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MULTIPLICITY RESULTS FOR FRACTIONAL p-LAPLACIAN PROBLEMS WITH HARDY TERM AND HARDY–SOBOLEV CRITICAL EXPONENT IN $\mathbb{R}^N$

Hadi Mirzaee

Abstract. This paper is devoted to the study of a class of singular fractional p-Laplacian problems of the form

$$(-\Delta)_p^s u - \mu \frac{|u|^{p-2}u}{|x|^{ps}} = \alpha \frac{|u|^{p_s^*(b)-2}u}{|x|^b} + \beta f(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^N$$

where $0 < s < 1$, $0 \le b < ps < N$, $1 < q < p_s^*(b)$, $\alpha, \beta > 0$, $\mu \in \mathbb{R}$, and $f(x)$ is a given function which satisfies some appropriate condition. By using variational methods, we prove the existence of infinitely many solutions under different conditions.

1. Introduction and statement of main result

In this article, we consider the following fractional p-Laplacian equations with Hardy term and Hardy–Sobobev critical exponent:

$$(-\Delta)_p^s u - \mu \frac{|u|^{p-2}u}{|x|^{ps}} = \alpha \frac{|u|^{p_s^*(b)-2}u}{|x|^b} + \beta f(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^N \tag{1.1}$$

where $0 < s < 1$, $0 \le b < ps < N$, $1 < q < p_s^*(b) = p(N-b)/(N-ps)$, $\alpha, \beta > 0$ and $\mu \in \mathbb{R}$. The operator $(-\Delta)_p^s$ is the fractional p-Laplacian, which up to

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normalization factors, may be defined, for $x \in \mathbb{R}^N$, by

$$(-\Delta)_p^s \varphi(x) := 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|\varphi(x) - \varphi(y)|^{p-2}(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}}\, dy, \quad x \in \mathbb{R}^N,$$

along any function $\varphi \in C_0^\infty(\mathbb{R}^N)$, where $B_\varepsilon(x) := \{y \in \mathbb{R}^N : |x - y| < \varepsilon\}$. The fractional p-Laplacian $(-\Delta)_p^s$ reduces to the fractional Laplacian $(-\Delta)^s$ if $p = 2$. For more details on the fractional p-Laplacian, we refer to [7].

In recent years, much attention is given to the study of the nonlocal elliptic problems involving singular nonlinearity (e.g. [9], [2], [3], [16], [8], [15]). Fractional p-Laplacian problems involving Hardy term and critical exponent have been also investigated (e.g. [10], [12], [19], [5]). However, as far as we know, there is no work about multiplicity results for fractional p-Laplacian problems with Hardy term and Hardy–Sobolev critical exponent in unbounded domains. In the present paper, we investigate multiplicity results of solutions for some fractional p-Laplacian problems involving Hardy term and Hardy–Sobolev exponent in $\mathbb{R}^N$. Since we deal with a singular problem in the unbounded domain $\mathbb{R}^N$, the lack of compactness of the Sobolev embedding presents an appropriate variational technique which make the problem more attractive. For this purpose we first need to verify a new version of the Rellich–Kondrachov compactness theorem which has a crucial role in verifying our results. Using variational techniques and the theory of genus we obtain infinitely many solutions under different conditions.

In order to state main results of this paper, we introduce some Sobolev and weighted function spaces. Let $L^q(\mathbb{R}^N; w)$ be the weighted Lebesgue space endowed with the norm

$$\|u\|_{q,w}^q = \int_{\mathbb{R}^N} w(x)|u(x)|^q\, dx.$$

Then it follows from Proposition A.6 of [1] that the Banach space $L^q(\mathbb{R}^N; w) = (L^q(\mathbb{R}^N; w); \|u\|_{q,w})$ is uniformly convex. Let $0 < s < 1 < p < \infty$ be real numbers. The Gagliardo seminorm is defined for all measurable function $u\colon \mathbb{R}^N \to \mathbb{R}$ by

$$[u]_{s,p} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}}\, dx\, dy \right)^{1/p}. \tag{1.2}$$

The fractional Sobolev space is defined as

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}}\, dx\, dy < \infty \right\},$$

equipped with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \left(\|u\|_p^p + [u]_{s,p}^p \right)^{1/p} \tag{1.3}$$

where $\|\cdot\|_p$ denotes the usual L_p norm.

The fractional Sobolev space $D^{s,p}(\mathbb{R}^N) := X$ is defined as the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_X = \|u\|_{D^{s,p}(\mathbb{R}^N)} = \bigg(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\bigg)^{1/p}.$$

From Theorems 1 and 2 of [14] we have

$$\|u\|_{p_s^*}^p \le C_{N,p}\frac{s(1-s)}{(N-ps)^{p-1}}\,[u]_{s,p}^p, \qquad \int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}\,dx \le C_{N,p}\,\frac{s(1-s)}{(N-ps)^{p-1}}[u]_{s,p}^p$$

for all $u \in D^{s,p}(\mathbb{R}^N)$, where $C_{N,p}$ is a positive constant depending only on N and p. As in [5] we introduce the best fractional critical Sobolev and Hardy constant $S = S(N,p,s)$ and $\overline{\mu} = \overline{\mu}(N,p,s)$ given by

$$(1.4)\qquad S = \inf_{u\in D^{s,p}(\mathbb{R}^N)\setminus\{0\}} \frac{[u]_{s,p}^p}{\|u\|_{p_s^*}^p}, \qquad \overline{\mu} = \inf_{u\in D^{s,p}(\mathbb{R}^N)\setminus\{0\}} \frac{[u]_{s,p}^p}{\int_{\mathbb{R}^N}|u(x)|^p/|x|^{ps}\,dx}.$$

We conclude from (1.4) that if $\mu < 0$, then the fractional Sobolev space $D^{s,p}(\mathbb{R}^N)$ has the equivalent norm $\|u\|_\mu$, where

$$\|u\|_\mu^p = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \mu\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}\,dx, \quad \mu \in (-\infty,0).$$

We now establish the following fractional Hardy–Sobolev inequality due to Pucci et al. [10]:

$$(1.5)\qquad H_b\bigg(\int_{\mathbb{R}^N}\frac{u^{p_s^*(b)}}{|x|^b}\,dx\bigg) \le \bigg(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\bigg)^{p_s^*(b)/p}$$

where $0 < b < ps$ and H_b is the best constant in the fractional Hardy–Sobolev inequality. Hence, we can define the following best Sobolev constant:

$$S_\mu = \inf_{u\in D^{s,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \mu\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}\,dx}{\displaystyle\bigg(\int_{\mathbb{R}^N}\frac{|u|^{p_s^*(b)}}{|x|^b}\,dx\bigg)^{p/p_s^*(b)}},$$

for $\mu \in (-\infty,0)$. Throughout this paper, we make the following assumptions on the function $f\colon \mathbb{R}^N \to \mathbb{R}$:

- (f_1) $f(x) > 0$ and $f(x)|x|^{sq} \in L^{\varrho_1}(\mathbb{R}^N) \cap L^\infty_{\rm loc}(\mathbb{R}^N)$ where $\varrho_1 = p/(p-q)$ and $1 < q < p$;
- (f_2) $f(x) > 0$ and $f(x)|x|^{bq/p_s^*(b)} \in L^{\varrho_2}(\mathbb{R}^N) \cap L^\infty_{\rm loc}(\mathbb{R}^N)$ where $\varrho_2 = p_s^*(b)/(p_s^*(b)-q)$ and $p < q < p_s^*(b)$.

