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A THREE SOLUTION THEOREM FOR A SINGULAR DIFFERENTIAL EQUATION WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. We study positive solutions to singular boundary value problems of the form:

$$\begin{cases} -u'' = h(t)\,\dfrac{f(u)}{u^\alpha} & \text{for } t \in (0,1), \\ u(0) = 0, \\ u'(1) + c(u(1))u(1) = 0, \end{cases}$$

where $0 < \alpha < 1$, $h\colon (0,1] \to (0,\infty)$ is continuous such that $h(t) \le d/t^\beta$ for some $d > 0$ and $\beta \in [0, 1-\alpha)$ and $c\colon [0,\infty) \to [0,\infty)$ is continuous such that $c(s)s$ is nondecreasing. We assume that $f\colon [0,\infty) \to (0,\infty)$ is continuously differentiable such that $[(f(s) - f(0))/s^\alpha] + \tau s$ is strictly increasing for some $\tau \ge 0$ for $s \in (0,\infty)$. When there exists a pair of sub-supersolutions (ψ,ϕ) such that $0 \le \psi \le \phi$, we first establish a minimal solution $\underline{u}$ and a maximal solution $\overline{u}$ in $[\psi,\phi]$. When there exist two pairs of sub-supersolutions (ψ_1,ϕ_1) and (ψ_2,ϕ_2) where $0 \le \psi_1 \le \psi_2 \le \phi_1$, $\psi_1 \le \phi_2 \le \phi_1$ with $\psi_2 \not\le \phi_2$, and ψ_2, ϕ_2 are not solutions, we next establish the existence of at least three solutions u_1, u_2 and u_3 satisfying $u_1 \in [\psi_1,\phi_2], u_2 \in [\psi_2,\phi_1]$ and $u_3 \in [\psi_1,\phi_1] \setminus ([\psi_1,\phi_2] \cup [\psi_2,\phi_1])$.

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1. Introduction

We study positive solutions to singular boundary value problems of the form:

$$
\begin{cases} -u'' = h(t)\,\dfrac{f(u)}{u^{\alpha}} & \text{for } t \in (0,1), \\ u(0) = 0, \\ u'(1) + c(u(1))u(1) = 0, \end{cases} \tag{1.1}
$$

where $0 < \alpha < 1$. Here functions f, h and c satisfy the following properties:

(H1) $f\colon [0,\infty) \to (0,\infty)$ is continuously differentiable,

(H2) there exists $\tau \geq 0$ such that $g(s) := [(f(s) - f(0))/s^{\alpha}] + \tau s$ is strictly increasing for $s \in (0,\infty)$,

(H3) $h\colon (0,1] \to (0,\infty)$ is continuous such that $\inf_{t\in(0,1)} h(t) > 0$ and $h(t) \leq d/t^{\beta}$ for some $d > 0$ and $\beta \in [0, 1-\alpha)$,

(H4) $c\colon [0,\infty) \to [0,\infty)$ is continuous such that $c(s)s$ is nondecreasing.

The boundary value problem (1.1) also arises in the study of radial solutions to the following exterior domain problem:

$$
\begin{cases} -\Delta u = K(|x|)\,\dfrac{f(u)}{u^{\alpha}} & \text{for } x \in \Omega, \\ \dfrac{\partial u}{\partial \eta} + c(u)u = 0 & \text{if } |x| = r_0, \\ u(x) \to 0 & \text{if } |x| \to \infty, \end{cases} \tag{1.2}
$$

where Δu is the Laplacian of u, $\Omega := \{x \in \mathbb{R}^N \mid N > 2,\ |x| > r_0 > 0\}$, $\partial u/\partial\eta$ is the outward normal derivative of u on $|x| = r_0$ and $K\colon [r_0,\infty) \to (0,\infty)$ is a continuous function such that $K(|x|) \to 0$ as $|x| \to \infty$. By a Kelvin type transformation, namely the change of variable $r = |x|$ and $t = (r/r_0)^{2-N}$, (1.2) reduces to anaylzing the singular boundary value problem (1.1). We also note that such nonlinear boundary conditions arise naturally in applications, see [6], [16] and [18] where they discuss models arising in chemical reactor theory, and see [5], [4] and [7] where they discuss models arising in population dynamics.

In [9], for classes of nonlinearities f of the form $f(s) = \lambda g(s)$ where $\lambda > 0$ is a parameter, for a certain range of λ, the authors discuss the existence of two solutions by creating two pairs of sub-supersolutions (ψ_1, ϕ_1) and (ψ_2, ϕ_2) as described in the abstract. However, they could not conclude the existence of a third solution since a three solution theorem for such singular problems with nonlinear boundary conditions (such as Theorem 1.2 in this paper) was not available in the literature.

