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GENERALIZED FRACTIONAL DIFFERENTIAL EQUATIONS AND INCLUSIONS EQUIPPED WITH NONLOCAL GENERALIZED FRACTIONAL INTEGRAL BOUNDARY CONDITIONS

Sotiris K. Ntouyas — Bashir Ahmad
Madeaha Alghanmi — Ahmed Alsaedi

Abstract. In this paper, we establish sufficient criteria for the existence of solutions for generalized fractional differential equations and inclusions supplemented with generalized fractional integral boundary conditions. We make use of the standard fixed point theorems for single-valued and multivalued maps to obtain the desired results, which are well illustrated with the aid of examples.

1. Introduction

Fractional calculus is a branch of mathematical analysis dealing with the study of derivatives and integrals of non-integer order. Differential equations involving fractional order derivatives are termed as fractional order differential equations and find useful applications in physics, chemical technology, population dynamics, biotechnology, economics, viscoelasticity, control theory of dynamical systems, electrical networks optics and signal processing, rheology etc. For details and examples, we refer the reader to the books [3], [12], [17], [20], [23], [24] and a series of articles [1], [2], [4]–[8] and the references cited therein.

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Fractional derivatives appear in terms of fractional integrals and have different forms in contrast to the classical integer order derivative. Riemann–Liouville and Hadamard fractional integrals are the well-known examples of fractional integrals used for defining several types of fractional derivatives. A new fractional integral, known as *generalized Riemann–Liouville fractional integral* or *Katugampola fractional integral*, unifies the Riemann–Liouville and Hadamard integrals into a single integral [15]. The fractional derivative expressed in terms of generalized fractional integral is known as generalized fractional derivative [16] (see definitions in Section 2). Recently, Lupinska and Odzijewicz [22] obtained a Lyapunov-type inequality for fractional boundary value problem with Katugampola fractional derivative.

In this paper, we initiate the study of a new class of boundary value problems involving generalized fractional derivatives and integrals. Precisely, we investigate the existence and uniqueness of solutions for the following generalized fractional differential equation and inclusions:

$$ {}^{\rho}D^{\alpha}y(t) = f(t, y(t)), \quad t \in [0, T], \tag{1.1} $$

$$ {}^{\rho}D^{\alpha}y(t) \in F(t, y(t)), \quad t \in [0, T], \tag{1.2} $$

supplemented with the following nonlocal boundary conditions:

$$ y(0) = 0, \quad y(T) = \lambda \frac{\rho^{1-\beta}}{\Gamma(\beta)} \int_0^{\xi} \frac{s^{\rho-1}}{(\xi^{\rho} - s^{\rho})^{1-\beta}} y(s)\, ds := \lambda\, {}^{\rho}I^{\beta} y(\xi), \tag{1.3} $$

for $\xi \in (0, T)$,

$$ y(0) = 0, \quad y(T) = \sum_{i=1}^{n} \lambda_i y(\xi_i), \quad \xi_i \in (0, T), \tag{1.4} $$

and

$$ y(0) = 0, \quad y(T) = \sum_{i=1}^{n} \lambda_i\, {}^{\rho}I^{\beta} y(\xi_i), \quad \xi_i \in (0, T), \tag{1.5} $$

where ${}^{\rho}D^{\alpha}$ is the generalized (Katugampola) fractional derivative of order $1 < \alpha \leq 2$, $\rho > 0$, ${}^{\rho}I^{\beta}$ is the generalized (Katugampola) type fractional integral of order $\beta > 0, \rho > 0$, $f\colon [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\lambda, \lambda_i \in \mathbb{R}$, $i = 1, \ldots, n$, and $F\colon [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued function ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subjects of $\mathbb{R}$). Note that the boundary conditions (1.3) and (1.5) involve generalized fractional integrals.

The rest of the paper is organized as follows. In Subsection 3.1, we present the existence and uniqueness results for the problem (1.1) and (1.3) by using Banach's contraction mapping principle, Krasnoselskii's fixed point theorem and Leray-Schauder nonlinear alternative. The existence of solutions for the equation (1.1) equipped with the boundary conditions (1.4) and (1.5) is outlined in Subsection 3.2. Existence results for the inclusions problem (1.2) and (1.3) are studied in Section 4 via Leray–Schauder nonlinear alternative for multi-valued maps and Covitz and Nadler fixed point theorem for multi-valued contractions. Examples illustrating the obtained results are also included.

2. Preliminaries

For $c \in \mathbb{R}$, $1 \le p \le \infty$, let $X_c^p(a,b)$ denote the space of all complex-valued Lebesgue measurable functions ϕ on (a,b) with $\|\phi\|_{X_c^p} < \infty$, and

$$\|\phi\|_{X_c^p} = \left(\int_a^b |x^c \phi(x)|^p \, \frac{dx}{x} \right)^{1/p} < \infty.$$

In particular, when $c = 1/p$ and $p = 1$, the space $X_c^p(a,b)$ coincides with the $L^1(a,b)$-space ($L^1(a,b)$ denote the space of all Lebesgue measurable functions φ on (a,b) equipped with the norm

$$\|\varphi\|_{L^1} = \int_a^b |\varphi(x)| \, dx < \infty.$$

Definition 2.1 [15]. The generalized fractional integral of order $\alpha > 0$ and $\rho > 0$ of $f \in X_c^p(a,b)$ for $-\infty < a < t < b < \infty$, is defined by

$$({}^\rho I_{a+}^\alpha f)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} \, f(s) \, ds. \tag{2.1}$$

Note that the integral in (2.1) is called the left-sided fractional integral.

Example 2.2. Let $\alpha > 0$ and $\rho \in \mathbb{R}$. Then

$${}^\rho I^\alpha t^q = \frac{t^{\rho\alpha+q}}{\rho^\alpha} \frac{\Gamma\left(\dfrac{q}{\rho}+1\right)}{\Gamma\left(\dfrac{q}{\rho}+\alpha+1\right)}.$$

Similarly we can define right-sided fractional integral ${}^\rho I_{b-}^\alpha f$ as

$$({}^\rho I_{b-}^\alpha f)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b \frac{s^{\rho-1}}{(s^\rho - t^\rho)^{1-\alpha}} \, f(s) \, ds. \tag{2.2}$$

Remark 2.3. The above definitions for generalized fractional integrals reduce to the ones for the standard Riemann–Liouville fractional integrals and Hadamard fractional integrals (see [17]) for $\rho = 1$ and $\rho \to 0$ respectively [15].

Definition 2.4 ([16]). The generalized fractional derivatives associated with the generalized fractional integrals (2.1) and (2.2) are defined, for $0 \le a < x < b < \infty$ as follows:

$$\begin{aligned}({}^\rho D_{a+}^\alpha f)(t) &= \left(t^{1-\rho} \frac{d}{dt}\right)^n ({}^\rho I_{a+}^{n-\alpha} f)(t) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(t^{1-\rho} \frac{d}{dt}\right)^n \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{\alpha-n+1}} \, f(s) \, ds,\end{aligned}$$

$$
\begin{aligned}
({}^\rho D^\alpha_{b-} f)(t) &= \bigg(-t^{1-\rho}\frac{d}{dt}\bigg)^n ({}^\rho I^{n-\alpha}_{b-} f)(t) \\
&= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \bigg(-t^{1-\rho}\frac{d}{dt}\bigg)^n \int_t^b \frac{s^{\rho-1}}{(s^\rho - t^\rho)^{\alpha-n+1}} f(s)\, ds.
\end{aligned}
$$

Lemma 2.5 ([16]). *Let $0<\alpha<1$ and $f \in X^p_c(a,b), \rho>0$. Then, for $a>0$, $\rho>0$,*

$$\big({}^\rho D^\alpha_{a+}\, {}^\rho I^\alpha_{a+}\big) f(t) = f(t).$$

Lemma 2.6 ([16]). *Let $\alpha,\beta \in \mathbb{C}$. Let $0<a<b<\infty$ and $1 \le p \le \infty$. Then, for $f \in X^p_c(a,b)$, $\rho>0$,*

$$ {}^\rho I^\alpha_{a+}\, {}^\rho I^\beta_{a+} f = {}^\rho I^{\alpha+\beta}_{a+} f \quad \text{and} \quad {}^\rho D^\alpha_{a+}\, {}^\rho D^\beta_{a+} f = {}^\rho D^{\alpha+\beta}_{a+} f.$$

We remark in passing that the the generalized fractional integral and the generalized fractional derivative of f with $a=0$ are denoted by ${}^\rho I^\alpha f$ and ${}^\rho D^\alpha f$, respectively.