Now, we state main results of this paper

THEOREM 1.1. *Suppose $\mu \le 0$, $1 < q < p < p_s^*(b)$, and (f_1) hold. Then*

(a) *for each* $\beta > 0$ *there exists* $\alpha_0 > 0$ *such that if* $0 < \alpha < \alpha_0$, *then* (1.1) *has a sequence of solutions* $\{u_n\}$ *with* $I(u_n) < 0$, *and* $I(u_n) \to 0$ *as* $n \to \infty$, *where* $I\colon X \to \mathbb{R}$ *is the energy functional associated with* (1.1) *and defined in the Section* 2,

(b) *for all* $\alpha > 0$ *there exists* $\beta_0 > 0$ *such that if* $0 < \beta < \beta_0$, *then* (1.1) *has a sequence of solutions* $\{u_n\}$ *with* $I(u_n) < 0$, *and* $I(u_n) \to 0$ *as* $n \to \infty$.

Theorem 1.2. *Suppose* $\alpha = 0$, $0 < \mu < \overline{\mu}$, $1 < p < q < p_s^*(b)$, *and* (f_2) *hold. Then, for each* $\beta > 0$, (1.1) *has a sequence of solutions* $\{u_n\}$, *such that* $I(u_n) \to +\infty$ *as* $n \to \infty$, *where* $I\colon X \to \mathbb{R}$ *is the energy functional associated with this problem and defined in Section* 5.

The rest of this paper is organized as follows. Some compactness results and preliminaries are given in Section 2. In Section 3, we investigate the behaviour of Palais–Smale sequence which can be used in the proof of Theorem 1.1. The proof of Theorem 1.1 is given in Section 4. Finally, in Section 5, we prove Theorem 1.2.

2. Preliminaries

In deriving the following Theorem we have been inspired by [21]. In particular, Theorem 2.1 implies the compact imbedding from the spcae $D^{s,p}(\Omega)$ into some weighted Lebesgue spaces, and gives us a new version of the Rellich–Kondrachov compactness theorem:

Theorem 2.1. *Assume that* $0 < b < ps$, *and that* $\Omega \subset \mathbb{R}^N$ *is an open bounded domain with smooth boundary and* $0 \in \Omega$. *The embedding* $D^{s,p}(\Omega) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$ *is compact if*

$$1 \le r < \frac{p(N-b)}{N-ps}, \qquad \alpha < sr + N\left(1 - \frac{r}{p}\right).$$

Proof. Claim A. There are constants $c_*, c_\vartheta > 0$ such that for each $u \in D^{s,p}(\Omega)$ we have:

$$\int_\Omega |x|^{-\alpha}|u|^r\,dx \le c_*\left(\int_\Omega |x|^{-b}|u|^{p_s^*(b)}\,dx\right)^{r/p_s^*(b)} \le c_*c_\vartheta\big([u]_{s,p}\big)^r. \tag{2.1}$$

Hence, it suffices to prove the compactness part of the theorem. Let $\{u_m\}$ be a bounded sequence in $D^{s,p}(\Omega)$. For any $\eta > 0$, let $B_\eta(0) \subset \Omega$ be a closed ball centered at the origin with radius η. In view of claim (A), $\{u_m\} \subset L^p(\Omega \setminus B_\eta(0))$ is bounded. One can easily see that $\{u_m\} \subset W^{s,p}(\Omega \setminus B_\eta(0))$. Since

$$1 < r < \frac{p(N-b)}{N-ps} < \frac{pN}{N-ps},$$

the Rellich–Kondrachov compactness theorem (see [7]) implies the existence of a convergent subsequence of $\{u_m\}$ in $L^r(\Omega \setminus B_\eta(0))$. By taking a diagonal sequence one may assume, without loss of generality, that $\{u_m\}$ converges in $L^r(\Omega \setminus B_\eta(0))$ for any $\eta > 0$. Since

$$r < q = p_s^*(b) = \frac{p(N-b)}{N-ps},$$

From the Hölder inequality and the fractional Hardy–Sobolev inequality (1.5), for any $\eta > 0$ we have

$$\begin{aligned}(2.2)\quad &\int_{|x|<\eta} |x|^{-\alpha}|u_m - u_j|^r\, dx \\ &\le \left(\int_{|x|<\eta} |x|^{-(\alpha - br/q)(q/(q-r))}\, dx\right)^{(q-r)/q} \left(\int_{|x|<\eta} |x|^{-b}|u_m - u_j|^q\, dx\right)^{r/q} \\ &\le C\left(\int_0^\eta t^{N-1-(\alpha-br/q)(q/(q-r))}\, dt\right)^{(q-r)/q} \\ &= C\eta^{[N-(\alpha-br/q)(q/(q-r))](q-r)/q},\end{aligned}$$

for some constant C independent of m and j. The assumption $\alpha < sr + N(1 - r/p)$ implies that:

$$\begin{aligned}(2.3)\quad N - \left(\alpha - \frac{br}{q}\right)\frac{q}{q-r} &> N - \left(\left(sr + N\left(1 - \frac{r}{p}\right)\right) - \frac{br}{q}\right)\frac{q}{q-r} \\ &= N - \left(\left(sr + N\left(1 - \frac{r}{p}\right)\right) - b + \left(b - \frac{br}{q}\right)\right)\frac{q}{q-r} \\ &= N - \left(\left(sr + N\left(1 - \frac{r}{p}\right)\right) - b\right)\frac{q}{q-r} - b \\ &= N - \left(\left(sr + N\left(1 - \frac{r}{p}\right)\right) - b\right)\frac{p(N-b)}{p(N-b) - rN + rps} - b = 0.\end{aligned}$$

Thus, for a given $\varepsilon > 0$, we can choose $\eta > 0$ such that

$$\int_{|x|<\eta} |x|^{-\alpha}|u_m - u_j|^r\, dx \le \varepsilon \quad \text{for all } m, j \in \mathbb{N}.$$

Now, let $N \in \mathbb{N}$ be such that, for all $m, j \ge \mathbb{N}$,

$$\int_{\Omega\setminus B_\eta(0)} |x|^{-\alpha}|u_m - u_j|^r\, dx \le C_\alpha \int_{\Omega\setminus B_\eta(0)} |u_m - u_j|^r\, dx \le \varepsilon,$$

where $C_\alpha = \eta^{-\alpha}$ for $\alpha \ge 0$ and $C_\alpha = (\operatorname{diam}(\Omega))^{-\alpha}$ for $\alpha < 0$. Thus

$$\int_\Omega |x|^{-\alpha}|u_m - u_j|^r\, dx \le 2\varepsilon \quad \text{for all } m, j \ge \mathbb{N}.$$

Therefore, $\{u_m\}$ is a Cauchy sequence in $L^r(\Omega, |x|^{-\alpha})$. Now, by considering the proof of compactness part of the theorem, one can easily verify Claim A. Hence, the proof of Claim A is omitted. □

Next we prove the following lemma:

Lemma 2.2.