The main goal of this paper is to establish Theorem 1.2 via fixed point arguments. First we define sub-supersolutions of (1.1). By a subsolution of (1.1), we mean a function $\psi \in C^2(0,1) \cap C^1[0,1]$ that satisfies

$$\begin{cases} -\psi'' \le h(t)\,\dfrac{f(\psi)}{\psi^\alpha} & \text{for } t \in (0,1), \\ \psi(t) > 0 & \text{for } t \in (0,1], \\ \psi(0) = 0, & \\ \psi'(1) + c(\psi(1))\psi(1) \le 0. & \end{cases}$$

By a supersolution of (1.1), we mean a function $\phi \in C^2(0,1) \cap C^1[0,1]$ that satisfies

$$\begin{cases} -\phi'' \ge h(t)\,\dfrac{f(\phi)}{\phi^\alpha} & \text{for } t \in (0,1), \\ \phi(t) > 0 & \text{for } t \in (0,1], \\ \phi(0) = 0, & \\ \phi'(1) + c(\phi(1))\phi(1) \ge 0. & \end{cases}$$

We establish the following results:

Theorem 1.1 (Minimal and maximal solutions). *Let* (H1)–(H4) *hold. Suppose there exist a subsolution ψ and a supersolution ϕ of* (1.1) *satisfying* $0 \le \psi \le \phi$. *Then there exist a minimal solution $\underline{u}$ and a maximal solution $\overline{u}$ for* (1.1) *in the ordered interval* $[\psi, \phi]$, *which belong to* $C^2(0,1] \cap C^{1,\kappa}[0,1]$ *where* $\kappa = 1 - \alpha - \beta$.

Theorem 1.2 (A three solution theorem). *Let* (H1)–(H4) *hold. Suppose there exist two pairs of ordered sub-supersolutions* (ψ_1, ϕ_1) *and* (ψ_2, ϕ_2) *of* (1.1) *such that* $0 \le \psi_1 \le \psi_2 \le \phi_1$, $\psi_1 \le \phi_2 \le \phi_1$ *and* $\psi_2 \not\le \phi_2$. *Additionally assume that* ψ_2 *and* ϕ_2 *are not solutions of* (1.1). *Then there exist at least three solutions* u_1, u_2 *and* u_3 *for* (1.1) *belonging to* $C^2(0,1] \cap C^{1,\kappa}[0,1]$ *such that* $u_1 \in [\psi_1, \phi_2], u_2 \in [\psi_2, \phi_1]$ *and* $u_3 \in [\psi_1, \phi_1] \setminus ([\psi_1, \phi_2] \cup [\psi_2, \phi_1])$ *where* $\kappa = 1 - \alpha - \beta$.

For problems with nonsingular reaction terms ($\alpha = 0$), there is a rich history of such three solution theorems based on sub-supersolutions. See [1] and [17] for the case of linear boundary conditions, and see [3] and [14] for the case of nonlinear boundary conditions. Recently, for problems with singular reaction terms with Dirichlet boundary condition, such three solution theorems were discussed in [10] and [11]. Here we enrich the literature by establishing an extention of a three solution theorem for singular reaction terms ($\alpha \in (0,1)$) with nonlinear boundary conditions. Such three solution theorems are useful in analyzing models arising in combustion theory and population dynamics where the bifurcation diagram of positive solutions related to a parameter exhibits a S-shaped behavior, see [2] and [8].

To establish our results, our first step is to make a translation so that we can obtain a monotone operator. Namely, we rewrite (1.1) as:

$$(1.3)\qquad \begin{cases} -u'' - h(t)\bigg(\dfrac{f(0)}{u^\alpha} - \tau u\bigg) = h(t)g(u) & \text{for } t \in (0,1), \\ u(0) = 0, \\ u'(1) + c(u(1))u(1) = 0, \end{cases}$$

where g is as in (H2). Here we extend g to be identically zero for $s \le 0$, and extend c as an even extension for $s < 0$ whenever necessary. We note that a positive solution of (1.3) is a positive solution of (1.1) and also vice versa. The same is true for positive sub-supersolutions.

In Section 2, a Banach space $C_e[0,1]$ is introduced. In Section 3, we construct a priori lower bound for solutions of (1.3). In Section 4, we study a crucial boundary value problem (related to (1.3)), and observe useful properties of its solution. In Section 5, we construct an increasing completely continuous operator associated to (1.3). We prove Theorems 1.1 and 1.2 in Section 6.

