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SYMMETRIC TOPOLOGICAL COMPLEXITY FOR FINITE SPACES AND CLASSIFYING SPACES

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Abstract. We present a combinatorial approach to the symmetric motion planning in polyhedra using finite spaces. For a finite space P and a positive integer k, we introduce two types of combinatorial invariants, $\mathrm{CC}^S_k(P)$ and $\mathrm{CC}^\Sigma_k(P)$, that are closely related to the design of symmetric robotic motions in the k-iterated barycentric subdivision of the associated simplicial complex $\mathcal{K}(P)$. For the geometric realization $\mathcal{B}(P) = |\mathcal{K}(P)|$, we show that the first $\mathrm{CC}^S_k(P)$ converges to Farber–Grant's symmetric topological complexity $\mathrm{TC}^S(\mathcal{B}(P))$ and the second $\mathrm{CC}^\Sigma_k(P)$ converges to Basabe–González–Rudyak–Tamaki's symmetrized topological complexity $\mathrm{TC}^\Sigma(\mathcal{B}(P))$ as k becomes larger.

1. Introduction

The *topological complexity* $\mathrm{TC}(X)$ of a space X is a homotopy invariant introduced by Farber [8] to study the robotic motion planning in X. For a positive integer $n \geq 1$, the equality $\mathrm{TC}(X) = n$ implies that we need at least n local motion planning rules on open sets covering X to design continuous robotic motions in X. Farber and Grant considered additional practical motion planning rules [9] by taking symmetricity into account, and extended TC to the *symmetric topological complexity* TC^S. Another symmetrization TC^Σ of the topological

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complexity was presented by Basabe, González, Rudyak, and Tamaki [2]. These two concepts, TC^S and TC^Σ, are closely related to each other.

On the other hand, combinatorial approaches to the topological complexity have recently been developed for simplicial complexes [12], [10] and finite spaces [20]. This paper provides a combinatorial description of the symmetric motion planning for (realized) simplicial complexes in the context of finite spaces. We present two types of invariants, $\mathrm{CC}_k^S(P)$ and $\mathrm{CC}_k^\Sigma(P)$, for a finite space P by using the k-iterated barycentric subdivision of the product $P \times P$. Our objective is to establish the equalities $\mathrm{CC}_k^S(P) = \mathrm{TC}^S(\mathcal{B}(P))$ and $\mathrm{CC}_k^\Sigma(P) = \mathrm{TC}^\Sigma(\mathcal{B}(P))$ for sufficiently large k, where $\mathcal{B}(P)$ is the *classifying space* of P. Every finite cell complex is homotopy equivalent to the classifying space of some finite space. Hence, both $\mathrm{TC}^S(X)$ and $\mathrm{TC}^\Sigma(X)$ for a finite cell complex X can be described in purely combinatorial terms.

The rest of this paper is organized as follows. Section 2 is devoted to the fundamental $\mathbb{Z}_2$-homotopy theory for finite spaces and simplicial complexes.

In Section 3, we first consider a combinatorial analog of the symmetrized topological complexity TC^Σ introduced by Basabe, González, Rudyak and Tamaki [2]. For a finite space P and $k \geq 0$, we define an invariant $\mathrm{CC}_k^\Sigma(P)$ by using the k-iterated barycentric subdivision and establish the equality $\mathrm{CC}_k^\Sigma(P) = \mathrm{TC}^\Sigma(\mathcal{B}(P))$ for sufficiently large k. We essentially use an argument similar to that presented in the author's previous paper [20] with respect to the standard topological complexity from the viewpoint of finite spaces.

In Section 4, we focus on a combinatorial analog of the symmetric topological complexity TC^S established by Farber and Grant [9]. Similar to the case of CC_k^Σ, we introduce an invariant $\mathrm{CC}_k^S(P)$ for a finite space P, considering the k-iterated barycentric subdivision, and establish the equality $\mathrm{CC}_k^S(P) = \mathrm{TC}^S(\mathcal{B}(P))$ for sufficiently large k. Unlike the case of CC_k^Σ, it is difficult to consider CC_k^S for only finite spaces. The symmetric topological complexity TC^S is defined using the quotient spaces by $\mathbb{Z}_2$-actions. However, the quotient space of a finite space (in the category of spaces) is not compatible with the classifying space. Babson and Kozlov pointed out this fact in [3], and they treated the categorical quotient instead of the topological quotient. For this reason, we consider the categorical quotient of a finite space P as well. It should be noted that the categorical quotient of a finite space is not a finite space, but rather an acyclic category in general. Hence, we need an acyclic category model of (unordered) configuration spaces of 2-points to define CC_k^S and to relate it with TC^S.

2. Homotopy theory of $\mathbb{Z}_2$-finite spaces and $\mathbb{Z}_2$-simplicial complexes

In this section, we consider $\mathbb{Z}_2$-spaces and their homotopy theory for the symmetric motion planning. A $\mathbb{Z}_2$-space X is a topological space equipped with an action of $\mathbb{Z}_2 = \{e, r \mid r^2 = e\}$, and we will express $\overline{x} = rx$ for $x \in X$.

Definition 2.1. Let X be a $\mathbb{Z}_2$-space and let Y be an arbitrary space. Two continuous maps $f, g\colon X \to Y$ are called *symmetrically homotopic* if there is a homotopy $H\colon X \times I \to Y$ between f and g satisfying $H(\overline{x}, t) = H(x, 1-t)$ for any $x \in X$ and $t \in I = [0,1]$. We call H a *symmetric homotopy* between f and g.

Note that if $f, g\colon X \to Y$ are symmetrically homotopic, then $f(\overline{x}) = g(x)$ for any $x \in X$. When we consider the path space Y^I as a $\mathbb{Z}_2$-space by defining $\overline{\gamma}(t) = \gamma(1-t)$ for $\gamma \in Y^I$, a symmetric homotopy can be regarded as a $\mathbb{Z}_2$-map $X \to Y^I$.

2.1. Homotopy theory of finite spaces. We recall the homotopy theory of finite spaces established by Stong [18]. Throughout this paper, a finite space means a T_0 space that consists of finitely many points. Every point x of a finite space X has the minimal open neighbourhood U_x, defined as the intersection of all open neighbourhoods of x. We can define a partial order $x \leq y$ on X by $x \in U_y$. Conversely, a poset P admits a topology called the *Alexandroff topology.* An open set of P consists of ideals that are subsets closed under the lower order. From this viewpoint, we can identify finite spaces and finite posets.

The *order complex* $\mathcal{K}(P)$ of a finite space P is a simplicial complex that consists of totally ordered subsets of P. The geometric realization of $\mathcal{K}(P)$ is called the *classifying space* of P and is denoted $\mathcal{B}(P)$. The (geometric) realization $|K|$ of a finite simplicial complex K is constructed by gluing the topological standard simplices indexed by simplices of K. This is isomorphic to the classifying space of some finite space. Indeed, we have the *face poset* $\mathcal{X}(K)$, which consists of simplices of K with the inclusion order relation, and the classifying space $\mathcal{B}(\mathcal{X}(K))$ is naturally homeomorphic to $|K|$.