(a) *Assume that* $1<q<p<p_s^*(b)$ *and that* (f_1) *hold. Then the functional*

$$\mathcal{F}(u) := \int_{\mathbb{R}^N} f|u|^q\,dx$$

from X *to* $\mathbb{R}$ *is well defined and weakly continuous.*

(b) *Assume that* $p<q<p_s^*(b)$ *and that* (f_2) *hold. Then the functional*

$$\mathcal{F}(u) := \int_{\mathbb{R}^N} f|u|^q\,dx$$

from X *to* $\mathbb{R}$ *is well defined and weakly continuous.*

Proof. (a) It follows from Hölder inequality and (f_1) that

$$\begin{aligned}\int_{\mathbb{R}^N} f|u|^q\,dx &\le \left(\int_{\mathbb{R}^N} |x|^{-ps}|u|^p\,dx\right)^{q/p} \|f|x|^{qs}\|_{L^{\varrho_1}(\mathbb{R}^N)} \\ &\le \overline{\mu}^{\,-q/p}\|u\|_X^q \|f|x|^{sq}\|_{L^{\varrho_1}(\mathbb{R}^N)}.\end{aligned} \tag{2.4}$$

For any $u\in X$. Hence, in view of (f_1), the functional $\mathcal{F}(u)$ is well defined on X. Note that $f|x|^{sq}\in L^{\varrho_1}(\mathbb{R}^N)$. Thus for any $\varepsilon>0$, there exists $R_0>0$, such that

$$\int_{\mathbb{R}^N\setminus B_{R_0}} (f|x|^{sq})^{\varrho_1}\,dx<\varepsilon \tag{2.5}$$

where $B_r=\{x\in\mathbb{R}^{\mathbb{N}} : |x|\le r\}$ for any $r>0$. Now, assume $u_n\rightharpoonup u$ weakly in X. Hence, $\{u_n\}$ is bounded in X. From (2.4) and (2.5) we deduce there exists $C_3>0$, such that

$$\int_{\mathbb{R}^N\setminus B_{R_0}} f|u_n|^q\,dx<C_3\varepsilon, \qquad \int_{\mathbb{R}^N\setminus B_{R_0}} f|u|^q\,dx<C_3\varepsilon \tag{2.6}$$

for all $n\in\mathbb{N}$. Note that $f|x|^{sq}\in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$. On the other hand, we take $r=q$, $\alpha=sq$ in Theorem 2.1 to obtain that there exists $N_0\in\mathbb{N}$, such that

$$\begin{aligned}&\int_{\mathbb{R}^N\cap B_{R_0}} f\big(|u_n|^q-|u|^q\big)\,dx \\ &\quad\le \|f|x|^{sq}\|_{L^\infty(\mathbb{R}^N\cap B_{R_0})}\left(\int_{(\mathbb{R}^N\cap B_{R_0})} |x|^{-sq}\big(|u_n|^q-|u|^q\big)\,dx\right)\le\varepsilon\end{aligned} \tag{2.7}$$

for all $n>N_0$. Therefore by (2.6) and (2.7),

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f|u_n|^q\,dx=\int_{\mathbb{R}^N} f|u|^q\,dx.$$

This completes the proof of (a).

(b) For any $u\in X$, by Hölder inequality and (f_2), we have that

$$\begin{aligned}\int_{\mathbb{R}^N} f|u|^q\,dx &\le \left(\int_{\mathbb{R}^N} |x|^{-b}|u|^{p_s^*(b)}\,dx\right)^{q/p_s^*(b)} \big\|f|x|^{bq/p_s^*(b)}\big\|_{L^{\varrho_2}(\mathbb{R}^N)} \\ &\le C_1^{q/p}\|u\|_X^q\big\|f|x|^{bq/p_s^*(b)}\big\|_{L^{\varrho_2}(\mathbb{R}^N)}.\end{aligned} \tag{2.8}$$

Thus, by (f_2), $\mathcal{F}(u)$ is well defined on X.

Since $f|x|^{bq/p_s^*(b)} \in L^{\varrho_2}(\mathbb{R}^N)$, for any $\varepsilon > 0$, there exists $R_1 > 0$ such that

$$\int_{\mathbb{R}^N \setminus B_{R_1}} \big(f|x|^{bq/p_s^*(b)}\big)^{\varrho_2}\, dx < \varepsilon. \tag{2.9}$$

Now, assume $u_n \rightharpoonup u$ weakly in X, then $\{u_n\}$ is bounded in X. Thus, (2.8) and (2.9) yield that there exists $C_4 > 0$, such that

$$\int_{\mathbb{R}^N \setminus B_{R_1}} f|u_n|^q dx < C_4\varepsilon, \quad \int_{\mathbb{R}^N \setminus B_{R_1}} f|u|^q dx < C_4\varepsilon \tag{2.10}$$

for all $n \in \mathbb{N}$. On the other hand, we set $\alpha = bq/p_s^*(b)$. Then, since $p < q < p_s^*(b)$, there is $0 < t < 1$ such that $q = tp + (1-t)p_s^*(b)$. Thus we have

$$\begin{aligned} sq + N\bigg(1 - \frac{q}{p}\bigg) &= s(tp + (1-t)p_s^*(b)) + N\bigg(1 - \frac{(tp + (1-t)p_s^*(b))}{p}\bigg) \\ &= t\bigg(sp + N\bigg(1 - \frac{p}{p}\bigg)\bigg) + (1-t)(sp_s^*(b) + N\bigg(1 - \frac{p_s^*(b)}{p}\bigg)\bigg) \\ &= tsp + (1-t)b > b > \frac{bq}{p_s^*(b)} = \alpha. \end{aligned} \tag{2.11}$$

Hence, we can use Theorem 2.1. Therefore, by Theorem 2.1 and (f_2), we deduce that there exists $N_1 \in \mathbb{N}$, such that

$$\begin{aligned} &\int_{\mathbb{R}^N \cap B_{R_1}} f\big(|u_n|^q - |u|^q\big)\, dx \\ &\le \|f|x|^{bq/p_s^*(b)}\|_{L^\infty(\mathbb{R}^N \cap B_{R_1})}\bigg(\int_{(\mathbb{R}^N \cap B_{R_1})} |x|^{-bq/p_s^*(b)}\big(|u_n|^q - |u|^q\big)\bigg) \le \varepsilon \end{aligned} \tag{2.12}$$

for all $n > N_1$. It follows from (2.10) and (2.12) that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f|u_n|^q\, dx = \int_{\mathbb{R}^N} f|u|^q\, dx.$$

This completes the proof of (b). $\square$

The energy functional associated with (1.1) is defined on $X = D^{s,p}(\mathbb{R}^N)$ by

$$\begin{aligned} I(u) = \frac{1}{p}\|u\|_X^p - \frac{\mu}{p}\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}}\, dx \\ - \frac{\alpha}{p_s^*(b)}\int_{\mathbb{R}^N} \frac{|u(x)|^{p_s^*(b)}}{|x|^b}\, dx - \frac{\beta}{q}\int_{\mathbb{R}^N} f(x)|u(x)|^q\, dx \end{aligned} \tag{2.13}$$

Obviously, the functional I is well defined and of class $C^1(D^{s,p}(\mathbb{R}^N))$.

We say that $u \in D^{s,p}(\mathbb{R}^N)$ is a weak solution of (1.1) if $I'(u) = 0$, that is,

$$\begin{aligned} 0 = \langle I'(u), \varphi\rangle = \langle u, \varphi\rangle_{s,p} - \mu\int_{\mathbb{R}^N} \frac{|u(x)|^{p-2}u(x)\varphi(x)}{|x|^{ps}}\, dx \\ - \alpha\int_{\mathbb{R}^N} \frac{|u(x)|^{p_s^*(b)-2}u(x)\varphi(x)}{|x|^b}\, dx - \beta\int_{\mathbb{R}^N} f(x)|u(x)|^{q-2}u(x)\varphi(x)\, dx \end{aligned} \tag{2.14}$$

for all $\varphi \in D^{s,p}(\mathbb{R}^N)$, where

$$\langle u, \varphi\rangle_{s,p} := \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy.$$

Given E a real Banach space and $I \in C^1(E,R)$, we recall that I satisfies the Palais–Smale condition on the level $c \in \mathbb{R}$ denoted by $(\mathrm{PS})_c$, if every sequence $\{u_n\} \subset E$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence.