2. The Banach space $C_e[0,1]$

Consider the following boundary value problem:

$$(2.1)\qquad \begin{cases} -e'' = 1 & \text{for } t \in (0,1), \\ e(0) = 0, \\ e'(1) + c(e(1))e(1) = 0. \end{cases}$$

Define the functional $J\colon \widetilde{H} \to \mathbb{R}$ by

$$J(z) := \frac{1}{2}\int_0^1 |z'|^2 - \int_0^1 z + p(z(1)),$$

where $\widetilde{H} := \{z \in H^1(0,1) \mid z(0) = 0\}$ and $p(s) := \int_0^s c(r)r\,dr$. We note that $z \mapsto (\int_0^1 z'^2)^{1/2}$ is equivalent to the standard norm in the space $\widetilde{H}$, and p is convex. Then J is weakly lower semicontinuous and coercive. Hence there exists $e \in \widetilde{H}$ such that $J(e) = \min_{u\in\widetilde{H}} J(u)$. Since $J(|e|) \le J(e)$, without loss of generality, we can assume that e is nonnegative in $(0,1)$. Note that J is a C^1 functional on $\widetilde{H}$. Therefore e is a critical point of J, i.e.

$$0 = \langle J'(e), \varphi\rangle = \int_0^1 e'\varphi' - \int_0^1 \varphi + p'(e(1))\,\varphi(1)$$

for $\varphi \in \widetilde{H}$. Thus e is a weak solution of (2.1). In a standard way, we can also show that $e \in C^2[0,1]$ and $e(t) > 0$ for $t \in (0,1]$. Finally we note that the solution for (2.1) is unique. If not, there exist two solutions e and $\widetilde{e}$. Since e and

$\widetilde{e}$ satisfy the weak formulation, we have

$$\int_0^1 (e-\widetilde{e})'\varphi' + \big(p'(e(1)) - p'(\widetilde{e}(1))\big)\varphi(1) = 0$$

for $\varphi \in \widetilde{H}$. Taking $\varphi = (e-\widetilde{e})^+$ in the above identity, we find $(e-\widetilde{e})^+ = 0$. Similarly, we obtain $(e-\widetilde{e})^- = 0$. This is a contradiction. Hence the solution is unique.

Now we define $C_e[0,1]$ as the set of functions $u \in C[0,1]$ such that $-le \le u \le le$ for some $l > 0$. It is well-known that $C_e[0,1]$ equipped with a norm $\|u\|_e := \inf\{l > 0 \mid -le \le u \le le\}$ is a Banach space. Let $P_e := \{u \in C_e[0,1] \mid u \ge 0\}$ be a positive cone of $C_e[0,1]$ and P_e^0 be the set of all interior points of P_e. We note that $(C_e[0,1], P_e)$ is an ordered Banach space. Further, P_e^0 is the set of $u \in C_e[0,1]$ such that $u \ge l_1 e$ for some $l_1 > 0$.

3. A priori lower bound for solutions of (1.3)

Lemma 3.1. *There exists a unique positive weak solution $\theta \in \widetilde{H}$ to the boundary value problem:*

$$\begin{cases} -\theta'' = h(t)\bigg(\dfrac{f(0)}{\theta^\alpha} - \tau\theta\bigg) & \text{for } t \in (0,1), \\ \theta(0) = 0, \\ \theta'(1) + c(\theta(1))\theta(1) = 0. \end{cases} \tag{3.1}$$

Further, this solution θ belongs to $C^2(0,1] \cap C^1[0,1]$ and satisfies (3.1) *in the classical sense.*

Proof. We extend here the proof of Lemma 2.1 in [11] for the nonlinear boundary condition case. By a weak solution we mean $\theta \in \widetilde{H}$ such that

$$\int_0^1 \theta'\varphi' - f(0)\int_0^1 \frac{h(t)}{\theta^\alpha}\varphi + \tau\int_0^1 h(t)\theta\varphi + p'(\theta(1))\varphi(1) = 0$$

for $\varphi \in \widetilde{H}$. We define the functional $E_1 \colon \widetilde{H} \to \mathbb{R}$ associated to the problem (3.1) by

$$E_1(z) := \frac{1}{2}\int_0^1 |z'|^2 - \frac{f(0)}{1-\alpha}\int_0^1 h(t)(z^+)^{1-\alpha} + \frac{\tau}{2}\int_0^1 h(t)z^2 + p(z(1)).$$

Let $\widetilde{H}^+ := \{z \in \widetilde{H} \mid z \ge 0\}$. Then E_1 is weakly lower semicontinuous and coercive on $\widetilde{H}^+$. Thus E_1 admits a minimizer, say θ, in the space $\widetilde{H}^+$. Note that $p(\varepsilon e(1)) \le L\varepsilon^2$ for some $L > 0$ when $\varepsilon \approx 0$. Hence for $\varepsilon \approx 0$, we have

$$E_1(\varepsilon e) = \frac{\varepsilon^2}{2}\int_0^1 |e'|^2 - \frac{\varepsilon^{1-\alpha}f(0)}{1-\alpha}\int_0^1 h(t)e^{1-\alpha} + \frac{\tau\varepsilon^2}{2}\int_0^1 h(t)e^2 + p(\varepsilon e(1)) < 0 = E_1(0).$$

This implies that the minimizer θ is nonzero. We also note that θ is a global minimizer in $\widetilde{H}$ since $E_1(|u|) \leq E_1(u)$ for any $u \in \widetilde{H}$.

