to u \quad \text{a.e. in } \mathbb{R}^N \text{ as } n\to\infty,$$

and by [7, Theorem IV-9] there exists $\ell \in L^{q_1}(\mathbb{R}^N)\cap L^{q_2}(\mathbb{R}^N)$ such that

$$|u_n(x)| \le \ell(x) \quad \text{a.e. in } \mathbb{R}^N.$$

First, we show that Ψ is weakly strongly continuous on $W_V^{s,p}(\mathbb{R}^N)$. Since $F \in C^1(\mathbb{R},\mathbb{R})$, we see that $F(u_n)\to F(u)$ as $n\to\infty$ for almost all $x\in\mathbb{R}^N$, and so $(\mathcal{I}_\mu * F(u_n))F(u_n) \to (\mathcal{I}_\mu * F(u))F(u)$ as $n\to\infty$ for almost all $x\in\mathbb{R}^N$. From Lemma 2.3 and (F_2), we have

$$\big|(\mathcal{I}_\mu * F(u_n))F(u_n)\big| \le Kc_0\left(\frac{|u_n(x)|^{q_1}}{q_1} + \frac{|u_n(x)|^{q_2}}{q_2} \right) \in L^1(\mathbb{R}^N).$$

By Lebesgue dominated convergence theorem, we get

$$\int_{\mathbb{R}^N} (\mathcal{I}_\mu * F(u_n))F(u_n)\,dx \to \int_{\mathbb{R}^N} (\mathcal{I}_\mu * F(u))F(u)\,dx \quad \text{as } n\to\infty,$$

which implies that $\Psi(u_n)\to\Psi(u)$ as $n\to\infty$. Thus Ψ is weakly strongly continuous on $W_V^{s,p}(\mathbb{R}^N)$.

We next prove that Ψ' is weakly strongly continuous on $W_V^{s,p}(\mathbb{R}^N)$. Since $u_n(x) \to u(x)$ as $n \to \infty$ for almost all $x \in \mathbb{R}^N$, $f(u_n) \to f(u)$ for almost all $x \in \mathbb{R}^N$ as $n \to \infty$. Then

$$(\mathcal{I}_\mu * F(u_n))f(u_n) \to (\mathcal{I}_\mu * F(u))f(u) \quad \text{a.e. in } \mathbb{R}^N, \text{ as } n \to \infty.$$

By (F_2) and the Hölder inequality, we have that, for any $\varphi \in W_V^{s,p}(\mathbb{R}^N)$,

$$\begin{aligned}\int_{\mathbb{R}^N} &\big|(\mathcal{I}_\mu * F(u_n))f(u_n)\varphi(x)\big|\,dx \\ &\le c_0 K \int_{\mathbb{R}^N} \big|\big(|u_n|^{q_1-1} + |u_n|^{q_2-1}\big)\varphi(x)\big|\,dx \\ &\le c_0 K\big(\|u_n\|_{q_1}^{q_1-1}\|\varphi\|_{q_1} + \|u_n\|_{q_2}^{q_2-1}\|\varphi\|_{q_2}\big) \\ &\le c_0 K\big(C_{q_1}\|\ell(x)\|_{q_1}^{q_1-1} + C_{q_2}\|\ell(x)\|_{q_2}^{q_2-1}\big)\|\varphi\|_W.\end{aligned}$$

Then, by Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}\|\Psi'(u_n) - \Psi'(u)\|_{(W_V^{s,p}(\mathbb{R}^N))'} &= \sup_{\|\varphi\|_{W_V^{s,p}(\mathbb{R}^N)}=1} |\langle \Psi'(u_n) - \Psi'(u), \varphi\rangle| \\ = \sup_{\|\varphi\|_{W_V^{s,p}(\mathbb{R}^N)}=1} \int_{\mathbb{R}^N} &\big|(\mathcal{I}_\mu * F(u_n))f(u_n)\varphi(x) - (\mathcal{I}_\mu * F(u))f(u)\varphi(x)\big|\,dx \to 0\end{aligned}$$

as $n \to \infty$. Therefore, we get that $\Psi'(u_n) \to \Psi'(u)$ in $\big(W_V^{s,p}(\mathbb{R}^N)\big)'$ as $n \to \infty$.□

3. Proof of the main result

In this section, we will prove our main result. First, we introduce the following definition.