3. Behavior of (PS) sequences

In this section, we study the behavior of the Palais–Smale sequence and prove some compactness results which will be used in the next section.

Lemma 3.1. *Assume that $1 < q < p < p_s^*(b)$, $\mu \le 0$ and that* (f_1) *holds. Then:*

(a) *for each $\alpha > 0$ there exists $\beta_* > 0$ such that if $0 < \beta < \beta_*$ and $\{u_n\} \subset X$ is a $(\mathrm{PS})_c$-sequence for I with $c < 0$, then $\{u_n\}$ has a convergent subsequence in X;*

(b) *for each $\beta > 0$ there exists $\alpha_* > 0$ such that if $0 < \alpha < \alpha_*$, and $\{u_n\} \subset X$ is a $(\mathrm{PS})_c$-sequence for I with $c < 0$, then $\{u_n\}$ has a convergent subsequence in X.*

Proof. Let $\{u_n\}$ be a sequence in $D^{s,p}(\mathbb{R}^N)$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$. Following the idea used in [18] and [5], we prove (a) and (b). We first prove that $\{u_n\}_n$ is bounded in $D^{s,p}(\mathbb{R}^N)$. We have

$$I(u_n) = \frac{1}{p}\|u_n\|_X^p - \frac{\mu}{p}\int_{\mathbb{R}^N} \frac{|u_n(x)|^p}{|x|^{ps}}\,dx - \frac{\alpha}{p_s^*(b)}\int_{\mathbb{R}^N} \frac{|u_n(x)|^{p_s^*(b)}}{|x|^b}\,dx - \frac{\beta}{q}\int_{\mathbb{R}^N} f(x)|u_n(x)|^q\,dx = c + o_n(1) \tag{3.1}$$

and

$$\begin{aligned} \langle I'(u_n),\varphi\rangle &= \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy \\ &\quad - \mu\int_{\mathbb{R}^N} \frac{|u_n(x)|^{p-2}u_n(x)\varphi(x)}{|x|^{ps}}\,dx \\ &\quad - \alpha\int_{\mathbb{R}^N} \frac{|u_n(x)|^{p_s^*(b)-2}u_n(x)\varphi(x)}{|x|^b}\,dx \\ &\quad - \beta\int_{\mathbb{R}^N} f(x)|u_n(x)|^{q-2}u_n(x)\varphi(x)\,dx = o_n(1) \end{aligned} \tag{3.2}$$

for any $\varphi \in D^{s,p}(\mathbb{R}^N)$.

By (3.1) and (3.2), we obtain

$$\begin{aligned}(3.3)\quad c+o_n(1)(\|u_n\|_X+1) &\geq I(u_n)-\frac{1}{p_s^*(b)}\langle I'(u_n),u_n\rangle \\ &\geq \left(\frac{1}{p}-\frac{1}{p_s^*(b)}\right)\|u_n\|_\mu^p-\beta\left(\frac{1}{q}-\frac{1}{p_s^*(b)}\right)(\overline{\mu})^{-q/p}\|u_n\|_X^q\,\|f|x|^{sq}\|_{L^{\varrho_1}(\mathbb{R}^N)}.\end{aligned}$$

Thus $\{u_n\}$ is bounded in X. Hence, following an arguments similar to Lemma 2.1 of [5], we can assume, going if necessary to a subsequence,

$$(3.4)\quad \begin{aligned} &u_n \rightharpoonup u \quad \text{in } X, && \|u_n\|_X\to\eta,\\ &u_n \rightharpoonup u \quad \text{in } L^{p_s^*(b)}(\mathbb{R}^N;|x|^{-b}), && \|u_n-u\|_{p_s^*(b),|x|^{-b}}\to\xi\\ &u_n \rightharpoonup u \quad \text{in } L^{p}(\mathbb{R}^N;|x|^{-ps}), && \|u_n-u\|_{p,|x|^{-ps}}\to\tau,\\ &u_n \to u \quad \text{in } L^{q}(\mathbb{R}^N,f) && u_n(x)\to u(x)\quad \text{a.e. in } \mathbb{R}^N.\end{aligned}$$

Then, in view of the proof of Lemma 2.4 of [5], the sequence $\{\mathcal{U}_n\}_n$, defined in $\mathbb{R}^{2N}\setminus \operatorname{Diag}\mathbb{R}^{2N}$ by

$$(x,y)\mapsto \mathcal{U}_n(x,y):=\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{(N+ps)/p'}}$$

is bounded in $L^{p'}(\mathbb{R}^{2N})$ and $\mathcal{U}_n\to\mathcal{U}$ almost everywhere in $\mathbb{R}^{2N}$, where

$$(x,y)\mapsto \mathcal{U}(x,y):=\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{(N+ps)/p'}}.$$

Hence, going if necessary to a subsequence, we obtain $\mathcal{U}_n\rightharpoonup\mathcal{U}$ in $L^{p'}(\mathbb{R}^{2N})$, and thus

$$(3.5)\qquad \langle u_n,\varphi\rangle_{s,p}\to\langle u,\varphi\rangle_{s,p}\quad \text{as } n\to\infty$$

for any $\varphi\in X$ because $|\varphi(x)-\varphi(y)|\cdot|x-y|^{-(N+ps)/p}\in L^p(\mathbb{R}^{2N})$. From, (3.4) and Proposition A.8 of [1] we conclude that $|u_n|^{q-2}u_n\rightharpoonup|u|^{q-2}u$ in $L^{q'}(\mathbb{R}^N,f)$, and $|u_n|^{p-2}u_n\rightharpoonup|u|^{p-2}u$ in $L^{p'}(\mathbb{R}^N,|x|^{-ps})$, and $|u_n|^{p_s^*(b)-2}u_n\rightharpoonup|u|^{p_s^*(b)-2}u$ in the space $L^{p_s^*(b)'}(\mathbb{R}^N,|x|^{-b})$, consequently

$$(3.6)\qquad \int_{\mathbb{R}^N}\frac{|u_n|^{p-2}u_n\varphi}{|x|^{ps}}\,dx\to\int_{\mathbb{R}^N}\frac{|u(x)|^{p-2}u(x)\varphi}{|x|^{ps}}\,dx,$$

$$(3.7)\qquad \int_{\mathbb{R}^N}\frac{|u_n|^{p_s^*(b)-2}u_n\varphi}{|x|^{b}}\,dx\to\int_{\mathbb{R}^N}\frac{|u|^{p_s^*(b)-2}u\varphi}{|x|^{b}}\,dx,$$

$$(3.8)\qquad \int_{\mathbb{R}^N}f(x)|u_n|^{q-2}u_n\varphi\,dx\to\int_{\mathbb{R}^N}f(x)|u|^{q-2}u\varphi\,dx,\quad \text{as } n\to\infty,$$

for any $\varphi \in X$. Recall that $\{u_n\}_n$ satisfies the Palais–Smale condition. Therefore, for any $\varphi \in X$, by (3.5), (3.6), (3.7) and (3.8) we obtain

$$\begin{aligned}(3.9)\quad \langle u,\varphi\rangle_{s,p} = \mu\int_{\mathbb{R}^N}\frac{|u(x)|^{p-2}u(x)\varphi(x)}{|x|^{ps}}\,dx + \alpha\int_{\mathbb{R}^N}\frac{|u(x)|^{p_s^*(b)-2}u(x)\varphi(x)}{|x|^{b}}\,dx \\ + \beta\int_{\mathbb{R}^N} f(x)|u(x)|^{q-2}u(x)\varphi(x)\,dx.\end{aligned}$$