It is important to observe that the functional E_1 is not differentiable in the entire space $\widetilde{H}$ because of the presence of the term $\int_0^1 h(t)(u^+)^{1-\alpha}$. Hence we cannot directly conclude that θ is a critical point of E_1. Following the proof of Lemma A.2 in [13], we infer that E_1 is Gateaux differentiable at any $u \in \widetilde{H}$ which additionally satisfies $u \geq \varepsilon_0\phi_1$ for some $\varepsilon_0 > 0$ where ϕ_1 is a positive principal eigenfunction corresponding eigenvalue problem: $-\phi'' = \lambda_1\phi$ in $(0,1)$ and $\phi(0) = \phi(1) = 0$. Further, for any such a u and any $\varphi \in \widetilde{H}$,

$$\langle E_1'(u), \varphi\rangle = \int_0^1 u'\varphi' - f(0)\int_0^1 \frac{h(t)}{u^\alpha}\,\varphi + \tau\int_0^1 h(t)u\varphi + p'(u(1))\varphi(1).$$

As in [11], we can show that $\theta \geq \varepsilon_0\phi_1$ for some $\varepsilon_0 > 0$. This implies that θ is a critical point of E_1, and hence θ is a weak solution of (3.1).

In order to prove the uniqueness, we can argue by contradiction. If θ and $\widetilde{\theta}$ are two weak solutions, then

$$\begin{aligned}\int_0^1 (\theta' - \widetilde{\theta}')\varphi' - f(0)\int_0^1 h(t)\bigg(\frac{1}{\theta^\alpha} - \frac{1}{\widetilde{\theta}^\alpha}\bigg)\varphi \\ + \tau\int_0^1 h(t)\big(\theta - \widetilde{\theta}\big)\varphi + \big(p'\big(\theta(1)\big) - p'\big(\widetilde{\theta}(1)\big)\big)\varphi(1) = 0\end{aligned}$$

for $\varphi \in \widetilde{H}$. Choosing $\varphi = \big(\theta - \widetilde{\theta}\big)^+$ as a test function in the above identity, we observe that $\big(\theta - \widetilde{\theta}\big)^+ = 0$. Similarly, we can show that $\big(\theta - \widetilde{\theta}\big)^- = 0$. Hence the solution is unique.

Further, $\theta \in W^{2,p}(0,1)$ for some $p > 1$ since $\theta \geq \varepsilon_0\phi_1$. Thus the weak solution θ satisfies $-\theta'' = h(t)[(f(0)/\theta^\alpha) - \tau\theta]$ almost everywhere. By the embedding $W^{2,p}(0,1) \subset C^1[0,1]$ and using integration by parts, one can prove that the boundary condition $\theta'(1) + c(\theta(1))\theta(1) = 0$ is satisfied in the pointwise sense. Further, we can show that $\theta \in C^2(0,1] \cap C^1[0,1]$ and solves (3.1) in the classical sense. For complete details, see Lemma 7 in [12]. □

Remark 3.2. Note that θ is a subsolution of (1.3).

Lemma 3.3. *Any positive solution u (or supersolution u) of (1.3), if it exists, must satisfy $u \geq \theta$ on $[0,1]$.*

Proof. Let u be a positive solution or supersolution of (1.3). Assume to the contrary that $\Omega := \{t \in [0,1] \mid u(t) < \theta(t)\} \neq \emptyset$. Then there exists $[a,b] \subset [0,1]$ such that $u(a) - \theta(a) = 0$ and $u(t) - \theta(t) < 0$ for $t \in (a,b)$. We note that u and

θ satisfy

$$\begin{cases} -(u-\theta)'' - h(t)\bigg(f(0)\bigg(\dfrac{1}{u^\alpha} - \dfrac{1}{\theta^\alpha}\bigg) - \tau(u-\theta)\bigg) \geq 0 & \text{for } t \in (0,1), \\ u(0) - \theta(0) = 0, \\ u'(1) - \theta'(1) + c(u(1))u(1) - c(\theta(1))\theta(1) \geq 0. \end{cases}$$

Then we have $-(u-\theta)'' \geq 0$ on (a,b), and thus $u(b) - \theta(b) < 0$. It follows that $u(t) - \theta(t) < 0$ for $t \in (a,1]$ and $u'(1) - \theta'(1) < 0$. However, $u'(1) - \theta'(1) \geq -c(u(1))u(1) + c(\theta(1))\theta(1) \geq 0$ by (H_4). This is a contradiction. Hence $\Omega = \emptyset$.□

4. Perron's method with nonlinear boundary condition

Proposition 4.1. *Let $v \in C(0,1] \cap L^\infty(0,1)$ and $v \geq 0$ on $(0,1]$. Then there exists a unique positive weak solution $w \in \widetilde{H}$ solving*:

$$\begin{cases} -w'' - h(t)\bigg(\dfrac{f(0)}{w^\alpha} - \tau w\bigg) = v & \textit{for } t \in (0,1), \\ w(0) = 0, \\ w'(1) + c(w(1))w(1) = 0. \end{cases} \tag{4.1}$$

