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THE WEAK FIXED POINT PROPERTY OF DIRECT SUMS OF SOME BANACH SPACES

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Abstract. We prove that if a Banach space X has the weak fixed point property and Y satisfies the condition $M(Y) > 1$, then the direct sum $X \oplus Y$ with a uniformly convex norm has the weak fixed point property.

1. Introduction

A Banach space X has the fixed point property if for every nonempty closed convex and bounded set K every nonexpansive mapping $T\colon K \to K$, i.e. a mapping such that $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$, has a fixed point. Similarly, the space X has the weak fixed point property if for every nonempty weakly compact convex set K every nonexpansive mapping $T\colon K \to K$ has a fixed point. In 1965 Browder [4] proved that every uniformly convex Banach space has the fixed point property. Since then, many papers about geometric conditions of a space implying the fixed point property have been published. In 1996 Domínguez Benavides [8] introduced the coefficient $M(X)$ of a Banach space X and proved that if $M(X) > 1$, then X has the weak fixed point property. Using this result García Falset, Llorens Fuster and Mazcuñan Navarro [9] solved a long-standing problem: every uniformly nonsquare space has the fixed point property.

One of research directions in the fixed point theory is to study conditions under which a direct sum of spaces has the fixed point property. The simplest case is when a geometric property, which implies the fixed point property, is

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preserved under passing to direct sums. In 1943 Day [5] showed that the direct sum of a family of Banach spaces $\{X_i\}$ with respect to a proper function space Z is uniformly convex if and only if Z is uniformly convex and the spaces X_i have a common modulus of convexity. In 1968 Belluce, Kirk and Steiner [2] showed that the direct sum of two Banach spaces with normal structure, endowed with the maximum norm, also has normal structure. Permanence properties of normal structure and the weakly convergent sequence coefficient are given in [12] and [7], respectively. Further research in this direction can be found in [14]. In 2014 Wiśnicki [16] proved that the direct sum $X_1 \oplus \ldots \oplus X_n$ with a strictly monotone norm has the weak fixed point property for nonexpansive mappings whenever $M(X_i) > 1$ for each $i = 1, \ldots, n$. A brief survey of the geometry of direct sums of finitely many Banach spaces with a variety of generalizations of uniform convexity and uniform smoothness is given in [6].

In 2011 Wiśnicki [15] proved that if a Banach space X has the weak fixed point property and Y has the generalized Gossez–Lami Dozo property or is uniformly convex in every direction, then the direct sum $X \oplus Y$ with a strictly monotone norm has the weak fixed point property.

2. Preliminaries

A Banach space X is said to be uniformly convex if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in B_X$ and $\|x - y\| \geq \varepsilon$, then $\|x + y\| \leq 2(1 - \delta)$. The modulus of uniform convexity of X is defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x + y\| : x, y \in B_X,\ \|x - y\| \geq \varepsilon\right\},$$

where $\varepsilon \in [0, 2]$. Clearly, X is uniformly convex if and only if $\delta_X(\varepsilon) > 0$ for every $\varepsilon > 0$.

Given $a \geq 0$, we put

$$R(a, X) = \sup \liminf_{n\to\infty} \|x + x_n\|,$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq a$, and all weakly null sequences $(x_n) \subset B_X$ such that $\lim_{m,n\to\infty, n\neq m} \|x_n - x_m\| \leq 1$. We define

$$M(X) = \sup\left\{\frac{1 + a}{R(a, X)} : a > 0\right\}.$$

The above two coefficients were introduced in [8].

By $\widetilde{X}$ we denote the quotient space $l_\infty(X)/c_0(X)$. For every coset $(x_n) + c_0(X) \in \widetilde{X}$ we put $[(x_n)] = (x_n) + c_0(X)$. It is easy to prove that the quotient norm of a vector $[(x_n)]$ is equal to $\limsup_{n\to\infty} \|x_n\|$. We identify an element $x \in X$ with $[(x, x, \ldots)] \in \widetilde{X}$. Let K be a subset of X. We define

$$\widetilde{K} = \big\{[(x_n)] \in \widetilde{X} : x_n \in K\big\}.$$

Given a nonexpansive mapping $T\colon K \to K$, we define $\widetilde{T}[(x_n)] = [(Tx_n)]$.