Let $C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with the norm defined by $\|y\| = \sup_{t\in[0,T]} |y(t)|$. Let $AC[0,T]$ denotes the space of all absolutely continuous real valued function on $[0,T]$. Moreover, let us introduce the space AC^2_ρ by

$$AC^2_\rho[0,T] = \bigg\{ f\colon [0,T] \to \mathbb{R} : \bigg(t^{1-\rho}\frac{d}{dt}\bigg) f(t) \in AC[0,T] \bigg\}.$$

Lemma 2.7 [22]. *Let $\alpha,\rho>0$, $n=[\alpha]+1$, $u \in C(0,T) \cap X^p_c(0,T)$ and ${}^\rho I^{n-\alpha} u \in AC^2_\rho$. Then the general solution of the fractional differential equation ${}^\rho D^\alpha u(t) = 0$ is*

$$u(t) = c_1 t^{\rho(\alpha-1)} + \ldots + c_n t^{\rho(\alpha-n)},$$

where $c_i \in \mathbb{R}$, $i=1,\ldots,n$. Moreover,

$${}^\rho I^\alpha\, {}^\rho D^\alpha u(t) = u(t) + c_1 t^{\rho(\alpha-1)} + \ldots + c_n t^{\rho(\alpha-n)}.$$

Definition 2.8. A function $y \in C([0,T],\mathbb{R})$ is said to be a solution of (1.1) and (1.3) if y satisfies the equation ${}^\rho D^\alpha y(t) = f(t,y(t))$ on $[0,T]$, and the condition $y(0)=0$, $y(T) = \lambda\, {}^\rho I^\beta y(\xi)$.

Relative to the problem (1.1) and (1.3), we consider the following lemma.

Lemma 2.9. *Let $h \in C(0,T) \cap L^1(0,T)$, $y \in C([0,T],\mathbb{R})$ such that ${}^\rho I^{2-\alpha} y \in AC^2_\rho[0,T]$ and*

$$\Lambda_1 = T^{\rho(\alpha-1)} - \lambda \frac{\Gamma(\alpha)}{\rho^\beta \Gamma(\alpha+\beta)} \xi^{\rho(\alpha+\beta-1)} \neq 0. \tag{2.3}$$

Then the solution of the linear equation

$${}^\rho D^\alpha y(t) = h(t), \quad t \in [0,T], \tag{2.4}$$

supplemented with the conditions (1.3) *is equivalent to the integral equation*:

$$y(t) = {}^\rho I^\alpha h(t) + \frac{t^{\rho(\alpha-1)}}{\Lambda_1}\big\{\lambda\, {}^\rho I^{\alpha+\beta} h(\xi) - {}^\rho I^\alpha h(T)\big\}. \tag{2.5}$$

PROOF. Applying the operator ${}^\rho I^\alpha$ on the generalized linear fractional differential equation (2.4) and using Lemma 2.7, we obtain

$$y(t) = {}^\rho I^\alpha h(t) + c_1 t^{\rho(\alpha-1)} + c_2 t^{\rho(\alpha-2)}, \tag{2.6}$$

where c_1 and c_2 are arbitrary unknown constants. Using the first boundary condition $y(0) = 0$ in (2.6), we get $c_2 = 0$. Applying the generalized integral to (2.6) after inserting the value of c_2, we get

$${}^\rho I^\beta y(t) = {}^\rho I^{\alpha+\beta} h(t) + c_1 \frac{\Gamma(\alpha)}{\rho^\beta \Gamma(\alpha+\beta)}\, t^{\rho(\alpha+\beta-1)}. \tag{2.7}$$

From the second boundary condition $y(T) = \lambda\, {}^\rho I^\beta y(\xi)$, we have

$${}^\rho I^\alpha h(T) + c_1 T^{\rho(\alpha-1)} = \lambda\, {}^\rho I^{\alpha+\beta} h(\xi) + c_1 \lambda \frac{\Gamma(\alpha)}{\rho^\beta \Gamma(\alpha+\beta)}\, \xi^{\rho(\alpha+\beta-1)},$$

which yields

$$c_1 = \frac{1}{\Lambda_1}\big\{\lambda\, {}^\rho I^{\alpha+\beta} h(\xi) - {}^\rho I^\alpha h(T)\big\}.$$

Substituting the value of c_1 in (2.6) we get the equation (2.5). The converse follows by direct computation. □

Now we give two lemmas associated with the nonlinear boundary value problems (1.1) and (1.4), and (1.1) and (1.5). One can obtain the proof of these results by using the arguments employed in Lemma 2.9.

LEMMA 2.10. *For any* $h \in C(0,T) \cap L(0,T)$, $y \in C([0,T],\mathbb{R})$ *such that* ${}^\rho I^{2-\alpha} y \in AC^2_\rho[0,T]$ *and*

$$\Lambda_2 = T^{\rho(\alpha-1)} - \sum_{i=1}^n \lambda_i \xi_i^{\rho(\alpha-1)} \neq 0, \tag{2.8}$$

the solution of the equation (2.4) *with the boundary conditions* (1.4) *is given by*

$$y(t) = {}^\rho I^\alpha h(t) + \frac{t^{\rho(\alpha-1)}}{\Lambda_2}\left\{\sum_{i=1}^n \lambda_i\, {}^\rho I^\alpha h(\xi_i) - {}^\rho I^\alpha h(T)\right\}. \tag{2.9}$$

LEMMA 2.11. *For any* $h \in C(0,T) \cap L(0,T)$, $y \in C([0,T],\mathbb{R})$ *such that* ${}^\rho I^{2-\alpha} y \in AC^2_\rho[0,T]$ *and*

$$\Lambda_3 = T^{\rho(\alpha-1)} - \sum_{i=1}^n \lambda_i \frac{\Gamma(\alpha)}{\rho^\beta \Gamma(\alpha+\beta)}\, \xi_i^{\rho(\alpha+\beta-1)} \neq 0, \tag{2.10}$$

the solution of the equation (2.4) *with the boundary conditions* (1.5) *is given by*

$$y(t) = {}^\rho I^\alpha h(t) + \frac{t^{\rho(\alpha-1)}}{\Lambda_3}\left\{\sum_{i=1}^n \lambda_i\, {}^\rho I^{\alpha+\beta} h(\xi_i) - {}^\rho I^\alpha h(T)\right\}. \tag{2.11}$$

3. Main results for the single valued problems

This section is devoted to the existence of solutions for the fractional differential equation (1.1) supplemented with the boundary conditions (1.3)–(1.5).

3.1. Existence results for the problem (1.1) and (1.3). Here we discuss the existence and uniqueness of solutions for the problem (1.1) and (1.3). In relation to the problem (1.1) and (1.3), we define an operator $\mathbf{F_1}\colon C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ by

$$\mathbf{F_1}(y)(t) = {}^{\rho}I^{\alpha} f(t,y(t)) + \frac{t^{\rho(\alpha-1)}}{\Lambda_1}\big\{\lambda\, {}^{\rho}I^{\alpha+\beta} f(\xi,y(\xi)) - {}^{\rho}I^{\alpha} f(T,y(T))\big\}, \tag{3.1}$$

For computational convenience, we set

$$A_1 = \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{T^{\rho(\alpha-1)}}{|\Lambda_1|}\bigg\{|\lambda|\frac{\xi^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\bigg\}. \tag{3.2}$$

In the first result we establish the uniqueness of solutions for the problem (1.1) and (1.3) via Banach's fixed point theorem.

Theorem 3.1. *Let $f\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the assumption*:

(H_1) $|f(t,x)-f(t,y)| \le \vartheta\|x-y\|$, *for all* $t\in[0,T]$, $\vartheta>0$, $x,y\in\mathbb{R}$.

Then the problem (1.1) *and* (1.3) *has a unique solution on* $[0,T]$ *if*

$$\vartheta A_1 < 1, \tag{3.3}$$

where A_1 is given by (3.2).

Proof. In view of the condition (3.2), consider the set

$$B_r = \{y\in C([0,T],\mathbb{R}) : \|y\|\le r\}$$

with $r > f_0A_1/(1-\vartheta A_1)$, $\sup\limits_{t\in[0,T]}|f(t,0)| = f_0$, and show that $\mathbf{F_1}B_r\subset B_r$, where $\mathbf{F_1}$ is defined by (3.1). For $y\in B_r$, we have

$$\begin{aligned}
|\mathbf{F_1}(y)(t)| &= \bigg|{}^{\rho}I^{\alpha} f(t,y(t)) + \frac{t^{\rho(\alpha-1)}}{\Lambda_1}\big\{\lambda\, {}^{\rho}I^{\alpha+\beta} f(\xi,y(\xi)) - {}^{\rho}I^{\alpha} f(T,y(T))\big\}\bigg| \\
&\le {}^{\rho}I^{\alpha}\big(|f(t,y(t))-f(t,0)| + |f(t,0)|\big) \\
&\quad + \frac{t^{\rho(\alpha-1)}}{|\Lambda_1|}\big\{|\lambda|\, {}^{\rho}I^{\alpha+\beta}\big(|f(\xi,y(\xi))-f(\xi,0)| + |f(\xi,0)|\big) \\
&\quad + {}^{\rho}I^{\alpha}\big(|f(T,y(T))-f(T,0)| + |f(T,0)|\big)\big\} \\
&\le (\vartheta r + f_0)\bigg[\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{T^{\rho(\alpha-1)}}{|\Lambda_1|}\bigg\{|\lambda|\frac{\xi^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} \\
&\quad + \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\bigg\}\bigg] = (\vartheta r + f_0)A_1 < r,
\end{aligned}$$