Proof. We extend here the proof of Lemma 3.2 in [10] for the nonlinear boundary condition case. Note that $w_0(t) = t(3-t)/2$ uniquely solves:

$$\begin{cases} -w_0'' = 1 & \text{for } t \in (0,1), \\ w_0(0) = 0, \\ w_0(1) = 2w_0'(1). \end{cases}$$

Let $\underline{w} := \theta$ (where θ is as in Lemma 3.1) and let $\overline{w} := \theta + Mw_0$ where $M \geq \|v\|_\infty := \sup_{t\in(0,1]} |v(t)|$ is a constant. Then $\underline{w}$ and $\overline{w}$ are a subsolution and a supersolution of (4.1), respectively.

Let $\mathcal{M} := \{z \in \widetilde{H} \mid \underline{w} \leq z \leq \overline{w}\}$. Define the functional $E\colon \mathcal{M} \to \mathbb{R}$ by

$$E(z) := \frac{1}{2}\int_0^1 |z'|^2 - \frac{f(0)}{1-\alpha}\int_0^1 h(t)z^{1-\alpha} + \frac{\tau}{2}\int_0^1 h(t)z^2 - \int_0^1 vz \,+\, p(z(1)).$$

Then E is weakly lower semicontinuous and coercive. Hence E admits a minimizer, say w, in $\mathcal{M}$. To prove that w is a weak solution of (4.1), rest of the proof is aimed towards showing

$$\int_0^1 w'\varphi' - f(0)\int_0^1 \frac{h(t)}{w^\alpha}\varphi + \tau\int_0^1 h(t)w\varphi - \int_0^1 v\varphi + p'(w(1))\varphi(1) = 0$$

for $\varphi \in C_c^\infty(0,1]$. Let $\varphi \in C_c^\infty(0,1]$, $\varepsilon > 0$ and $v_\varepsilon := \min\{\overline{w}, \max\{\underline{w}, w + \varepsilon\varphi\}\}$, Then $v_\varepsilon = w + \varepsilon\varphi - \varphi^\varepsilon + \varphi_\varepsilon$, where $\varphi^\varepsilon := \max\{0, w + \varepsilon\varphi - \overline{w}\}$ and $\varphi_\varepsilon := -\min\{0, w + \varepsilon\varphi - \underline{w}\}$. Note that $\varphi^\varepsilon, \varphi_\varepsilon \in \widetilde{H}$ and $v_\varepsilon \in \mathcal{M}$. Since $\mathcal{M}$ is convex,

$w+t(v_\varepsilon-w)\in\mathcal{M}$. Thus the limit $\lim_{t\to 0^+}(E(w+t(v_\varepsilon-w))-E(w))/t$ exists and is nonnegative, which we denote by $\langle DE(w),v_\varepsilon-w\rangle$. Then we have

$$\begin{aligned}0&\le\langle DE(w),v_\varepsilon-w\rangle\\&=\int_0^1 w'(v_\varepsilon-w)'-f(0)\int_0^1\frac{h(t)(v_\varepsilon-w)}{w^\alpha}+\tau\int_0^1 h(t)w(v_\varepsilon-w)\\&\quad-\int_0^1 v(v_\varepsilon-w)+p'(w(1))(v_\varepsilon(1)-w(1)).\end{aligned}$$

Substituting for $v_\varepsilon-w$ in the above expression, we can rewrite

$$\langle DE(w),v_\varepsilon-w\rangle=\varepsilon\big\langle\widetilde{D}E(w),\varphi\big\rangle-\big\langle\widetilde{D}E(w),\varphi^\varepsilon\big\rangle+\big\langle\widetilde{D}E(w),\varphi_\varepsilon\big\rangle,$$

where

$$\big\langle\widetilde{D}E(w),\widetilde{\varphi}\big\rangle:=\int_0^1 w'\widetilde{\varphi}'-f(0)\int_0^1\frac{h(t)\widetilde{\varphi}}{w^\alpha}+\tau\int_0^1 h(t)w\widetilde{\varphi}-\int_0^1 v\widetilde{\varphi}+p'(w(1))\widetilde{\varphi}(1)$$

for $\widetilde{\varphi}\in\widetilde{H}$. This implies

$$\big\langle\widetilde{D}E(w),\varphi\big\rangle\ge\frac{1}{\varepsilon}\big[\big\langle\widetilde{D}E(w),\varphi^\varepsilon\big\rangle-\big\langle\widetilde{D}E(w),\varphi_\varepsilon\big\rangle\big].\tag{4.2}$$