Let Z be a Banach lattice. We say that Z is strictly monotone if $\|x\| < \|y\|$ provided that $0 \le x \le y$ and $x \ne y$. A modulus of uniform monotonicity of Z is defined by

$$\sigma_Z(\varepsilon) = \inf\{\|x+y\| - 1 : x, y \ge 0,\ \|x\| \ge 1,\ \|y\| \ge \varepsilon\},$$

where $\varepsilon \in [0,\infty]$. We say that Z is uniformly monotone if $\sigma_Z(\varepsilon) > 0$ for every $\varepsilon > 0$. Obviously, $\delta_X(\varepsilon)$ and $\sigma_Z(\varepsilon)$ are nondecreasing functions and $\sigma_Z(\varepsilon)$ is continuous. Note that if Z is uniformly convex, then Z is uniformly monotone (see [1]). In this paper we consider only two dimensional Banach lattices. Recall that a norm in $\mathbb{R}^2$ is monotone if

$$x_1 \le y_1 \wedge x_2 \le y_2 \ \Rightarrow\ \|(x_1,x_2)\| \le \|(y_1,y_2)\|,$$

is strictly monotone if

$$x_1 \le y_1 \wedge x_2 \le y_2 \wedge (x_1,x_2) \ne (y_1,y_2) \ \Rightarrow\ \|(x_1,x_2)\| < \|(y_1,y_2)\|,$$

and is absolute if

$$\|(x_1,x_2)\| = \|(|x_1|,|x_2|)\|,$$

where $x_1, x_2, y_1, y_2 \in \mathbb{R}$. It is well known that a norm is absolute if and only if it is monotone.

Let X_1 and X_2 be Banach spaces and let $Z = \mathbb{R}^2$ be endowed with a monotone norm $\|\cdot\|_Z$. We shall write $X_1 \oplus_Z X_2$ for the direct sum of X_1 and X_2 endowed with the norm $\|(\|x_1\|,\|x_2\|)\|_Z$, where $(x_1,x_2) \in X_1 \times X_2$. Elements of the direct sum $X_1 \oplus_Z X_2$ are denoted by $x = (x(1), x(2))$, $y = (y(1), y(2))$, etc.

3. Results

We need the following lemmas and theorem.

Lemma 3.1 (Goebel–Karlovitz Lemma [10], [11]). *Let K be a weakly compact convex and minimal invariant set for a nonexpansive mapping T. If a sequence (x_n) in K is an approximate fixed point sequence, i.e.*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0,$$

then

$$\lim_{n\to\infty} \|z - x_n\| = \operatorname{diam} K$$

for every $z \in K$.

Lemma 3.2 (Lin's Lemma [13]). *Let K be a weakly compact convex and minimal invariant set for a nonexpansive mapping T, and $W \subset \widetilde{K}$ be a nonempty closed convex and invariant set for $\widetilde{T}$. Then*

$$\sup\{\|[(y_n)] - x\| : [(y_n)] \in W\} = \operatorname{diam} K$$

for every $x \in K$.

Lemma 3.3 ([15]). *Let X be a direct sum $X_1 \oplus_Z X_2$, where Z has a strictly monotone norm. Assume that $T\colon K \to K$ is a nonexpansive mapping defined on a weakly compact convex subset K of X which is minimal invariant for T, $\operatorname{diam} K = 1$, and $0 \in K$. Fix $k \in \{1,2\}$. Let $(x_n) = (x_n(1), x_n(2))$ be a weakly null approximate fixed point sequence for T such that $\lim_{n\to\infty} \|x_n(k)\| = 0$. Then for every positive integer p there exist a subsequence (v_n) of (x_n) and a family $\{D_j^i : j \in \{1,\dots,p\}, i \in \mathbb{N}\}$ of relatively weakly compact subsets of K such that $D_1^i = \operatorname{conv}(D_1^{i-1} \cup \{v_i\})$, $D_{j+1}^i = \operatorname{conv}\big(D_j^i \cup T(D_j^i)\big)$ for $j \in \{1,\dots,p-1\}$, $i \in \mathbb{N}$, $D_1^0 = \emptyset$, and $\|x(k)\| < 1/p$ for every $x \in \bigcup_{i=1}^{\infty} D_p^i$.*