Thus, u is a critical point of the I. In view of (3.4) we deduce that

$$(3.10)\qquad \int_{\mathbb{R}^N} f(x)(|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n-u)\,dx \to 0 \quad \text{as } n\to\infty.$$

Now we apply the Brézis–Lieb lemma [4] to obtain that

$$(3.11)\qquad \begin{aligned}\|u_n\|_X^p &= \|u_n-u\|_X^p + \|u\|_X^p + o_n(1),\\ \|u_n\|_{p,|x|^{-ps}}^p &= \|u_n-u\|_{p,|x|^{-ps}}^p + \|u\|_{p,|x|^{-ps}}^p + o_n(1),\end{aligned}$$

and

$$(3.12)\qquad \|u_n\|_{p_s^*(b),|x|^{-b}}^{p_s^*(b)} = \|u_n-u\|_{p_s^*(b),|x|^{-b}}^{p_s^*(b)} + \|u\|_{p_s^*(b),|x|^{-b}}^{p_s^*(b)} + o_n(1).$$

Since $\{u_n\}_n$ satisfies the Palais–Smale condition, it follows from (3.4), (3.9), (3.10) and (3.11)

$$\begin{aligned}(3.13)\quad o_n(1) &= \langle I'(u_n) - I'(u), u_n - u\rangle\\ &= \|u_n\|_X^p + \|u\|_X^p - \langle u_n,u\rangle_{s,p} - \langle u,u_n\rangle_{s,p}\\ &\quad - \mu\int_{\mathbb{R}^N}\frac{(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)}{|x|^{ps}}\,dx\\ &\quad - \alpha\int_{\mathbb{R}^N}\frac{(|u_n|^{p_s^*(b)-2}u_n - |u|^{p_s^*(b)-2}u)(u_n-u)}{|x|^{b}}\,dx\\ &\quad - \beta\int_{\mathbb{R}^N} f(x)\big(|u_n|^{q-2}u_n - |u|^{q-2}u\big)(u_n-u)\,dx\\ &= \big(\eta^p - \|u\|_X^p\big) - \alpha\|u_n\|_{p_s^*(b),|x|^{-b}}^{p_s^*(b)} + \alpha\|u\|_{p_s^*(b),|x|^{-b}}^{p_s^*(b)}\\ &\quad - \mu\|u_n\|_{p,|x|^{-ps}}^p + \mu\|u\|_{p,|x|^{-ps}}^p + o_n(1)\\ &= \|u_n-u\|_X^p - \alpha\|u_n-u\|_{p_s^*(b),|x|^{-b}}^{p_s^*(b)} - \mu\|u_n-u\|_{p,|x|^{-ps}}^p + o_n(1).\end{aligned}$$

Thus

$$(3.14)\qquad \|u_n-u\|_X^p - \mu\|u_n-u\|_{p,|x|^{-ps}}^p = \alpha\|u_n-u\|_{p_s^*(b),|x|^{-b}}^{p_s^*(b)} + o_n(1).$$

From (3.14) we obtain

$$(3.15)\qquad \xi^{p_s^*(b)} \ge \alpha^{-1}S_\mu\xi^p.$$

When $\xi = 0$, because $\alpha > 0$, it follows from (3.14) that $\|u_n-u\|_\mu \to 0$ and, in view of the fact that $\mu \le 0$, we deduce $\|u_n-u\|_X \to 0$. Therefore, let us assume by contradiction that $\xi > 0$. Thus

$$\xi \geq (\alpha^{-1} S_\mu)^{1/(p_s^*(b)-p)}. \tag{3.16}$$

We claim that (3.16) cannot occur if α and β are chosen properly. Otherwise, by Lemma 2.2, we have

$$\begin{aligned} 0 > c &= \lim_{n\to\infty} \left(I(u_n) - \frac{1}{p_s^*(b)} \langle I'(u_n), u_n \rangle \right) \\ &= \lim_{n\to\infty} \left(\left(\frac{1}{p} - \frac{1}{p_s^*(b)} \right) \|u_n\|_\mu^p - \beta \left(\frac{1}{q} - \frac{1}{p_s^*(b)} \right) \int_{\mathbb{R}^N} f(x)|u_n|^q \, dx \right) \\ &\geq \left(\frac{1}{p} - \frac{1}{p_s^*(b)} \right) \|u\|_\mu^p - \beta \left(\frac{1}{q} - \frac{1}{p_s^*(b)} \right) (\overline{\mu})^{-q/p} \|u\|_X^q \|f|x|^{sq}\|_{L^{\varrho_1}(\mathbb{R}^N)}. \end{aligned} \tag{3.17}$$

Therefore there exists $C_2 > 0$, independent of the choice of the $(\mathrm{PS})_c$ sequence $\{u_n\}$, such that $\|u\|_X^q \leq C_2 \beta^{q/(p-q)}$. Thus by considering equation (3.12), we may have

$$\begin{aligned} 0 > c &= \lim_{n\to\infty} \left(I(u_n) - \frac{1}{p_s^*(b)} \langle I'(u_n), u_n \rangle \right) \\ &\geq \lim_{n\to\infty} \left(\frac{1}{p} - \frac{1}{p_s^*(b)} \right) \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \, dy \, dx - \mu \int_{\mathbb{R}^N} \frac{|u_n(x)|^p}{|x|^{ps}} \, dx \right) \\ &\quad - \beta \left(\frac{1}{q} - \frac{1}{p_s^*(b)} \right) (\overline{\mu})^{-q/p} \|u\|_X^q \|f|x|^{sq}\|_{L^{\varrho_1}(\mathbb{R}^N)} \\ &\geq S_\mu \lim_{n\to\infty} \left(\frac{1}{p} - \frac{1}{p_s^*(b)} \right) \left(\int_{\mathbb{R}^N} \frac{|u_n(x)|^{p_s^*(b)}}{|x|^b} \, dx \right)^{p/p_s^*(b)} - C_3 \beta^{p/(p-q)} \\ &\geq \left(\frac{1}{p} - \frac{1}{p_s^*(b)} \right) S_\mu \big(\alpha^{-1} S_\mu \big)^{p/(p_s^*(b)-p)} - C_3 \beta^{p/(p-q)}, \end{aligned}$$

where

$$C_3 = C_2 \left(\frac{1}{q} - \frac{1}{p_s^*(b)} \right) (\overline{\mu})^{-q/p} \|f|x|^{sq}\|_{L^{\varrho_1}(\mathbb{R}^N)}.$$

Therefore

$$0 > \left(\frac{1}{p} - \frac{1}{p_s^*(b)} \right) S_\mu \big(\alpha^{-1} S_\mu \big)^{p/(p_s^*(b)-p)} - C_3 \beta^{p/(p-q)}$$

and C_3 is independent of the choice of the $(\mathrm{PS})_c$ sequence $\{u_n\}$. Now, we can choose β_* so small that if $0 < \beta < \beta_*$, then the term on the right hand side above is greater than zero, which is a contradiction. Similarly, we can choose α_* so small that if $0 < \alpha < \alpha_*$, then the term on the right hand side above is greater than zero. Hence, we have shown that $\xi = 0$, which is a contradiction, and thus $\|u_n - u\|_X \to 0$. □

4. Proof of Theorem 1.1

In this section, we will use minimax procedure to prove the existence of infinitely many solutions of problem 1.1. Let E be a Banach space, we denote

$$\Sigma = \{A \subset E \setminus \{0\} : \\ A \text{ is closed in } E \text{ and symmetric with respect to the origin}\}$$

For $A \in \Sigma$, we define $\gamma(A)$ as

$$\gamma(A) = \inf\{m \in \mathbb{N} : \exists\, \varphi \in C(A, \mathbb{R}^m \setminus \{0\}), \varphi(-x) = -\varphi(x)\}.$$

If there is no mapping as above for any $m \in \mathbb{N}$, then $\gamma(A) = \infty$. We list the following main properties of the genus (cf. [13]).