where A_1 is given by (3.2). For the above inequality, it follows that $\|\mathbf{F_1}y\| < r$, for any $y \in B_r$. Thus $\mathbf{F_1}B_r \subset B_r$. Now, for $y, z \in C([0,T],\mathbb{R})$ and for each $t \in [0,T]$, we obtain

$$\begin{aligned}
|\mathbf{F_1}(y)(t) - \mathbf{F_1}(z)(t)| &\le {}^{\rho}I^{\alpha}|f(t,y(t)) - f(t,z(t))| \\
&\quad + \frac{t^{\rho(\alpha-1)}}{\Lambda_1}\big\{|\lambda|\,{}^{\rho}I^{\alpha+\beta}|f(\xi,y(\xi)) - f(\xi,z(\xi))| + {}^{\rho}I^{\alpha}|f(T,y(T)) - f(T,z(T))|\big\} \\
&\le \vartheta\bigg[\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{T^{\rho(\alpha-1)}}{|\Lambda_1|}\bigg\{|\lambda|\frac{\xi^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\bigg\}\bigg]\|y-z\| \\
&= \vartheta A_1\|y-z\|,
\end{aligned}$$

which, by taking the norm for $t \in [0,T]$, yields

$$\|\mathbf{F_1}(y) - \mathbf{F_1}(z)\| \le \vartheta A_1\|y-z\|. \tag{3.4}$$

By the condition $\vartheta A_1 < 1$, it follows from (3.4) that the operator $\mathbf{F_1}$ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

In the next result, we prove the existence of solutions for the problem (1.1) and (1.3) by applying Leray–Schauder nonlinear alternative.

Lemma 3.2 (Nonlinear alternative for single valued maps, [13]). *Let $\mathcal{S}$ be a closed, convex nonempty subset of a Banach space E, U an open subset of $\mathcal{S}$ and $0 \in U$. Suppose that $\mathcal{F}\colon \overline{U} \to \mathcal{S}$ is a continuous, compact (that is, $\mathcal{F}(\overline{U})$ is a relatively compact subset of $\mathcal{S}$) map. Then, either $\mathcal{F}$ has a fixed point in $\overline{U}$ or there is a $u \in \partial U$ (the boundary of U in $\mathcal{S}$) and $\lambda \in (0,1)$ with $u = \lambda\mathcal{F}(u)$.*

Theorem 3.3. *Assume that:*

- (H_2) *there exist a function $p \in L^1([0,T],\mathbb{R}^+)$ and a nondecreasing function $\Omega\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t,y)| \le p(t)\Omega(\|y\|)$, for all $(t,y) \in [0,T]\times\mathbb{R}$;*
- (H_3) *there exists a constant $K > 0$ such that*

$$\frac{K}{\Omega(K)\Big({}^{\rho}I^{\alpha}p(T) + \dfrac{T^{\rho(\alpha-1)}}{|\Lambda_1|}\big\{|\lambda|^{\rho}I^{\alpha+\beta}p(\xi) +^{\rho} I^{\alpha}p(T)\big\}\Big)} > 1.$$

Then the problem (1.1) and (1.3) has at least one solution on $[0,T]$.

Proof. First of all, we show that the operator $\mathbf{F_1}\colon C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ defined by (3.1) is continuous and completely continuous. This will be established in several steps.

Step 1. $\mathbf{F_1}$ is continuous. Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([0,T],\mathbb{R})$. Then

$$
\begin{aligned}
|\mathbf{F_1}(y_n)(t)-\mathbf{F_1}(y)(t)| &\leq {}^{\rho}I^{\alpha}|f(t,y_n(t))-f(t,y(t))| \\
&+\frac{t^{\rho(\alpha-1)}}{\Lambda_1}\big\{|\lambda|\,{}^{\rho}I^{\alpha+\beta}|f(\xi,y_n(\xi))-f(\xi,y(\xi))|+{}^{\rho}I^{\alpha}|f(T,y_n(T))-f(T,y(T))|\big\} \\
&\leq A_1\|f(\,\cdot\,,y_n)-f(\,\cdot\,,y)\|.
\end{aligned}
$$

Since f is continuous functions, therefore, we have

$$
\|\mathbf{F_1}(y_n)-\mathbf{F_1}(y)\|\leq A_1\|f(\,\cdot\,,y_n)-f(\,\cdot\,,y)\|\to 0, \quad \text{as } n\to\infty.
$$

Step 2. $\mathbf{F_1}$ *maps bounded sets into bounded sets in* $C([0,T],\mathbb{R})$. Indeed, it is enough to show that for any $\mu>0$ there exists a positive constant ℓ such that for $y\in B_\mu=\{y\in C([0,T],\mathbb{R}):\|y\|\leq\mu\}$, we have $\|\mathbf{F_1}(y)\|\leq\ell$. By ($H_2$), for each $t\in[0,T]$, we have

$$
\begin{aligned}
|\mathbf{F_1}(y)(t)| &\leq {}^{\rho}I^{\alpha}|f(t,y(t))|+\frac{t^{\rho(\alpha-1)}}{|\Lambda_1|}\big\{|\lambda|\,{}^{\rho}I^{\alpha+\beta}|f(\xi,y(\xi))|+{}^{\rho}I^{\alpha}|f(T,y(T))|\big\} \\
&\leq {}^{\rho}I^{\alpha}p(T)\Omega(\|y\|)+\frac{t^{\rho(\alpha-1)}}{|\Lambda_1|}\big\{|\lambda|\,{}^{\rho}I^{\alpha+\beta}p(\xi)\Omega(\|y\|)+{}^{\rho}I^{\alpha}p(T)\Omega(\|y\|)\big\} \\
&\leq \Omega(\|y\|)\bigg({}^{\rho}I^{\alpha}p(T)+\frac{T^{\rho(\alpha-1)}}{|\Lambda_1|}\big\{|\lambda|\,{}^{\rho}I^{\alpha+\beta}p(\xi)+{}^{\rho}I^{\alpha}p(T)\big\}\bigg),
\end{aligned}
$$

Thus

$$
\|\mathbf{F_1}(y)\|\leq\Omega(\|\mu\|)\bigg({}^{\rho}I^{\alpha}p(T)+\frac{T^{\rho(\alpha-1)}}{|\Lambda_1|}\big\{|\lambda|\,{}^{\rho}I^{\alpha+\beta}p(\xi)+{}^{\rho}I^{\alpha}p(T)\big\}\bigg):=\ell.
$$

Step 3. F *maps bounded sets into equicontinuous sets of* $C([0,T],\mathbb{R})$. Let $0<t_1<t_2<T$, B_μ be a bounded set of $C([0,T],\mathbb{R})$ as in Step 2, and let $y\in B_\mu$. Then

$$
\begin{aligned}
|\mathbf{F}_1(y)(t_2) &- \mathbf{F}_1(y)(t_1)| \\
\leq& \big|{}^{\rho}I^{\alpha}|f(t_2,y(t_2))|-{}^{\rho}I^{\alpha}|f(t_1,y(t_1))|\big| \\
&+\frac{\Omega(\mu)|t_2^{\rho(\alpha-1)}-t_1^{\rho(\alpha-1)}|}{|\Lambda_1|}\big\{|\lambda|\,{}^{\rho}I^{\alpha+\beta}p(\xi)+{}^{\rho}I^{\alpha}p(T)\big\} \\
\leq& \frac{\rho^{1-\alpha}\Omega(\mu)}{\Gamma(\alpha)}\bigg|\int_0^{t_1}\bigg[\frac{s^{\rho-1}}{(t_2^{\rho}-s^{\rho})^{1-\alpha}}-\frac{s^{\rho-1}}{(t_1^{\rho}-s^{\rho})^{1-\alpha}}\bigg]p(s)\,ds \\
&+\int_{t_1}^{t_2}\frac{s^{\rho-1}}{(t_2^{\rho}-s^{\rho})^{1-\alpha}}p(s)\,ds\bigg| \\
&+\frac{\Omega(\mu)\big|t_2^{\rho(\alpha-1)}-t_1^{\rho(\alpha-1)}\big|}{|\Lambda_1|}\big\{|\lambda|\,{}^{\rho}I^{\alpha+\beta}p(\xi)+{}^{\rho}I^{\alpha}p(T)\big\}.
\end{aligned}
$$

As $t_1\to t_2$, the right-hand side of the above inequality is independent of y and tends to zero. In view of the foregoing three steps, the Arzelá–Ascoli theorem applies. In consequence, we conclude that $\mathbf{F}_1\colon C([0,T],\mathbb{R})\to C([0,T],\mathbb{R})$ is completely continuous.