Theorem 3.4 ([8]). *A Banach space X has the weak fixed point property whenever $M(X) > 1$ (or equivalently $R(1,X) < 2$ [3, Lemma 4.4]).*

The following theorem is the main result of the paper.

Theorem 3.5. *Let $X = X_1 \oplus_Z X_2$, where $Z = \big(\mathbb{R}^2, \|\cdot\|_Z\big)$ is uniformly convex, $M(X_1) > 1$, and X_2 has the weak fixed point property. Then X has the weak fixed point property.*

Proof. We can assume that $\|(1,0)\|_Z = \|(0,1)\|_Z = 1$. Suppose that X lacks the weak fixed point property. Then there exists a convex weakly compact set $K \subset X$ with diameter 1 which is minimal invariant for a nonexpansive mapping $T\colon K \to K$. We can assume that T has an approximate fixed point sequence (x_n) in K such that $x_n \rightharpoonup 0 \in K$ and the following limits exist

$$\lim_{n,m\to\infty,\, n\neq m} \|x_n - x_m\|,$$

$$a(k) = \lim_{n\to\infty} \|x_n(k)\|, \qquad b(k) = \lim_{n,m\to\infty,\, n\neq m} \|x_n(k) - x_m(k)\|$$

for $k \in \{1,2\}$. Since

$$x_n(k) - x_m(k) \xrightharpoonup{m\to\infty} x_n(k),$$

by the weak lower semicontinuity of the norm, we get

$$\|x_n(k)\| \le \liminf_{m\to\infty} \|x_n(k) - x_m(k)\|,$$

and therefore $a(k) \le b(k)$. From Goebel–Karlovitz Lemma, it follows that

$$\lim_{n,m\to\infty,\, n\neq m} \|x_n - x_m\| = \lim_{n\to\infty} \|x_n\| = 1.$$

Moreover, by the continuity of the norm, $\|a\|_Z = 1$, $\|b\|_Z = 1$, therefore, since Z is uniformly monotone, $a = b$.

Assume that $a(k) = 0$ for some $k \in \{1,2\}$. Define

$$C_0 = \{0\} \subset K, \quad C_j = \operatorname{conv}(C_{j-1} \cup T(C_{j-1})) \quad \text{for } j \in \mathbb{N} \quad \text{and} \quad C = \overline{\bigcup_{j=1}^{\infty} C_j}.$$

Then C is a closed and convex subset of K which is invariant for T, hence $C = K$. Fix $p \in \mathbb{N}$. By Lemma 3.3, there exist a subsequence (v_n) of (x_n) and a family $\{D_j^i : j \in \{1, \dots, p\}, i \in \mathbb{N}\}$ of relatively weakly compact subsets of K such that such that $D_1^i = \operatorname{conv}\big(D_1^{i-1} \cup \{v_i\}\big)$, $D_{j+1}^i = \operatorname{conv}\big(D_j^i \cup T(D_j^i)\big)$ for $j \in \{1, \dots, p-1\}$, $i \in \mathbb{N}$, $D_1^0 = \emptyset$, and $\|x(k)\| < 1/p$ for every $x \in \bigcup_{i=1}^{\infty} D_p^i$. Since $v_n \rightharpoonup 0$, $0 \in \overline{\bigcup_{i=1}^{\infty} D_1^i}$. Moreover, for $j \in \{1, \dots, p-1\}$,

$$T\left(\overline{\bigcup_{i=1}^{\infty} D_j^i}\right) \subset \overline{\bigcup_{i=1}^{\infty} T(D_j^i)} \subset \overline{\bigcup_{i=1}^{\infty} D_{j+1}^i},$$

therefore, by induction on j,

$$C_j \subset \overline{\bigcup_{i=1}^{\infty} D_{j+1}^i} \subset \overline{\bigcup_{i=1}^{\infty} D_p^i}.$$