A map of finite spaces is continuous if and only if it preserves the order. We define a partial order $f \leq g$ on the set of continuous maps Q^P between finite spaces P and Q by $f(p) \leq g(p)$ for any $p \in P$. Two continuous maps $f, g\colon P \to Q$ on finite spaces are homotopic if and only if there is a sequence of maps $f = h_0, \ldots, h_n = g$ such that either $h_i \leq h_{i+1}$ or $h_i \geq h_{i+1}$ holds for each i. In other words, two maps f and g are homotopic if and only if there is a map (combinatorial homotopy) $H\colon P \times J^n \to Q$ for some $n \geq 0$ with $H_0 = f$ and $H_n = g$, where J^n is the finite space described as follows:

$$0 < 1 > 2 < \ldots > (<)\, n.$$

When n is even, the group $\mathbb{Z}_2$ acts on J^n continuously (preserving the order) by defining $\bar{i} = n - i$.

PROPOSITION 2.2. *Let P be a finite $\mathbb{Z}_2$-space and let Q be an arbitrary finite space. Two maps $f, g\colon P \to Q$ are symmetrically homotopic if and only if there exists a symmetric combinatorial homotopy $H\colon P \times J^{2m} \to Q$ for some $m \geq 0$ such that $H_0 = f$ and $H_{2m} = g$ and $H(\overline{x}, i) = H(x, 2m - i)$ for any $x \in P$ and $0 \leq i \leq 2m$.*

PROOF. We assume that there is a symmetric combinatorial homotopy $H\colon P \times J^{2m} \to Q$ with length $2m$ between f and g. The map $\mu\colon [0, 2m] = \mathcal{B}(J^{2m}) \to J^{2m}$ introduced in [14] is defined by

$$\mu(t) = \begin{cases} 2i+1 & \text{if } t = 2i+1, \\ 2i & \text{if } 2i-1 < t < 2i+1. \end{cases}$$

This map and the $2m$-times isomorphism $I \cong [0, 2m]$ are $\mathbb{Z}_2$-maps. The composition of these maps

$$P \times I \cong P \times [0, 2m] \xrightarrow{\mathrm{id}_P \times \mu} P \times J^{2m} \xrightarrow{H} Q$$

determines a symmetric homotopy between f and g.

Conversely, suppose that we have a symmetric homotopy $H\colon P \times I \to Q$ between f and g. We define a map $h\colon P \to Q$ by $h(x) = H(x, 1/2)$. By the homotopy theory of finite spaces in [18, Theorem 7], we have a combinatorial homotopy $G\colon P \times J^m \to Q$ between f and h.

Let φ_i denote the map $G(\,\cdot\,, i)\colon P \to Q$ for each i. We should note that $\varphi_m(x) = h(x) = h(\overline{x}) = \varphi_m(\overline{x})$ for any $x \in P$. Thus, we have a symmetric combinatorial homotopy $\widetilde{G}\colon P \times J^{2m} \to Q$ between f and g defined by

$$\widetilde{G}(x, i) = \begin{cases} \varphi_i(x) & \text{if } 0 \leq i \leq m, \\ \varphi_{2m-i}(\overline{x}) & \text{if } m \leq i \leq 2m. \end{cases}$$

□

2.2. Homotopy theory of simplicial complexes. We consider only finite simplicial complexes throughout this paper. A simplicial complex K consists of a finite set of vertices $V(K)$ and a set of simplices $\Sigma(K) \subset 2^{V(K)}$ satisfying the face relation. A simplicial map $K \to L$ between simplicial complexes K and L is a map on vertices $V(K) \to V(L)$ sending a simplex of K into a simple of L. Two simplicial maps $f, g\colon K \to L$ are *contiguous*, denoted by $f \simeq_c g$, if $f(\sigma) \cup g(\sigma)$ is a simplex of L for any simplex σ in K. We say that two simplicial maps f, g are *in the same contiguity class* if there is a sequence of contiguous simplicial maps

$$f = h_0 \simeq_c h_1 \simeq_c \ldots \simeq_c h_m = g \quad \text{for some } m \geq 0.$$

A $\mathbb{Z}_2$-*simplicial complex* is a simplicial complex K with a $\mathbb{Z}_2$-action on the set of vertices $V(K)$ such that for any simplex $\sigma = \{v_0, \ldots, v_n\}$ of K, the subset

$\overline{\sigma} = \{\overline{v}_0, \dots, \overline{v}_n\}$ is also a simplex of K. For a $\mathbb{Z}_2$-simplicial complex K, the realization $|K|$ and the face poset $\mathcal{X}(K)$ are $\mathbb{Z}_2$-spaces in the natural way.

Definition 2.3. Let K be a $\mathbb{Z}_2$-simplicial complex and let L be an arbitrary simplicial complex. Two simplicial maps $f, g\colon K \to L$ are *symmetrically contiguous* if they are in the same contiguity class by a sequence of contiguous simplicial maps

$$f = h_0 \simeq_c \dots \simeq_c h_m = g$$

such that $h_i(\overline{v}) = h_{m-i}(v)$ for any i and vertex $v \in V(K)$.

A simplicial map $f\colon K \to L$ induces a map $\mathcal{X}(f)\colon \mathcal{X}(K) \to \mathcal{X}(L)$ on the face posets, sending $\sigma \in \mathcal{X}(K)$ to $f(\sigma) \in \mathcal{X}(L)$. The next lemma follows immediately from [5, Proposition 4.12].

Lemma 2.4. *If two simplicial maps $f, g\colon K \to L$ are symmetrically contiguous, then the induced maps $\mathcal{X}(f), \mathcal{X}(g)\colon \mathcal{X}(K) \to \mathcal{X}(L)$ on the face posets are symmetrically homotopic.*

Proof. We have a sequence of contiguous simplicial maps $f = h_0, \dots, h_m = g$ such that $h_i(\overline{v}) = h_{m-i}(v)$ for any i and vertex $v \in V(K)$. We define a continuous map $H_i\colon \mathcal{X}(K) \to \mathcal{X}(L)$ by $H_i(\sigma) = h_i(\sigma) \cup h_{i-1}(\sigma)$ for $i = 1, \dots, m$. The inequality $\mathcal{X}(h_{i-1}) \leq H_i \geq \mathcal{X}(h_i)$ holds, and $H_i(\overline{\sigma}) = H_{m-i+1}(\sigma)$ for each i. This gives rise to a symmetric homotopy $G\colon \mathcal{X}(K) \times J^{2m} \to \mathcal{X}(L)$ between $\mathcal{X}(f)$ and $\mathcal{X}(g)$. $\square$

A simplicial map $f\colon K \to L$ induces a map $|f|\colon |K| \to |L|$ sending $\sum_i t_i v_i$ in $|K|$ to $\sum_i t_i f(v_i)$ in $|L|$. The next lemma follows immediately from [17, Lemma 3.5.2].