Proposition 4.1. *Let $A, B \in \Sigma$. Then:*

(a) *If there exists an odd map $g \in C(A, B)$ then $\gamma(A) \le \gamma(B)$;*
(b) *If $A \subset B$, then $\gamma(A) \le \gamma(B)$;*
(c) $\gamma(A \cup B) \le \gamma(A) + \gamma(B)$;
(d) *If $\gamma(B) < \infty$, then $\gamma(\overline{A \setminus B}) \ge \gamma(A) - \gamma(B)$;*
(e) *n-dimensional sphere S_n has a genus of $n+1$ by the Borsuk–Ulam Theorem;*
(f) *If A is compact, then $\gamma(A) < +\infty$ and there exists $\delta > 0$ such that*

$$N_\delta(A) \subset \Sigma \quad \text{and} \quad \gamma(N_\delta(A)) = \gamma(A),$$

here $N_\delta(A) = \{x \in E : \operatorname{dist}(x, A) \le \delta\}$.

Let $I(u)$ be the functional defined as before. Assume $0 < q < p < p_s^*(b)$, $\alpha, \beta > 0$, $\mu \le 0$. Then we have

$$\begin{aligned} I(u) &= \frac{1}{p}\,\|u\|_\mu^p - \frac{\alpha}{p_s^*(b)} \int_{\mathbb{R}^N} \frac{|u(x)|^{p_s^*(b)}}{|x|^b}\,dx - \frac{\beta}{q} \int_{\mathbb{R}^N} f(x)|u(x)|^q\,dx \\ &\ge \frac{1}{p}\,\|u\|_\mu^p - \alpha C_3 \|u\|_\mu^{p_s^*(b)} - \beta C_4 \|u\|_\mu^q \end{aligned} \tag{4.1}$$

for some positive constants C_3 and C_4. Using the same idea as in [11], we define

$$Q(t) = \frac{1}{p}\,t^p - \alpha C_3 t^{p_s^*(b)} - \beta C_4 t^q.$$

Then, we have $I(u) \ge Q(\|u\|_\mu)$. If $q < p < p_s^*(b)$, then we have $\lim_{t\to+\infty} Q(t) = -\infty$. Thus I is not bounded from below. It is easy to see that, given $\beta > 0$, there exists $\alpha_1 > 0$ so small that for every $0 < \alpha < \alpha_1$, there exist $0 < t_0 < t_1$ such that $Q(t) < 0$ for $0 < t < t_0$, $Q(t) > 0$ for $t_0 < t < t_1$ and $Q(t) < 0$ for $t > t_1$. Similarly, given $\alpha > 0$, we can choose $\beta_1 > 0$ with the property that t_0,

t_1 as above exist for $0 < \beta < \beta_1$. Then, following the same idea as in [11], we define the following auxiliary functional on X by

$$\widetilde{I}(u) = \frac{1}{p}\,\|u\|_\mu^p - \frac{\alpha}{p_s^*(b)}\,\psi(u)\int_{\mathbb{R}^N} \frac{|u(x)|^{p_s^*(b)}}{|x|^b}\,dx - \frac{\beta}{q}\int_{\mathbb{R}^N} f(x)|u(x)|^q\,dx, \tag{4.2}$$

where $\psi(u) = \tau(\|u\|_\mu)$ and $\tau\colon \mathbb{R}^+ \to [0,1]$ is a non-increasing C^∞ function such that $\tau(t) = 1$ if $t \le t_0$ and $\tau(t) = 0$ if $t \ge t_1$. Obviously, $\widetilde{I}(u)$ is coercive on X and even. Therefore, by considering Lemma 3.1, one can easily verify the following propositions:

PROPOSITION 4.2. *Assume that $\alpha_1 > 0$ is as above. Then we have*:

(a) *If $\widetilde{I}(u) < 0$, then $\|u\|_\mu < t_0$ and $\widetilde{I}(u) = I(u)$;*
(b) *for each $\beta > 0$ there exists $0 < \overline{\alpha} < \alpha_1$ such that if $0 < \alpha < \overline{\alpha}$ and $c < 0$, then $\widetilde{I}$ satisfies $(PS)_c$ condition.*

PROPOSITION 4.3. *Assume that $\beta_1 > 0$ is as above. Then we have*:

(a) *If $\widetilde{I}(u) < 0$, then $\|u\|_\mu < t_0$ and $\widetilde{I}(u) = I(u)$;*
(b) *for each $\alpha > 0$ there exists $0 < \overline{\beta} < \beta_1$ such that if $0 < \beta < \overline{\beta}$ and $c < 0$, then $\widetilde{I}$ satisfies $(\mathrm{PS})_c$ condition.*

Now, we prove the following lemma:

LEMMA 4.4. *Denote $\widetilde{I}^c := \{u \in X : \widetilde{I}(u) \le c\}$. Given $m \in \mathbb{N}$, there exists $\varepsilon_m < 0$, such that $\gamma(\widetilde{I}^{\varepsilon_m}) \ge m$.*

PROOF. Let X_m be a m-dimensional subspace of X. For any $u \in X_m \setminus \{0\}$, write $u = r_m w$ with $\|w\|_\mu = 1$ and $r_m = \|u\|_\mu$. Then, there exists $d_m > 0$ such that, for every $w \in X_m$ with $\|w\|_\mu = 1$, we have

$$\int_{\mathbb{R}^N} f(x)|w|^q\,dx \ge d_m > 0. \tag{4.3}$$

Thus, for $0 < r_m < t_0$, we have

$$\begin{aligned}\widetilde{I}(u) = I(u) = \frac{1}{p}\,\|u\|_\mu^p - \frac{\alpha}{p_s^*(b)}\int_{\mathbb{R}^N} \frac{|u(x)|^{p_s^*(b)}}{|x|^b}\,dx - \frac{\beta}{q}\int_{\mathbb{R}^N} f(x)|u(x)|^q\,dx \\ \le \frac{1}{p}\,r_m^p - \beta\frac{d_m}{q}\,r_m^q := \varepsilon_m.\end{aligned}$$

Hence, we can choose $r_m \in (0, t_0)$ so small that $\widetilde{I}(u) \le \varepsilon_m < 0$. Let

$$S_{r_m} = \{u \in X : \|u\|_\mu = r_m\}. \tag{4.4}$$

Then $S_{r_m} \cap X_m \subset \widetilde{I}^{\varepsilon_m}$. Thus, it follows from Proposition 4.1 that

$$\gamma(\widetilde{I}^{\varepsilon_m}) \ge \gamma(S_{r_m} \cap X_m) = m.$$

□

Now, we denote

$$K_c = \{u \in X : \widetilde{I}'(u) = 0,\ \widetilde{I}(u) = c\},$$
$$\Sigma_m = \{A \in \Sigma : \gamma(A) \ge m\} \quad \text{and} \quad c_m = \inf_{A\in\Sigma_m} \sup_{u\in A} \widetilde{I}(u).$$

Since $\widetilde{I}^{\varepsilon_m} \in \Sigma_m$ and $\widetilde{I}$ is bounded from below, we conclude that

$$-\infty < c_m \le \varepsilon_m < 0, \quad m \in \mathbb{N}. \tag{4.5}$$

Next we are going to prove the following lemma.