DEFINITION 3.1. For $c \in \mathbb{R}$, we say that $\mathcal{J}_\lambda$ satisfies the $(C)_c$ condition if for any sequence $\{u_n\} \subset W_V^{s,p}(\mathbb{R}^N)$ with

$$\mathcal{J}_\lambda(u_n) \to c, \qquad \|\mathcal{J}_\lambda'(u_n)\|(1 + \|u_n\|_W) \to 0,$$

there is a subsequence $\{u_n\}$ such that $\{u_n\}$ converges strongly in $W_V^{s,p}(\mathbb{R}^N)$.

We will use the following mountain pass theorem to prove our result.

LEMMA 3.2 ([11, Theorem 1]). *Let E be a real Banach space, $I \in C^1(E, \mathbb{R})$ satisfies the $(C)_c$ condition for any $c \in \mathbb{R}$, and*

(a) *there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \ge \alpha$,*
(b) *there is an $e \in E \setminus B_\rho$ such that $I(e) \le 0$.*

Then,

$$c = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} I(\gamma(t)) \ge \alpha$$

is a critical value of I, where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}$.

We first show that the energy functional $\mathcal{J}_\lambda$ satisfies the geometric structure.

LEMMA 3.3. *Assume that* (V), (M_1)–(M_2) *and* (F_1)–(F_3) *hold. Then*:

(a) *There exists* $\alpha, \rho > 0$ *such that* $\mathcal{J}_\lambda(u) \geq \alpha$ *for all* $u \in W_V^{s,p}(\mathbb{R}^N)$ *with* $\|u\|_W = \rho$.

(b) $\mathcal{J}_\lambda(u)$ *is unbounded from below on* $W_V^{s,p}(\mathbb{R}^N)$.

PROOF. (a) From Lemma 2.3 and (M_1)–(M_2), (F_2), we have

$$\begin{aligned}\mathcal{J}_\lambda(u) &= \frac{1}{p}\mathscr{M}(\|u\|_W^p) - \frac{\lambda}{2}\iint_{\mathbb{R}^{2N}} \frac{F(u(x))F(u(y))}{|x-y|^\mu}\,dx\,dy \\ &\geq \frac{1}{p\theta}M(\|u\|_W^p)\|u\|_W^p - \frac{\lambda c_0 K}{2}\int_{\mathbb{R}^N}\bigg(\frac{|u|^{q_1}}{q_1} + \frac{|u|^{q_2}}{q_2}\bigg)\,dx \\ &\geq \bigg[\frac{m_0}{p\theta} - \frac{\lambda c_0 K}{2}\big(C_{q_1}^{q_1}\|u\|_W^{q_1-p} + C_{q_2}^{q_2}\|u\|_W^{q_2-p}\big)\bigg]\|u\|_W^p.\end{aligned}$$

Since $q_2 \geq q_1 > p$, the claim follows if we choose ρ small enough.

(b) From (M_2), we have

$$\mathscr{M}(t) \leq \mathscr{M}(1)t^\theta \quad \text{for all } t \geq 1. \tag{3.1}$$

By the assumption (F_3), we take t_0 such that $F(t_0) \neq 0$, we find

$$\int_{\mathbb{R}^N} (\mathcal{I}_\mu * F(t_0\chi_{B_1}))F(t_0\chi_{B_1})\,dx = F(t_0)^2 \int_{B_1}\int_{B_1} \mathcal{I}_\mu(x-y)\,dx\,dy > 0,$$

where B_r denotes the open ball centered at the origin with radius r and χ_{B_1} denotes the standard indicator function of set B_1. By the density theorem, there will be $v_0 \in W_V^{s,p}(\mathbb{R}^N)$ with

$$\int_{\mathbb{R}^N} (\mathcal{I}_\mu * F(v_0))F(v_0)\,dx > 0.$$

Define the function $v_t(x) = v_0(x/t)$, then

$$\begin{aligned}\mathcal{J}_\lambda(v_t) &= \frac{1}{p}\mathscr{M}(\|v_t\|_W^p) - \frac{\lambda}{2}\iint_{\mathbb{R}^{2N}} \frac{F(v_t(x))F(v_t(y))}{|x-y|^\mu}\,dx\,dy \\ &\leq \frac{1}{p}\mathscr{M}(1)\|v_t\|_W^{p\theta} - \frac{\lambda}{2}\iint_{\mathbb{R}^{2N}} \frac{F(v_t(x))F(v_t(y))}{|x-y|^\mu}\,dx\,dy \\ &= \frac{1}{p}\mathscr{M}(1)\bigg[t^{N-ps}\|v_0\|_W^p + t^N\int_{\mathbb{R}^N} V(tx)|v_0|^p\,dx\bigg]^\theta \\ &\quad - t^{2N-\mu}\frac{\lambda}{2}\iint_{\mathbb{R}^{2N}} \frac{F(v_0(x))F(v_0(y))}{|x-y|^\mu}\,dx\,dy,\end{aligned}$$