Lemma 2.5. *If two simplicial maps $f, g\colon K \to L$ are symmetrically contiguous, then the induced maps $|f|, |g|\colon |K| \to |L|$ on the realizations are symmetrically homotopic.*

Proof. We have a sequence of contiguous simplicial maps $f = h_0, \dots, h_m = g$ such that $h_i(\overline{v}) = h_{m-i}(v)$ for any i and vertex $v \in V(K)$. For each $1 \leq i \leq m$, the maps $|h_i|$ and $|h_{i-1}|$ are homotopic by a homotopy $H_i\colon |K| \times I \to L$, defined by $H_i(x,t) = (1-t)|h_i|(x) + t|h_{i-1}|(x)$. Direct calculation shows that

$$\begin{aligned} H_i(\overline{x}, t) &= (1-t)|h_i|(\overline{x}) + t|h_{i-1}|(\overline{x}) \\ &= (1-t)|h_{m-i}|(x) + t|h_{m-i+1}|(x) = H_{m-i+1}(x, 1-t). \end{aligned}$$

By concatenating these homotopies H_i, we obtain a symmetric homotopy $|K| \times I \to |L|$ between $|f|$ and $|g|$. $\square$

3. Combinatorial analog of TC^Σ for finite spaces

3.1. Topological complexity for finite spaces. The topological complexity $\mathrm{TC}(X)$ of a space X was introduced by Farber [8] to study robotic motion planning. In this paper, let X be a path-connected space.

Definition 3.1. Let $p\colon E \to B$ be a continuous map on spaces. The *Schwarz genus* $g(p)$ of p is the smallest number n such that there is an open cover $\{U_i\}_{i=1}^n$ of B, where each U_i admits a local section $s_i\colon U_i \to E$ of p, i.e., the composition $p \circ s_i$ coincides with the inclusion $U_i \hookrightarrow B$.

Remark 3.2. In [2], the authors adopted the normalized version of Definition 3.1 using homotopy sections (that is, the composition $p \circ s_i$ is homotopic to the inclusion) instead of genuine sections. When $p\colon E \to B$ is a fibration, a homotopy section of p can be replaced with a genuine section by the homotopy lifting property. For this reason, the normalized definition agrees with the Schwarz genus $g(p)$ for a fibration p.

The topological complexity is defined as a special case of Schwarz genus for the path fibration.

Definition 3.3. For a space X, let $\pi\colon X^I \to X \times X$ denote the path fibration defined by $\pi(\gamma) = (\gamma(0), \gamma(1))$. The *topological complexity* $\mathrm{TC}(X)$ is defined as the Schwarz genus $g(\pi)$ of the path fibration.

In this paper we consider the unreduced version of topological complexity, not the reduced version, which is one less than the above definition. We now consider a combinatorial analog of the topological complexity for a finite space. Let P be a (path-connected) finite space and consider the combinatorial path space P^{J_m} with length $m \geq 0$. This is equipped with the canonical source-target map $\pi_m\colon P^{J_m} \to P \times P$ given by $\pi_m(\gamma) = (\gamma(0), \gamma(m))$. Note that this is not a fibration in general.

In the author's previous work [20], the Schwarz genus $g(\pi_m)$ was compared with $\mathrm{TC}(\mathcal{B}(P))$ and the inequality $\mathrm{TC}(\mathcal{B}(P)) \leq g(\pi_m)$ was proved for any $m \geq 0$. However, even for sufficiently large $m \geq 0$, the Schwarz genus $g(\pi_m)$ is not a good estimate for $\mathrm{TC}(\mathcal{B}(P))$ because of the small amount of open sets in a finite space P. This problem can be fixed by taking the barycentric subdivision. For a finite space P, we call $\mathrm{sd}(P) = \mathcal{X}(\mathcal{K}(P))$ the *barycentric subdivision* of P and $\mathrm{sd}^k(P) = \mathrm{sd}^{k-1}(\mathrm{sd}(P))$ the k-iterated barycentric subdivision for $k \geq 1$. The barycentric subdivision $\mathrm{sd}(P)$ is equipped with the canonical map $\tau_P\colon \mathrm{sd}(P) \to P$, sending a totally ordered subset $p_0 < \ldots < p_n$ into the greatest element p_n. Let $\tau_P^k\colon \mathrm{sd}^k(P) \to P$ denote the composition of

$$\mathrm{sd}^k(P) \xrightarrow{\tau_{\mathrm{sd}^{k-1}(P)}} \mathrm{sd}^{k-1}(P) \longrightarrow \cdots \longrightarrow \mathrm{sd}(P) \xrightarrow{\tau_P} P.$$

DEFINITION 3.4. Let P be a finite space and let $k, m \geq 0$ be non-negative integers. We define $\mathrm{CC}_{k,m}(P)$ as the smallest number n such that there is an open cover $\{U_i\}_{i=1}^n$ of $\mathrm{sd}^k(P \times P)$, where each U_i admits a map $s_i\colon U_i \to P^{J_m}$ with $\pi_m \circ s_i = \left(\tau_{P\times P}^k\right)_{|U_i}$. We also call these maps s_i sections of π_m on U_i for simplicity, although they are not rigorously sections. For a fixed $k \geq 0$, we have the following monotonically decreasing sequence:

$$\mathrm{CC}_{k,0}(P) \geq \mathrm{CC}_{k,1}(P) \geq \ldots \geq 0.$$

Let $\mathrm{CC}_k(P)$ stand for the stable value of the above sequence. Moreover, we have the following monotonically decreasing sequence:

$$\mathrm{CC}_0(P) \geq \mathrm{CC}_1(P) \geq \ldots \geq 0.$$

Let $\mathrm{CC}(P)$ stand for the stable value of the above sequence.

By the homotopy theory of finite spaces, the equality $\mathrm{CC}_0(P) = \mathrm{TC}(P)$ holds for any finite space P [20, Theorem 3.2]. Moreover, the equality $\mathrm{CC}_k(P) = \mathrm{TC}(\mathcal{B}(P))$ holds for sufficiently large $k \geq 0$.

THEOREM 3.5 ([20, Corollary 4.10]). *For any finite space P, it holds that*

$$\mathrm{CC}(P) = \mathrm{TC}(\mathcal{B}(P)).$$

3.2. Symmetrized topological complexity for finite spaces. To study the symmetric robotic motion planning Farber and Grant developed the concept of the topological complexity [9]. Subsequently, Basabe, González, Rudyak, and Tamaki introduced a slightly different invariant from Farber and Grant's symmetric topological complexity [2]. We first focus on Basabe–González–Rudyak–Tamaki's symmetrized motion planning using symmetric subspaces and symmetric sections. For a $\mathbb{Z}_2$-space X, a subset A of X is called a *symmetric subspace* if $\overline{x} \in A$ for every $x \in A$.

DEFINITION 3.6. Let X be a space and consider the product $X \times X$ as a $\mathbb{Z}_2$-space by switching elements. A local section $s\colon U \to X^I$ of π on a symmetric subspace $U \subset X \times X$ is called a *symmetric section* if $s(y,x)(t) = s(x,y)(1-t)$ for any $(x,y) \in U$ and $t \in I$. We denote $\mathrm{TC}^{\Sigma}(X)$ as the smallest number n such that there are symmetric open subspaces $U_1, \ldots, U_n$ covering $X \times X$, where each U_i admits a symmetric section $U_i \to X^I$ of π. We call $\mathrm{TC}^{\Sigma}(X)$ the *symmetrized topological complexity* of X.

REMARK 3.7. As mentioned in Remark 3.2, if $s\colon U \to X^I$ is a homotopy section of π on an open set U in $X \times X$, then we can choose a genuine section on U by the homotopy lifting property of the path fibration $\pi\colon X^I \to X \times X$. However, it is not trivial that we can choose a "symmetric" section for a homotopy symmetric section on a symmetric open set U. The following steps explain how to construct a genuine symmetric section from a homotopy symmetric section.