Once again estimating the terms in RHS as in [10], we get

$$\big\langle\widetilde{D}E(w),\varphi^\varepsilon\big\rangle\ge o(\varepsilon)+\big(p'(w(1))-p'(\overline{w}(1))\big)\varphi^\varepsilon(1)$$

and

$$\big\langle\widetilde{D}E(w),\varphi_\varepsilon\big\rangle\le o(\varepsilon)+\big(p'(w(1))-p'(\underline{w}(1))\big)\varphi_\varepsilon(1),$$

where $o(\varepsilon)/\varepsilon\to 0$ as $\varepsilon\to 0$. Then, from (4.2), we obtain

$$\begin{aligned}&\big\langle\widetilde{D}E(w),\varphi\big\rangle\\&\quad\ge\frac{o(\varepsilon)}{\varepsilon}+\frac{\big(p'(w(1))-p'(\overline{w}(1))\big)\varphi^\varepsilon(1)-\big(p'(w(1))-p'(\underline{w}(1))\big)\varphi_\varepsilon(1)}{\varepsilon}.\end{aligned}\tag{4.3}$$

To estimate the last term in (4.3), we observe all cases of $\varphi^\varepsilon(1)$ and $\varphi_\varepsilon(1)$:

(a) $\varphi^\varepsilon(1)=0$ and $\varphi_\varepsilon(1)=0$,
(b) $\varphi^\varepsilon(1)>0$ and $\varphi_\varepsilon(1)=0$,
(c) $\varphi^\varepsilon(1)=0$ and $\varphi_\varepsilon(1)>0$,
(d) $\varphi^\varepsilon(1)>0$ and $\varphi_\varepsilon(1)>0$.

For the case (a), we have $\big\langle\widetilde{D}E(w),\varphi\big\rangle\ge o(\varepsilon)/\varepsilon$. Let us consider the case (b) for $\varepsilon\approx 0$ in detail. If $\varphi^\varepsilon(1)>0$ for $\varepsilon\approx 0$, then necessarily $w(1)=\overline{w}(1)$. This implies

$$\big(p'(w(1))-p'(\overline{w}(1))\big)\varphi^\varepsilon(1)-\big(p'(w(1))-p'(\underline{w}(1))\big)\varphi_\varepsilon(1)=0.$$

Thus we obtain $\big\langle\widetilde{D}E(w),\varphi\big\rangle\ge o(\varepsilon)/\varepsilon$ for $\varepsilon\approx 0$. Similar calculations lead to the same estimate for case (c) as well. Finally we note that the case (d) never happens by the definitions of φ^ε and φ_ε. Hence we have $\big\langle\widetilde{D}E(w),\varphi\big\rangle\ge o(\varepsilon)/\varepsilon$

for $\varepsilon \approx 0$. This implies $\langle \widetilde{D}E(w), \varphi \rangle \geq 0$. Reversing the sign of φ and using the density of $C_c^\infty(0,1]$ in $\widetilde{H}$, we conclude

$$\begin{aligned} 0 &= \langle \widetilde{D}E(w), \varphi \rangle \\ &= \int_0^1 w'\varphi' - f(0) \int_0^1 \frac{h(t)\varphi}{w^\alpha} + \tau \int_0^1 h(t) w\varphi - \int_0^1 v\varphi + p'(w(1))\varphi(1) \end{aligned}$$

for $\varphi \in \widetilde{H}$. Thus w is a weak solution of (4.1).

The uniqueness of the weak solution follows in a standard way. □

Lemma 4.2 (Regularity). *Let $v \in C(0,1] \cap L^\infty(0,1)$ and $v \geq 0$ on $(0,1]$ and let $w \in \widetilde{H}$ be the unique positive solution of (4.1). Then w belongs to $C^2(0,1) \cap C^{1,\kappa}[0,1]$ for $\kappa = 1 - \alpha - \beta$ and satisfies (4.1) in the classical sense.*

Proof. Since $w \in \mathcal{M}$, we can show that $w \in C^2(0,1) \cap C^1[0,1]$ and satisfies (4.1) in the classical sense. Further, $\|w\|_\infty \leq \|\overline{w}\|_\infty \leq C_1$, where the constant depends on α, β, c and v. By (H3), we estimate

$$\begin{aligned} |w'(t)| &= \left| w'(1) + \int_t^1 h(s)\left(\frac{f(0)}{w^\alpha} - \tau w\right) + \int_t^1 v \right| \\ &\leq |c(w(1))w(1)| + C_2 \int_t^1 s^{-\alpha-\beta} + \|v\|_\infty (1-t) \end{aligned}$$

for some $C_2 > 0$. Thus $\|w'\|_\infty \leq C_3$, where the constant depends on α, β, c and v. We also obtain

$$\begin{aligned} |w'(t_2) - w'(t_1)| &= \left| \int_{t_1}^{t_2} h(s)\left(\frac{f(0)}{w^\alpha} - \tau w\right) + \int_{t_1}^{t_2} v \right| \\ &\leq C_4 \left| \int_{t_1}^{t_2} s^{-\alpha-\beta} \right| \leq C_5 |t_2 - t_1|^{1-\alpha-\beta} \end{aligned}$$

for some $C_4 > 0$ and $C_5 > 0$, where the constants depend on α, β, c and v. Hence $w \in C^{1,\kappa}[0,1]$ and the required estimate holds. □