Lemma 4.5. *Assume that α and β are as in Proposition 4.2. Then all c_m are critical values of $\widetilde{I}$ and $c_m \to 0$ as $m \to \infty$.*

Proof. Since $\Sigma_{m+1} \subset \Sigma_m$, we deduce that $c_m \le c_{m+1}$ and from (4.5) it follows that $c_m < 0$. Hence there is a $\overline{c} \le 0$ such that $c_m \to \overline{c} \le 0$, as $m \to +\infty$. Also, because $(\mathrm{PS})_c$ is satisfied, it follows from a standard argument (see [17]) that all c_m are critical values of $\widetilde{I}$. We claim that $\overline{c} = 0$. If $\overline{c} < 0$, then by proposition 4.2, $K_{\overline{c}} = \{u \in X : \widetilde{I}'(u) = 0,\ \widetilde{I}(u) = \overline{c}\}$ is compact and $K_{\overline{c}} \in \Sigma$. From Proposition 4.1 we obtain that, there exists $\delta > 0$ such that $\gamma(K_{\overline{c}}) = \gamma(N_\delta(K_{\overline{c}})) = m_0 < +\infty$. By the deformation lemma (see [20]), there exist $\varepsilon > 0$ ($\overline{c} + \varepsilon < 0$) and an odd homeomorphism $\eta\colon X \to X$ such that

$$\eta\big(\widetilde{I}^{\overline{c}+\varepsilon} \setminus N_\delta(K_{\overline{c}})\big) \subset \widetilde{I}^{\overline{c}-\varepsilon}.$$

Because $\{c_m\}$ is increasing and converges to $\overline{c}$, there exists $m \in \mathbb{N}$ such that $c_m > \overline{c} - \varepsilon$ and $c_{m+m_0} \le \overline{c}$. Choose $A \in \Sigma_{m+m_0}$ such that $\sup_{u\in A} \widetilde{I}(u) < \overline{c} + \varepsilon$, that is $A \subset \widetilde{I}^{\overline{c}+\varepsilon}$. By the properties of γ, we have

$$\gamma\big(\overline{A \setminus N_\delta(K_{\overline{c}})}\big) \ge \gamma(A) - \gamma(N_\delta(K_{\overline{c}}))) \ge m, \qquad \gamma\big(\overline{\eta(A \setminus N_\delta(K_{\overline{c}}))}\big) \ge m.$$

Hence, we have $\overline{\eta(A \setminus N_\delta(K_{\overline{c}}))} \in \Sigma_m$. Consequently,

$$\sup_{u\in\overline{\eta(A\setminus N_\delta(K_{\overline{c}}))}} \widetilde{I}(u) \ge c_m > \overline{c} - \varepsilon,$$

a contradiction, hence $c_m \to 0$, as $m \to +\infty$. □

Proof of Theorem 1.1. Note that $\widetilde{I}(u) = I(u)$ if $\widetilde{I}(u) < 0$. Thus, combining Propositions 4.2, 4.3 and Lemmas 4.4, 4.5, we obtain the desired result. □

5. Hardy critical case

Consider the following equation

$$(-\Delta)^s_p u - \mu \frac{|u|^{p_s^*(c)-2}u}{|x|^c} = \alpha \frac{|u|^{p_s^*(b)-2}u}{|x|^b} + \beta f(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \tag{5.1}$$

where $c, b \le ps$ and $0 < \mu < \overline{\mu}$. In this section, we study the Hardy critical case. More precisely, we consider the case of $c = ps$ in the above equation, and also we assume that $\alpha = 0$. In the other words, we investigate existence and multiplicity results of solutions for the following equation with Hardy term

$$(-\Delta)_p^s u - \mu \frac{|u|^{p-2}u}{|x|^{ps}} = \beta f(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \tag{5.2}$$

where $0 < \mu < \overline{\mu}$ and $p < q < p_s^*(b)$. The energy functional associated with (5.2) is defined by

$$I(u) = \frac{1}{p}\|u\|_X^p - \frac{\mu}{p}\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}}\,dx - \frac{\beta}{q}\int_{\mathbb{R}^N} f(x)|u(x)|^q\,dx. \tag{5.3}$$

In the sequel, we will need the following lemma (see [17, Theorem 9.12]).

LEMMA 5.1. *Let I be an even C^1-functional satisfying the* (PS)*-condition on a Banach space $X = Y \oplus Z$ with $\dim(Y) < \infty$. Assume $I(0) = 0$, as well as the following conditions*:

(a) *There are constants $\rho, \delta > 0$ such that $\inf\limits_{S_\rho(Z)} I \geq \delta$;*
(b) *For any finite dimensional subspace $\overline{Y} \subset X$, there is $R = R(\overline{Y})$ such that $I \leq 0$ on $\overline{Y} \setminus B_R(\overline{Y})$.*

Then I has an unbounded sequence of critical values.

As in previous sections, firstly we prove the following lemma

LEMMA 5.2. *Assume $\beta > 0$, $1 < p < q < p_s^*(b)$, $0 < \mu < \overline{\mu}$ and* (f_2) *hold. Let $\{u_n\} \subset X$ be a* $(\mathrm{PS})_c$*-sequence for I where $c \in \mathbb{R}$. Then $\{u_n\}$ has a convergent subsequence in X.*

PROOF. Let $\{u_n\}$ be a sequence in X such that $I(u_n) \to c$, $I'(u_n) \to 0$. We first prove that $\{u_n\}$ is bounded in $D^{s,p}(\mathbb{R}^N)$. We have

$$I(u_n) = \frac{1}{p}\|u_n\|_X^p - \frac{\mu}{p}\int_{\mathbb{R}^N} \frac{|u_n(x)|^p}{|x|^{ps}}\,dx - \frac{\beta}{q}\int_{\mathbb{R}^N} f(x)|u_n(x)|^q\,dx = c + o_n(1) \tag{5.4}$$

and

$$\begin{aligned}\langle I'(u_n), \varphi\rangle &= \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+ps}}\,dx\,dy \\ &\quad - \mu\int_{\mathbb{R}^N} \frac{|u_n(x)|^{p-2}u_n(x)\varphi(x)}{|x|^{ps}}\,dx \\ &\quad - \beta\int_{\mathbb{R}^N} f(x)|u_n(x)|^{q-2}u_n(x)\varphi(x)\,dx = o_n(1),\end{aligned} \tag{5.5}$$

for any $\varphi \in D^{s,p}(\mathbb{R}^N)$. For n large enough, by (5.4) and (5.5), we obtain that

$$|c| + o_n(1)(\|u_n\|_X + 1) \geq I(u_n) - \frac{1}{q}\langle I'(u_n), u_n\rangle \geq \left(\frac{1}{p} - \frac{1}{q}\right)\left(1 - \frac{\mu}{\overline{\mu}}\right)\|u_n\|_X^p. \tag{5.6}$$