We take and fix a homotopy $H\colon U \times I \to X \times X$ between the inclusion $H_0\colon U \hookrightarrow X \times X$ and $H_1 = \pi \circ s$. By composing the j-th projection $X \times X \to X$ with H, we have two homotopies $\gamma_j\colon U \times I \to X$ for $j = 1, 2$. When we regard a homotopy symmetric section s as a symmetric homotopy $U \times I \to X$, the concatenation $\gamma_1 * s * \overline{\gamma_2}$ of the three homotopies determines a genuine symmetric section on U.

We can easily consider the combinatorial version of TC^Σ in Definition 3.6 based on CC in Definition 3.4 for finite spaces.

Definition 3.8. For a finite space P, we define $\mathrm{CC}^\Sigma_{k,2m}(P)$ as the smallest number n such that there are symmetric open subspaces $U_1, \ldots, U_n$ covering $\mathrm{sd}^k(P \times P)$, where each U_i admits a symmetric section $s_i\colon U_i \to P^{J_{2m}}$ of π_{2m}.

Lemma 3.9. *For any finite space P and $k, m \geq 0$, it holds that*

$$\mathrm{CC}^\Sigma_{k,2m}(P) \geq \mathrm{CC}^\Sigma_{k,2m+2}(P).$$

Proof. We define a $\mathbb{Z}_2$-map $r\colon J_{2m+2} \to J_{2m}$ by

$$r(j) = \begin{cases} j & \text{if } 0 \leq j \leq m-1, \\ m & \text{if } j = m, m+1, m+2, \\ j-2 & \text{if } m+3 \leq j \leq 2m+2. \end{cases}$$

This induces a map $r^*\colon P^{J_{2m}} \to P^{J_{2m+2}}$ on path spaces, which preserves both ends, i.e., $\pi_{2m+2} \circ r^* = \pi_{2m}$. For a symmetric section $s\colon U \to P^{J_{2m}}$ of π_{2m} on a symmetric open set U of $\mathrm{sd}^k(P \times P)$, the composition $r^* \circ s$ is a symmetric section on U of π_{2m+2}. This implies that the desired inequality $\mathrm{CC}^\Sigma_{k,2m}(P) \geq \mathrm{CC}^\Sigma_{k,2m+2}(P)$ holds. $\square$

For $k \geq 0$, let $\mathrm{CC}^\Sigma_k(P)$ denote the stable value of the monotonically decreasing sequence

$$\mathrm{CC}^\Sigma_{k,0}(P) \geq \mathrm{CC}^\Sigma_{k,2}(P) \geq \mathrm{CC}^\Sigma_{k,4}(P) \geq \ldots \geq 0.$$

The next proposition can be proved by Proposition 2.2 and the proof of [20, Theorem 3.2]. Note that a symmetric section $U \to P^{J_{2m}}$ on a symmetric subspace U of a finite space P agrees with a symmetric homotopy $U \times J_{2m} \to P$ between the projections $U \to P$.

Proposition 3.10. *For any finite space P, it holds that* $\mathrm{CC}^\Sigma_0(P) = \mathrm{TC}^\Sigma(P)$.

For a $\mathbb{Z}_2$-finite space P, the canonical map $\tau_P\colon \mathrm{sd}(P) \to P$ is a $\mathbb{Z}_2$-map. The next lemma follows immediately from this fact.

Lemma 3.11. *For any finite space P and $k \geq 0$, it holds that*

$$\mathrm{CC}^\Sigma_k(P) \geq \mathrm{CC}^\Sigma_{k+1}(P).$$

Proof. For a symmetric section $s\colon U \to P^{J_{2m}}$ on a symmetric open set U in $\mathrm{sd}^k(P \times P)$, the inverse image $\tau^{-1}_{\mathrm{sd}^k(P\times P)}(U)$ is a symmetric open set in $\mathrm{sd}^{k+1}(P \times P)$ with a symmetric section $s \circ \tau_{\mathrm{sd}^k(P\times P)}$. This implies that the desired inequality $\mathrm{CC}^\Sigma_k(P) \geq \mathrm{CC}^\Sigma_{k+1}(P)$ holds. ☐

Let $\mathrm{CC}^\Sigma(P)$ denote the stable value of the monotonically decreasing sequence

$$\mathrm{CC}^\Sigma_0(P) \geq \mathrm{CC}^\Sigma_1(P) \geq \ldots \geq 0.$$

The next proposition shows the inequality $\mathrm{TC}^\Sigma(\mathcal{B}(P)) \leq \mathrm{CC}^\Sigma(P)$ for any finite space P. We will use a fundamental theory of PL topology on simplicial complexes to prove the inequality. For a subcomplex L of a simplicial complex K, the realization $|L|$ has the (*open*) *regular neighbourhood* $N(L) = \bigcup\limits_{v\in V(L)} \mathrm{st}(v)$ in K, which consists of the open stars $\mathrm{st}(v)$ at $v \in V(L)$. It is important to note that the closed set $|L|$ is deformation retract of the regular neighbourhood $N(\mathrm{sd}(L))$ ([7, Lemma 2.9.3 and 2.9.4]). When K is a $\mathbb{Z}_2$-simplicial complex and L is a symmetric subcomplex, it is not difficult to verify that these regular neighbourhoods and deformation retractions are compatible with the $\mathbb{Z}_2$-action.

Proposition 3.12. *For any finite space P and $k, m \geq 0$, it holds that*

$$\mathrm{TC}^\Sigma(\mathcal{B}(P)) \leq \mathrm{CC}^\Sigma_{k,2m}(P).$$

Proof. We assume that $\mathrm{CC}^\Sigma_{k,2m}(P) = n$ with symmetric open sets $U_1, \ldots, U_n$ covering $\mathrm{sd}^k(P \times P)$ and symmetric sections $s_i\colon U_i \to P^{J_{2m}}$. The classifying space $\mathcal{B}(U_i)$ is a subcomplex of $\mathcal{B}(\mathrm{sd}^k(P \times P))$ for each i. We have the regular neighbourhood V_i of $\mathcal{B}(U_i)$, which consists of the open stars $\mathrm{st}(v)$ at $v \in V(\mathrm{sd}(\mathcal{K}(U_i)))$. This V_i is a symmetric open subspace and equipped with a $\mathbb{Z}_2$-deformation retraction $r_i\colon V_i \to \mathcal{B}(U_i)$. The evaluation map $P^{J_{2m}} \times J_{2m} \to P$ induces a map

$$\mathcal{B}(P^{J_{2m}}) \times I \cong \mathcal{B}(P^{J_{2m}}) \times \mathcal{B}(J_{2m}) \cong \mathcal{B}(P^{J_{2m}} \times J_{2m}) \to \mathcal{B}(P).$$