5. Properties of the associated operator

Definition 5.1. Let $A\colon C_e[0,1] \to C^{1,\kappa}[0,1]$ be such that $A(v) := w$, where w is the unique positive solution of

$$\begin{gathered} -w'' - h(t)[(f(0)/w^\alpha) - \tau w] = h(t)g(v) \quad \text{in } (0,1), \\ w(0) = 0 = w'(1) + c(w(1))w(1). \end{gathered}$$

For a given $v \in C_e[0,1]$, let $\widetilde{v}(t) := h(t)g(v(t))$ for $t \in (0,1]$. By (H1)–(H3), we have

$$|\widetilde{v}(t)| \leq h(t)g(|v(t)|) = h(t)\big(|f'(\zeta)||v(t)|^{1-\alpha} + \tau|v(t)|\big) \leq M_1 t^{1-\alpha-\beta},$$

where $\zeta(t) \in [0, |v(t)|]$ and for some $M_1 > 0$. Thus $\widetilde{v} \in C(0,1] \cap L^\infty(0,1)$. Then $A(v) \in C^{1,\kappa}[0,1]$ by Lemma 4.2. Hence $A\colon C_e[0,1] \to C^{1,\kappa}[0,1]$ is well-defined. Next we shall prove some more properties of this operator.

Proposition 5.2. *$A\colon C_e[0,1] \to C_e[0,1]$ and is completely continuous. Further, if $0 \le v_1 \le v_2$ and $v_1 \not\equiv v_2$, then $A(v_1) < A(v_2)$. i.e. A is strictly increasing.*

Proof. We first show that $A\colon C_e[0,1] \to C_e[0,1]$ and is completely continuous. Let $v, v_0 \in C_e[0,1]$ and $w, w_0 \in C^{1,\kappa}[0,1]$ be such that $A(v) = w$ and $A(v_0) = w_0$. From the definition of the solutions w and w_0, we have

$$\begin{aligned}\int_0^1 |w' - w_0'|^2 = f(0)\int_0^1 h(t)\bigg(\frac{1}{w^\alpha} - \frac{1}{w_0^\alpha}\bigg)(w - w_0) - \tau\int_0^1 h(t)(w-w_0)^2 \\ + \int_0^1 h(t)(g(v) - g(v_0))(w - w_0) - \big(p'(w(1)) - p'(w_0(1))\big)(w(1) - w_0(1)) \\ \le \int_0^1 h(t)|g(v) - g(v_0)||w - w_0|.\end{aligned}$$

Let $\varepsilon > 0$. We note that

$$|g(v(t)) - g(v_0(t))| < \varepsilon \quad \text{for } t \in [0,1]$$

provided $\|v - v_0\|_e \approx 0$. This implies that

$$\|w - w_0\|_{\widetilde{H}}^2 = \int_0^1 |w' - w_0'|^2 \le \varepsilon \int_0^1 h(t)|w - w_0| \le \varepsilon M_2 \|w - w_0\|_{\widetilde{H}}$$

for some $M_2 > 0$. Thus if $v_n \to v_0$ in $C_e[0,1]$ then $A(v_n) = w_n \to w_0 = A(v_0)$ in $\widetilde{H}$. Since $\{v_n\}$ is bounded in $C_e[0,1]$, $\{\widetilde{v}_n\}$ is uniformly bounded in $C[0,1]$. Then it follows that $\{w_n\}$ is bounded in $C^{1,\kappa}[0,1]$ (see the proof of Lemma 4.2). This implies that $\{w_n\}$ has a subsequence converging to w_0 in $C^{1,\kappa'}[0,1]$ since $w_n \to w_0$ in $\widetilde{H}$ and $C^{1,\kappa}[0,1] \subset\subset C^{1,\kappa'}[0,1]$ for $0 < \kappa' < \kappa$. Thus $A\colon C_e[0,1] \to C^{1,\kappa'}[0,1]$ is continuous. We note that $C^{1,\kappa'}[0,1] \subset\subset C^1[0,1]$ and $\{z \in C^1[0,1] \mid z(0) = 0\} \hookrightarrow C_e[0,1]$. Hence $A\colon C_e[0,1] \to C_e[0,1]$ and is completely continuous. Let $0 \le v_1 \le v_2$ be such that $v_1 \not\equiv v_2$. Since g is strictly increasing, we have $g(v_1) \le g(v_2)$ and $g(v_1) \not\equiv g(v_2)$.