Step 4. We show that there exists an open set $U \subseteq C([0,T],\mathbb{R})$ with $y \neq \nu \mathbf{F}_1(y)$ for $\nu \in (0,1)$ and $y \in \partial U$. Let $y \in C([0,T],\mathbb{R})$ be a solution of $y = \nu \mathbf{F_1} y$ for $\nu \in [0,1]$. Then, for $t \in [0,T]$, we have

$$\begin{aligned}
|y(t)| &= |\nu(\mathbf{F_1}y)(t)| \\
&\leq {}^{\rho}I^{\alpha}|f(t,y(t))| + \frac{t^{\rho(\alpha-1)}}{|\Lambda_1|}\big\{|\lambda|\,{}^{\rho}I^{\alpha+\beta}|f(\xi,y(\xi))| + {}^{\rho}I^{\alpha}|f(T,y(T))|\big\} \\
&\leq {}^{\rho}I^{\alpha}p(T)\Omega(\|y\|) + \frac{t^{\rho(\alpha-1)}}{|\Lambda_1|}\big\{|\lambda|\,{}^{\rho}I^{\alpha+\beta}p(\xi)\Omega(\|y\|) + {}^{\rho}I^{\alpha}p(T)\Omega(\|y\|)\big\} \\
&\leq \Omega(\|y\|)\bigg({}^{\rho}I^{\alpha}p(T) + \frac{T^{\rho(\alpha-1)}}{|\Lambda_1|}\big\{|\lambda|\,{}^{\rho}I^{\alpha+\beta}p(\xi) + {}^{\rho}I^{\alpha}p(T)\big\}\bigg),
\end{aligned}$$

which, by taking the norm for $t \in [0,T]$, implies that

$$\frac{\|y\|}{\Omega(\|y\|)\bigg({}^{\rho}I^{\alpha}p(T) + \dfrac{T^{\rho(\alpha-1)}}{|\Lambda_1|}\big\{|\lambda|\ {}^{\rho}I^{\alpha+\beta}p(\xi) + {}^{\rho}I^{\alpha}p(T)\big\}\bigg)} \leq 1.$$

In view of (H_3), there exists K such that $\|y\| \neq \mathrm{K}$. Let us set

$$U = \{y \in C([0,T],\mathbb{R}) : \|y\| < \mathrm{K}\}.$$

Note that the operator $\mathbf{F_1} : \overline{U} \to C([0,T],\mathbb{R})$ is continuous and completely continuous.

From the choice of U, there is no $y \in \partial U$ such that $y = \nu \mathbf{F_1}(y)$ for some $\nu \in (0,1)$. Consequently, by the nonlinear alternative of Leray–Schauder type (Lemma 3.2), we deduce that $\mathbf{F_1}$ has a fixed point $y \in \overline{U}$ which is a solution of the problem (1.1) and (1.3). $\square$

Remark 3.4. The condition (H_2) can be modified by assuming $p \in C([0,T],\mathbb{R}^+)$ instead of $p \in L^1([0,T],\mathbb{R}^+)$. In this case, the constant K in condition (H_3) will take the form:

$$\frac{K}{\Omega(K)\|p\|A_1} > 1.$$

As a special case, when $p(t) = 1$ and $\Omega(\|y\|) = d_1\|y\| + d_2$, we have the following corollary.

Corollary 3.5. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(H_4) *there exist constants $0 \leq d_1 < 1/A_1$ and $d_2 > 0$ such that*

$$|f(t,y)| \leq d_1|y| + d_2 \quad \textit{for all } t \in [0,T],\ y \in \mathbb{R}.$$

Then the problem (1.1) *and* (1.3) *has at least one solution on* $[0,T]$.

In the last result, we prove the existence of solutions for the problem (1.1) and (1.3) by applying Krasnosel'skiĭ's fixed point theorem.

LEMMA 3.6 (Krasnosel'skiĭ's fixed point theorem, [19]). *Let M be a closed convex and nonempty subset of a Banach space X. Let A_1, A_2 be the operators such that*

(a) *$A_1x + A_2y \in M$ whenever $x, y \in M$,*
(b) *A_1 is compact and continuous, and A_2 is a contraction mapping.*

Then there exists $w \in M$ such that $w = A_1w + A_2w$.

THEOREM 3.7. *Let $f\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function, and the assumption* (H_1) *holds. In addition we assume that*

(H_5) *$|f(t,y)| \le \theta(t)$, for all $(t,y) \in [0,T] \times \mathbb{R}$ and $\theta \in C([0,T],\mathbb{R})$.*

Then the problem (1.1) *and* (1.3) *has at least one solution on* $[0,T]$, *provided that*

$$\vartheta \frac{T^{\rho(\alpha-1)}}{|\Lambda_1|}\left\{|\lambda|\frac{\xi^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\right\} < 1. \tag{3.5}$$

PROOF. Letting $\sup_{t\in[0,T]} |\theta(t)| = \|\theta\|$ and $\overline{r} \ge A_1\|\theta\|$, we define $B_{\overline{r}} = \{y \in C([0,T],\mathbb{R}) : \|y\| \le \overline{r}\}$. Introduce operators $\mathcal{P}$ and $\mathcal{Q}$ on $B_{\overline{r}}$ as

$$\begin{aligned}
\mathcal{P}(y)(t) &= {}^{\rho}I^{\alpha} f(s,y(s))(t),\\
\mathcal{Q}(y)(t) &= \frac{t^{\rho(\alpha-1)}}{\Lambda_1}\big\{\lambda\, {}^{\rho}I^{\alpha+\beta} f(s,y(s))(\xi) - {}^{\rho}I^{\alpha} f(s,y(s))(T)\big\}.
\end{aligned}$$

Note that $\mathbf{F_1} = \mathcal{P} + \mathcal{Q}$. For $y, z \in B_{\overline{r}}$, it easy to find that

$$\begin{aligned}
\|\mathcal{P}y + \mathcal{Q}z\| \le \|\theta\| \bigg[&\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\\
&+ \frac{T^{\rho(\alpha-1)}}{|\Lambda_1|}\left\{|\lambda|\frac{\xi^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\right\}\bigg] \le \overline{r}.
\end{aligned}$$

Thus, $\mathcal{P}y + \mathcal{Q}z \in B_{\overline{r}}$.

It follows from the assumptions (H_1) and (3.5) that $\mathcal{Q}$ is a contraction mapping, that is, for $y, z \in C([0,T],\mathbb{R})$ and for each $t \in [0,T]$, we have

$$\begin{aligned}
&\|\mathcal{Q}(y)(t) - \mathcal{Q}(z)(t)\|\\
&\quad\le \sup_{t\in[0,T]}\bigg\{\frac{t^{\rho(\alpha-1)}}{\Lambda_1}\big(|\lambda|\, {}^{\rho}I^{\alpha+\beta}|f(\xi,y(\xi)) - f(\xi,z(\xi))|\\
&\qquad + {}^{\rho}I^{\alpha}|f(T,y(T)) - f(T,z(T))|\big)\bigg\}\\
&\quad\le \vartheta\frac{T^{\rho(\alpha-1)}}{|\Lambda_1|}\left\{|\lambda|\frac{\xi^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\right\}\|y - z\|.
\end{aligned}$$

Continuity of f implies that the operator $\mathcal{P}$ is continuous. Also, $\mathcal{P}$ is uniformly bounded on $B_{\overline{r}}$ as

$$\|\mathcal{P}x\| \le \|\theta\|\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}.$$

Finally, we prove the compactness of the operator $\mathcal{P}$. For that we define

$$\sup_{(t,y)\in[0,T]\times B_{\overline{r}}} |f(t,y)| = \overline{f} < \infty.$$

Then, for $0 < t_1 < t_2 < T$, we have

$$\begin{aligned}
&|\mathcal{P}y(t_2) - \mathcal{P}y(t_1)| \\
&\quad \le \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\bigg|\int_0^{t_1}\bigg[\frac{s^{\rho-1}}{(t_2^\rho - s^\rho)^{1-\alpha}} - \frac{s^{\rho-1}}{(t_1^\rho - s^\rho)^{1-\alpha}}\bigg] f(s,y(s))\,ds \\
&\qquad + \int_{t_1}^{t_2}\frac{s^{\rho-1}}{(t_2^\rho - s^\rho)^{1-\alpha}} f(s,y(s))\,ds\bigg| \\
&\quad \le \frac{\rho^{1-\alpha}\overline{f}}{\Gamma(\alpha)}\bigg|\int_0^{t_1}\bigg[\frac{s^{\rho-1}}{(t_2^\rho - s^\rho)^{1-\alpha}} - \frac{s^{\rho-1}}{(t_1^\rho - s^\rho)^{1-\alpha}}\bigg]\,ds + \int_{t_1}^{t_2}\frac{s^{\rho-1}}{(t_2^\rho - s^\rho)^{1-\alpha}}\,ds\bigg| \\
&\quad \le \frac{\overline{f}}{\rho^\alpha\Gamma(\alpha+1)}\big|2(t_2^\rho - t_1^\rho)^\alpha + t_1^{\rho\alpha} - t_2^{\rho\alpha}\big|,
\end{aligned}$$

which is independent of y and tends to zero as $t_2 - t_1 \to 0$. Thus, $\mathcal{P}$ is equicontinuous. So $\mathcal{P}$ is relatively compact on $B_{\overline{r}}$. Hence, by the Arzelá–Ascoli theorem, $\mathcal{P}$ is compact on $B_{\overline{r}}$. Thus all the assumptions of Theorem 3.7 are satisfied. Hence, by Lemma 3.6, we deduce that the problem (1.1) and (1.3) has at least one solution on $[0,T]$. $\square$

EXAMPLE 3.8. Consider the following boundary value problem

$$\begin{cases} {}^{1/2}D^{7/5}y(t) = \dfrac{1}{20(t+1)}\bigg(\dfrac{|y|+2}{|y|+1} + \cos t\bigg), & t \in [0,2], \\[2ex] y(0) = 0, \quad y(2) = 1/3\,{}^{1/2}I^{4/5}y(3/2), \end{cases} \tag{3.6}$$

where $\rho = 1/2$, $\alpha = 7/5$, $\lambda = 1/3$, $\beta = 4/5$, $\xi = 3/2$ and $T = 2$. Using the given data, we find that $|\Lambda_1| = 0.5526156533$, $A_1 = 12.67699740$, where Λ_1 and A_1 are respectively given by (2.3) and (3.2). One can easily check that $f(t,y)$ is continuous and the condition (H_1) of Theorem 3.1 holds true with $\vartheta = 1/20$. Also $\vartheta A_1 \approx 0.63384987 < 1$. Thus all the conditions of Theorem 3.1 are satisfied. So, by the conclusion of theorem 3.1, the problem (3.6) has a unique solution on $[0,2]$.