for sufficiently large t. Therefore, we have that $\mathcal{J}_\lambda(v_t) \to -\infty$ as $t \to \infty$ since $1 \leq \theta < (2N-\mu)/N$ gives that $2N-\mu > N\theta > (N-ps)\theta$. Hence the functional $\mathcal{J}_\lambda$ is unbounded from below on $W_V^{s,p}(\mathbb{R}^N)$. □

LEMMA 3.4. *Assume that* (V), (M_1)–(M_2) *and* (F_1)–(F_4) *hold. Then* $(\mathrm{C})_c$-*sequence of* $\mathcal{J}_\lambda$ *is bounded for any* $\lambda > 0$.

PROOF. Suppose that $\{u_n\} \subset W_V^{s,p}(\mathbb{R}^N)$ is a $(\mathrm{C})_c$-sequence for $\mathcal{J}_\lambda(u)$, that is

$$\mathcal{J}_\lambda(u_n) \to c, \qquad \|\mathcal{J}_\lambda'(u_n)\|_W(1 + \|u_n\|_W) \to 0,$$

which shows that

$$c = \mathcal{J}_\lambda(u_n) + o(1), \qquad \langle \mathcal{J}_\lambda'(u_n), u_n\rangle = o(1) \tag{3.2}$$

where $o(1) \to 0$ as $n \to \infty$. We now prove that $\{u_n\}$ is bounded in $W_V^{s,p}(\mathbb{R}^N)$. We argue by contradiction. Suppose that the sequence $\{u_n\}$ is unbounded in $W_V^{s,p}(\mathbb{R}^N)$, then we may assume that

$$\|u_n\|_W \to \infty, \quad \text{as } n \to \infty. \tag{3.3}$$

Let $\omega_n(x) = u_n/\|u_n\|_W$, then $\omega_n \in W_V^{s,p}(\mathbb{R}^N)$ with $\|\omega_n\|_W = 1$. Hence, up to a subsequence, still denoted by itself, there exists a function $\omega \in W_V^{s,p}(\mathbb{R}^N)$ such that

$$\omega_n(x) \to \omega(x) \quad \text{a.e. in } \mathbb{R}^N \quad \text{and} \quad \omega_n(x) \to \omega(x) \quad \text{a.e. in } L^r(\mathbb{R}^N) \tag{3.4}$$

as $n \to \infty$, for $p \le r < Np/(N - ps)$.

Let $\Omega_1 = \{x \in \mathbb{R}^N : \omega(x) \ne 0\}$, then

$$\lim_{n\to\infty} \omega_n(x) = \lim_{n\to\infty} \frac{u_n(x)}{\|u_n\|_W} = \omega(x) \ne 0 \quad \text{in } \Omega_1,$$

and (3.3) implies that

$$|u_n| \to \infty \quad \text{a.e. in } \Omega_1. \tag{3.5}$$

From the assumption (F_3) and Lemma 2.3, we have

$$\lim_{n\to\infty} \frac{(\mathcal{I}_\mu * F(u_n(x)))F(u_n(x))}{|u_n(x)|^{p\theta}}\,|\omega_n(x)|^{p\theta} = \infty, \quad \text{for a.e. } x \in \Omega_1. \tag{3.6}$$

Moreover, by (F_3), there exists $t_0 > 0$ such that

$$\frac{F(t)}{|t|^{p\theta}} > 1 \quad \text{for all } |t| > t_0.$$

Since F is continuous, then there exists $\mathcal{C} > 0$ such that $|F(t)| \le \mathcal{C}$ for all $t \in [-t_0, t_0]$. Thus, we see that there is a constant C_0 such that for any $t \in \mathbb{R}$, we have $F(t) \ge C_0$, which show that there is a constant C such that

$$\frac{(\mathcal{I}_\mu * F(u_n))F(u_n) - C}{\|u_n\|_W^{p\theta}} \ge 0.$$

This means that

$$\frac{(\mathcal{I}_\mu * F(u_n))F(u_n(x))}{|u_n(x)|^{p\theta}}\,|\omega_n(x)|^{p\theta} - \frac{C}{\|u_n\|_W^{p\theta}} \ge 0. \tag{3.7}$$