Let $w_i = A(v_i)$ for $i = 1, 2$. Then

$$\begin{cases} -(w_2 - w_1)'' - h(t)\bigg(f(0)\bigg(\dfrac{1}{w_2^\alpha} - \dfrac{1}{w_1^\alpha}\bigg) - \tau(w_2 - w_1)\bigg) \ge 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t \in (0,1), \\ w_2(0) - w_1(0) = 0, \\ w_2'(1) - w_1'(1) + c(w_2(1))w_2(1) - c(w_1(1))w_1(1) = 0. \end{cases} \tag{5.1}$$

By the similar argument in the proof of Lemma 3.3, we can easily prove $w_1 \le w_2$. Now we can directly apply Corollary in [15] (see page 7) to obtain $w_1 < w_2$ in $(0,1)$. Hence A is strictly increasing. □

Remark 5.3. It is also clear from the Hopf maximum principle that $w_1(1) < w_2(1)$.

Lemma 5.4. *A is strongly increasing, i.e. $A(v_2) - A(v_1) \in P_e^0$ whenever $0 \le v_1 \le v_2$ and $v_1 \not\equiv v_2$.*

Proof. Let $v_1 \le v_2$, $w_i = A(v_i)$ for $i = 1, 2$ and denote $\widetilde{w} = w_2 - w_1$. By Proposition 5.2 and Remark 5.3, $\widetilde{w} > 0$ in $(0, 1]$. From (5.1), we have

$$-\widetilde{w}'' + h(t)\bigg[\bigg(\frac{\alpha f(0)}{\xi^{\alpha+1}}\bigg) + \tau\bigg]\widetilde{w} \ge 0 \quad \text{for some } \xi \in [w_1, w_2].$$

Note that, when $t \approx 0$,

$$h(t)\bigg[\bigg(\frac{\alpha f(0)}{\xi^{\alpha+1}}\bigg) + \tau\bigg] \le \frac{\widetilde{c}}{d(t)^{\alpha+\beta+1}} \quad \text{for some } \widetilde{c} > 0.$$

Let $\beta' = \alpha + \beta$ and $\varepsilon \approx 0$. Then we have

$$\begin{cases} -\widetilde{w}'' + \dfrac{\widetilde{c}}{d(t)^{\beta'+1}}\,\widetilde{w} \ge 0 & \text{in } (0, \varepsilon), \\ \widetilde{w}(0) = 0, \\ \widetilde{w}(\varepsilon) > 0. \end{cases}$$

Let $v := e + e^{\gamma}$ for some $\gamma \in (1, 2 - \beta')$, where e is as defined in Section 2. Noting $\varepsilon \approx 0$, an explicit calculation yields

$$-v'' + \frac{\widetilde{c}}{d(t)^{\beta'+1}}\,v = 1 + \gamma e^{\gamma-1} - \gamma(\gamma-1)e^{\gamma-2}(e')^2 + \frac{\widetilde{c}}{d(t)^{\beta'+1}}\,v \le 0$$

for $t \in (0, \varepsilon)$ since $e'(0) > 0$. Now we choose $k_1 > 0$ so that $k_1 v(\varepsilon) < \widetilde{w}(\varepsilon)$. Then we have

$$\begin{cases} -(\widetilde{w} - k_1 v)'' + \dfrac{\widetilde{c}}{d(t)^{\beta'+1}}(\widetilde{w} - k_1 v) \ge 0 & \text{for } t \in (0, \varepsilon), \\ \widetilde{w}(0) - k_1 v(0) = 0, \\ \widetilde{w}(\varepsilon) - k_1 v(\varepsilon) > 0. \end{cases}$$

By the maximum principle, $\widetilde{w}(t) \ge k_1 e(t)$ for $t \in (0, \varepsilon)$. Since $\widetilde{w}(t) > 0$ for $t \in [\varepsilon, 1]$, we also obtain that $\widetilde{w}(t) \ge k_2 e(t)$ for $t \in [\varepsilon, 1]$, where $k_2 := \inf\limits_{t \in [\varepsilon, 1]} \widetilde{w}(t)/e(t)$. Hence the result follows directly by choosing $k = \min\{k_1, k_2\}$. □

6. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Let $E := C_e[0, 1]$ and P_e be the positive cone of E. Then (E, P_e) is an ordered Banach space. Let $X := [\psi_1, \phi_1]$. Then $A\colon X \to E$ is an increasing completely continuous map by Proposition 5.2. Now from Corollary 6.2 in [1], the proof of Theorem 1.1 easily follows. □

Proof of Theorem 1.2. Let $X := [\psi_1, \phi_1]$, $X_1 := [\psi_1, \phi_2]$ and $X_2 := [\psi_2, \phi_1]$. Then $A\colon X \to X$ is completely continuous and $A(X_i) \subset X_i$ for $i = 1, 2$. Hence the proof of Theorem 1.2 follows by Lemma 14.1 in [1] and Theorem 1.4 in [17]. □

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