Further, the hypothesis of Theorem 3.7 is satisfied with

$$\theta(t) = \frac{2+\cos t}{20(1+t)}.$$

Also

$$\vartheta\,\frac{T^{\rho(\alpha-1)}}{|\Lambda_1|}\bigg\{|\lambda|\frac{\xi^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\bigg\} \approx 0.4612850793 < 1.$$

Thus the conclusion of Theorem 3.7 applies to the problem (3.6).

3.2. Existence results for the problems (1.1) with (1.4) and (1.5). We introduce an operator $\mathbf{F_2}\colon C([0,1],\mathbb{R}) \to C([0,T],\mathbb{R})$, associated with the problem (1.1) and (1.4), defined as

$$\begin{aligned}(3.7)\quad \mathbf{F_2}(y)(t) &= {}^{\rho}I^{\alpha} f(t,y(t)) \\ &\quad + \frac{t^{\rho(\alpha-1)}}{\Lambda_2}\bigg\{\sum_{i=1}^{n}\lambda_i\, {}^{\rho}I^{\alpha} f(\xi_i,y(\xi_i)) - {}^{\rho}I^{\alpha} f(T,y(T))\bigg\},\end{aligned}$$

where Λ_2 is defined by (2.8). Furthermore, we set

$$(3.8)\qquad A_2 = \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{T^{\rho(\alpha-1)}}{|\Lambda_2|}\bigg\{\sum_{i=1}^{n}|\lambda_i|\frac{\xi_i^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\bigg\}.$$

The existence results for the problem (1.1) and (1.4), similar to ones for the problem (1.1) and (1.3) obtained in Section 3, can be proved with the aid of the operator $\mathbf{F_2}$ and the constant A_2 given by (3.7) and (3.8) respectively. So, we formulate the results without proof.

Theorem 3.9. *Suppose that the condition* (H$_1$) *holds. Then the problem* (1.1) *and* (1.4) *has a unique solution on* $[0,T]$ *if* $\vartheta A_2 < 1$, *where* A_2 *is given by* (3.8).

Theorem 3.10. *Assume that* (H$_2$) *and the following condition hold*:

(H$_6$) *there exists a constant* $K > 0$ *such that*

$$\frac{K}{\Omega(K)\bigg({}^{\rho}I^{\alpha}p(T) + \dfrac{T^{\rho(\alpha-1)}}{|\Lambda_2|}\bigg\{\displaystyle\sum_{i=1}^{n}|\lambda_i|{}^{\rho}I^{\alpha}p(\xi_i) + {}^{\rho}I^{\alpha}p(T)\bigg\}\bigg)} > 1.$$

Then the problem (1.1) *and* (1.4) *has at least one solution on* $[0,T]$.

Theorem 3.11. *Assume that the conditions* (H$_1$) *and* (H$_5$) *hold. Then the problem* (1.1) *and* (1.4) *has at least one solution on* $[0,T]$, *provided that*

$$(3.9)\qquad \vartheta\,\frac{T^{\rho(\alpha-1)}}{|\Lambda_1|}\bigg\{\sum_{i=1}^{n}|\lambda_i|\frac{\xi_i^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\bigg\} < 1.$$

Next, for the problem (1.1) and (1.5), we define an operator $\mathbf{F_3}\colon C([0,1],\mathbb{R}) \to C([0,T],\mathbb{R})$ by

$$\begin{aligned}(3.10)\qquad \mathbf{F_3}(y)(t) &= {}^{\rho}I^{\alpha} f(t,y(t)) \\ &\quad + \frac{t^{\rho(\alpha-1)}}{\Lambda_3}\bigg\{\sum_{i=1}^{n}\lambda_i\, {}^{\rho}I^{\alpha+\beta} f(\xi_i,y(\xi_i)) - {}^{\rho}I^{\alpha} f(T,y(T))\bigg\}\end{aligned}$$

where Λ_3 is defined by (2.10). Let us set

$$(3.11)\quad A_3 = \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{T^{\rho(\alpha-1)}}{|\Lambda_3|}\bigg\{\sum_{i=1}^{n}|\lambda_i|\frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\bigg\}.$$

The existence results for the problem (1.1) and (1.5) can be obtained with the help of the operator $\mathbf{F_3}$ and the constant A_3. As before, we only state these results.

THEOREM 3.12. *Suppose that the condition* (H_1) *holds. Then the problem* (1.1) *and* (1.5) *has a unique solution on* $[0,T]$ *if* $\vartheta A_3 < 1$, *where* A_3 *is given by* (3.11).

THEOREM 3.13. *Assume that* (H_2) *and the following condition hold*:

(H_7) *there exists a constant* $K > 0$ *such that*

$$\frac{K}{\Omega(K)\bigg({}^\rho I^\alpha p(T) + \dfrac{T^{\rho(\alpha-1)}}{|\Lambda_3|}\bigg\{\displaystyle\sum_{i=1}^{n}|\lambda_i|{}^\rho I^{\alpha+\beta}p(\xi_i) +{}^\rho I^\alpha p(T)\bigg\}\bigg)} > 1.$$

Then the problem (1.1) *and* (1.5) *has at least one solution on* $[0,T]$.

THEOREM 3.14. *Assume that the conditions* (H_1) *and* (H_5) *hold. Then the problem* (1.1) *and* (1.5) *has at least one solution on* $[0,T]$, *provided that*

$$\vartheta\,\frac{T^{\rho(\alpha-1)}}{|\Lambda_3|}\bigg\{\sum_{i=1}^{n}|\lambda_i|\frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\bigg\} < 1. \tag{3.12}$$

EXAMPLE 3.15. Let us consider the following boundary value problem

$$\begin{cases} {}^{1/2}D^{7/5}y(t) = f(t,y(t)), \quad t\in[0,2],\\ y(0)=0, \quad y(2) = y(3/4) + 1/3\,y(3/2), \end{cases} \tag{3.13}$$

where $\rho = 1/2$, $\alpha = 7/5$, $\lambda_1 = 1$, $\lambda_2 = 1/3$, $\xi_1 = 3/4$, $\xi_2 = 3/2$ and $T = 2$ and $f(t,y(t))$ will be fixed later.

Using the given values, we find that $|\Lambda_2| = 0.1568797472$, $A_2 = 48.32814774$, where Λ_2 and A_2 are respectively given by (2.8) and (3.8). For illustrating Theorem 3.9, we take

$$f(t,y) = \frac{1}{3\sqrt{900+t^2}}\,e^{-t}\big(\tan^{-1}y + \cos t\big). \tag{3.14}$$

Notice that $f(t,y)$ is continuous and satisfies the condition (H_1) with $\vartheta = 1/90$. Also $\vartheta A_2 \approx 0.5369794193 < 1$. Thus all the conditions of Theorem 3.9 are satisfied. Hence the conclusion of Theorem 3.9 applies to the problem (3.13) with $f(t,y)$ given by (3.14).

In order to illustrate Theorem 3.10, we take

$$f(t,y(t)) = \frac{(1+t)}{90}\left(\tan^{-1}y + y + \frac{1}{8}\right), \tag{3.15}$$

where $p(t) = (1+t)/90$ and $\psi(||y||) = ||y|| + (4\pi+1)/8$. By condition (H_6), we have $K > 5.021036057$. Thus all the conditions of Theorem 3.10 are satisfied and consequently the problem (3.13) with $f(t,y)$ given by (3.15) has at least one solution on $[0,2]$.

Example 3.16. Consider the following boundary value problem

$$\begin{cases} {}^{1/2}D^{7/5}y(t) = f(t,y(t)), \quad t \in [0,2], \\ y(0) = 0, \quad y(2) = {}^{1/2}I^{4/5}y(3/4) + 1/3\,{}^{1/2}I^{4/5}y(3/2), \end{cases} \tag{3.16}$$

where $\rho = 1/2$, $\alpha = 7/5$, $\lambda_1 = 1$, $\lambda_2 = 1/3$, $\beta = 4/5$, $\xi_1 = 3/4$, $\xi_2 = 3/2$ and $T = 2$ and $f(t,y(t))$ will be defined later.