By (3.2) we have that

$$c = \mathcal{J}_\lambda(u_n) + o(1) = \frac{1}{p}\mathscr{M}(\|u_n\|_W^p) - \frac{\lambda}{2}\int_{\mathbb{R}^N} (\mathcal{I}_\mu * F(u_n))F(u_n)\,dx + o(1). \tag{3.8}$$

Using this and (M_1)–(M_2), we find

$$\begin{aligned}\text{(3.9)}\qquad \frac{1}{2}\int_{\mathbb{R}^N}(\mathcal{I}_\mu * F(u_n))F(u_n)\,dx &= \frac{1}{p\lambda}\mathscr{M}(\|u_n\|_W^p) - \frac{c}{\lambda} + \frac{o(1)}{\lambda}\\ &\geq \frac{m_0}{p\theta\lambda}\|u_n\|_W^p - \frac{c}{\lambda} + \frac{o(1)}{\lambda} \to \infty,\end{aligned}$$

as $n \to \infty$. We claim that $\operatorname{meas}(\Omega_1) = 0$. Indeed, if $\operatorname{meas}(\Omega_1) \neq 0$, from (3.1), (3.6)–(3.8) and Fatou's lemma, we have

$$\begin{aligned}\text{(3.10)}\quad +\infty &= \int_{\Omega_1} \liminf_{n\to\infty} \frac{(\mathcal{I}_\mu * F(u_n(x)))F(u_n(x))}{|u_n(x)|^{p\theta}}\,|\omega_n(x)|^{p\theta}\,dx\\ &\quad - \int_{\Omega_1} \limsup_{n\to\infty} \frac{C}{\|u_n\|_W^{p\theta}}\,dx\\ &\leq \int_{\Omega_1} \liminf_{n\to\infty} \bigg(\frac{(\mathcal{I}_\mu * F(u_n(x)))F(u_n(x))}{|u_n(x)|^{p\theta}}|\omega_n(x)|^{p\theta} - \frac{C}{\|u_n\|_W^{p\theta}}\bigg)\,dx\\ &\leq \liminf_{n\to\infty} \int_{\Omega_1} \bigg(\frac{(\mathcal{I}_\mu * F(u_n(x)))F(u_n(x))}{|u_n(x)|^{p\theta}}|\omega_n(x)|^{p\theta} - \frac{C}{\|u_n\|_W^{p\theta}}\bigg)\,dx\\ &= \liminf_{n\to\infty} \int_{\Omega_1} \bigg(\frac{(\mathcal{I}_\mu * F(u_n))F(u_n)}{\|u_n\|_W^{p\theta}} - \frac{C}{\|u_n\|_W^{p\theta}}\bigg)\,dx\\ &\leq \liminf_{n\to\infty} \int_{\Omega_1} \frac{\mathscr{M}(1)(\mathcal{I}_\mu * F(u_n))F(u_n)}{\mathscr{M}(\|u_n\|_W^p)}\,dx - \liminf_{n\to\infty}\int_{\Omega_1} \frac{C}{\|u_n\|_W^{p\theta}}dx\\ &\leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{\mathscr{M}(1)(\mathcal{I}_\mu * F(u_n))F(u_n)}{\mathscr{M}(\|u_n\|_W^p)}\,dx - \liminf_{n\to\infty}\int_{\Omega_1} \frac{C}{\|u_n\|_W^{p\theta}}\,dx\\ &= \frac{\mathscr{M}(1)}{p} \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{(\mathcal{I}_\mu * F(u_n))F(u_n)}{\frac{1}{p}\mathscr{M}(\|u_n\|_W^p)}\,dx\\ &= \frac{\mathscr{M}(1)}{p} \liminf_{n\to\infty} \frac{\displaystyle\int_{\mathbb{R}^N}(\mathcal{I}_\mu * F(u_n))F(u_n)\,dx}{\displaystyle\frac{\lambda}{2}\int_{\mathbb{R}^N}(\mathcal{I}_\mu * F(u_n))F(u_n)\,dx + c - o(1)}.\end{aligned}$$

So, by (3.9) and (3.10), we get $+\infty \leq 2\mathscr{M}(1)/(p\lambda)$. This is a contradiction. This shows that $\operatorname{meas}(\Omega_1) = 0$. Hence $\omega(x) = 0$ for almost all $x \in \mathbb{R}^N$. The convergence in (3.4) means that

$$\text{(3.11)}\qquad \omega_n(x) \to 0 \quad \text{a.e. in } \mathbb{R}^N \quad \text{and} \quad \omega_n(x) \to 0 \quad \text{a.e. in } L^r(\mathbb{R}^N),$$

as $n \to \infty$, for $p \leq r < Np/(N - ps)$.