This map determines a $\mathbb{Z}_2$-map $\varphi\colon \mathcal{B}(P^{J_{2m}}) \to \mathcal{B}(P)^I$ by the exponential law. We have the following diagrams, of which the middle and right-hand squares are commutative and the left-hand triangle is commutative up to homotopy:

$$\begin{tikzcd}
V_i \arrow[r, "r_i"] \arrow[dr, hook] & \mathcal{B}(U_i) \arrow[r, "\mathcal{B}(s_i)"] \arrow[d, hook] & \mathcal{B}(P^{J_{2m}}) \arrow[r, "\varphi"] \arrow[d, "\mathcal{B}(\pi_{2m})"] & \mathcal{B}(P)^I \arrow[d, "\pi"] \\
& \mathcal{B}(\mathrm{sd}^k(P \times P)) \arrow[r, "\simeq", "\mathcal{B}(\tau^k_{P\times P})"'] & \mathcal{B}(P \times P) \arrow[r, "\cong"] & \mathcal{B}(P) \times \mathcal{B}(P)
\end{tikzcd}$$

Note that $\mathcal{B}(\tau^k_{P\times P})$ is homotopic to the canonical $\mathbb{Z}_2$-homeomorphism

$$\mathcal{B}(\mathrm{sd}^k(P \times P)) \cong B(P \times P)$$

because $\mathcal{K}(\tau^k_{P\times P})$ is a simplicial approximation to the homeomorphism. When we regard V_i as a symmetric open subspace in $\mathcal{B}(P)\times\mathcal{B}(P)$ under the homeomorphism

$$\mathcal{B}(\mathrm{sd}^k(P\times P))\cong\mathcal{B}(P\times P)\cong\mathcal{B}(P)\times\mathcal{B}(P),$$

we obtain a homotopy symmetric section $V_i\to\mathcal{B}(P)^I$ of π. By Remark 3.7, the desired inequality $\mathrm{TC}^\Sigma(\mathcal{B}(P))\le n=\mathrm{CC}^\Sigma_{k,2m}(P)$ holds. □

To show the converse inequality of the above for sufficiently large $k,m\ge 0$, we need a $\mathbb{Z}_2$-equivariant version of the simplicial approximation theorem. The following lemma is a simple symmetric analog of [17, Theorem 3.5.6].

Lemma 3.13. *Let K be a $\mathbb{Z}_2$-simplicial complex and L be an arbitrary simplicial complex. If two maps $f,g\colon|K|\to|L|$ are symmetrically homotopic, then there are simplicial approximations $\varphi,\chi\colon\mathrm{sd}^k(K)\to L$ to f and g, respectively, that are symmetrically contiguous.*

Proof. We have a symmetric homotopy $H\colon|K|\times I\to|L|$ between f and g. Because $|K|$ is compact, we have points $0=t_0<t_1<\ldots<t_m=1/2$ in the half interval $[0,1/2]$ such that there exists a vertex $v\in V(L)$ satisfying both $H(x,t_i)$ and $H(x,t_{i-1})$ belong to the open star $\mathrm{st}(v)$ for each $x\in|K|$ and $i=1,\ldots,m$. We express the map $h_i=H_{t_i}\colon|K|\to|L|$ for each i. For sufficiently large $k\ge 0$, we have a simplicial approximation $\varphi_i\colon\mathrm{sd}^k(K)\to L$ to h_i and to h_{i-1} by [17, Theorem 3.5.6]. We define a simplicial map $\overline{\varphi}_i\colon\mathrm{sd}^k(K)\to L$ by $\overline{\varphi}_i(v)=\varphi_i(\overline{v})$, which is a simplicial approximation to $\overline{h}_i$ and $\overline{h}_{i-1}$. Note that φ_m and $\overline{\varphi}_m$ are contiguous because they are simplicial approximations to the same continuous map $h_m=\overline{h}_m=H_{1/2}$ ([17, Lemma 3.5.4]). Thus, we have the following sequence of contiguous simplicial maps:

$$\varphi_1\simeq_c\varphi_2\simeq_c\cdots\simeq_c\varphi_m\simeq_c\overline{\varphi}_m\simeq_c\cdots\simeq_c\overline{\varphi}_2\simeq_c\overline{\varphi}_1.$$

The simplicial maps φ_1 and $\overline{\varphi}_1$ are symmetrically contiguous and are simplicial approximations to $h_0=H_0=f$ and $\overline{h}_0=H_1=g$, respectively. □

Lemma 3.14. *For a finite space P, it holds that*

$$\mathrm{sd}(\tau_P)\le\tau_{\mathrm{sd}(P)}\colon\mathrm{sd}^2(P)\to\mathrm{sd}(P).$$

Proof. We can express an element S in $\mathrm{sd}^2(P)$ as an increasing sequence $\sigma_0\subset\ldots\subset\sigma_n$, where $\sigma_i=\{p_{i_0},\ldots,p_{i_m}\mid p_{i_j}<p_{i_{j+1}}\}$ is a totally ordered subsets in P. We have

$$\mathrm{sd}(\tau_P)(S)=\{p_{i_m}\mid 0\le i\le n\}\subset\sigma_n=\tau_{\mathrm{sd}(P)}(S).$$

This implies that the desired inequality $\mathrm{sd}(\tau_P)\le\tau_{\mathrm{sd}(P)}$ holds. □

Theorem 3.15. *For any finite space P, it holds that $\mathrm{CC}^\Sigma(P)=\mathrm{TC}^\Sigma(\mathcal{B}(P))$.*

Proof. By Proposition 3.12, we need only to show the inequality $\mathrm{CC}^{\Sigma}(P) \leq \mathrm{TC}^{\Sigma}(\mathcal{B}(P))$. We assume that $\mathrm{TC}^{\Sigma}(\mathcal{B}(P)) = n$ with symmetric open subspaces $U_1, \ldots, U_n$ covering $\mathcal{B}(P) \times \mathcal{B}(P) \cong \mathcal{B}(P \times P)$ and symmetric sections $s_i \colon U_i \to \mathcal{B}(P)^I$ for $1 \leq i \leq n$. For large $k \geq 0$, the realization $|\sigma|$ of any simplex σ in $\mathrm{sd}^k(\mathcal{K}(P \times P))$ is contained in some U_i. Let K_i denote the symmetric subcomplex of $\mathrm{sd}^k(\mathcal{K}(P \times P))$, which consists of simplices σ with $|\sigma| \subset U_i$. Then, the family $\{K_i\}_{i=1}^n$ of subcomplexes covers $\mathrm{sd}^k(\mathcal{K}(P \times P))$, and we have a symmetric homotopy $|K_i| \times I \to \mathcal{B}(P)$ between $\mathcal{B}(\mathrm{pr}_1)$ and $\mathcal{B}(\mathrm{pr}_2)$ for each i. Here, we express the j-th projection $\mathrm{pr}_j \colon P \times P \to P$ for $j = 1, 2$. There are simplicial approximations $p_1, p_2 \colon \mathrm{sd}^{\ell}(K_i) \to \mathcal{K}(P)$ to $\mathcal{B}(\mathrm{pr}_1)$ and $\mathcal{B}(\mathrm{pr}_2)$, respectively, which are symmetrically contiguous for sufficiently large $\ell \geq 0$ by Lemma 3.13. Note that

$$q_j = \mathcal{K}\big(\mathrm{pr}_j \circ \tau_{P \times P}^{k+\ell}\big) \colon \mathcal{K}\big(\mathrm{sd}^{\ell-1}(\mathcal{X}(K_i))\big) = \mathrm{sd}^{\ell}(K_i) \to \mathcal{K}(P)$$