Using the given data, we find that $|\Lambda_3| = 0.6271881077$, $A_3 = 14.11003320$, where Λ_3 and A_3 are respectively given by (2.10) and (3.11). For illustrating Theorem 3.12, we take

$$f(t,y(t)) = \frac{1}{20(t+1)}\left(\frac{|y|+2}{|y|+1} + \cos t\right). \tag{3.17}$$

Observe that $f(t,y)$ is continuous and satisfies the condition (H_1) with $\vartheta = 1/20$. Also $\vartheta A_3 \approx 0.70550166 < 1$. Thus all the conditions of Theorem 3.12 are satisfied and hence its conclusion applies to the problem (3.16) with $f(t,y)$ given by (3.17).

Next we illustrate Theorem 3.14 by taking $f(t,y)$ given by (3.17). Clearly, the hypothesis of Theorem 3.14 holds true with $\theta(t) = (2+\cos t)/(20(1+t))$. Also

$$\vartheta\,\frac{T^{\rho(\alpha-1)}}{|\Lambda_3|}\left\{\sum_{i=1}^{n}|\lambda_i|\frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\right\} \approx 0.5329368695 < 1.$$

Hence we deduce by Theorem 3.14 that there exist at least one solution for the problem (3.16) with $f(t,y)$ given by (3.17).

4. Main results for the multivalued problem (1.2) and (1.3)

This section is devoted to the existence of solutions for the problem (1.2) and (1.3). The existence of solutions for the problems (1.2) and (1.4), and (1.2) and (1.5) can be studied by employing the strategy used in this section. Let us first turn to some preliminary concepts of multivalued analysis [11], [14].

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{\mathrm{cl}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_{\mathrm{b}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{\mathrm{cl,b}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed and bounded}\}$, $\mathcal{P}_{\mathrm{cp}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{\mathrm{cp,c}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

A multi-valued map $G\colon X \to \mathcal{P}(X)$ is

(a) *convex* (*closed*) *valued* if $G(x)$ is convex (closed) for all $x \in X$.

(b) *bounded* on bounded sets if $G(Y) = \bigcup_{x\in Y} G(x)$ is bounded in X for all $Y \in \mathcal{P}_b(X)$ $\left(\text{i.e. } \sup_{x\in Y}\{\sup\{|y| : y \in G(x)\}\} < \infty\right)$.

(c) *upper semi-continuous* (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$.

(d) *completely continuous* if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e. $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$.

(e) *measurable* if for every $y \in X$, the function

$$t \mapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Recall that G *has a fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$.

4.1. The Carathéodory case. In this subsection, we consider the case when F has convex values and is of Carathéodory type, and prove an existence result for the problem (1.2) and (1.3) by applying nonlinear alternative of Leray-Schauder type.

Definition 4.1. A multivalued map $F\colon J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be *Carathéodory* if

(a) $t \mapsto F(t, y)$ is measurable for each $x \in \mathbb{R}$;

(b) $y \mapsto F(t, y)$ is upper semicontinuous for almost all $t \in J$.

Further, a Carathéodory function F is called L^1-*Carathéodory* if

(c) for each $\beta > 0$, there exists $\varphi_\beta \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, y)\| = \sup\{|v| : v \in F(t, y)\} \le \varphi_\beta(t)$$

for all $\|y\| \le \beta$ and for almost every $t \in J$.

For each $y \in C([0, T], \mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \big\{v \in L^1([0, T], \mathbb{R}) : v(t) \in F(t, y(t)) \text{ on } [0, T]\big\}.$$

We define the graph of G to be the set $\mathrm{Gr}(G) = \{(x, y) \in X \times Y,\ y \in G(x)\}$ and state a known result for closed graphs and upper-semicontinuity.

Lemma 4.2 ([11, Proposition 1.2]). *If* $G\colon X \to \mathcal{P}_{\mathrm{cl}}(Y)$ *is u.s.c., then* $\mathrm{Gr}(G)$ *is a closed subset of* $X \times Y$, *i.e. for every sequence* $\{x_n\}_{n\in\mathbb{N}} \subset X$ *and* $\{y_n\}_{n\in\mathbb{N}} \subset Y$, *if* $x_n \to x_*$, $y_n \to y_*$ *as* $n \to \infty$ *and* $y_n \in G(x_n)$, *then* $y_* \in G(x_*)$. *Conversely, if G is completely continuous and has a closed graph, then it is upper semicontinuous.*

We also need the following lemmas in the sequel.

Lemma 4.3 ([21]). *Let X be a separable Banach space. Let $F\colon J\times\mathbb{R}\to\mathcal{P}_{\mathrm{cp,c}}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then the operator*

$$\Theta\circ S_F\colon C(J,X)\to\mathcal{P}_{\mathrm{cp,c}}(C(J,X)),\qquad x\mapsto(\Theta\circ S_F)(x)=\Theta(S_{F,x})$$

is a closed graph operator in $C(J,X)\times C(J,X)$.

Lemma 4.4 (Nonlinear alternative for Kakutani maps, [13]). *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0\in U$. Suppose that $F\colon\overline{U}\to\mathcal{P}_{\mathrm{cp,c}}(C)$ is a upper semicontinuous compact map. Then either*

(a) *F has a fixed point in $\overline{U}$, or*
(b) *there is a $u\in\partial U$ and $\lambda\in(0,1)$ with $u\in\lambda F(u)$.*

Definition 4.5. A function $y\in C([0,T],\mathbb{R})$ is said to be a solution of the boundary value problem (1.2)–(1.3) if $y(0)=0, y(T)=\lambda\,{}^\rho I^\beta y(\xi)$ and there exists function $v\in L^1([0,1],\mathbb{R})$ such that $v(t)\in F(t,y(t))$ almost everywhere on $[0,T]$ and

$$y(t)={}^\rho I^\alpha v(t)+\frac{t^{\rho(\alpha-1)}}{\Lambda_1}\big\{\lambda\,{}^\rho I^{\alpha+\beta}v(\xi)-{}^\rho I^\alpha v(T)\big\}.$$

Theorem 4.6. *Assume that*

(B_1) *$F\colon[0,T]\times\mathbb{R}\to\mathcal{P}_{\mathrm{cp,c}}(\mathbb{R})$ is L^1-Carathéodory,*
(B_2) *there exists a continuous nondecreasing function $\Phi\colon[0,\infty)\to(0,\infty)$ and a function $p\in L^1([0,T],\mathbb{R}^+)$ such that*

$$\|F(t,y)\|_{\mathcal{P}}:=\sup\{|x|:x\in F(t,y)\}\le p(t)\Phi(\|y\|)\quad \textit{for each } (t,y)\in[0,T]\times\mathbb{R},$$

(B_3) *there exists a constant $\widehat{K}>0$ such that*

$$\frac{\widehat{K}}{\Phi(\widehat{K})\bigg({}^\rho I^\alpha p(T)+\dfrac{T^{\rho(\alpha-1)}}{\Lambda_1}\big(|\lambda|\,{}^\rho I^{\alpha+\beta}p(\xi)+{}^\rho I^\alpha p(T)\big)\bigg)}>1.$$

Then the problem (1.2) *and* (1.3) *has at least one solution on* $[0,T]$.

Proof. To transform the problem (1.2) and (1.3) into a fixed point problem, we define an operator $\mathcal{F}\colon C([0,T],\mathbb{R})\to\mathcal{P}(C([0,T],\mathbb{R}))$ by

$$\mathcal{F}(y)=\{h\in C([0,T],\mathbb{R}):h(t)=N(y)(t)\},\tag{4.1}$$

where

$$N(y)(t)={}^\rho I^\alpha v(t)+\frac{t^{\rho(\alpha-1)}}{\Lambda_1}\big\{\lambda\,{}^\rho I^{\alpha+\beta}v(\xi)+{}^\rho I^\alpha v(T)\big\},$$

for $v\in S_{F,x}$. It is obvious that the fixed points of $\mathcal{F}$ are solutions of the problem (1.2) and (1.3).

We will show that $\mathcal{F}$ satisfies the assumptions of Leray-Schauder nonlinear alternative (Lemma 4.4) in several steps.

Step 1. $\mathcal{F}(y)$ is convex for each $y \in C([0,T],\mathbb{R})$. This step is obvious since $S_{F,y}$ is convex (F has convex values).