Using (3.2) and (M_2), for n large enough, we get

$$\begin{aligned}\text{(3.12)}\qquad c + 1 &\geq \mathcal{J}_\lambda(u_n) - \frac{1}{p\theta}\,\langle \mathcal{J}_\lambda'(u_n), u_n\rangle\\ &= \frac{1}{p}\mathscr{M}(\|u_n\|_W^p) - \frac{1}{p\theta}\,M(\|u_n\|_W^p)\|u_n\|_W^p\end{aligned}$$

$$
\begin{aligned}
&+\lambda\int_{\mathbb{R}^N}(\mathcal{I}_\mu * F(u_n))\left(\frac{1}{p\theta}\, f(u_n)u_n-\frac{1}{2}\,F(u_n)\right)dx\\
&\geq\lambda\int_{\mathbb{R}^N}(\mathcal{I}_\mu * F(u_n))\left(\frac{1}{p\theta}f(u_n)u_n-\frac{1}{2}F(u_n)\right)dx\\
&=\lambda\int_{\mathbb{R}^N}(\mathcal{I}_\mu * F(u_n))\mathscr{F}(u_n)\,dx.
\end{aligned}
$$

Let us define $\Omega_n(a,b) := \{x\in\mathbb{R}^N : a\leq |u_n(x)|\leq b\}$ for $a,b\geq 0$. From (M_1) and (M_2), we have that

$$
\mathscr{M}(\|u_n\|_W^p)\geq\frac{1}{\theta}\,M(\|u_n\|_W^p)\|u_n\|_W^p\geq\frac{m_0}{\theta}\|u_n\|_W^p. \tag{3.13}
$$

This together with (3.3) and (3.8) yields that

$$
\begin{aligned}
0<\frac{2}{p\lambda}&\leq\limsup_{n\to\infty}\frac{1}{\mathscr{M}(\|u_n\|_W^p)}\int_{\mathbb{R}^N}(\mathcal{I}_\mu * F(u_n))F(u_n)\,dx\\
&=\limsup_{n\to\infty}\int_{\mathbb{R}^N}\frac{(\mathcal{I}_\mu * F(u_n))F(u_n)}{\mathscr{M}(\|u_n\|_W^p)}\,dx\\
&=\limsup_{n\to\infty}\left(\int_{\Omega_n(0,r_0)}+\int_{\Omega_n(r_0,\infty)}\right)\frac{(\mathcal{I}_\mu * F(u_n))F(u_n)}{\mathscr{M}(\|u_n\|_W^p)}\,dx.
\end{aligned} \tag{3.14}
$$

On the one hand, by Lemma 2.3, (3.13), (F_2) and (3.11), we obtain

$$
\begin{aligned}
\int_{\Omega_n(0,r_0)}\frac{(\mathcal{I}_\mu * F(u_n))F(u_n)}{\mathscr{M}(\|u_n\|_W^p)}\,dx&\leq\frac{K\theta}{m_0}\int_{\Omega_n(0,r_0)}\frac{|F(u_n)|}{\|u_n\|_W^p}\,dx\\
&\leq\frac{c_0K\theta}{m_0}\int_{\Omega_n(0,r_0)}\left(\frac{|u_n|^{q_1}}{q_1\|u_n\|_W^p}+\frac{|u_n|^{q_2}}{q_2\|u_n\|_W^p}\right)dx\\
&=\frac{c_0K\theta}{m_0}\int_{\Omega_n(0,r_0)}\left(\frac{|u_n|^{q_1-p}}{q_1}|\omega_n|^p+\frac{|u_n|^{q_2-p}}{q_2}|\omega_n|^p\right)dx\\
&\leq\frac{c_0K\theta}{m_0}\left(\frac{r_0^{q_1-p}}{q_1}+\frac{r_0^{q_2-p}}{q_2}\right)\int_{\Omega_n(0,r_0)}|\omega_n|^p\,dx\to 0,
\end{aligned} \tag{3.15}
$$

as $n\to\infty$. On the other hand, using the Hölder inequality, (3.11), (3.12) and (F_4), we find