Thus, if $x = (x(1), x(2)) \in C_j$ for some $j \in \{1, \dots, p-1\}$, then $\|x(k)\| \le 1/p$. Since p is arbitrary, $x(k) = 0$ for every $x = (x(1), x(2)) \in K$. Therefore, K is isometric to a subset of X_{3-k}. Since X_{3-k} has the weak fixed point property, it follows that T has a fixed point in K which contradicts our assumption. Thus, we can assume that $a(1) \neq 0$ and $a(2) \neq 0$. Let

$$\varepsilon \in \left(0, \frac{1}{4}\min\{a(1), a(2)\}\right), \quad \alpha \in (1, 1 + \sigma_Z(\varepsilon)), \quad \beta = 2\alpha,$$

and

$$\delta = \min\left\{\delta_Z(\varepsilon), \frac{1}{2}\left(1 - \frac{1}{\alpha}\right)\right\}.$$

Note that $\delta > 0$. We define

$$W = \left\{[(y_n)] \in \widetilde{K} : \big\|[(y_n)] - [(x_n)]\big\| \le \frac{1}{2} \wedge \limsup_{n\to\infty} \limsup_{m\to\infty} \|y_n - y_m\| \le \frac{1}{2}\right\}.$$

It is easy to check that W is closed convex and invariant for $\widetilde{T}$. It is a nonempty set because it contains $[(x_n/2)]$. By Lin's Lemma, there exists $[(y_n)] \in W$ such that $\|[(y_n)]\| > 1 - \delta$. Now we pass to subsequences of (x_n) and (y_n) simultaneously to keep the inequality

$$\|[(y_n)] - [(x_n)]\| \le \frac{1}{2}$$

in two steps. Passing to subsequences for the first time, we can assume that

$$\inf_{n\in\mathbb{N}} \|y_n\| > 1 - \delta.$$

Passing to subsequences for the second time, we can assume that $y_n \rightharpoonup y$ and there exist the following limits:

$$c(k) = \lim_{n\to\infty} \|y_n(k)\|, \qquad d(k) = \lim_{n,m\to\infty,\, n\neq m} \|y_n(k) - y_m(k)\|,$$

$$e(k)=\lim_{n\to\infty}\|y(k)-y_n(k)\|,\qquad f(k)=\lim_{n\to\infty}\|x_n(k)-y_n(k)\|,$$

for $k\in\{1,2\}$, and $\big|\|y_n(k)-y(k)\|-e(k)\big|\le\varepsilon$ for every $n\in\mathbb{N}$ and $k\in\{1,2\}$. We also define

$$g(k)=\|y(k)\|,\quad k\in\{1,2\}.$$

We have $\|d\|_Z\le 1/2$, $\|f\|_Z\le 1/2$, and $\|c\|_Z>1-\delta$. Since

$$\begin{aligned} y_n(k) &\xrightarrow{n\to\infty} y(k),\\ y_m(k)-y_n(k) &\xrightarrow{m\to\infty} y(k)-y_n(k),\\ y_n(k)-x_n(k) &\xrightarrow{n\to\infty} y(k), \end{aligned}$$

we obtain

$$\begin{aligned} \|y(k)\| &\le \lim_{n\to\infty}\|y_n(k)\|,\\ \|y(k)-y_n(k)\| &\le \liminf_{m\to\infty}\|y_m(k)-y_n(k)\|,\\ \|y(k)\| &\le \lim_{n\to\infty}\|x_n(k)-y_n(k)\|. \end{aligned}$$