Step 2. $\mathcal{F}$ maps bounded sets (balls) into bounded sets in $C([0,T],\mathbb{R})$. For a positive number r, let $B_r = \{y \in C([0,T],\mathbb{R}) : \|y\| \le r\}$ be a bounded ball in $C([0,T],\mathbb{R})$. Then, for each $h \in \mathcal{F}(y)$, $y \in B_r$, there exists $v \in S_{F,y}$ such that

$$h(t) = {}^\rho I^\alpha v(t) + \frac{t^{\rho(\alpha-1)}}{\Lambda_1}\big\{\lambda\, {}^\rho I^{\alpha+\beta} v(\xi) + {}^\rho I^\alpha v(T)\big\}.$$

Then, for $t \in [0,T]$, we have

$$\begin{aligned}
|h(t)| &\le {}^\rho I^\alpha |v(T)| + \frac{t^{\rho(\alpha-1)}}{|\Lambda_1|}\big\{|\lambda|\, {}^\rho I^{\alpha+\beta}|v(\xi)| + {}^\rho I^\alpha |v(T)|\big\}\\
&\le \Phi(\|y\|)\bigg({}^\rho I^\alpha p(T) + \frac{T^{\rho(\alpha-1)}}{|\Lambda_1|}\big\{|\lambda|\ {}^\rho I^{\alpha+\beta} p(\xi) + {}^\rho I^\alpha p(T)\big\}\bigg).
\end{aligned}$$

Thus

$$\|h\| \le \Phi(r)\bigg({}^\rho I^\alpha p(T) + \frac{T^{\rho(\alpha-1)}}{|\Lambda_1|}\big\{|\lambda|\, {}^\rho I^{\alpha+\beta} p(\xi) + {}^\rho I^\alpha p(T)\big\}\bigg) := \ell.$$

Step 3. $\mathcal{F}$ maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$. Let $t_1, t_2 \in (0,T]$, $t_1 < t_2$ and let $y \in B_r$. Then

$$\begin{aligned}
|h(t_2) - h(t_1)| &\le \big|{}^\rho I^\alpha |v(t_2)| - {}^\rho I^\alpha |v(t_1)|\big|\\
&\qquad + \frac{\Phi(r)\big|t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}\big|}{\Lambda_1}\big\{|\lambda| {}^\rho I^{\alpha+\beta} p(\xi) + {}^\rho I^\alpha p(T)\big\}\\
&\le \frac{\rho^{1-\alpha}\Phi(r)}{\Gamma(\alpha)}\bigg|\int_0^{t_1}\bigg[\frac{s^{\rho-1}}{(t_2^\rho - s^\rho)^{1-\alpha}} - \frac{s^{\rho-1}}{(t_1^\rho - s^\rho)^{1-\alpha}}\bigg]p(s)\,ds\\
&\qquad + \int_{t_1}^{t_2}\frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}}p(s)\,ds\bigg|\\
&\qquad + \frac{\Phi(r)|t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}|}{\Lambda_1}\big\{|\lambda| {}^\rho I^{\alpha+\beta} p(\xi) + {}^\rho I^\alpha p(T)\big\}.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $y \in B_r$ as $t_2 - t_1 \to 0$. Therefore it follows by the Arzelá–Ascoli theorem that $\mathcal{F}\colon C([0,T],\mathbb{R}) \to \mathcal{P}(C([0,T],\mathbb{R}))$ is completely continuous.

In our next step, we show that $\mathcal{F}$ is u.s.c. Since $\mathcal{F}$ is completely continuous, it is enough to establish that it has a closed graph.

Step 4. $\mathcal{F}$ has a closed graph. Let $y_n \to y_*$, $h_n \in \mathcal{F}(y_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{F}(y_*)$. Associated with $h_n \in \mathcal{F}(y_n)$, there exists $v_n \in S_{F,y_n}$ such that, for each $t \in [0,T]$,

$$h_n(t) = {}^\rho I^\alpha v_n(t) + \frac{t^{\rho(\alpha-1)}}{\Lambda_1}\big\{\lambda\, {}^\rho I^{\alpha+\beta} v_n(\xi) - {}^\rho I^\alpha v_n(s) v_n(T)\big\}.$$

Thus it suffices to show that there exists $v_* \in S_{F,y_*}$ such that, for each $t \in [0,T]$,

$$h_*(t) = {}^\rho I^\alpha v_*(t) + \frac{t^{\rho(\alpha-1)}}{\Lambda_1}\big\{\lambda\, {}^\rho I^{\alpha+\beta} v_*(\xi) - {}^\rho I^\alpha v_*(T)\big\}.$$

Let us consider the linear operator $\Theta\colon L^1([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ given by

$$v \mapsto \Theta v(t) = {}^\rho I^\alpha v(t) + \frac{t^{\rho(\alpha-1)}}{\Lambda_1}\big\{\lambda\, {}^\rho I^{\alpha+\beta} v(\xi) - {}^\rho I^\alpha v(T)\big\}.$$

Observe that $\|h_n(t) - h_*(t)\| \to 0$ as $n \to \infty$, so it follows by Lemma 4.3 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,y_n})$. Since $y_n \to y_*$, therefore, we have

$$h_*(t) = {}^\rho I^\alpha v_*(t) + \frac{t^{\rho(\alpha-1)}}{\Lambda_1}\big\{\lambda\, {}^\rho I^{\alpha+\beta} v_*(\xi) - {}^\rho I^\alpha v_*(T)\big\},$$

for some $v_* \in S_{F,y_*}$.

Step 5. We show that there exists an open set $U \subseteq C([0,T],\mathbb{R})$ with $y \notin \nu\mathcal{F}(y)$ for any $\nu \in (0,1)$ and all $y \in \partial U$. Let $\nu \in (0,1)$ and $y \in \nu\mathcal{F}(y)$. Then, there exists $v \in L^1([0,T],\mathbb{R})$ with $v \in S_{F,y}$ such that, for $t \in [0,T]$, we have

$$y(t) = {}^\rho I^\alpha v(t) + \frac{t^{\rho(\alpha-1)}}{\Lambda_1}\big\{\lambda\, {}^\rho I^{\alpha+\beta} v(\xi) - {}^\rho I^\alpha v(T)\big\}.$$

As in the second step, one can obtain

$$\begin{aligned}|y(t)| &\le {}^\rho I^\alpha |v(T)| + \frac{t^{\rho(\alpha-1)}}{|\Lambda_1|}\big\{|\lambda|\, {}^\rho I^\alpha |v(\xi)| + {}^\rho I^\alpha |v(T)|\big\} \\ &\le \Phi(\|y\|)\bigg({}^\rho I^\alpha p(T) + \frac{T^{\rho(\alpha-1)}}{|\Lambda_1|}\big\{|\lambda|\, {}^\rho I^{\alpha+\beta} p(\xi) + {}^\rho I^\alpha p(T)\big\}\bigg),\end{aligned}$$

which implies that

$$\frac{\|y\|}{\Phi(\|y\|)\bigg({}^\rho I^\alpha p(T) + \dfrac{T^{\rho(\alpha-1)}}{|\Lambda_1|}\big\{|\lambda|\, {}^\rho I^{\alpha+\beta} p(\xi) + {}^\rho I^\alpha p(T)\big\}\bigg)} \le 1.$$

In view of (B_3), there exists $\widehat{K}$ such that $\|y\| \neq \widehat{K}$. Let us set

$$U = \big\{y \in C(I,\mathbb{R}) : \|y\| < \widehat{K}\big\}.$$

Note that the operator $\mathcal{F}\colon \overline{U} \to \mathcal{P}(C(I,\mathbb{R}))$ is a compact multi-valued map, u.s.c. with convex closed values. From the choice of U, there is no $y \in \partial U$ such that $y \in \nu\mathcal{F}(y)$ for some $\nu \in (0,1)$. Consequently, by the nonlinear alternative of Leray–Schauder type (Lemma 4.4), we deduce that $\mathcal{F}$ has a fixed point $y \in \overline{U}$ which is a solution of the problem (1.2) and (1.3). $\square$

4.2. The Lipschitz case. In this subsection we prove the existence of solutions for the problem (1.2) and (1.3) with nonconvex valued right hand side by applying a fixed point theorem for multivalued maps due to Covitz and Nadler [10].

Let (X,d) be a metric space induced from the normed space $(X;\|\cdot\|)$. Consider $H_d\colon \mathcal{P}(X)\times\mathcal{P}(X)\to\mathbb{R}\cup\{\infty\}$ given by

$$H_d(A,B)=\max\Big\{\sup_{a\in A} d(a,B),\sup_{b\in B} d(A,b)\Big\},$$

where $d(A,b)=\inf_{a\in A} d(a;b)$ and $d(a,B)=\inf_{b\in B} d(a;b)$. Then $(\mathcal{P}_{\mathrm{cl,b}}(X),H_d)$ is a metric space (see [18]).

Definition 4.7. A multivalued operator $N\colon X\to\mathcal{P}_{cl}(X)$ is called

(a) γ-Lipschitz if and only if there exists $\gamma>0$ such that

$$H_d(N(x),N(y))\le \gamma\, d(x,y)\quad \text{for each } x,y\in X,$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma<1$.