$$
\begin{aligned}
\int_{\Omega_n(r_0,\infty)}\frac{|\mathcal{I}_\mu * F(u_n)|F(u_n)}{\mathscr{M}(\|u_n\|_W^p)}\,dx&\leq\frac{\theta}{m_0}\int_{\Omega_n(r_0,\infty)}\frac{|\mathcal{I}_\mu * F(u_n)|F(u_n)}{\|u_n\|_W^p}\,dx\\
&=\frac{\theta}{m_0}\int_{\Omega_n(r_0,\infty)}\frac{|\mathcal{I}_\mu * F(u_n)|F(u_n)}{|u_n|^p}|\omega_n(x)|^p\,dx\\
&\leq\frac{\theta}{m_0}\left(\int_{\Omega_n(r_0,\infty)}\left(\frac{|\mathcal{I}_\mu * F(u_n)|F(u_n)}{|u_n|^p}\right)^\kappa dx\right)^{1/\kappa}\\
&\quad\cdot\left(\int_{\Omega_n(r_0,\infty)}|\omega_n(x)|^{\kappa p/(\kappa-1)}\,dx\right)^{(\kappa-1)/\kappa}
\end{aligned} \tag{3.16}
$$

$$\begin{aligned}
&\le \frac{\theta}{m_0} c_1^{1/\kappa}\bigg(\int_{\Omega_n(r_0,\infty)} |\mathcal{I}_\mu * F(u_n)|^\kappa \mathscr{F}(u_n)\,dx\bigg)^{1/\kappa} \\
&\quad\cdot\bigg(\int_{\Omega_n(r_0,\infty)} |\omega_n(x)|^{\kappa p/(\kappa-1)}\,dx\bigg)^{(\kappa-1)/\kappa} \\
&\le \frac{\theta}{m_0} c_1^{1/\kappa} K^{(\kappa-1)/\kappa}\bigg(\int_{\Omega_n(r_0,\infty)} |\mathcal{I}_\mu * F(u_n)| \mathscr{F}(u_n)\,dx\bigg)^{1/\kappa} \\
&\quad\cdot\bigg(\int_{\Omega_n(r_0,\infty)} |\omega_n(x)|^{\kappa p/(\kappa-1)}\,dx\bigg)^{(\kappa-1)/\kappa} \\
&\le \frac{\theta}{m_0} c_1^{1/\kappa} K^{(\kappa-1)/\kappa}\bigg(\frac{c+1}{\lambda}\bigg)^{1/\kappa} \qquad (3.17)\\
&\quad\cdot\bigg(\int_{\Omega_n(r_0,\infty)} |\omega_n(x)|^{\kappa p/(\kappa-1)}\,dx\bigg)^{(\kappa-1)/\kappa} \to 0,
\end{aligned}$$

as $n \to \infty$. Here we used the fact that $\kappa p/(\kappa-1) \in (p, Np/(N-ps))$ if $\kappa > N/(ps)$. Thus, we get a contradiction from (3.14)–(3.16). □

Lemma 3.5. *Assume that* (V), (M_1)–(M_2) *and* (F_1)–(F_4) *hold. Then the functional* $\mathcal{J}_\lambda$ *satisfies* $(C)_c$*-condition for any* $\lambda > 0$*.*

Proof. Suppose that $\{u_n\} \subset W_V^{s,p}(\mathbb{R}^N)$ is a $(C)_c$-sequence for $\mathcal{J}_\lambda(u)$, from Lemma 3.4, we have that $\{u_n\}$ is bounded in $W_V^{s,p}(\mathbb{R}^N)$, then if necessary to a subsequence, we have

$$\begin{aligned}
&u_n \rightharpoonup u && \text{in } W_V^{s,p}(\mathbb{R}^N), \qquad u_n \to u \quad \text{a.e. in } \mathbb{R}^N,\\
&u_n \to u && \text{in } L^{q_1}(\mathbb{R}^N)\cap L^{q_2}(\mathbb{R}^N), \qquad (3.18)\\
&|u_n| \le \ell(x) && \text{a.e. in } \mathbb{R}^N, \quad \text{for some } \ell(x) \in L^{q_1}(\mathbb{R}^N)\cap L^{q_2}(\mathbb{R}^N).
\end{aligned}$$

For simplicity, let $\varphi \in W_V^{s,p}(\mathbb{R}^N)$ be fixed and B_φ be the linear functional on $W_V^{s,p}(\mathbb{R}^N)$ defined by

$$B_\varphi(v) = \iint_{\mathbb{R}^{2N}} \frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,(v(x)-v(y))\,dx\,dy.$$

for all $v \in W_V^{s,p}(\mathbb{R}^N)$. By the Hölder inequality, we have

$$|B_\varphi(v)| \le [\varphi]_{s,p}^{p-1}[v]_{s,p} \le \|\varphi\|_W^{p-1}\|v\|_W,$$

for all $v \in W_V^{s,p}(\mathbb{R}^N)$. Hence, (3.18) gives that

$$\lim_{n\to\infty} \big(M(\|u_n\|_W^p) - M(\|u\|_W^p)\big) B_u(u_n-u) = 0, \tag{3.19}$$

since $\big\{M(\|u_n\|_W^p) - M(\|u\|_W^p)\big\}_n$ is bounded in $\mathbb{R}$.