is also a simplicial approximation to $\mathcal{B}(\mathrm{pr}_j)$. By [17, Lemma 3.5.4], the two simplicial maps p_j and q_j are contiguous for $j = 1, 2$. Let V_i denote the face poset $\mathcal{X}(\mathrm{sd}^{\ell}(K_i))$ as an open set in $\mathcal{X}\big(\mathrm{sd}^{k+\ell}(\mathcal{K}(P \times P))\big) = \mathrm{sd}^{k+\ell+1}(P \times P)$ and consider the map $H_j \colon V_i \to \mathrm{sd}(P)$ defined by $H_j(\sigma) = p_j(\sigma) \cup q_j(\sigma)$. We have $\mathcal{X}(p_j) \leq H_j \geq \mathcal{X}(q_j)$, moreover,

$$\tau_P \circ \mathcal{X}(p_j) \leq \tau_P \circ H_j \geq \tau_P \circ \mathcal{X}(q_j) = \mathrm{pr}_j \circ \tau_{P \times P} \circ \mathrm{sd}\big(\tau_{P \times P}^{k+\ell}\big) \leq \mathrm{pr}_j \circ \tau_{P \times P}^{k+\ell+1}$$

by Lemma 3.14. Note that $p_1(\overline{x}) = p_2(x)$ and $q_1(\overline{x}) = q_2(x)$ for any $x \in \mathrm{sd}^{\ell}(K_i)$. Hence, $H_1(\overline{y}) = H_2(y)$ for any $y \in V_i$. We obtain a symmetric homotopy between $\mathrm{pr}_1 \circ \tau_{P \times P}^{k+\ell+1}$ and $\mathrm{pr}_2 \circ \tau_{P \times P}^{k+\ell+1}$ as

$$\begin{aligned} \mathrm{pr}_1 \circ \tau_{P \times P}^{k+\ell+1} &\leq \mathrm{pr}_1 \circ \tau_{P \times P}^{k+\ell+1} \geq \tau_P \circ \mathcal{X}(q_1) \leq \tau_P \circ H_1 \geq \tau_P \circ \mathcal{X}(p_1) \leq \ldots \\ &\geq \tau_P \circ \mathcal{X}(p_2) \leq \tau_P \circ H_2 \geq \tau_P \circ \mathcal{X}(q_2) \leq \mathrm{pr}_2 \circ \tau_{P \times P}^{k+\ell+1} \geq \mathrm{pr}_2 \circ \tau_{P \times P}^{k+\ell+1}. \end{aligned}$$

This sequence determines a symmetric section $V_i \to P^{J_{2m}}$ of π_{2m} for some $m \geq 0$. Thus, the desired inequality $\mathrm{CC}^{\Sigma}(P) \leq \mathrm{CC}^{\Sigma}_{k+\ell+1,2m}(P) \leq n = \mathrm{TC}^{\Sigma}(\mathcal{B}(P))$ holds. $\square$

4. Combinatorial analog of TC^S for finite spaces

Next, we focus on Farber's and Grant's idea of the symmetric motion planning. Farber and Grant originally worked with the quotient space of the path space without loops and the unordered configuration space.

4.1. Configuration space of 2-points in finite space. For a space X, the *ordered configuration space* of n-points in X consists of distinguished n-tuples of points in X. This is a subspace of the product X^n defined as:

$$F_n(X) = X^n - \Delta_X,$$

where $\Delta_X = \{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j \text{ for some } i \neq j\}$. The ordered configuration space $F_n(X)$ is equipped with the permutation $\mathbb{Z}_n$-action, and the *unordered configuration space* $C_n(X)$ is defined as the quotient space $F_n(X)/\mathbb{Z}_n$.

We address the configuration space of 2-points in a finite space P to study the symmetric motion planning in finite spaces. We focus on the relationship between $F_2(\mathcal{B}(P))$ and $\mathcal{B}(F_2(P))$. We have a natural homeomorphism $\mathcal{B}(P \times P) \cong \mathcal{B}(P) \times \mathcal{B}(P)$ induced by the projections $P \times P \to P$. This is restricted to a homeomorphism between $\mathcal{B}(P \times P) - \mathcal{B}(\Delta_P)$ and $F_2(\mathcal{B}(P))$. By identifying them, we have the inclusion $\mathcal{B}(F_2(P)) \hookrightarrow F_2(\mathcal{B}(P))$.

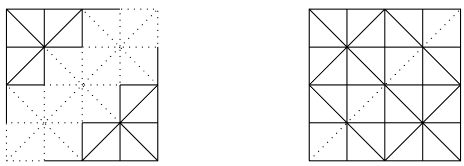


Figure 1. $\mathcal{B}(F_2(J^4))$ (left-hand) and $F_2(\mathcal{B}(J^4))$ (right-hand).

For example, Figure 1 depicts $\mathcal{B}(F_2(J^4))$ in the left-hand and $F_2(\mathcal{B}(J^4))$ in the right-hand for the finite space $J^4 = 0 < 1 > 2 < 3 > 4$. We construct a natural $\mathbb{Z}_2$-deformation retraction $F_2(\mathcal{B}(P)) \to \mathcal{B}(F_2(P))$ for an arbitrary finite space P.

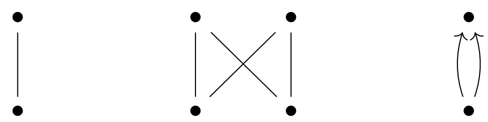
Proposition 4.1. *For a finite space P, the classifying space $\mathcal{B}(F_2(P))$ is $\mathbb{Z}_2$-deformation retract of the ordered configuration space $F_2(\mathcal{B}(P))$.*

Proof. Any point in $\mathcal{B}(P \times P)$ is included in the interior of a unique n-simplex $\Delta^n_{\mathbf{p}}$, indexed by a totally ordered subset $\mathbf{p} = \{(p_i, q_i) \mid (p_i, q_i) < (p_{i+1}, q_{i+1}), i = 0, 1, \ldots, n\}$ in $P \times P$. This index set $\mathbf{p}$ can be decomposed into two subsets $\mathbf{p}_1 = \{(p_i, q_i) \in \mathbf{p} \mid p_i \neq q_i\}$ and $\mathbf{p}_2 = \{(p_i, q_i) \in \mathbf{p} \mid p_i = q_i\}$. Note that if $F_2(\mathcal{B}(P)) \cap \Delta^n_{\mathbf{p}} \neq \emptyset$, then $\mathbf{p}_1 \neq \emptyset$. We regard the simplex $\Delta^n_{\mathbf{p}}$ as the join of the two faces $\Delta^{n_1}_{\mathbf{p}_1}$ and $\Delta^{n_2}_{\mathbf{p}_2}$. Then, any point in $\Delta^n_{\mathbf{p}}$ is expressed as $tx + sy$ for $x \in \Delta^{n_1}_{\mathbf{p}_1}$, $y \in \Delta^{n_2}_{\mathbf{p}_2}$, and $t + s = 1$. We have a deformation retraction

$$\Delta^n_{\mathbf{p}} \cap F_2(\mathcal{B}(P)) \to \Delta^{n_1}_{\mathbf{p}_1}$$

sending $tx + sy$ to x. This gives rise to a $\mathbb{Z}_2$-deformation retraction $F_2(\mathcal{B}(P)) \to \mathcal{B}(F_2(P))$. $\square$

The deformation retraction $F_2(\mathcal{B}(P)) \to \mathcal{B}(F_2(P))$ in the above proof induces a homotopy equivalence on the quotient spaces: $C_2(\mathcal{B}(P)) \to \mathcal{B}(F_2(P))/\mathbb{Z}_2$. We focus on the above right-hand space $\mathcal{B}(F_2(P))/\mathbb{Z}_2$. The relationship between classifying space and group action was studied by Babson and Kozlov [3]. Note that a finite space with a group action admits two types of quotients: the topological quotient and the categorical quotient. For example, let P be a finite space consisting of 4-points $\{a, a', b, b'\}$ with the partial order $a < b, b'$ and $a' < b, b'$. This can be expressed as the Hasse diagram in the following middle figure.