Lemma 4.8. *([10]) Let (X,d) be a complete metric space. If $N:X\to\mathcal{P}_{cl}(X)$ is a contraction, then $FixN\neq\emptyset$.*

Theorem 4.9. *Assume that*

(C_1) *$F\colon[0,T]\times\mathbb{R}\to\mathcal{P}_{\mathrm{cp}}(\mathbb{R})$ is such that $F(\,\cdot\,,y)\colon[0,T]\to\mathcal{P}_{\mathrm{cp}}(\mathbb{R})$ is measurable for each $y\in\mathbb{R}$,*

(C_2) *$H_d(F(t,y),F(t,\overline{y}))\le m(t)|y-\overline{y}|$ for almost all $t\in[0,T]$ and $y,\overline{y}\in\mathbb{R}$ with $m\in C([0,T],\mathbb{R}^+)$ and $d(0,F(t,0))\le m(t)$ for almost all $t\in[0,T]$.*

Then the problem (1.2) *and* (1.3) *has at least one solution on* $[0,T]$ *if* $\|m\|A_1<1$, *i.e.*

$$\delta:=\|m\|\bigg[\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}+\frac{T^{\rho(\alpha-1)}}{|\Lambda_1|}\bigg\{|\lambda|\frac{\xi^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}+\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\bigg\}\bigg]<1. \tag{4.2}$$

Proof. Notice that the set $S_{F,y}$ is nonempty for each $y\in C([0,T],\mathbb{R})$ by the assumption (C_1). So F has a measurable selection (see [9, Theorem III.6]). Now we show that the operator $\mathcal{F}$ defined by (4.1) satisfies the assumptions of Lemma 4.8. Firstly we show that $\mathcal{F}(y)\in\mathcal{P}_{\mathrm{cl}}((C[0,T],\mathbb{R}))$ for each $y\in C([0,T],\mathbb{R})$. Let $\{u_n\}_{n\ge0}\in\mathcal{F}(y)$ be such that $u_n\to u$ $(n\to\infty)$ in $C([0,T],\mathbb{R})$. Then $u\in C([0,T],\mathbb{R})$ and there exists $v_n\in S_{F,y_n}$ such that, for each $t\in[0,T]$,

$$u_n(t)={}^{\rho}I^{\alpha}v_n(t)+\frac{t^{\rho(\alpha-1)}}{\Lambda_1}\big\{\lambda\,{}^{\rho}I^{\alpha+\beta}v_n(\xi)-{}^{\rho}I^{\alpha}v_n(T)\big\}.$$

As F has compact values, passing to subsequences if necessary, we obtain that v_n converges to v in $L^1([0,T],\mathbb{R})$. Thus, $v \in S_{F,y}$ and for each $t \in [0,T]$, we have

$$u_n(t) \to u(t) = {}^\rho I^\alpha v(t) + \frac{t^{\rho(\alpha-1)}}{\Lambda_1}\big\{\lambda\, {}^\rho I^{\alpha+\beta} v(\xi) - {}^\rho I^\alpha v(T)\big\}.$$

Hence, $u \in \mathcal{F}(y)$.

Next we show that there exists $\delta < 1$ (defined by (4.2)) such that

$$H_d(\mathcal{F}(y), \mathcal{F}(\overline{y})) \le \delta \|y - \overline{y}\| \quad \text{for each } y, \overline{y} \in C([0,T],\mathbb{R}).$$

Let $y, \overline{y} \in C([0,T],\mathbb{R})$ and $h_1 \in \mathcal{F}(y)$. Then there exists $v_1(t) \in F(t, y(t))$ such that, for each $t \in [0,T]$,

$$h_1(t) = {}^\rho I^\alpha v_1(t) + \frac{t^{\rho(\alpha-1)}}{\Lambda_1}\big\{\lambda\, {}^\rho I^{\alpha+\beta} v_1(\xi) - {}^\rho I^\alpha v_1(T)\big\}.$$

By (C_2), we have

$$H_d(F(t,y), F(t,\overline{y})) \le m(t)|y(t) - \overline{y}(t)|.$$

So, there exists $w \in F(t, \overline{y}(t))$ such that

$$|v_1(t) - w| \le m(t)|y(t) - \overline{y}(t)|, \quad t \in [0,T].$$

Define $U\colon [0,T] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \le m(t)|y(t) - \overline{y}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t, \overline{y}(t))$ is measurable ([9, Proposition III.4]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t, \overline{y}(t))$ and for each $t \in [0,T]$, we have $|v_1(t) - v_2(t)| \le m(t)|y(t) - \overline{y}(t)|$.

For each $t \in [0,T]$, let us define

$$h_2(t) = {}^\rho I^\alpha v_2(t) + \frac{t^{\rho(\alpha-1)}}{\Lambda_1}\big\{\lambda\, {}^\rho I^{\alpha+\beta} v_2(\xi) - {}^\rho I^\alpha v_2(T)\big\}.$$

Then

$$\begin{aligned}
|h_1(t) - h_2(t)| \le{}& {}^\rho I^\alpha |v_1(t) - v_2(t)| \\
&+ \frac{T^{\rho(\alpha-1)}}{|\Lambda_1|}\big\{|\lambda|\, {}^\rho I^{\alpha+\beta}|v_1(\xi) - v_2(\xi)| + {}^\rho I^\alpha |v_1(T) - v_2(T)|\big\} \\
\le{}& \|m\| \bigg[\frac{T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \\
&+ \frac{T^{\rho(\alpha-1)}}{|\Lambda_1|}\bigg\{|\lambda| \frac{\xi^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\bigg\}\bigg] \|y - \overline{y}\|.
\end{aligned}$$

Hence

$$\begin{aligned}
\|h_1 - h_2\| \le \|m\| \bigg[&\frac{T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \\
&+ \frac{T^{\rho(\alpha-1)}}{|\Lambda_1|}\bigg\{|\lambda| \frac{\xi^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\bigg\}\bigg] \|y - \overline{y}\|.
\end{aligned}$$

Analogously, interchanging the roles of y and $\overline{y}$, we can obtain

$$H_d(\mathcal{F}(y),\mathcal{F}(\overline{y})) \le \|m\| \bigg[\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{T^{\rho(\alpha-1)}}{|\Lambda_1|} \bigg\{ |\lambda| \frac{\xi^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} \bigg\} \bigg] \|y-\overline{y}\|.$$

So $\mathcal{F}$ is a contraction. Therefore, it follows by Lemma 4.8, that $\mathcal{F}$ has a fixed point y which is a solution of (1.2) and (1.3). □

Example 4.10. Consider the following boundary value problem

$$\begin{cases} {}^{1/2}D^{7/5}y(t) \in F(t,y(t)), & t \in [0,2], \\ y(0)=0, \quad y(2) = 1/3\ {}^{1/2}I^{4/5}y(3/2), \end{cases} \tag{4.3}$$

where $\rho = 1/2$, $\alpha = 7/5$, $\lambda = 1/3$, $\beta = 4/5$, $\xi = 3/2$ and $T = 2$ and $F(t,y(t))$ will be fixed later.

Using the given data, we find that $|\Lambda_1| = 0.5526156533$, $A_1 = 12.67699740$, where Λ_1 and A_1 are respectively given by (2.3) and (3.2). For illustrating Theorem 4.6, we take

$$F(t,y(t)) = \left[\frac{1}{250+e^t}\left(\frac{|y|}{|y|+1}+\frac{1}{2}\right), \frac{(1+t)}{30}\left(\tan^{-1}y + y + \frac{1}{8}\right)\right], \tag{4.4}$$

which, in view of (B_2), implies that $p(t) = (1+t)/30$ and $\Phi(\|y\|) = ||y|| + (4\pi+1)/8$. By condition ($B_3$), it is found that $\widehat{K} > 2.668611421$. Thus all conditions of Theorem 4.6 are satisfied and consequently, there exists at least one solution for the problem (4.3) with $F(t,y(t))$ given by (4.4) on $[0,2]$.

In order to demonstrate the application of Theorem 4.9, let us consider

$$F(t,y(t)) = \left[\frac{\sin t}{250}\left(\frac{|y|}{|y|+1}+1\right), \frac{(t+1)\tan^{-1}y + t}{50}\right]. \tag{4.5}$$

Clearly

$$H_d(F(t,y),F(t,\overline{y})) \le \frac{(t+1)}{50}\|y-\overline{y}\|.$$

Letting $m(t) = (t+1)/50$, it is easy to check that $d(0,F(t,0)) \le m(t)$ holds for almost all $t \in [0,2]$ and that $\delta \approx 0.760619844 < 1$ (δ is given by (4.2)). As the hypotheses of Theorem 4.9 are satisfied, we conclude that the problem (4.3) with $F(t,y(t))$ given by (4.5) has at least one solution on $[0,2]$.

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Sotiris K. Ntouyas
Department of Mathematics
University of Ioannina
451 10 Ioannina, GREECE
and
Nonlinear Analysis and Applied Mathematics
(NAAM)-Research Group
Department of Mathematics
Faculty of Science
King Abdulaziz University
P.O. Box 80203
Jeddah 21589, SAUDI ARABIA

E-mail address: sntouyas@uoi.gr

Bashir Ahmad, Madeaha Alghanmi and Ahmed Alsaedi
Nonlinear Analysis and Applied Mathematics
(NAAM)-Research Group
Department of Mathematics, Faculty of Science
King Abdulaziz University
P.O. Box 80203
Jeddah 21589, SAUDI ARABIA

E-mail address: bashirahmad_qau@yahoo.com
madeaha@hotmail.com
aalsaedi@hotmail.com