Since $\mathcal{J}_\lambda'(u_n) \to 0$ in $(W_V^{s,p}(\mathbb{R}^N))'$ and $u_n \rightharpoonup u$ in $W_V^{s,p}(\mathbb{R}^N)$, we have

$$\langle \mathcal{J}_\lambda'(u_n) - \mathcal{J}_\lambda'(u), u_n - u\rangle \to 0 \quad \text{as } n\to\infty.$$

That is,

$$
\begin{aligned}
o(1) &= \langle \mathcal{J}_\lambda'(u_n) - \mathcal{J}_\lambda'(u), u_n - u\rangle \\
&= M(\|u_n\|_W^p)\bigg(B_{u_n}(u_n-u) + \int_{\mathbb{R}^N} V(x)|u_n|^{p-2}u_n(u_n-u)\,dx\bigg) \\
&\quad - M(\|u\|_W^p)\bigg(B_u(u_n-u) + \int_{\mathbb{R}^N} V(x)|u|^{p-2}u(u_n-u)\,dx\bigg) \\
&\quad - \lambda \int_{\mathbb{R}^N} \big[(\mathcal{I}_\mu * F(u_n))f(u_n) - (\mathcal{I}_\mu * F(u))f(u)\big](u_n-u)\,dx \\
&= M(\|u_n\|_W^p)\big[B_{u_n}(u_n-u) - B_u(u_n-u)\big] \\
&\quad + \big(M(\|u_n\|_W^p) - M(\|u\|_W^p)\big)B_u(u_n-u) \\
&\quad + M(\|u_n\|_W^p)\int_{\mathbb{R}^N} V(x)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)(u_n-u)\,dx \\
&\quad + \big[M(\|u_n\|_W^p) - M(\|u\|_W^p)\big]\int_{\mathbb{R}^N} V(x)|u|^{p-2}u(u_n-u)\,dx \\
&\quad - \lambda \int_{\mathbb{R}^N} \big[(\mathcal{I}_\mu * F(u_n))f(u_n) - (\mathcal{I}_\mu * F(u))f(u)\big](u_n-u)\,dx.
\end{aligned} \tag{3.20}
$$

From Lemma 2.4, we have

$$
\int_{\mathbb{R}^N} \big[(\mathcal{I}_\mu * F(u_n))f(u_n) - (\mathcal{I}_\mu * F(u))f(u)\big](u_n-u)\,dx \to 0, \tag{3.21}
$$

as $n \to \infty$. Moreover, using the Hölder inequality and (3.18), we have

$$
\big[M(\|u_n\|_W^p) - M(\|u\|_W^p)\big]\int_{\mathbb{R}^N} V(x)|u|^{p-2}u(u_n-u)\,dx \to 0, \tag{3.22}
$$

as $n \to \infty$. From (3.19)–(3.22) and (M_1), we obtain

$$
\begin{aligned}
\lim_{n\to\infty} M(\|u_n\|_W^p)\bigg(&\big[B_{u_n}(u_n-u) - B_u(u_n-u)\big] \\
&+ \int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)\,dx\bigg) = 0.
\end{aligned}
$$

Since

$$
\begin{aligned}
M(\|u_n\|_W^p)\big[B_{u_n}(u_n-u) - B_u(u_n-u)\big] &\ge 0, \\
V(x)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)(u_n-u) &\ge 0,
\end{aligned}
$$

for all n by convexity, (M_1) and (V_1), we have

$$
\begin{gathered}
\lim_{n\to\infty} \big[B_{u_n}(u_n-u) - B_u(u_n-u)\big] = 0, \\
\lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)(u_n-u)\,dx = 0.
\end{gathered} \tag{3.23}
$$

Let us now recall the well-known Simon inequalities. There exist positive numbers c_p and C_p, depending only on p, such that

$$
(3.24)\qquad |\xi-\eta|^p \le \begin{cases} c_p(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)(\xi-\eta) & \text{for } p\ge 2,\\ C_p\big[(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)(\xi-\eta)\big]^{p/2}(|\xi|^p+|\eta|^p)^{(2-p)/2} & \\ & \text{for } 1<p<2, \end{cases}
$$

for all $\xi,\eta\in\mathbb{R}^N$. According to the Simon inequality, we divide the discussion into two cases.