The finite space P is equipped with the free $\mathbb{Z}_2$-action defined by $\overline{a} = a'$ and $\overline{b} = b'$. The topological quotient is the finite space $[a] < [b]$ described as the left-hand Hasse diagram in the above. On the other hand, the categorical quotient is the right-hand acyclic category with two objects and two parallel morphisms $[a] \rightrightarrows [b]$.

For the ordered configuration space $F_2(P)$ of a finite space with the $\mathbb{Z}_2$-action, the topological quotient is the unordered configuration space $C_2(P)$. The categorical quotient, denoted by $A_2(P)$, is defined as the colimit of the associated functor from $\mathbb{Z}_2$ to the category of small categories. We should note that $A_2(P)$ is not in general a finite space (poset), but an acyclic category. Because the $\mathbb{Z}_2$-action on $F_2(P)$ is free, we can apply [3, Theorem 3.4], and the canonical map $\mathcal{B}(F_2(P))/\mathbb{Z}_2 \to \mathcal{B}(A_2(P))$ is a homeomorphism. Here, we consider the classifying space $\mathcal{B}(A_2(P))$ of the acyclic category $A_2(P)$ as a generalization of the classifying space of a poset. See [15] for the general construction and properties of the classifying spaces of small categories. We will use the following homotopy equivalence in Proposition 4.8.

COROLLARY 4.2. *For a finite space P, there is a homotopy equivalence $h\colon C_2(\mathcal{B}(P)) \to \mathcal{B}(A_2(P))$.*

The acyclic category $A_2(P)$ can be considered as a categorical model of the unordered configuration space of 2-points in the classifying space $\mathcal{B}(P)$. Other combinatorial approaches to configuration space can be found in several studies [1], [13], [11].

4.2. Symmetric topological complexity for finite spaces. For a space X, let $L(X)$ denote the space of loops as a subspace of X^I and let X^I_L denote the complement $X^I - L(X)$. We express $Q(X)$ as the quotient space of X^I_L by the restricted $\mathbb{Z}_2$-action. The path fibration is restricted to a $\mathbb{Z}_2$-map $\pi\colon X^I_L \to F_2(X)$, which induces a fibration on the quotient spaces: $\rho\colon Q(X) \to C_2(X)$.

DEFINITION 4.3. Let X be a space. The *symmetric topological complexity* $\mathrm{TC}^S(X)$ is defined as one plus the Schwarz genus of ρ, i.e. $\mathrm{TC}^S(X) = 1 + g(\rho)$.

The symmetric topological complexity TC^S is closely related to the symmetrized topological complexity TC^Σ in the previous section.

Proposition 4.4 ([2, Proposition 4.2]). *For an Euclidean neighbourhood retract* (ENR) X, *it holds that*

$$\mathrm{TC}^S(X) - 1 \leq \mathrm{TC}^\Sigma(X) \leq \mathrm{TC}^S(X).$$

We now present a combinatorial analog of symmetric topological complexity for finite spaces. Recall the path space $P^{J_{2m}}$ with length $2m$ for a finite space P and the source-target map $\pi_{2m}\colon P^{J_{2m}} \to P \times P$. Let $L_{2m}(P)$ denote the finite space of loops in P with length $2m$ and let $P_L^{J_{2m}}$ denote the complement $P^{J_{2m}} - L_{2m}(P)$. We have the restricted free $\mathbb{Z}_2$-action on $P_L^{J_{2m}}$ and the $\mathbb{Z}_2$-map $\pi_{2m}\colon P_L^{J_{2m}} \to F_2(P)$. Let $Q_{2m}(P)$ denote the categorical quotient of $P_L^{J_{2m}}$ and let $\rho_{2m}\colon Q_{2m}(P) \to A_2(P)$ denote the induced functor by π_{2m}.

Definition 4.5. For a finite space P, we denote the categorical quotient $\mathrm{sd}^k(F_2(P))/\mathbb{Z}_2$ by $A_2^k(P)$ for $k \geq 0$. By [3, Proposition 3.14], the acyclic category $A_2^k(P)$ becomes a finite space (poset) for $k \geq 1$.

Remark 4.6. The canonical $\mathbb{Z}_2$-map $\tau^k_{F_2(P)}\colon \mathrm{sd}^k(F_2(P)) \to F_2(P)$ induces a functor

$$\widetilde{\tau}^k_P\colon A_2^k(P) \to A_2(P).$$

The following left diagram in small (acyclic) categories is commutative, and the right-hand is the diagram of their classifying spaces.

$$\begin{tikzcd}
\mathrm{sd}^k(F_2(P)) \arrow[r, "\tau^k_{F_2(P)}"] \arrow[d] & F_2(P) \arrow[d] \\
A_2^k(P) \arrow[r, "\widetilde{\tau}^k_P"'] & A_2(P)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B}\big(\mathrm{sd}^k(F_2(P))\big) \arrow[r, "\mathcal{B}\left(\tau^k_{F_2(P)}\right)"] \arrow[d] & \mathcal{B}(F_2(P)) \arrow[d] \\
\mathcal{B}\big(A_2^k(P)\big) \arrow[r, "\mathcal{B}\left(\widetilde{\tau}^k_P\right)"'] & \mathcal{B}(A_2(P))
\end{tikzcd}$$

The vertical functors in the above left diagram are the quotient projections. Because both $\mathbb{Z}_2$-actions on $F_2(P)$ and $\mathrm{sd}^k(F_2(P))$ are free, these projections become $\mathbb{Z}_2$-coverings in small categories [6]. Moreover, the classifying space preserves coverings [19]. Hence, the vertical maps in the above right diagram are also $\mathbb{Z}_2$-coverings. The map $\mathcal{B}\big(\tau^k_{F_2(P)}\big)$ is a homotopy equivalence [14] and so is $\mathcal{B}\big(\widetilde{\tau}^k_P\big)$. We will use this fact in Proposition 4.8.

Definition 4.7. For a finite space P and $k \geq 1$ and $m \geq 0$, we define $\mathrm{CC}^S_{k,2m}(P)$ as the smallest number n such that there are open sets $U_1, \ldots, U_{n-1}$ covering $A_2^k(P)$, where each U_i admits a section $s_i\colon U_i \to Q_{2m}(P)$ of ρ_{2m}, i.e. $\rho_{2m} \circ s_i = \big(\widetilde{\tau}^k_P\big)_{|U_i}$. When P is a single point, $A_2^k(P)$ is empty for any $k \geq 1$. In this case, we set $\mathrm{CC}^S_{k,2m}(P) = 1$ for any k, m.