Therefore, $g\le c$, $e\le d$, $g\le f$. In consequence, $\|g\|_Z\le\|c\|_Z$, $\|e\|_Z\le\|d\|_Z\le 1/2$, $\|g\|_Z\le\|f\|_Z\le 1/2$. Moreover,

$$c(k)=\lim_{n\to\infty}\|y_n(k)\|\le\lim_{n\to\infty}\|y_n(k)-y(k)\|+\|y(k)\|=e(k)+g(k)$$

for $k\in\{1,2\}$, thus $1-\delta_Z(\varepsilon)<\|c\|_Z\le\|e+g\|_Z$, so $\|e-g\|_Z<\varepsilon/2$. We have

$$\begin{gathered} \|\alpha c\|_Z\ge\alpha(1-\delta)\ge\alpha(1-(1-1/\alpha))=1,\\ \|\alpha c+\alpha(e+g-c)\|_Z-1\le\alpha(\|e\|_Z+\|g\|_Z)-1<\sigma_Z(\varepsilon), \end{gathered}$$

therefore

$$\|e+g-c\|_Z<\|\alpha(e+g-c)\|_Z<\varepsilon.$$

We have

$$\frac{1}{\beta}=1-\frac12\left(1-\frac1\alpha\right)-\frac12<\|c\|_Z-\frac12\le\|e\|_Z+\|g\|_Z-\frac12\le\|g\|_Z,$$

thus $\|\beta g\|_Z\ge 1$. We obtain

$$\|\beta g+(\beta f-\beta g)\|_Z-1\le\frac12\beta-1=\alpha-1<\sigma_Z(\varepsilon),$$

therefore

$$\|f-g\|_Z<\frac12\|\beta f-\beta g\|_Z<\frac{\varepsilon}{2}.$$

Similarly, $\|d-e\|_Z\le\varepsilon/2$. We have

$$\begin{aligned} a(k)&=\lim_{n\to\infty}\|x_n(k)\|\le\lim_{n\to\infty}\|x_n(k)-y_n(k)\|+\lim_{n\to\infty}\|y_n(k)\|=f(k)+c(k)\\ &\le g(k)+\frac{\varepsilon}{2}+e(k)+g(k)\le 3g(k)+\varepsilon\le 3g(k)+\frac14 a(k), \end{aligned}$$

so $a(k)/4 \leq g(k)$. Note that

$$\frac{1}{g(k)+2\varepsilon} \lim_{n,m\to\infty,\, n\neq m} \|y_n(k) - y_m(k)\| = \frac{d(k)}{g(k)+2\varepsilon} \leq 1,$$

$$\left\| \frac{y(k)}{g(k)+2\varepsilon} \right\| = \frac{g(k)}{g(k)+2\varepsilon} \leq 1 \quad \text{and} \quad \left\| \frac{y_n(k) - y(k)}{g(k)+2\varepsilon} \right\| \leq \frac{e(k)+\varepsilon}{g(k)+2\varepsilon} \leq 1.$$

We obtain

$$\begin{aligned} R(1, X_k) &\geq \lim_{n\to\infty} \left\| \frac{y(k)}{g(k)+2\varepsilon} + \frac{y_n(k) - y(k)}{g(k)+2\varepsilon} \right\| = \lim_{n\to\infty} \left\| \frac{y_n(k)}{g(k)+2\varepsilon} \right\| \\ &= \frac{c(k)}{g(k)+2\varepsilon} \geq \frac{2g(k) - 2\varepsilon}{g(k)+2\varepsilon} = 2 - \frac{6\varepsilon}{g(k)+2\varepsilon}, \end{aligned}$$

thus

$$R(1, X_k) \geq 2 - \frac{6\varepsilon}{a(k)/4 + 2\varepsilon}$$

for $k \in \{1, 2\}$. Passing to the limit as $\varepsilon \to 0$, we get $R(1, X_1) = R(1, X_2) = 2$. In view of Theorem 3.4, this contradicts our assumption. □

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