Case 1. $p\ge 2$. From (3.23) and (3.24), as $n\to\infty$, we have

$$
\begin{aligned}
[u_n-u]_{s,p}^p &= \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u(x)-u_n(y)+u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\\
&\le c_p\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))-|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\\
&\qquad\cdot\big(u_n(x)-u(x)-u_n(y)+u(y)\big)\,dx\,dy\\
&= c_p\big[B_{u_n}(u_n-u)-B_u(u_n-u)\big]=o(1),
\end{aligned}
$$

and

$$
\|u_n-u\|_{p,V}^p \le c_p\int_{\mathbb{R}^N} V(x)\big(|u_n|^{p-2}u_n-|u|^{p-2}u\big)(u_n-u)\,dx=o(1).
$$

Consequently, $\|u_n-u\|_W\to 0$ as $n\to\infty$.

Case 2. $1<p<2$. Taking $\xi=u_n(x)-u_n(y)$ and $\eta=u(x)-u(y)$ in (3.24), as $n\to\infty$, we have

$$
\begin{aligned}
[u_n-u]_{s,p}^p &\le C_p\big[B_{u_n}(u_n-u)-B_u(u_n-u)\big]^{p/2}\big([u_n]_{s,p}^p+[u]_{s,p}^p\big)^{(2-p)/2}\\
&\le C_p\big[B_{u_n}(u_n-u)-B_u(u_n-u)\big]^{p/2}\big([u_n]_{s,p}^{p(2-p)/2}+[u]_{s,p}^{p(2-p)/2}\big)\\
&\le C\big[B_{u_n}(u_n-u)-B_u(u_n-u)\big]^{p/2}=o(1).
\end{aligned}
$$

Here we used the fact that $[u_n]_{s,p}$ and $[u]_{s,p}$ are bounded, and the elementary inequality

$$
(a+b)^{(2-p)/2}\le a^{(2-p)/2}+b^{(2-p)/2}\quad \text{for all } a,b\ge 0 \text{ and } 1<p<2.
$$

Moreover, by the Hölder inequality and (3.23), as $n\to\infty$,

$$
\begin{aligned}
\|u_n-u\|_{p,V}^p &\le C_p\int_{\mathbb{R}^N} V(x)\big[\big(|u_n|^{p-2}u_n-|u|^{p-2}u\big)(u_n-u)\big]^{p/2}\\
&\qquad\cdot\big(|u_n|^p+|u|^p\big)^{(2-p)/2}dx\\
&\le C_p\bigg(\int_{\mathbb{R}^N} V(x)\big(|u_n|^{p-2}u_n-|u|^{p-2}u\big)(u_n-u)\,dx\bigg)^{p/2}\\
&\qquad\cdot\bigg(\int_{\mathbb{R}^N} V(x)(|u_n|^p+|u|^p)\,dx\bigg)^{(2-p)/2}
\end{aligned}
$$

$$
\begin{aligned}
&\le C_p\big(\|u_n\|_{p,V}^{p(2-p)/2}+\|u\|_{p,V}^{p(2-p)/2}\big)\\
&\quad\cdot\bigg(\int_{\mathbb{R}^N}V(x)\big(|u_n|^{p-2}u_n-|u|^{p-2}u\big)(u_n-u)\,dx\bigg)^{p/2}\\
&\le C\bigg(\int_{\mathbb{R}^N}V(x)\big(|u_n|^{p-2}u_n-|u|^{p-2}u\big)(u_n-u)\,dx\bigg)^{p/2}\to 0.
\end{aligned}
$$

Thus $\|u_n-u\|_W\to 0$ as $n\to\infty$. □

Now we are ready to prove our main result.

Proof of Theorem 1.1. By Lemmas 3.2–3.5, we obtain that there exists a critical point of functional $\mathcal{J}_\lambda$, so problem (1.1) has a nontrivial weak solution for any $\lambda>0$. □

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Wenjing Chen
School of Mathematics and Statistics
Southwest University
Chongqing 400715, P.R. CHINA

E-mail address: wjchen@swu.edu.cn