Thus $\{u_n\}$ is bounded in X. Therefore we can assume, going if necessary to a subsequence,

$$
\begin{aligned}
&u_n \rightharpoonup u \quad \text{in } X, && \|u_n\|_X \to \eta,\\
&u_n \rightharpoonup u \quad \text{in } L^p\big(\mathbb{R}^N; |x|^{-ps}\big), && \|u_n - u\|_{p,|x|^{-ps}} \to \varsigma,\\
&u_n \to u \quad \text{in } L^q\big(\mathbb{R}^N, f\big), && u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^N.
\end{aligned} \tag{5.7}
$$

Then, by an argument similar to the proof of Lemma 3.1, we deduce

$$\langle u_n, \varphi\rangle_{s,p} \to \langle u, \varphi\rangle_{s,p} \quad \text{as } n \to \infty \tag{5.8}$$

and

$$\int_{\mathbb{R}^N} \frac{|u_n|^{p-2}u_n\varphi}{|x|^{ps}}\,dx \to \int_{\mathbb{R}^N} \frac{|u(x)|^{p-2}u(x)\varphi}{|x|^{ps}}\,dx, \tag{5.9}$$

$$\int_{\mathbb{R}^N} f(x)|u_n|^{q-2}u_n\varphi\,dx \to \int_{\mathbb{R}^N} f(x)|u|^{q-2}u\varphi\,dx \quad \text{as } n \to \infty, \tag{5.10}$$

for any $\varphi \in X$. Thus, by (5.8)–(5.10) we obtain

$$
\begin{aligned}
\langle u, \varphi\rangle_{s,p} = \mu \int_{\mathbb{R}^N} \frac{|u(x)|^{p-2}u(x)\varphi(x)}{|x|^{ps}}\,dx&\\
+ \beta \int_{\mathbb{R}^N} f(x)|u(x)|^{q-2}u(x)\varphi(x)\,dx&
\end{aligned} \tag{5.11}
$$

for any $\varphi \in X$. Thus u is a critical point of the I. From (5.7) we deduce that

$$\int_{\mathbb{R}^N} f(x)\big(|u_n|^{q-2}u_n - |u|^{q-2}u\big)(u_n - u)\,dx \to 0 \quad \text{as } n \to \infty. \tag{5.12}$$

Now we apply the Brézis–Lieb lemma [4] to obtain that

$$\|u_n\|_X^p = \|u_n - u\|_X^p + \|u\|_X^p + o_n(1), \tag{5.13}$$

and

$$\|u_n\|_{p,|x|^{-ps}}^p = \|u_n - u\|_{p,|x|^{-ps}}^p + \|u\|_{p,|x|^{-ps}}^p + o_n(1). \tag{5.14}$$

Since $\{u_n\}_n$ satisfies the Palais–Smale condition, by (3.4), (3.9), (3.10), and (3.11) we get

$$
\begin{aligned}
o_n(1) &= \langle I'(u_n) - I'(u), u_n - u\rangle\\
&= \|u_n\|_X^p + \|u\|_X^p - \langle u_n, u\rangle_{s,p} - \langle u, u_n\rangle_{s,p}\\
&\quad - \mu \int_{\mathbb{R}^N} \frac{\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)(u_n - u)}{|x|^{ps}}\,dx\\
&\quad - \beta \int_{\mathbb{R}^N} f(x)\big(|u_n|^{q-2}u_n - |u|^{q-2}u\big)(u_n - u)\,dx\\
&= (\eta^p - \|u\|_X^p) - \mu\|u_n\|_{p,|x|^{-ps}}^p + \mu\|u\|_{p,|x|^{-ps}}^p + o_n(1)\\
&= \|u_n - u\|_X^p - \mu\|u_n - u\|_{p,|x|^{-ps}}^p + o_n(1).
\end{aligned} \tag{5.15}
$$

Thus

$$\|u_n - u\|_X^p = \mu \|u_n - u\|_{p,|x|^{-ps}}^p + o_n(1). \tag{5.16}$$

From this we obtain

$$\varsigma^p \geq \frac{\overline{\mu}}{\mu} \varsigma^p. \tag{5.17}$$

It follows from $\overline{\mu} > \mu > 0$ that $\varsigma = 0$. Hence, from (5.16) we conclude that $\|u_n - u\|_X \to 0$. □

Proof of Theorem 1.2. We apply Lemma 5.1 to obtain the desired result. We prove that the functional I satisfies conditions (a) and (b) of Lemma 5.1.

Let V be a nontrivial finite dimensional subspace of X and Z be the complemented subspace of V in X. For each $u \in Z$, $u \neq 0$, $R > 0$,

$$\begin{aligned} I(Ru) =& \frac{R^p}{p}\left(\|u\|_X^p - \mu \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}}\, dx\right) - \frac{\beta}{q}\int_{\mathbb{R}^N} f(x)|Ru|^q\, dx \\ \geq& \frac{R^p}{p}\left(1 - \frac{\mu}{\overline{\mu}}\right)\|u\|_X^p - \frac{\beta}{q}\int_{\mathbb{R}^N} f(x)|Ru|^q\, dx \\ \geq& \frac{R^p}{p}\left(1 - \frac{\mu}{\overline{\mu}}\right)\|u\|_X^p - \frac{\beta}{q} R^q \|f|x|^{bq/p_s^*(b)}\|_{L^{\varrho_2}(\mathbb{R}^N)}\|u\|_X^q. \end{aligned} \tag{5.18}$$

Thus, the functional I satisfies condition (a).

Now, let X_m be an arbitrary m-dimensional subspace of X. Then, similar to the proof of Lemma 4.4, there exists $d_m' > 0$ such that, for every $w \in X_m$ with $\|w\|_X = 1$, we have

$$\int_{\mathbb{R}^N} f(x)|w|^q \mathrm{d}x \geq d_m' > 0. \tag{5.19}$$

Now, suppose $u \in X_m$, $\|u\|_X = 1$ and $R > 0$. Thus we get

$$I(Ru) = \frac{R^p}{p}\left(\|u\|_X^p - \mu \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}}\, dx\right) - \frac{\beta}{q}\int_{\mathbb{R}^N} f(x)|Ru|^q\, dx \tag{5.20}$$

$$\leq \frac{R^p}{p}\|u\|_X^p - \frac{\beta}{q}\int_{\mathbb{R}^N} f(x)|Ru|^q\, dx \leq \frac{R^p}{p} - \frac{\beta d_m'}{q} R^q \tag{5.21}$$

choosing R large enough, we conclude that functional I satisfies condition (b). Hence, in view of Lemma 5.1, we have proved that (5.2) has a unbounded sequence of critical values. Hence (5.2) has a sequence of solutions $\{u_n\}$, such that $I(u_n) \to \infty$ as $n \to \infty$. Now, we show more precisely that $I(u_n) \to +\infty$ as $n \to \infty$. Let $I(u_n) = c_n$ for any $n \in \mathbb{N}$. Any u_n is a critical point of the functional I. Hence $I'(u_n)u_n = 0$ for each $n \in \mathbb{N}$. From this we conclude that

$$c_n = I(u_n) = \left(\frac{1}{p} - \frac{1}{q}\right)\left(\|u\|_X^p - \mu \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}}\, dx\right).$$

Since $q > p$, we deduce $c_n \to +\infty$ as $n \to \infty$. □

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HADI MIRZAEE
Faculty of Mathematical and Computer Science
Kharazmi University
50 Taleghani Avenue
15618 Tehran, IRAN
E-mail address: hadimirzaee.278@gmail.com