Similarly to the case of CC^Σ, we can verify that

$$\mathrm{CC}^S_{k,0}(P) \geq \mathrm{CC}^S_{k,2}(P) \geq \mathrm{CC}^S_{k,4}(P) \geq \ldots \geq 0$$

for each $k \geq 1$. When we express $\mathrm{CC}_k^S(P)$ as the stable value of the above monotonically decreasing sequence, we have

$$\mathrm{CC}_1^S(P) \geq \mathrm{CC}_2^S(P) \geq \mathrm{CC}_3^S(P) \geq \ldots \geq 0.$$

The number $\mathrm{CC}^S(P)$ is defined as the stable value of the above monotonically decreasing sequence.

Proposition 4.8. *For any finite space P and $k \geq 1$ and $m \geq 0$, it holds that* $\mathrm{TC}^S(\mathcal{B}(P)) \leq \mathrm{CC}_{k,2m}^S(P)$.

Proof. We assume that $\mathrm{CC}_{k,2m}^S(P) = n+1$ with open sets $\{U_i\}_{i=1}^n$ covering $A_2^k(P)$ and sections $s_i \colon U_i \to Q_{2m}(P)$. The classifying space $\mathcal{B}(U_i)$ is a subcomplex of the CW complex $\mathcal{B}(A_2^k(P))$. Hence, we can take an open neighbourhood V_i of $\mathcal{B}(U_i)$ with a deformation retraction $r \colon V_i \to \mathcal{B}(U_i)$. Recall the homotopy equivalences $h \colon C_2(\mathcal{B}(P)) \to \mathcal{B}(A_2(P))$ given in Corollary 4.2 and $\ell = \mathcal{B}(\widetilde{\tau}_P^k) \colon \mathcal{B}(A_2^k(P)) \to \mathcal{B}(A_2(P))$ in Remark 4.6. The inverse images $W_i = (\ell' \circ h)^{-1}(V_i)$ constitute an open cover of $C_2(\mathcal{B}(P))$, where we denote $\ell' \colon \mathcal{B}(A_2(P)) \to \mathcal{B}(A_2^k(P))$ as a homotopy inverse of ℓ. The $\mathbb{Z}_2$-map

$$\varphi \colon \mathcal{B}\big(P_L^{J_{2m}}\big) \to \mathcal{B}(P)_L^I$$

given in the proof of Proposition 3.12 makes the following diagram commutative:

$$\begin{tikzcd}
\mathcal{B}\big(P_L^{J_{2m}}\big) \arrow[r, "\varphi"] \arrow[d, "\mathcal{B}(\pi_{2m})"'] & \mathcal{B}(P)_L^I \arrow[d, "\pi"] \\
\mathcal{B}(F_2(P)) \arrow[r, hook] & F_2(\mathcal{B}(P))
\end{tikzcd}$$

We obtain a map

$$\widetilde{\varphi} \colon \mathcal{B}(Q_{2m}(P)) \cong \mathcal{B}\big(P_L^{J_{2m}}\big)/\mathbb{Z}_2 \to Q(\mathcal{B}(P)),$$

making the following right-hand square diagram commute up to homotopy:

$$\begin{tikzcd}
\mathcal{B}(U_i) \arrow[rr, "\mathcal{B}(s_i)"] \arrow[dr, "\ell"] & & \mathcal{B}(Q_{2m}(P)) \arrow[dl, "\mathcal{B}(\rho_{2m})"'] \arrow[dd, "\widetilde{\varphi}"] \\
& \mathcal{B}(A_2(P)) & \\
W_i \arrow[uu, "r \circ \ell' \circ h"] \arrow[rr] \arrow[dr, hook] & & Q(\mathcal{B}(P)) \arrow[dl, "\rho"] \\
& C_2(\mathcal{B}(P)) \arrow[uu, "h"'] &
\end{tikzcd}$$

Moreover, the top triangle is commutative and the left-hand square is commutative up to homotopy. Therefore, the composition

$$\widetilde{\varphi} \circ \mathcal{B}(s_i) \circ r \circ \ell' \circ h \colon W_i \to Q(\mathcal{B}(P))$$

makes the bottom triangle commute up to homotopy in the above diagram. By Remark 3.2, the desired inequality $\mathrm{TC}^S(\mathcal{B}(P)) \leq n+1 = \mathrm{CC}^S_{k,2m}(P)$ holds. □

We will show the converse inequality to that from Proposition 4.8 for sufficiently large k, m.

Theorem 4.9. *For any finite space P, it holds that* $\mathrm{CC}^S(P) = \mathrm{TC}^S(\mathcal{B}(P))$.

Proof. By Proposition 4.8, we need only to show the inequality $\mathrm{CC}^S(P) \leq \mathrm{TC}^S(\mathcal{B}(P))$. We assume that $\mathrm{TC}^S(\mathcal{B}(P)) = n+1$ with an open cover $\{U_i\}_{i=1}^n$ of $C_2(\mathcal{B}(P))$ and local sections $s_i \colon U_i \to Q(\mathcal{B}(P))$ of ρ. By [9, Lemma 8], we have a symmetric section $\widetilde{s}_i \colon \widetilde{U}_i \to \mathcal{B}(P)^I_L$ of π over s_i, where $\widetilde{U}_i$ is the symmetric open set in $F_2(\mathcal{B}(P))$ over U_i (the inverse image of U_i by the quotient projection $F_2(\mathcal{B}(P)) \to C_2(\mathcal{B}(P))$). The open sets $V_i = \widetilde{U}_i \cap \mathcal{B}(F_2(P))$ constitute a symmetric open cover of $\mathcal{B}(F_2(P))$. The section $\widetilde{s}_i$ determines a symmetric homotopy between the maps $\mathcal{B}(\mathrm{pr}_1), \mathcal{B}(\mathrm{pr}_2) \colon V_i \to \mathcal{B}(P)$ induced by the projections $\mathrm{pr}_j \colon F_2(P) \to P$ for $j = 1, 2$. A similar argument in the proof of Theorem 3.15 provides a symmetric open cover $\{W_i\}_{i=1}^n$ of $\mathrm{sd}^k(F_2(P))$ for large $k \geq 1$ such that

$$\mathrm{pr}_1 \circ \tau^k_{F_2(P)}, \mathrm{pr}_2 \circ \tau^k_{F_2(P)} \colon W_i \to P$$

are symmetrically homotopic for each i. A symmetric homotopy between them determines a symmetric section $W_i \to P_L^{J_{2m}}$ of π_{2m} for some $m \geq 0$ and induces a section $W_i' \to Q_{2m}(P)$ of ρ_{2m} on the image W_i' of W_i by the quotient projection $\mathrm{sd}^k(F_2(P)) \to A_2^k(P)$. We can verify that W_i' is an open set in $A_2^k(P)$ for each i, and the family $\{W_i'\}_{i=1}^n$ constitutes an open cover of $A_2^k(P)$. This implies that the desired inequality $\mathrm{CC}^S(P) \leq \mathrm{CC}^S_{k,2m}(P) \leq n+1 = \mathrm{TC}^S(\mathcal{B}(P))$ holds. □

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